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On Definability of the Equality in Classes of Algebras with an Equivalence Relation*

Abstract. We present a finitary regularly algebraizable logic not finitely equivalential, for every similarity type. We associate to each of these logics a class of algebras with an equivalence relation, with the property that in this class, the identity is atomically definable but not finitely atomically definable.

Keywords: equivalential logics, algebraic logic, model theory.

Equivalential and finitely equivalential logics were introduced by Prucnal and Wroński in [14] and they were studied in depth by Czelakowski in [4]. They are the first class of logics considered in the literature to which the Lindembaum-Tarski method can be applied in a generalized form: the congruence associated to a theory is defined by a possibly infinite set of formulas in two variables instead of a single formula. The finitely equivalential logics being the ones for which a finite such set exists. This congruence turns out to be what is nowadays known as the Leibniz congruence of the theory.

The class of equivalential logics is one of the important classes in the well know hierarchy of logics defined according to the behaviour of the Leibniz operator that assigns, for every logic, to each theory its Leibniz congruence. In general, the Leibniz congruence of a theory T is the greatest congruence of the formula algebra that does not relate elements in T with elements not in T. The main classes considered from this point of view, apart from the class of equivalential logics, are the class of protoalgebraic logics, introduced in [1], the class of weakly algebraizable logics, studied in [6], and the class of algebraizable logics, introduced in [2].

Inside the class of equivalential logics Czelakowski investigated in [4] the equivalential logics with an algebraic semantics in the sense of Suszko, which are the equivalential logics that have an strongly adequate semantics whose members are matrices of the form $\langle \mathcal{A}, D \rangle$ where D is a singleton. These logics can equivalently be characterized as the equivalential logics that in addition satisfy the rule $p, q \vdash \Delta(p, q)$, where $\Delta(p, q)$ is the set of formulas that defines the Leibniz congruences. The implicative logics of Rasiowa,

* This work is partially supported by Spanish DGICYT grant PB94-0854.

Presented by **Jacek Malinowski**; *Received* May 19, 1999

Studia Logica **64**: 345–353, 2000.

introduced in [15], belong to that class. The equivalential logics with an algebraic semantics are known also as regularly algebraizable logics, see [5].

One interesting question is if there are finitary regularly algebraizable logics which are not finitely equivalential, that is, whose Leibniz congruences are definable by an infinite set of formulas in two variables but can not be defined by any finite such set. Herrmann formulated in his Ph.D. dissertation explicitly this problem as an open problem (Problem 3.18), calling the regularly algebraizable logics 1-equivalential.

In [7] an example of a finitary regularly algebraizable that is not finitely equivalential was presented in the similarity type $\{\leftrightarrow\}$. The purpose of this paper is twofold. On the one hand, to show that this logic is not an isolated case, on the contrary we can find a logic with these properties for arbitrary finite similarity types. On the other hand, to point out that, apart from its interest for algebraic logic, the study of the logics introduced contributes to the model-theoretic study of the definability of the identity relation in first-order logic. We can associate in a very natural way to each of these logics a class of algebras with an equivalence relation with the following interesting property: in this class the identity is atomically definable but not finitely atomically definable.

1. A generalization: the logics $\vdash_{D_\tau}$

In this section we will construct a finitary regularly algebraizable logic that is not finitely equivalential, for arbitrary similarity types. Let us start with some notation and definitions. Given an algebraic similarity type τ, the propositional language L_τ of type τ is constructed from a countable set of variables $\mathrm{Var} = \{p_0, q_0, \ldots\}$ by the connectives of τ in the usual way. By convention $p = p_0$ and $q = q_0$. By $\Delta(p_1, \ldots, p_n)$ we denote a set of formulas with at most the variables $p_1, \ldots, p_n$. Given a set of formulas $\Delta(p, q)$ of L_τ, an algebra of type τ, $\mathcal{A}$, and elements $c, d \in A$, let $\Delta^{\mathcal{A}}(c, d) = \{\varphi^{\mathcal{A}}(c, d) : \varphi \in \Delta\}$.

