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THE THEORY OF MODULES OF SEPARABLY CLOSED FIELDS 1

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Abstract. We consider separably closed fields of characteristic $p > 0$ and fixed imperfection degree as modules over a skew polynomial ring. We axiomatize the corresponding theory and we show that it is complete and that it admits quantifier elimination in the usual module language augmented with additive functions which are the analog of the p-component functions.

§1. Introduction. We will denote by $SCF_{p,e}$ the first-order theory of separably closed fields of characteristic $p > 0$ and imperfection degree $e \in \omega \cup \{\infty\}$ in the language of fields. Y. Ershov showed that $SCF_{p,e}$ is complete (see [5]). Moreover, whenever e is finite, if one adds new constants for the elements of a chosen p-basis and the p^e unary functions sending an element to its p-components over this basis, one gets quantifier elimination in this extended language (see for instance [4], Proposition 27).

Let K be a model of $SCF_{p,e}$; we will consider additive reducts of this field. This kind of structures was first considered in the light of Zil'ber conjecture on the possible geometries on strongly minimal sets. In the case where K is algebraically closed and $f(x)$ is any polynomial with coefficients in K, Gary Martin proved that either $f(x)$ is affine or multiplication is definable in an expansion of $(K, +, f)$ by finitely many scalar multiplications (see Theorem 3.7 in [7]). (A polynomial is *additive* if it is a linear combination of X^q's, where q is a power of p and it is affine if it differs by a constant from an additive polynomial.) This specific result is of interest to us because its proof uses only the fact that K is separably closed.

Here, we investigate the first-order theory of K regarded as a module over the skew polynomial ring $R = \mathbb{F}_p(B)[t; \alpha]$ (see [2]), where B is a p-basis of K (hence a set of cardinality e), $\mathbb{F}_p(B)$ is the fraction field of the polynomial ring $\mathbb{F}_p[B]$ and α the Frobenius map (i.e., p^{th} power). Note that, in the context of modules, the cardinality of B is important, also when infinite: it defines the language in which we work. We extract a series of properties of these fields when viewed as R-modules and we show that the corresponding theory of modules over R is model-complete (and decidable). In the resulting structures, the decomposition of an element over the p-basis can be expressed and we will extend the ordinary language of modules with the analog of the p-component functions. Let T_e denotes the

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theory in this expanded language of the structures we are considering. The analysis proceeds first in investigating the torsion part, which comes down to the question whether we can describe in that weaker language the behaviour of roots of additive polynomials. Second, we show that any positive primitive formula is equivalent to a positive quantifier-free formula. Then we show that the index of one p.p. definable subgroup in another is either 1 or ∞ in any torsion-free summand of a model of our theory. This suffices to prove on one hand that T_e admits quantifier elimination, and on the other hand that it is the theory of separably closed fields of characteristic p and imperfection degree e in that weaker language.

In a second article (see [3]), we will show that (as in the field case [11]) for $e > 0$ T_e is non-superstable (though stable) and we will give a partial description of the closed subset of the Ziegler spectrum corresponding to this theory. Then we will characterize the types as certain submodules, we will identify the types of (finite) U-rank and we will show that we don't have elimination of imaginaries but p.p. elimination of imaginaries.

§2. Preliminaries.

Fix a prime number p and a cardinal number e.

DEFINITION 1. Let $\mathbb{F}_p(B)$ be the fraction field of the polynomial ring $\mathbb{F}_p[B]$, where $B = \{b_i : i \in e\}$. Let $R = \mathbb{F}_p(B)[t; \alpha]$ be the skew polynomial ring in t over $\mathbb{F}_p(B)$ associated with the Frobenius map α (see [2]). The domain of R is the set of polynomials in t of the form $q(t) = t^n.a_n + \cdots + t.a_1 + a_0$, where the a_i's belong to $\mathbb{F}_p(B)$, with multiplication defined by the commutation rule $a.t = t.a^{\alpha}$, for any $a \in \mathbb{F}_p(B)$. If a_n is different from zero, we call n the degree $deg(q)$ of $q(t)$ and q is called monic if a_n equals 1. We also set $deg(0) = -\infty$. We have the usual rule $deg(q.r) = deg(q) + deg(r)$, and so the right Euclidean division algorithm holds.

Substituting x^{p^i} in $q(t)$ for t^i for all i (see Remark 1, item 3), the resulting ordinary polynomials are exactly the additive polynomials. They have been first considered by O. Ore who called them p-polynomials (see [8]). Equipped with addition and composition, the set of additive polynomials forms a ring. As noted by Ore, an ordinary polynomial having only simple roots is an additive polynomial iff its roots form an $\mathbb{F}_p$-vector space (see Theorem 8, p. 565 in [8]). This entails that any polynomial divides (in the ordinary sense) some additive polynomial (see Chapter 3, p. 581 in [8]).

Let us recall a few properties of this ring, which can be found in Chapter 2 of [2]. By Proposition 2.11 in [2], R is an integral domain where the right ideals are principal. The proof is straightforward using the right Euclidean division algorithm. This entails that this ring is right Ore (see Corollary 1.3.7 in [2]), but whenever $e \neq 0$, it is not left Ore i.e., there exist two non zero elements x_1 and x_2 such that $R.x_1 \cap R.x_2 = \{0\}$. (For instance take the elements t and $t.b$, for some $b \in B$.) So, it contains the free algebra over $\mathbb{F}_p$ on two generators x_1 and x_2 (see Proposition 1.6.6 in [2]). We don't know whether the theory T_R of all right R-modules is undecidable.

§3. Axiomatization. Finite imperfection degree.

For S a ring, the language of right S-modules is $\mathscr{L}_S = \{+, -, 0, \cdot r : r \in S\}$, where for any $r \in S$ and x in a right

S-module, $x \cdot r$ denotes the scalar multiplication of x by the ring element r. Let T_S be the theory of all right S-modules in this language.

We extend the usual R-module language by adding new unary functions which will be existentially $\mathscr{L}_R$-definable in the theory we will be considering (in case e is finite, they will be definable by a positive primitive formula).

Let e be finite and let $B = \{b_0, \dots, b_{e-1}\}$.

DEFINITION 2. Let $\mathscr{L} = \mathscr{L}_R \cup \{\lambda_i : i \in p^e\}$, where the λ_i's are unary functions.

Given $i \in p^e$, let m_i be the monomial $b_0^{i(0)} \cdot \dots \cdot b_{e-1}^{i(e-1)}$ (we identify here p^e with the set of all maps from $\{0, \dots, e-1\}$ to $\{0, \dots, p-1\}$). We will call the m_i's *p-monomials*.

DEFINITION 3. Let T_e be the following $\mathscr{L}$-theory:

1. T_R the theory of all right R-modules,
2. For each $q \in R$ with $q(0) \neq 0$, $\forall x \exists y, x = y \cdot q$,
3. $\exists^{=d} x_i$ (the x_i 's are linearly independent over $\mathbb{F}_p$ $\wedge \bigwedge_i x_i \cdot q = 0$), for each $q \in R$ of degree d with $q(0) \neq 0$,
4. $\forall x\ x = \sum_{i \in p^e} \lambda_i(x) \cdot t \cdot m_i$,
5. $\forall x \forall (x_i)_{i \in p^e} (x = \sum_{i \in p^e} x_i \cdot t \cdot m_i \rightarrow \bigwedge_i x_i = \lambda_i(x))$.