A *matrix* is a pair $\mathfrak{M} = (\mathcal{A}, F)$, where $\mathcal{A}$ is an algebra and $F \subseteq A$. Given a congruence θ of the algebra $\mathcal{A}$, it is said that θ is *compatible with* F if for any $\langle a, b\rangle \in \theta$, $a \in F$ iff $b \in F$. And it is said that θ *is a congruence of the matrix* $\mathfrak{M}$ if θ is compatible with F. Given a logic $\vdash$ and a matrix $\mathfrak{M} = (\mathcal{A}, F)$, it is said that $\mathfrak{M}$ is a $\vdash$-*matrix* if for any $\Gamma \cup \{\varphi\} \subseteq L_\tau$ and any interpretation $h \in \mathrm{Hom}(L_\tau, \mathcal{A})$, if $\Gamma \vdash \varphi$ and $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$. We assume that a logic is not necessarily finitary. It is a well known fact that given a matrix $\mathfrak{M}$, there is a greatest congruence of $\mathfrak{M}$, this congruence is called *the Leibniz congruence* of $\mathfrak{M}$ and it is denoted by $\boldsymbol{\Omega}_{\mathcal{A}}(F)$. The Leibniz congruence can be characterized in the following way (see [3], Th. 3.2): If $a, b \in A$, then $\langle a, b\rangle \in \mathbf{\Omega}_{\mathcal{A}}(F)$ if and only if $\forall \varphi(p, \vec{q}) \in L_\tau$, $\forall \vec{c} \in A^k$, $\varphi^{\mathcal{A}}(a, \vec{c}) \in F \Leftrightarrow \varphi^{\mathcal{A}}(b, \vec{c}) \in F$, where k is the length of the sequence of variables $\vec{q}$.

It is said that a set $\Delta(p, q)$ of formulas is an *equivalence in a logic* $\vdash$ if it satisfies the following rule schemes:

(R) $$\vdash \Delta(p, p),$$

(MP) $$p, \Delta(p, q) \vdash q.$$

(SR) For any k-adic function symbol $f \in \tau$,

$$\Delta(p_1, q_1), \ldots, \Delta(p_k, q_k) \vdash \Delta(f(p_1, \ldots, p_k), f(q_1, \ldots, q_k)).$$

We have the following characterization of an equivalence in a logic: A set $\Delta(p, q)$ of formulas is an equivalence in a logic $\vdash$ iff for every $\vdash$-matrix $\mathfrak{M} = (\mathcal{A}, F)$, the following condition holds: $\langle a, b\rangle \in \mathbf{\Omega}_{\mathcal{A}}(F)$ iff $\Delta^{\mathcal{A}}(a, b) \subseteq F$.

A logic $\vdash$ is *(finitely) equivalential* if there is some (finite) equivalence in $\vdash$ and $\vdash$ is *regularly algebraizable* if there is a nonempty equivalence $\Delta(p, q)$ in $\vdash$ such that

(G) $$p, q \vdash \Delta(p, q).$$

We give now a measure of the complexity of a formula of L_τ. We assign to any $\psi \in L_\tau$, a natural number, $\mathrm{rk}(\psi)$, the rank of ψ. Given a similarity type τ, the set L_τ can be defined in the following way:

$$L_\tau = \bigcup_{n \in \omega} L_\tau^n$$

where $L_\tau^0 = \mathrm{Var}$ and for any $n \in \omega$,

$$L_\tau^{n+1} = L_\tau^n \cup \{f(t_1, \ldots, t_k) : t_1, \ldots, t_k \in L_\tau^n, f \in \tau, k\text{-adic function symbol}\}.$$

we define *the rank of* ψ, $\mathrm{rk}(\psi)$, as the least $n \in \omega$ such that $\psi \in L_\tau^n$.

REMARK 1.1. Given a finite algebraic similarity type τ and a finite set of variables $X \subseteq \mathrm{Var}$, for any $n \in \omega$, the set of formulas of L_τ in the variables X with rank $\leq n$ is finite.

Let τ be a finite algebraic similarity type, $\tau \neq \emptyset$. Let $\tau' = \tau \cup \{\leftrightarrow\}$, where $\leftrightarrow \notin \tau$ is a binary function symbol. For any $n \in \omega$, $\Delta_n(p, q)$ is the following set of formulas

$$\Delta_n(p, q) = \{\varphi(p) \leftrightarrow \varphi(q) : \varphi(p) \in L_\tau(p), \mathrm{rk}(\varphi(p)) \leq n\},$$

where $\varphi(q)$ is obtained from $\varphi(p)$ by substituting the variable q for the variable p. Let

$$\Delta(p, q) = \bigcup_{n \in \omega} \Delta_n(p, q).$$

The logic $\vdash_{D_\tau}$ has τ' as set of connectives and is axiomatized by:

(1) $$\vdash_{D_\tau} p \leftrightarrow p.$$

(2) $$p, p \leftrightarrow q \vdash_{D_\tau} q.$$

(3) For any $\varphi(p) \in L_\tau(p)$,

$$p, q \vdash_{D_\tau} \varphi(p) \leftrightarrow \varphi(q).$$

(4) For any $\varphi(p) \in L_\tau(p)$,

$$p_1 \leftrightarrow q_1, p_2 \leftrightarrow q_2 \vdash_{D_\tau} \varphi(p_1 \leftrightarrow p_2) \leftrightarrow \varphi(q_1 \leftrightarrow q_2).$$

(5) For any $n, k \in \omega$, any $\varphi(p) \in L_\tau(p)$ with $\mathrm{rk}(\varphi(p)) = n$ and any k-adic function symbol $f \in \tau$,

$$\Delta_{n+1}(p_1, q_1), \ldots, \Delta_{n+1}(p_k, q_k) \vdash_{D_\tau} \varphi(f(p_1, \ldots, p_k)) \leftrightarrow \varphi(f(q_1, \ldots, q_k)).$$

By Remark 1.1 it is easy to see that $\vdash_{D_\tau}$ is finitary. Now we will show that $\Delta(p, q)$ is an equivalence in $\vdash_{D_\tau}$. By (1), since $p \leftrightarrow p \in \Delta(p, p)$, we have (R). By (2), since $\Delta(p, q) \vdash_{D_\tau} p \leftrightarrow q$, we have (MP). Let us see that (SR) holds. On the one hand we have

$$\Delta(p_1, q_1), \Delta(p_2, q_2) \vdash_{D_\tau} p_1 \leftrightarrow q_1, p_2 \leftrightarrow q_2,$$

therefore, by (4), for any $\varphi(p) \in L_\tau(p)$,

$$\Delta(p_1, q_1), \Delta(p_2, q_2) \vdash_{D_\tau} \varphi(p_1 \leftrightarrow p_2) \leftrightarrow \varphi(q_1 \leftrightarrow q_2).$$

Then,

$$\Delta(p_1, q_1), \Delta(p_2, q_2) \vdash_{D_\tau} \Delta((p_1 \leftrightarrow p_2), (q_1 \leftrightarrow q_2)).$$

And, on the other hand, by (5), for any $n, k \in \omega$, any $\varphi(p) \in L_\tau(p)$ with $\mathrm{rk}(\varphi(p)) = n$ and any k-adic function symbol $f \in \tau$,

$$\Delta_{n+1}(p_1, q_1), \ldots, \Delta_{n+1}(p_k, q_k) \vdash_{D_\tau} \varphi(f(p_1, \ldots, p_k)) \leftrightarrow \varphi(f(q_1, \ldots, q_k)),$$

therefore,

$$\Delta(p_1, q_1), \ldots, \Delta(p_k, q_k) \vdash_{D_\tau} \varphi(f(p_1, \ldots, p_k)) \leftrightarrow \varphi(f(q_1, \ldots, q_k)).$$