In the sequel, we find it useful to express some of our results for subtheories of T_e, namely:

NOTATION 3.1. Let T_{sep} be the $\mathscr{L}_R$-theory consisting of the axiom schemes 1, 2, 3 above. Let T_{free} be the $\mathscr{L}$-theory consisting of the axiom schemes 1, 2, 4, 5. Note that the class of models of T_{free} is closed under direct summands (in the class of $\mathscr{L}$-structures) and direct products (the axioms are Horn sentences). Let T_λ be the $\mathscr{L}$-theory consisting of the axiom schemes 1, 4, 5.

REMARK 1.

1. Note that we could have written axiom schemes 2 and 3 in a shorter way as: $\forall x \exists^{=p^d} y (x = y \cdot q)$, for each $q \in R$ of degree d with $q(0) \neq 0$.
2. Observe that if M is a model of T_λ, we have
$$M = \oplus_{i \in p^e} M \cdot t \cdot m_i,$$
as $\mathbb{F}_p$-vector-spaces. Note that an $\mathscr{L}_R$-substructure of a model of T_λ is itself a model of axioms 4, 5 if and only if it is an $\mathscr{L}$-substructure.
3. Let q be an element of R of the form $q = t^n.a_n + \dots + t.a_1 + a_0$. We denote by $q[x^p]$ the polynomial $a_n x^{p^n} + \dots + a_1 x^p + a_0 x$. We say that q is *separable* if $q(0) \neq 0$. This terminology is justified by the fact that q is separable as defined above if and only if $q[x^p]$ is separable in the classical sense, that is, its formal derivative is non-zero.

PROPOSITION 3.1. *T_e is consistent.*

PROOF. Let K be a separably closed field of characteristic p and imperfection degree e. We interpret B as a p-basis of K. We define the action of t on K as the Frobenius map and so for instance we get for $y \in K$ and $q \in R$, $y \cdot q = q[y^p]$. We interpret the unary functions λ_i as the p-components on B. Since B is a p-basis, K

is a model of axioms 4 and 5 and since K is separably closed, it satisfies T_{sep} and so K is a model of T_e. ⊣

Observe that with the same interpretation of the action of t and of the λ_i's, any field extension of $\mathbb{F}_p(B)$ admitting B as a p-basis, is a model of T_λ.

NOTATION 3.2.

1. Given $q \in R$, we will define $\sqrt[p]{q}$. First, for $a = \sum_{i\in p^e} a_i^p.m_i \in \mathbb{F}_p(B)$, where the a_i's belong to $\mathbb{F}_p(B)$, set $a^{1/p} := \sum_{i\in p^e} a_i.m_i$. (Observe that $(a^p)^{1/p} = a$, but except if $a \in \mathbb{F}_p(B)^p$, $(a^{1/p})^p$ and a are distinct.) Then, for $q = \sum_{i=0}^n t^j.a_j \in R$ with $a_j \in \mathbb{F}_p(B)$, set $\sqrt[p]{q} := \sum_{i=0}^n t^j.a_j^{1/p}$. Iteration m times of $\sqrt[p]{\ }$ is denoted $\sqrt[p^m]{\ }$.
2. Let $q = \sum_{j=0}^n t^j.a_j \in R$ with $a_j = \sum_{i\in p^e} a_{ji}^p.m_i$, $a_{ji} \in \mathbb{F}_p(B)$. For $i \in p^e$, set $q_{(i)} := \sum_{j=0}^n t^j.a_{ji}^p$. So, $q = \sum_{i\in p^e} q_{(i)}.m_i$.
3. Any nonzero $q \in R$ can be written $q = t^m.(t^n.a_n + \cdots + t.a_1 + a_0) \in R$, with $a_0, \ldots, a_n \in \mathbb{F}_p(B)$ and $a_n.a_0 \neq 0$. Define $\mathrm{v}(q) = m$ and write $q = t^{\mathrm{v}(q)}.q_s$. Note that q_s is separable.

REMARK 2. Let $q \in R$. Then

1. $t.q_{(i)} = \sqrt[p]{q_{(i)}}.t$, for each $i \in p^e$.
2. If q is separable, then for some $i \in p^e$, $\sqrt[p]{q_{(i)}}$ is also separable.
3. $t.q = \sum_{i\in p^e} \sqrt[p]{q_{(i)}}.t.m_i$.

We state now some preliminary lemmas on the $\mathscr{L}$-terms. First, we introduce a notation for the composite of the functions λ_i.

NOTATION 3.3. For any $i_1, \ldots, i_m \in p^e$, let $\lambda_{i_1,\ldots,i_m} := \lambda_{i_m} \circ \cdots \circ \lambda_{i_1}$. Each x in some model M of T_λ can be written uniquely as

$$x = \sum_{i_1,\ldots,i_m \in p^e} \lambda_{i_m}(\ldots(\lambda_{i_1}(x))\ldots) \cdot t^m \cdot m_{i_m}^{p^{m-1}} \cdot \ldots \cdot m_{i_1}.$$

We will index by $p^{em} = p^{me} = (p^m)^e$ the p-components iterated m times and letting

$$\begin{aligned} m_{(i_1,\ldots,i_m)} &= m_{i_m}^{p^{m-1}} \cdot \ldots \cdot m_{i_1} \\ &= b_0^{i_1(0)+i_2(0)\cdot p+\cdots+i_m(0)\cdot p^{m-1}} \cdot \ldots \cdot b_{e-1}^{i_1(e-1)+i_2(e-1)\cdot p+\cdots+i_m(e-1)\cdot p^{m-1}}, \end{aligned}$$

we get $x = \sum_{k\in p^{me}} \lambda_k(x) \cdot t^m \cdot m_k$. We extend our notation to write an element $q \in R$ as: $q = \sum_{k\in p^{me}} q_{(k)}.m_k$. Iterating the above Remark, we get that $t^m.q = \sum_{k\in p^{me}} \sqrt[p^m]{q_{(k)}}.t^m.m_k$.

LEMMA 3.2. *Let M be a model of T_λ. Then,*

1. *For any $u, v \in M$ and $i \in p^e$, $\lambda_i(u+v) = \lambda_i(u) + \lambda_i(v)$.*
2. *For any $u \in M$, $d = \sum_{s\in p^e} d_s^p.m_s \in \mathbb{F}_p(B)$ and $i \in p^e$,*

$$\lambda_i(u \cdot d) = \sum_{\delta\in 2^e} \left(\sum_{\substack{r,s\in p^e \\ r+s=i+p\delta}} \lambda_r(u) \cdot d_s \cdot \prod_{j\in e} b_j^{\delta(j)} \right).$$

3. *For any* $u \in M$, $d = \sum_{s \in p^e} d_s^p . m_s \in \mathbb{F}_p(B)$, *$k$ an integer* ≥ 1 *and* $i \in p^e$,

$$\lambda_i(u \cdot t^k \cdot d) = u \cdot t^{k-1} \cdot d_i.$$

Proof. Immediate calculation from the definition of the functions λ_i. ⊣

The above Lemma shows in fact that any model of T_λ is a module over the ring $R[\lambda_i : i \in p^e]$ where the multiplication rules are given by 2. and 3. above (see [1]).