Consequently,

$$\Delta(p_1, q_1), \ldots, \Delta(p_k, q_k) \vdash_{D_\tau} \Delta(f(p_1, \ldots, p_k), f(q_1, \ldots, q_k)).$$

Thus (SR) holds. Hence Δ is a nonempty equivalence in $\vdash_{D_\tau}$. Moreover, by (3), Δ satisfies the property (G). Thus $\vdash_{D_\tau}$ is regularly algebraizable. Let us show that $\vdash_{D_\tau}$ is not finitely equivalential. Suppose that $\Gamma \subseteq \Delta$ is finite and let m be the least number such that $\Gamma \subseteq \Delta_m(p, q)$. Now we construct in a similar way as in [7], a $\vdash_{D_\tau}$-matrix $\mathfrak{M} = (\mathcal{A}, F)$ and we find elements $c, d \in A$ such that $\Gamma^{\mathcal{A}}(c,d) \subseteq F$ but $\langle c, d\rangle \notin \Omega_{\mathcal{A}}(F)$. Thus, we can conclude that Γ is not an equivalence in $\vdash_{D_\tau}$.

Let us consider the matrix $\mathfrak{N} = (A, f^{\mathfrak{N}}, \leftrightarrow^{\mathfrak{N}}, F)_{f\in\tau}$, where $A = \omega \times \omega$ and $F = \{\langle 1, 0\rangle\}$. In order to define the interpretation of $\leftrightarrow$ in $\mathfrak{N}$, we first define the following equivalence relation, R, on A:

$$R = \mathrm{Id_A} \cup \{\langle\langle i, j\rangle, \langle l, m\rangle\rangle : i = l, i < j, l < m\},$$

and then we define

$$\leftrightarrow^{\mathfrak{N}} (x, y) = \begin{cases} \langle 1, 0\rangle & \text{if } \langle x, y\rangle \in R, \\ \langle 0, 0\rangle & \text{otherwise,} \end{cases}$$

for any $x, y \in A$. And for any k-adic function symbol $f \in \tau$, any $\langle i_1, j_1\rangle$, $\ldots$, $\langle i_k, j_k\rangle \in A$,

$$f^{\mathfrak{N}}(\langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle) = \langle i_l + 1, j_l\rangle,$$

where l is the least subindex such that $i_l = \max\{i_1, \ldots, i_k\}$. First we prove that $\mathfrak{N}$ is a $\vdash_{D_\tau}$-matrix. We will use the following fact that is easy to prove by induction.

REMARK 1.2. For any $n \in \omega$, any $\varphi(p) \in L_\tau(p)$ with $\mathrm{rk}(\varphi(p)) = n$ and any $\langle i, j\rangle \in A$, $\varphi^{\mathfrak{N}}(\langle i, j\rangle) = \langle i + n, j\rangle$.

It is routine to check that $\mathfrak{N}$ is a model of (1)–(4), let us see now that $\mathfrak{N}$ is also a model of (5). Let $n, k \in \omega$, $\varphi(p) \in L_\tau(p)$ with $\mathrm{rk}(\varphi(p)) = n$ and $f \in \tau$ a k-adic function symbol. Suppose that $\langle i_1, j_1\rangle$, $\ldots$, $\langle i_k, j_k\rangle$, $\langle i'_1, j'_1\rangle$, $\ldots$, $\langle i'_k, j'_k\rangle \in A$ and

$$\Delta^{\mathfrak{N}}_{n+1}(\langle i_1, j_1\rangle, \langle i'_1, j'_1\rangle), \ldots, \Delta^{\mathfrak{N}}_{n+1}(\langle i_k, j_k\rangle, \langle i'_k, j'_k\rangle) \in F. \tag{a}$$