Corollary 3.3. *Let M be a model of T_λ. Then any $\mathscr{L}$-term $t(x_0, \ldots, x_k)$ is in M equal to a term of the form*

$$\sum_{j=0}^{k} \sum_{i \in p^{em}} \lambda_i(x_j) \cdot d_{ji}$$

for some integer $m \geq 1$ and d_{ji}'s in R.

Proof. Using Lemma 3.2 we show, by induction on the length m of the tuple $j \in p^{em}$, that $\lambda_j(x \cdot r) = \sum_{i \in p^{em}} \lambda_i(x) \cdot r_i$, for some r_i's $\in R$. ⊣

Lemma 3.4. *Let $m \in \mathbb{N}, m > 0$, and $q_j, q_j' \in R$, with $q_j = t^m . q_j'$. Then the equation $\sum_j y_j \cdot q_j = u$ is equivalent to*

$$\bigwedge_{i \in p^{em}} \sum_j y_j \cdot \sqrt[p^m]{(q_j')_{(i)}} = \lambda_i(u)$$

in any model M of T_λ.

Proof. By Remark 2 and Notation 3.3, $q_j' = \sum_{i \in p^{em}} (q_j')_{(i)} . m_i$ and

$$\sum_j y_j \cdot t^m \cdot q_j' = \sum_{i \in p^{em}} \left(\sum_j y_j \cdot \sqrt[p^m]{(q_j')_{(i)}} \right) \cdot t^m \cdot m_i.$$

Therefore, if $\sum_j y_j \cdot q_j = u$, since M is model of T_λ, we have

$$\bigwedge_{i \in p^{em}} \sum_j y_j \cdot \sqrt[p^m]{(q_j')_{(i)}} = \lambda_i(u).$$

The converse is clear. ⊣

Proposition 3.5. *Let M be a model of T_λ, then the torsion part M_{tor} of M is:*

$$\{x \in M : x \cdot q = 0, \textit{ for some separable } q \in R\}$$

and M_{tor} is an $\mathscr{L}$-substructure of M. Moreover, if M is a model of T_{free} (respectively T_e), then M_{tor} is a model of T_{free} (respectively T_e).

Proof.

1. Since R is a right Ore ring, the set of torsion elements forms a submodule. Let x_1 and x_2 belong to M_{tor} with $x_1 \cdot q_1 = 0$, $x_2 \cdot q_2 = 0$, $q_1, q_2 \in R - \{0\}$. Then there exist $a_1, a_2 \in R - \{0\}$ such that $q_1.a_1 = q_2.a_2 = q \neq 0$ and so $(x_1 + x_2) \cdot q = 0$. For $r \in R$, let $s = lcm(r, q_1) = r.r_1$ for some $r_1 \in R - \{0\}$, then $x_1 \cdot r \cdot r_1 = x_1 \cdot s = 0$ and so $x_1 \cdot r$ belongs to M_{tor}.

2. Assume that $x \in M_{tor}$ with $x \cdot q = 0$, for some $q \in R - \{0\}$ with $\mathrm{v}(q) > 0$. Let $q = t^{\mathrm{v}(q)}.q_s$. By Remark 2(3) and axioms 4, 5, $\bigwedge_{i \in p^e} x \cdot t^{\mathrm{v}(q)-1} \cdot \sqrt[p^e]{(q_s)_{(i)}} = 0$. Now, since q_s is separable, by Remark 2(2), for some i_0, $\sqrt[p^e]{(q_s)_{(i_0)}}$ is separable. By iterating this procedure we obtain $x \cdot q' = 0$, for some q' separable.
3. Let $q = \sum_i t^i.a_i$ be a separable element of R and suppose that $x \cdot q = 0$. Then writing $q.a_0^{-1} = 1 + t.q'$, we get $x = -\sum_{i \in p^e} x \cdot \sqrt[p^e]{q'_{(i)}} \cdot t \cdot m_i$, so $\lambda_i(x) = x \cdot \sqrt[p^e]{q'_{(i)}}$ belongs to the R-submodule generated by x.
4. So, M_{tor} is an $\mathscr{L}$-substructure of M. To check that it is a model of T_{free} (respectively T_e) whenever M is, in both cases, it is enough to check axiom scheme 2. Let $x \in M_{tor}$, $q \in R - \{0\}$ and y such that $x = y \cdot q$. Since R is a domain, y belongs to M_{tor}. ⊣

COROLLARY 3.6. *Let K be a separably closed field with p-basis B. Let us denote by $\mathbb{F}_p(B)^{sep}$ the separable closure of $\mathbb{F}_p(B)$. Then $K_{tor} = \mathbb{F}_p(B)^{sep}$.*

PROOF. Since any polynomial divides an additive polynomial (see [8] Chapter 3), we have that $\mathbb{F}_p(B)^{sep} \subseteq K_{tor}$. Now $K_{tor} \subseteq \mathbb{F}_p(B)^{sep}$ since Proposition 3.5 tells us that $x \in K_{tor}$ implies that $x \cdot q = 0$ for some separable q. ⊣

REMARK 3. The proof of Proposition 3.5 may be generalized as follows. Let A, C be two $\mathscr{L}$-structures models of T_λ and suppose that $A \subseteq C$. Let u belong to C and assume that there exists $q \in R - \{0\}$ such that $u.q$ belongs to A. Then there exists a separable element $r \in R$ such that $u \cdot r$ belongs to A.

§4. Axiomatization. Infinite imperfection degree. In this section, we will extend our previous results to the infinite imperfection case. Let B be infinite. Let $\mathscr{L}_\infty = \mathscr{L}_R \cup \{\lambda_i^{\bar{b}} : i \in p^n, \bar{b} \in B^n, n \in \omega - \{0\}\}$, where the $\lambda_i^{\bar{b}}$'s are unary functions. We fix an enumeration $m_{n,i}(\bar{x})$ of the monomials of the form $x_0^{i(0)} \dots x_{n-1}^{i(n-1)}$, with $n \in \mathbb{N}$ and $i \in p^n$. As in the finite imperfection degree case, we will call the elements of the form $m_{n,i}(\bar{b})$ where $\bar{b} \in B^n$, p-monomials.

DEFINITION 4. Let T_∞ be the axiom schemes 1, 2, 3 as in the finite imperfection degree case together with the following schemes:

4_∞. $\forall x \left(\bigvee_{i \in p^n} \lambda_i^{\bar{b}}(x) \neq 0 \rightarrow x = \sum_{i \in p^n} \lambda_i^{\bar{b}}(x) \cdot t \cdot m_{n,i}(\bar{b})\right)$,
for each $\bar{b} = (b_0, \dots, b_{n-1}) \in B^n$, $n \in \mathbb{N} - \{0\}$.

5_∞. $\forall x \forall (x_i)_{i \in p^n} \left(x = \sum_{i \in p^n} x_i \cdot t \cdot m_{n,i}(\bar{b}) \rightarrow \bigwedge_{i \in p^n} x_i = \lambda_i^{\bar{b}}(x)\right)$,
for each $\bar{b} = (b_0, \dots, b_{n-1}) \in B^n$, $n \in \mathbb{N} - \{0\}$.

A model of T_∞ is $K_0 = \mathbb{F}_p(B)^{sep}$, where the functions $\lambda_i^{\bar{b}}$ are interpreted as follows. If $x \notin K_0^p(\bar{b})$, then $\lambda_i^{\bar{b}}(x) = 0$, for any $i \in p^n$. For $x = \sum_i x_i{}^p.m_{n,i}(\bar{b})$ for some x_i's in K_0, then $\lambda_i^{\bar{b}}(x) = x_i$. So, T_∞ is consistent.

Similarly to the finite imperfection case, we will denote by $T_{\lambda,\infty}$ the $\mathscr{L}_\infty$-theory consisting of axiom scheme 1 together with axiom schemes 4_∞ and 5_∞ and by $T_{free,\infty}$ the $\mathscr{L}_\infty$-theory consisting of $T_{\lambda,\infty}$ together with axiom scheme 2. Observe that if M is a model of $T_{\lambda,\infty}$, then for any integer n and $\bar{b} \in B^n$ the sum of the $\mathbb{F}_p$-vector spaces $M \cdot t \cdot m_{n,i}(\bar{b})$ in M is direct. And, for $x \in M$, $\lambda_i^{\bar{b}}(x) \neq 0$ for some $\bar{b} \in B^n$ and some $i \in p^n$ if and only if x is a nonzero element of the direct sum $\oplus_{i \in p^n} M \cdot t \cdot m_{n,i}(\bar{b})$ for some $\bar{b} \in B^n$.

In order to extend Notation 3.2 to the case when e is infinite, note that each $q \in R$ belongs to $\mathbb{F}_p(\bar{b})[t;\alpha]$ for some $n \in \mathbb{N}$ and $\bar{b} \in B^n$. Then we obtain analogous statements to Remark 2 and Proposition 3.5.

Now we need to introduce a "tree property" which indicates for a specific element the width of the tree of its (non-trivial) p-components.

NOTATION 4.1. For any sequences $\bar{b} \in B^n$ and $\bar{\imath} = (i_1, \dots, i_m) \in p^n \times \cdots \times p^n$, set $\lambda_{\bar{\imath}}^{\bar{b}} := \lambda_{i_m}^{\bar{b}} \circ \cdots \circ \lambda_{i_1}^{\bar{b}}$. As in Notation 3.3, we will identify the set $(p^n)^m$ with the set of maps from n to p^m, and for i such map, define $m_{n,i}(\bar{b}) := \prod_{j \in n} b_j^{i(j)}$.

From now on, we drop the "bar" above the index "i".

DEFINITION 5. Let M be an $\mathscr{L}_\infty$-structure, $u \in M$, $\bar{b} \in B^n$, m an integer ≥ 1. We say that u has the $(\bar{b}, m)$-*tree property* (t.p.): if $u = \sum_{i \in p^{nm}} \lambda_i^{\bar{b}}(u) \cdot t^m \cdot m_{n,i}(\bar{b})$. Note that this property is expressible by a positive quantifier-free formula.

REMARK 4. Let M be a model of $T_{\lambda,\infty}$, $u \in M$, $\bar{b} = (b_0, \dots, b_{n-1}) \in B^n$, m an integer ≥ 1. The following are equivalent:

1. u has the $(\bar{b}, m+1)$-t.p.
2. u has the $(\bar{b}, 1)$-t.p. and for any $k \leq m$, any $i = (i_1, \dots, i_k)$, $\lambda_i^{\bar{b}}(u)$ has the $(\bar{b}, 1)$-t.p.
3. u has the $(\bar{b}, 1)$-t.p. and for any $i \in p^n$, $\lambda_i^{\bar{b}}(u)$ has the $(\bar{b}, m)$-t.p.

The proofs of the next three Lemmas are immediate using the definition of the functions $\lambda_i^{\bar{b}}$.

LEMMA 4.1. *Let M be a model of $T_{\lambda,\infty}$, $u, v \in M$ and $d \in \mathbb{F}_p(B)$, $d = \sum_{i \in p^n} d_i^p . m_{n,i}(\bar{b})$, for some $\bar{b} = (b_0, \dots, b_{n-1}) \in B^n$ and d_i 's in $\mathbb{F}_p(B)$. Then,*

1. *If u and v have the $(\bar{b}, 1)$-tp, then for any $i \in p^n$,*

$$\lambda_i^{\bar{b}}(u+v) = \lambda_i^{\bar{b}}(u) + \lambda_i^{\bar{b}}(v).$$

2. *For any $i \in p^n$, k integer ≥ 1,*

$$\lambda_i^{\bar{b}}(u \cdot t^k \cdot d) = u \cdot t^{k-1} \cdot d_i.$$

3. *For any $i \in p^n$,*

$$\lambda_i^{\bar{b}}(u \cdot d) = \sum_{\delta \in 2^n} \sum_{\substack{r,s \in p^n \\ r+s=i+p\delta}} \lambda_r^{\bar{b}}(u) \cdot d_s \cdot \prod_{j \in n} b_j^{\delta(j)}$$

⊣

LEMMA 4.2. *Let M be a model of $T_{\lambda,\infty}$ and $u \in M$. Let $\bar{b} \in B^n$ and $q \in \mathbb{F}_p(\bar{b})[t;\alpha]$ be separable. Then, for any positive integer m, u has the $(\bar{b}, m)$-t.p. iff $u \cdot q$ has the $(\bar{b}, m)$-t.p.* ⊣

LEMMA 4.3. *Let $q_j, q'_j \in R$, $q_j = t^m.q'_j$ where $m > 0$. Assume that the q_j's belong to $\mathbb{F}_p(\bar{b})[t;\alpha]$ where $\bar{b} \in B^n$. Then the equation $\sum_j y_j \cdot q_j = u$ is equivalent to*

$$\bigwedge_{i \in p^{nm}} \sum_j y_j \cdot \sqrt[p^m]{(q'_j)_{(i)}} = \lambda_i^{\bar{b}}(u) \text{ and } (u \text{ has the } (\bar{b}, m)\text{-t.p. property})$$

in any model M of $T_{\lambda,\infty}$. ⊣

§5. The torsion part and divisible closure over a substructure. The main part of this section will be devoted to the proof that the torsion submodules of any two models of T_e are isomorphic. Then we will see that we can adapt our proofs to show that any $\mathscr{L}$-substructure of a model of T_e can be extended in a canonical way to a submodel.

DEFINITION 6. Let q be an element of R, q is said to be *prime* if it is non invertible, separable and if it cannot be written as a product of two non invertible elements of R.

Fix an integer $n \geq 1$. Let N be an $\mathscr{L}_R$-structure, N_0 a substructure of N and $u \in N - N_0$ be such that $u \cdot q = 0$, for some separable $q \in R$ of degree n.