Let $a = f^{\mathfrak{N}}(\langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle)$ and $b = f^{\mathfrak{N}}(\langle i'_1, j'_1\rangle, \ldots, \langle i'_k, j'_k\rangle)$, we have to show that $\langle \varphi^{\mathfrak{N}}(a), \varphi^{\mathfrak{N}}(b)\rangle \in R$. In order to prove that we choose $\psi(p) \in L_\tau(p)$ with $\mathrm{rk}(\psi(p)) = n + 1$. Therefore, by (a), for any $1 \le l \le k$,

$$\langle \psi^{\mathfrak{N}}(\langle i_l, j_l\rangle), \psi^{\mathfrak{N}}(\langle i'_l, j'_l\rangle)\rangle \in R.$$

By Remark 1.2,

$$\psi^{\mathfrak{N}}(\langle i_l, j_l\rangle) = \langle i_l + n + 1, j_l\rangle$$

and

$$\psi^{\mathfrak{N}}(\langle i'_l, j'_l\rangle) = \langle i'_l + n + 1, j'_l\rangle.$$

Consequently, we have that

$$\langle\langle i_l + n + 1, j_l\rangle, \langle i'_l + n + 1, j'_l\rangle\rangle \in R \tag{b}$$

and thus, $i_l = i'_l$. By definition of $f^{\mathfrak{N}}$,

$$a = f^{\mathfrak{N}}(\langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle) = \langle i_l + 1, j_l\rangle,$$

where l is the least subindex such that $i_l = \max\{i_1, \ldots, i_k\}$ and

$$b = f^{\mathfrak{N}}(\langle i'_1, j'_1\rangle, \ldots, \langle i'_k, j'_k\rangle) = \langle i'_l + 1, j'_l\rangle.$$

Hence, by Remark 1.2, since $\mathrm{rk}(\varphi(p)) = n$, we have $\varphi^{\mathfrak{N}}(a) = \langle i_l + n + 1, j_l\rangle$ and $\varphi^{\mathfrak{N}}(b) = \langle i'_l + n + 1, j'_l\rangle$. Consequently, by (b), $\langle\varphi^{\mathfrak{N}}(a), \varphi^{\mathfrak{N}}(b)\rangle \in R$. We can conclude that $\mathfrak{N}$ is a model of (5).

Let $\mathcal{A} = (A, f^{\mathfrak{N}}, \leftrightarrow^{\mathfrak{N}})_{f\in\tau}$ be the algebraic reduct of $\mathfrak{N}$ and let $c = \langle 0, m+1\rangle$ and $d = \langle 0, m+2\rangle$, where m is the least number such that $\Gamma \subseteq \Delta_m(p, q)$. Observe that, by Remark 1.2 and the definition of R, for any $\varphi(p) \in L_\tau(p)$, with $\mathrm{rk}(\varphi) \le m$,

$$(\varphi(p) \leftrightarrow \varphi(q))^{\mathcal{A}}(\langle 0, m+1\rangle, \langle 0, m+2\rangle) \in F,$$

and therefore, $\Gamma^{\mathcal{A}}(c, d) \subseteq F$. But for any $\varphi(p) \in L_\tau(p)$, with $\mathrm{rk}(\varphi) > m$,

$$(\varphi(p) \leftrightarrow \varphi(q))^{\mathcal{A}}(\langle 0, m+1\rangle, \langle 0, m+2\rangle) \notin F.$$

Then $\langle c, d\rangle \notin \Omega_{\mathcal{A}}(F)$, so Γ is not an equivalence in $\vdash_{D_\tau}$ and hence $\vdash_{D_\tau}$ is not finitely equivalential.