LEMMA 5.1. *There is a monic separable polynomial $q_u \in R$ with degree $\leq n$ such that:*

1. *q_u is of degree minimal such that $q_u \neq 0$ and $u \cdot q_u \in N_0$;*
2. *For any $r \in R$ such that $u \cdot r \in N_0$, q_u divides r on the right.*

Therefore, there is an unique monic separable polynomial $q_u \in R$ of minimal degree such that $u \cdot q_u \in N_0$.

Suppose furthermore that N_0 contains all elements of N_{tor} annihilated by some separable element of R of degree $< n$, then q_u is prime.

PROOF. Let I be the set of polynomials r such that $u \cdot r \in N_0$. By hypothesis, I contains a separable polynomial q. Since N_0 is a right R-module, I is a right ideal and since R is right principal, there is q_u a monic polynomial generating I, moreover such polynomial is unique. Since q is separable, q_u is separable.

The last assertion under the additional hypothesis on N_0 is clear. ⊣

For $c \in N$, or $c \subseteq N$, we denote by $\mathbb{F}_p(B)[c]$ the $\mathbb{F}_p(B)$-vector subspace of N generated by c and we denote by $N_0\langle c\rangle_R$ (respectively $\langle c\rangle_R$) the R-submodule of N generated by $N_0 \cup \{c\}$ or $N_0 \cup c$ (respectively c).

COROLLARY 5.2. *Let $u \in N - N_0$, as above, and let n_1 be the degree of q_u, then $N_0\langle u\rangle_R = N_0 \oplus \mathbb{F}_p(B)[u] \oplus \mathbb{F}_p(B)[u \cdot t] \oplus \cdots \oplus \mathbb{F}_p(B)[u \cdot t^{n_1-1}]$ as $\mathbb{F}_p(B)$-vector spaces, i.e., $N_0\langle u\rangle_R$ is isomorphic to the direct sum of N_0 and n_1 copies of $\mathbb{F}_p(B)$.*

PROOF. Since $u \cdot q_u \in N_0$ and q_u is of degree minimal such, one has $N_0\langle u\rangle_R = N_0 + \mathbb{F}_p(B)[u, u \cdot t, \ldots, u \cdot t^{n_1-1}]$ and $\mathbb{F}_p(B)[u \cdot t^d] \cap (N_0 + \mathbb{F}_p(B)[u, u \cdot t, \ldots, u \cdot t^{d-1}]) = \{0\}$ for any $d < n_1$. ⊣

Let N, M be two R-modules, models of T_{sep}, and suppose that N_0 and M_0 are two isomorphic submodules of N and M respectively which we assume to contain all the elements (in respectively N, M) annihilated by some separable element of R of degree $< n$. We denote by j this isomorphism and we want to extend it. Let $u \in N_{tor} - N_0$ be such that $u \cdot q = 0$, for some separable $q \in R$ of degree n. Let

q_u be the monic prime polynomial of degree $\leq n$ of Lemma 5.1. Note that $u \cdot q_u$ belongs to N_{tor} (N_{tor} is a submodule). Take $\overline{u}$ to be an element of M such that $\overline{u} \cdot q_u = j(u \cdot q_u)$. The existence of such an element is insured by axiom schemes 2 of T_{sep}. Moreover, since $j(u \cdot q_u)$ belongs to M_{tor}, $\overline{u}$ also belongs to M_{tor}. Moreover, since N_0 and M_0 are isomorphic, q_u has the same number of roots in each of them, and since M and N are models of axiom schemes 3 of T_{sep}, q_u has a root in $N - N_0$ iff it has one in $M - M_0$. So, we may choose $\overline{u}$ in $M_{tor} - M_0$ and note that since q_u is prime, it is of minimal degree such that $\overline{u} \cdot q_u \in M_0$.

LEMMA 5.3. *The mapping $u_0 + u \cdot r \rightarrow j(u_0) + \overline{u} \cdot r$, for $u_0 \in N_0$ and $r \in R$, is well defined and is an isomorphism of R-modules between $N_0\langle u\rangle_R$ and $M_0\langle\overline{u}\rangle_R$.*

PROOF. We will show only that it is well-defined. Using analogous arguments one can show that it is an isomorphism. Suppose that $u_0 + u \cdot r = u_0' + u \cdot r'$, we have $u \cdot (r - r') \in N_0$. Therefore, by Lemma 5.1, $r - r' = q_u.s$, for some $s \in R$. Then, $0 = (u_0 - u_0') + u \cdot (q_u.s)$ and, since j is an isomorphism and by choice of $\bar{u}$,

$$\begin{aligned} 0 &= j(u_0 - u_0' + u \cdot q_u \cdot s) = j(u_0) - j(u_0') + j(u \cdot q_u) \cdot s \\ &= j(u_0) - j(u_0') + \overline{u} \cdot q_u \cdot s = j(u_0) - j(u_0') + \overline{u} \cdot (r - r'). \end{aligned}$$

Consequently $j(u_0) + \overline{u} \cdot r = j(u_0') + \overline{u} \cdot r'$. ⊣

NOTATION 5.1. Let N be an R-module. Let $N_{tor,sep} = \{n \in N : n \cdot q = 0 \text{ for some } q \in R, q(0) \neq 0\}$.

PROPOSITION 5.4. *Let N be any model of T_{sep}. Then $N_{tor,sep}$ is isomorphic as an R-module to $\mathbb{F}_p(B)^{sep}$. Given two models N, M of T_{sep} with $N \subseteq M$, we have $N_{tor,sep} = M_{tor,sep}$.*

PROOF. By induction we build two chains of submodules of $N_{tor,sep}$ and $\mathbb{F}_p(B)^{sep}$ respectively, $(N_l : l \in \omega)$ and $(M_l : l \in \omega)$, such that for any $l \in \omega$, N_l and M_l contains all the elements of respectively N and $\mathbb{F}_p(B)^{sep}$ annihilated by some separable element of R of degree $< l$. We define simultaneously a chain of isomorphisms $(h_l : l \in \omega)$, $h_l : N_l \rightarrow M_l$. Take $N_0 = M_0 = \{0\}$ and h_0 the identity. Suppose inductively that we have defined N_l, M_l and h_l. Let $(u_i : i \in \omega)$ be an enumeration of the elements of N_{tor} annihilated by the separable elements of R of degree l. Using 5.3 we build by induction two chains of submodules of $N_{tor,sep}$ and $\mathbb{F}_p(B)^{sep}$ respectively, $(N_{l+1}^i : i \in \omega)$ and $(M_{l+1}^i : i \in \omega)$, and a chain of isomorphisms $(h_{l+1}^i : i \in \omega)$ such that $N_{l+1}^0 = N_l$, $M_{l+1}^0 = M_l$, $h_{l+1}^0 = h_l$ and for any $i \in \omega$, $h_{l+1}^i : N_{l+1}^i \rightarrow M_{l+1}^i$, $u_i \in N_{l+1}^{i+1}$. Let N_{l+1}, M_{l+1} and h_{l+1} be the union of these chains. Since N is a model of axiom schemes 2, 3 of T_{sep}, it is easy to check that $N_{tor,sep} = \bigcup_{l \in \omega} N_l$, $\mathbb{F}_p(B)^{sep} = \bigcup_{l \in \omega} M_l$ and $\bigcup_{l \in \omega} h_l : N_{tor,sep} \rightarrow \mathbb{F}_p(B)^{sep}$ is an isomorphism.

The number of elements in a model annihilated by a given separable element in R is finite and described by T_{sep}; this also gives the second assertion of the statement. ⊣

COROLLARY 5.5. *Let N be a model of T_e. Then, $N_{tor} \simeq \mathbb{F}_p(B)^{sep}$.*

PROOF. By Proposition 3.5, we have that $N_{tor} = N_{tor,sep}$. So, we may apply the preceding proposition. ⊣

Let $\mathscr{P}$ be the set of elements of R corresponding to primitive additive polynomials (see [8]) w.r. to $\mathbb{F}_p(B)^{sep}$; i.e., $q \in \mathscr{P}$ iff $q(0) \neq 0$ and there is an element $u \in \mathbb{F}_p(B)^{sep}$

such that $u \cdot q = 0$ but, for all polynomial $r[x^p]$ properly dividing $q[x^p]$ in the usual sense, one has $u \cdot r \neq 0$. Such an element u is called a primitive root of q. Note that O. Ore (see p. 582 in [2]) asked for necessary and sufficient conditions for an additive polynomial to have primitive roots. In particular, he gave an example of an additive polynomial without any.

COROLLARY 5.6. *Let q be a separable element of R and let the $r_i[x^p]$'s to be all monic additive polynomials properly dividing $q[x^p]$. Then q belongs to $\mathscr{P}$ iff $T_{sep} \vdash \exists x(x \cdot q = 0 \wedge \bigwedge_i x \cdot r_i \neq 0)$.* ⊣

Now we want to generalize Proposition 5.5 to the following situation. Let A be an $\mathscr{L}_R$-substructure (respectively $\mathscr{L}$-substructure) of a model M of T_{sep} (respectively T_e).

We will show in both cases that there is a canonical extension of A to a model of T_{sep} (respectively T_e). Even though in the first case it will be an $\mathscr{L}_R$-structure and the second case an $\mathscr{L}$-structure, the domain of both extensions will be the same and so we will denote it by the same letter.

By abuse of language, one might say that Proposition 5.4 was the case $A = \{0\}$.

We need the following Lemma.

LEMMA 5.7. *The set R_0 of separable elements of the skew polynomial ring R is a right denominator set: i.e., $\forall r \in R, \forall r_0 \in R_0, r.R_0 \cap r_0.R \neq \emptyset$.*

PROOF. Given two elements r and q of R, O. Ore constructs explicitly their least common multiple (as in the commutative case) (see [9]). One then checks that if r is separable, then $lcm(r, q) = r.r_1 = q.q_1$ with $r_1, q_1 \in R$ and q_1 separable. ⊣

PROPOSITION 5.8. *Let M, N be models of T_{sep} containing, respectively, isomorphic $\mathscr{L}_R$-substructures A, B. Then we may extend this partial isomorphism to a minimal submodel A^{sep} of M containing A.*

PROOF. By Proposition 5.5, we know that the separable torsion submodules of M and N are isomorphic. Since $A_{tor,sep} = M_{tor,sep} \cap A$, we may extend this isomorphism to an isomorphism from $A + M_{tor,sep}$ to $B + N_{tor,sep}$.

To get a model of T_{sep}, we need to close these substructures by adding solutions of equations of the form (*) $u \cdot s = a$, where $a \in A$ and s is a separable element of R. Setting $A_0 = A + M_{tor}$, then A_1 is constructed solving equations of the form (*) with $a \in A_0$.

Let ϕ be the isomorphism between A_0 and B_0, and suppose that there exists an element a in A_0 which is not divisible in A_0 by a separable polynomial, say q. Since M is a model of T_{sep}, there exists u belonging to M such that $u \cdot q = a$. Let now s' be a separable polynomial of degree minimal such that $u \cdot s'$ belongs to A_0. Let $a' = u \cdot s'$. The set of solutions of the equation $a' = x \cdot s'$ is equal to $u + \{x : x \cdot s' = 0\}$, which means that A_0 contains none of them. Let $v \in N$ be such that $v \cdot s' = \phi(a')$. Since A_0 is isomorphic to B_0, v does not belong to B_0.

CLAIM. *s' is of degree minimal such that $v \cdot s'$ belongs to B_0.*

PROOF. Let r be an element of R of degree minimal such that $v \cdot r$ belongs to B_0. Then r divides s' i.e., there exists $r_1 \in R$ such that $s' = r.r_1$. Set $b_0 = v \cdot r$. Since ϕ is an isomorphism between A_0 and B_0, there exists c_0 in A_0 with $\phi(c_0) = b_0$. Now $a' = u \cdot s' = u \cdot r \cdot r_1 = c_0 \cdot r_1$. So, $u \cdot r - c_0$ belongs to $ker(r_1)$ which is included in A_0. Therefore, $u \cdot r$ belongs to A_0 and by choice of s', $deg(r) = deg(s')$. ⊣

Extend ϕ on $\langle u\rangle_R + A_0$ by setting $\bar{\phi}(u\cdot r + a_0) = v\cdot r + \phi(a_0)$, where r belongs to R and a_0 to A_0. This is well defined: suppose that $u\cdot r + a_0 = u\cdot r' + a_1$ i.e., $u\cdot(r-r')$ belongs to A_0. Then s' divides $r-r'$ i.e., $(r-r') = s'.s_1$. Therefore, $u\cdot(r-r') = a'\cdot s_1 = a_1 - a_0$, so $\phi(a'\cdot s_1) = \phi(a_1-a_0)$ and $v\cdot(r-r') = v\cdot s'\cdot s_1 = \phi(a')\cdot s_1 = \phi(a'\cdot s_1)$. Thus, $v\cdot(r-r') = \phi(a_1)-\phi(a_0)$. This map is injective by the claim.

We repeat this process in order to get an R-module A_1 in which all elements of A_0 are divisible by each separable element of R.