2. Leibniz congruences and definability of the identity

Let $\mathcal{M}$ be a first-order structure of type τ with domain A. The *Leibniz congruence* of $\mathcal{M}$ is defined as the following binary relation $\boldsymbol{\Omega}(\mathcal{M})$ on A: if $a, b \in A$, then $\langle a, b\rangle \in \boldsymbol{\Omega}(\mathcal{M})$ iff for any atomic equality-free first-order formula $\varphi(p, q_1, \ldots, q_n)$ of type τ, any $d_1, \ldots, d_n \in A$,

$$\mathcal{M} \vDash \varphi[a, d_1, \ldots, d_n] \quad \text{iff} \quad \mathcal{M} \vDash \varphi[b, d_1, \ldots, d_n].$$

This notion is a generalization of the notion of Leibniz congruence of a matrix, introduced in the previous section. Given a class $\boldsymbol{K}$ of first-order structures and a set $\Phi(p, q)$ of first-order formulas, it is said that *Leibniz congruences are definable in* $\boldsymbol{K}$, if for any $\mathcal{M} \in \boldsymbol{K}$,

$$\boldsymbol{\Omega}(\mathcal{M}) = \{\langle a, b\rangle : \mathcal{M} \vDash \Phi[a, b]\},$$

And it is said that *Leibniz congruences are atomically definable in* $\boldsymbol{K}$ in case that all the formulas in Φ are atomic.

Now, given an algebraic similarity type τ and one binary relation symbol E, consider the class $\boldsymbol{K}$ of all $\tau \cup \{E\}$-structures where E is interpreted as an equivalence relation. It is proved in [10] that Leibniz congruences are definable in $\boldsymbol{K}$. However, in general, Leibniz congruences are not atomically definable. If τ contains only unary function symbols, Leibniz congruences are atomically definable in $\boldsymbol{K}$ by the following set of formulas

$$\Phi(p,q) = \{Ef^n(p)f^n(q) : f \text{ in } \tau,\ n \geq 0\},$$

where $f^0(p) = p$ and $f^{n+1}(p) = f(f^n(p))$. But, if there are function symbols in τ which are not unary, it was proved in [10] that, in $\boldsymbol{K}$, Leibniz congruences are not atomically definable.

It is easy to construct trivial classes of algebras with an equivalence relation in which we can define Leibniz congruences using only a finite number of atomic formulas. For example, let $\boldsymbol{S}$ be any class of algebras of an arbitrary type τ and take

$$\boldsymbol{S}' = \{(\mathcal{A}, \mathrm{Id}_A) : \mathcal{A} \in \boldsymbol{S}\}$$

and $\Phi(p,q) = \{Epq\}$. Now the following natural question arises:

> Question: For arbitrary similarity types, are there classes of algebras with an equivalence relation in which Leibniz congruences are atomically definable but not finitely atomically definable?

We can place this question in a broader context, asking for the existence of certain classes of structures in which the identity relation is atomically definable but not by a finite set of formulas. The reason of that, as it is pointed out in [10], is the following:

Remark 2.1. Leibniz congruences are definable in a class $\boldsymbol{K}$ by a set of first-order formulas $\Phi(p,q)$ if and only if the identity relation is definable in $\boldsymbol{K}^* = \{\mathcal{M}^* : \mathcal{M} \in \boldsymbol{K}\}$ by $\Phi(p,q)$, where $\mathcal{M}^*$ is the quotient structure $\mathcal{M}/\boldsymbol{\Omega}(\mathcal{M})$.

Now we prove the existence of this kind of structures. Let τ be a finite nonempty algebraic similarity type and E a binary relation symbol. Let $\tau' = \tau \cup \{\leftrightarrow\}$, where $\leftrightarrow$ is a binary function symbol, $\leftrightarrow \notin \tau$. Given a matrix $\mathfrak{M} = (\mathcal{A}, F)$, where $\mathcal{A} = (A, f^{\mathcal{A}}, \leftrightarrow^{\mathcal{A}})_{f\in\tau}$, we can define the $\tau \cup \{E\}$-structure $\mathcal{M} = (A, f^{\mathcal{A}}, E^{\mathcal{M}})_{f\in\tau}$, where

$$\langle a,b\rangle \in E^{\mathcal{M}} \quad \text{iff} \quad \leftrightarrow^{\mathcal{A}} (a,b) \in F, \tag{c}$$

for all $a, b \in A$. Let $\mathbf{K} = \{\mathcal{M} : \mathfrak{M} \text{ is a matrix model of } \vdash_{D_\tau}\}$. From (1)–(3) of the definition of $\vdash_{D_\tau}$ it follows that in any $\mathcal{M} \in \mathbf{K}$, E is interpreted as an equivalence relation.