Claim. *A_1 is a model of T_{sep}.*

Proof. Let $v\in A_1$ and $u\in M$ be such that $u\cdot q = v$, where $q\in R$ and $q(0)\neq 0$. We are going to show that u belongs to A_1. By construction of A_1, there exists u_i's in M, r_i, q_i's in R with $q_i(0)\neq 0$ such that $v=\sum_{i\geq 1} u_i\cdot r_i$ and $u_i\cdot q_i\in A_0$. Since R is right Ore, there exist $s_1\in R$ with $s_1(0)\neq 0$ (see Lemma 5.7). $t_1\in R-\{0\}$ such that $r_1.s_1 = q_1.t_1$. So we have $v\cdot s_1 = \sum_{i\geq 1} u_i\cdot r_i\cdot s_1 = u_1\cdot q_1\cdot t_1 + \sum_{i\geq 2} u_i\cdot r_i\cdot s_1$. Iterating, we get first that there exist $s_2, t_2\in R-\{0\}$ with $s_2(0)\neq 0$ such that $r_2.s_1.s_2 = q_2.t_2$. So we get $v\cdot s_1\cdot s_2 = a + \sum_{i\geq 3} u_i\cdot r_i\cdot s_1\cdot s_2$, where $a\in A_0$. Finally we get that there exists $s\in R$ with $s(0)\neq 0$ such that $v\cdot s\in A_0$. So $u\cdot q\cdot s\in A_0$, which implies that $u\in A_1$. ⊣

Corollary 5.9. *Let M, N be two models of T_e containing two isomorphic $\mathscr{L}$-substructures A, B, respectively. Then we may extend this partial isomorphism to A^{sep} which is a minimal submodel of M containing A.*

Proof. We keep the same notations as in the proof above. We assume now that A is an $\mathscr{L}$-structure and M is a model of T_e. We first check that A_1 is an $\mathscr{L}$-structure: if $u\cdot s = a$ for some separable element s of R, then $\lambda_i(u)$ belongs to the R-submodule generated by u and $\lambda_j(a)$, see proof (item 3) of Proposition 3.5, which also implies that the isomorphism $\bar{\phi}$ above commutes with the functions λ_i's. Thus $A_1 = A^{sep}$ is the required minimal submodel. ⊣

Remark 5. Note that in the above Corollary, we could have replaced T_e by any extension of T_{free} which specifies for each separable polynomial of R the number of elements annihilated by it.

§6. Quantifier elimination in the case of finite imperfection degree.

First, we note that the proof of the Proposition in [6] p. 176, adapts to right Euclidean rings. Let us recall the definition.

Definition 7. Let S be a domain, it is a right Euclidean ring if there exists a function δ from S to $\mathbb{N}$ such that $\forall p_1\forall p_2\exists q\exists r(p_1 = p_2.q + r$ with $\delta(r)<\delta(p_2))$ (see [6] p. 143).

Proposition 6.1. *Let S be a right Euclidean ring and A be a matrix with coefficients in S. Then there exist invertible matrices P, Q (P with coefficients in $\{0,1\}$), such that $P.A.Q$ is lower triangular.*

Proof. For convenience of the reader, let us indicate the main steps of the proof, which goes by induction on the number of lines of the $n\times m$ matrix A. As usual, a lower triangular matrix C is of the form $C=(c_{ij})$, where $c_{ij}=0$ if $i<j$.

1. Let $P_{ij} = 1 - e_{ii} - e_{jj} + e_{ij} + e_{ji}$, where 1 is the identity matrix and e_{ij} the matrix with 1 in position (i, j) and 0 in the other positions. Note that P_{ij} is invertible, $P_{ij}^2 = 1$.
2. Let $T_{ij}(c) = 1+e_{ij}.c$, where $c \in S$ and $i \neq j$. The inverse of $T_{ij}(c)$ is $T_{ij}(-c)$.
3. $P_{ij}.A$: exchange the lines i, j in A.
4. $A.P_{ij}$: exchange the columns i, j in A.
5. $A.T_{ij}(c)$: the j^{th} column of A is replaced by the sum of the i^{th} column $\times c$ of A and the j^{th} column of A.

If A is non zero, let a_{ij} be a non zero element of minimal δ in matrix A. In the product $P_{i1}.A.P_{1j}$, the element a_{ij} is in the $(1, 1)$-position. Thus, we may assume that the initial matrix A has the property that the element in the $(1, 1)$-position is of minimal δ. Then we will show that by multiplying $P_{i1}.A.P_{1j}$ by matrices of the form P_{sr} and $T_{hk}(c)$ (on the right), we will obtain a matrix which first line only consists of zero's except at the $(1, 1)$-position. So let us consider an element a_{1k}, with $k \neq 0$. Performing the Euclidean division, there exist q_k, r_k in S such that $a_{1k} = a_{11}.q_k + r_k$ with $\delta(r_k) < \delta(a_{11})$. In the product $P_{i1}.A.P_{1j}.T_{1k}(-q_k)$ the element at the $(1, k)$-position is r_k. Either it is equal to 0, so we go on applying the process above to another element on the first line, or one multiplies the matrix obtained so far by P_{1k}. Now the element at the $(1, 1)$-position is the (nonzero) element r_k with $\delta(r_k) < \delta(a_{11})$. Therefore, the process terminates after finitely many steps and finally one obtains a matrix A' of the form

$$\begin{pmatrix} a & \bar{0} \\ . & B \end{pmatrix}$$

with a a nonzero element and B a $(n-1) \times (m-1)$ matrix to which one applies the induction hypothesis. Note that A' is equal to the product of A by invertible matrices of the form P_{ij} or $T_{1k}(c)$, $c \in S$, and note that on the right we only multiply by matrices of the form P_{ij}. Note that the reasoning does not depend on whether a_{ij} is a non zero element of minimal δ within the i^{th} line, or in the whole matrix A. ⊣

DEFINITION 8. A lower triangular $n \times m$-matrix of *co-rank* ℓ is of the form $(A_1, 0)$ where A_1 is a lower triangular $n \times k$-matrix $(k \leq n, m)$ with only non zero elements on its diagonal and 0 is a zero $n \times (m-k)$-matrix, with $\ell = n - k$.

Proposition 6.1 may be reformulated as follows.

COROLLARY 6.2. *Any $n \times m$-matrix $A \neq 0$ is equivalent to a lower triangular matrix of co-rank ℓ, with $n - \ell > 0$.* ⊣

Now we will apply the above proposition in our setting, namely to the skew polynomial ring R. We note the following.

COROLLARY 6.3. *Suppose that some line of A contains a separable element of R, then the element d_{11} on the diagonal of the lower triangular matrix $P.A.Q$, given by Proposition 6.1, is separable.*

PROOF. It suffices to note that while performing the Euclidean algorithm: $a_{1k} = a_{11}.q_k + r_k$, if $a_{11}(0) = 0$ and $a_{1k}(0) \neq 0$, then $r_k(0) \neq 0$ and if $a_{11}(0) \neq 0$, either $r_k = 0$ and the element at the $(1, k)$-position is zero, so we keep a_{11} which is separable, or we switch the positions of r_k and a_{11} and we are back to the first case whenever $r_k(0) = 0$. ⊣

DEFINITION 9. An $n \times m$-matrix is *lower triangular separable* (*l.t.s.*) if it is of the form $(A_1, 0)$ where A_1 is a lower triangular $n \times k$-matrix $(k \leq n, m)$ with only separable elements on its diagonal and 0 is a zero $n \times (m-k)$-matrix.

LEMMA 6.4. *Let A be an $n \times k$-matrix with coefficients in R. Then the system of equations $\bar{y} \cdot A = \bar{u}$ is equivalent in any model M of T_λ, to a system*

$$\bar{y} \cdot P \cdot B = \overline{t(\bar{u})},$$

where P is a permutation matrix, B is a lower triangular separable $n \times k'$-matrix and $\bar{t}$ is a tuple of $\mathscr{L}$-terms.