Let $\Phi(p, q) = \{Et(p)t(q) : t \text{ is a term of } \tau\}$. We will prove that $\Phi(p, q)$ defines Leibniz congruences in $\mathbf{K}$. In order to do that we show that for any $\mathcal{M} \in \mathbf{K}$ and $a, b \in A$,

$$\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{M}) \quad \text{iff} \quad \mathfrak{M} \vDash \Phi[a, b].$$

By definition of Leibniz congruence, it is clear that if $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{M})$, then $\mathfrak{M} \vDash \Phi[a, b]$. Assume now that $\mathfrak{M} \vDash \Phi[a, b]$. Let $\Delta(p, q)$ be as in the previous section. By (c), we have $\Delta^{\mathcal{A}}(a, b) \subseteq F$. Since $\Delta(p, q)$ is an equivalence in $\vdash_{D_\tau}$, for any term $t(p, q_1, \dots, q_n)$ of τ and any $\bar{d} = d_1, \dots, d_n \in A$,

$$\Delta^{\mathcal{A}}(t^{\mathcal{A}}(a, \bar{d}), t^{\mathcal{A}}(b, \bar{d})) \subseteq F$$

and then by (c),

$$\langle t^{\mathcal{A}}(a, \bar{d}), t^{\mathcal{A}}(b, \bar{d})\rangle \in E^{\mathcal{M}}. \tag{d}$$

Let now φ be an atomic equality-free formula of $\tau \cup \{E\}$. It is of the form Et_1t_2, where t_1 and t_2 are terms of τ. Assume that the variables of t_1 and t_2 are among $p, q_1, \dots, q_n$. Then, for any $\bar{d} = d_1, \dots, d_n \in A$,

$$\mathcal{M} \vDash Et_1t_2[a, \bar{d}] \quad \text{iff} \quad \langle t_1^{\mathcal{A}}(a, \bar{d}), t_2^{\mathcal{A}}(a, \bar{d})\rangle \in E^{\mathcal{M}}.$$

By (d), $\langle t_1^{\mathcal{A}}(a, \bar{d}), t_1^{\mathcal{A}}(b, \bar{d})\rangle \in E^{\mathcal{M}}$ and $\langle t_2^{\mathcal{A}}(a, \bar{d}), t_2^{\mathcal{A}}(b, \bar{d})\rangle \in E^{\mathcal{M}}$. Consequently

$$\langle t_1^{\mathcal{A}}(a, \bar{d}), t_2^{\mathcal{A}}(a, \bar{d})\rangle \in E^{\mathcal{M}} \quad \text{iff} \quad \langle t_1^{\mathcal{A}}(b, \bar{d}), t_2^{\mathcal{A}}(b, \bar{d})\rangle \in E^{\mathcal{M}}.$$

Hence

$$\mathcal{M} \vDash Et_1t_2[a, \bar{d}] \quad \text{iff} \quad \mathcal{M} \vDash Et_1t_2[b, \bar{d}].$$

Therefore $\langle a, b\rangle \in \mathbf{\Omega}(\mathfrak{M})$.

We can conclude that $\Phi(p, q)$ defines Leibniz congruences in $\mathbf{K}$. Using the structure $\mathfrak{M} = (\mathcal{A}, F)$ and the elements $c, d \in A$ of the counterexample of the previous section it is easy to see that Leibniz congruences are not finitely atomically definable in $\mathbf{K}$. Finally, by Remark 2.1, the identity relation is atomically definable in $\mathbf{K}^*$ but not by means of a finite set of formulas.

Acknowledgments. The author wishes to thank Ramon Jansana for his commentaries on a preliminary version of this paper.

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