PROOF. We will prove the Lemma by induction on the number n of lines of matrix A. Let $a_{h\ell}$ be a coefficient of minimal valuation in matrix A, say w, and assume it is strictly positive. Multiplying A on the left by the matrix P_{1h}, we get this element on the first line of the resulting product.

Set $\bar{z} = \bar{y} \cdot P_{1h}^{-1}$. Now, $\sum_j z_j \cdot a_{j\ell} = u_\ell$ is equivalent by Lemma 3.4 to

$$\bigwedge_{i \in p^{ew}} \lambda_i(u_\ell) = \sum_j z_j \cdot \sqrt[p^w]{(a'_{j\ell})_{(i)}},$$

with $a'_{j\ell} \in R$ defined by $a_{j\ell} = t^w.a'_{j\ell}$. We have

$$a_{j\ell} = \sum_i (a_{j\ell})_{(i)}.m_i = t^w.\sum_i (a'_{j\ell})_{(i)}.m_i = \sum_i \sqrt[p^w]{(a'_{j\ell})_{(i)}}.t^w.m_i$$

and $\sqrt[p^w]{(a'_{h\ell})_{(i)}}$ is separable for at least one i. This amounts to replacing each column of $(P_{1h}.A)$ by p^{ew} columns and u_ℓ by the p^{ew}-tuple $\lambda_*(u_\ell) = (\lambda_i(u_\ell))_{i \in p^{ew}}$. The system $\bar{z} \cdot P_{1h} \cdot A = \bar{u}$ is equivalent to $\bar{z} \cdot \tilde{A} = \lambda_*(\bar{u})$, where the matrix $\tilde{A}$ has a separable element on its first line.

Now by the above Corollary, one obtains a matrix $\tilde{C}$ of the form

$$\begin{pmatrix} a & \bar{0} \\ \bar{a} & C \end{pmatrix}$$

with a a separable element, by multiplying the matrix $\tilde{A}$ on the right by matrices of the form P_{ij} and $T_{ij}(c)$, for some $c \in R$ (since the non zero element of $\tilde{A}$ is already on the first line we do not need to multiply $\tilde{A}$ on the left by some P_{ij}'s). Let Q be the product of these invertible matrices and set $Q = (Q_1, Q_2)$, where Q_1 is the first column of Q. So, the system $\bar{y} \cdot A = \bar{u}$ is equivalent to $\bar{z} \cdot \tilde{C} = \lambda_*(\bar{u}) \cdot Q$, which in turn is equivalent to

$$\sum_i z_i \cdot a_i = \lambda_*(\bar{u}) \cdot Q_1 \wedge (z_2, \ldots, z_n) \cdot C = \lambda_*(\bar{u}) \cdot Q_2,$$

where $a_1 = a$, $(a_i)_{2 \leq i \leq n} = \bar{a}$ and the matrix C has $n-1$ lines. We apply the induction hypothesis to C. ⊣

COROLLARY 6.5. *Every p.p. $\mathscr{L}_R$-formula is equivalent to a conjunction of atomic $\mathscr{L}$-formulas modulo T_{free}.*

Proof. Let $\phi(x_1, \dots, x_m)$ be a p.p. formula of the form $\exists y_1, \dots, y_k(\overline{x} \cdot B = \overline{y} \cdot A)$, where A, B are non zero matrices with coefficients in R. By Lemma 6.4, there exist a permutation matrix P, an l.t.s. $k \times n$-matrix $\tilde{A}$ and n $\mathscr{L}$-terms t_i, $1 \leq i \leq n$, such that:

$$\phi(\overline{x}) \leftrightarrow \exists y_1, \dots, y_k(t_1(\overline{x}), \dots, t_n(\overline{x})) = \overline{y} \cdot P \cdot \tilde{A}.$$

Set $\overline{y} \cdot P = \bar{y}'$. Assume that matrix $\tilde{A}$ is of co-rank ℓ and write it as $\tilde{A} = (A_1, 0)$, where A_1 is an l.t.s. $k \times (k - \ell)$-matrix.

We have

$$\begin{aligned}\phi(\overline{x}) &\leftrightarrow \exists y'_1, \dots, y'_k(t_1(\overline{x}), \dots, t_n(\overline{x})) = \overline{y}' \cdot \tilde{A})\\ &\leftrightarrow \exists y'_1, \dots, y'_k[(t_1(\overline{x}), \dots, t_{k-\ell}(\overline{x})) = \overline{y}' \cdot A_1 \wedge (t_{k-\ell+1}(\overline{x}), \dots, t_n(\overline{x})) = \bar{0}]\\ &\leftrightarrow (t_{k-\ell+1}(\overline{x}), \dots, t_n(\overline{x})) = \bar{0}.\end{aligned}$$ ⊣

Corollary 6.6. *Let M be a model of T_{free}. Then M_{tor} is a pure submodule.*

Proof. We have shown (see Proposition 3.5) that M_{tor} is an $\mathscr{L}$-substructure. So, we may apply the above Corollary. ⊣

In the next Lemma, we will use the following terminology. An $\mathscr{L}_R$-*inequation* in variables $x_1, \dots, x_n$ and parameters $\bar{v}$ is a basic formula of the form $\sum_i x_i \cdot r_i \neq t(\bar{v})$, where $r_i \in R$, $1 \leq i \leq n$, and t is an $\mathscr{L}_R$-term. We will say that this inequation is *nontrivial* if for some i, $r_i \neq 0$.

Lemma 6.7. *Let T be the theory of all torsion-free right R-modules satisfying axiom scheme 2. Let $\Sigma(\bar{x}, \bar{u})$ be a system of equations in $\mathscr{L}_R$ of the form $\bar{x} \cdot A = \bar{u}$, where A is a non zero l.t.s. $m \times k$-matrix of co-rank ℓ, and $\bar{x} = (x_1, \dots, x_m)$, $\bar{u} = (u_1, \dots, u_{m-\ell}, \bar{0})$ are variables. Then, for any system $\Gamma(\bar{x}, \bar{v})$ of $\mathscr{L}_R$-inequations in $\bar{x}$ and parameters $\bar{v}$, there exists a system of $\mathscr{L}_R$-inequations $\Gamma'(x_{m-\ell+1}, \dots, x_m, \bar{u}, \bar{v})$ such that $\Sigma(\bar{x}, \bar{u}) \wedge \Gamma(\bar{x}, \bar{v})$ is equivalent in T to $\Sigma(\bar{x}, \bar{u}) \wedge \Gamma'(x_{m-\ell+1}, \dots, x_m, \bar{u}, \bar{v})$, with the convention that if $\ell = 0$, the system of inequations is of the form $\Gamma'(\bar{u}, \bar{v})$.*

Moreover, we can split the set Γ' into two subsets, $\Gamma'_0(x_{m-\ell+1}, \dots, x_m, \bar{u}, \bar{v})$ consisting of non-trivial inequations and $\Gamma'_1(\bar{u}, \bar{v})$ of trivial ones such that for any model M of T, for any $\bar{a}$, $\bar{c}$ in M satisfying $\Gamma'_1(\bar{a}, \bar{c})$, $\Sigma(\bar{x}, \bar{a}) \wedge \Gamma(\bar{x}, \bar{c})$ has realizations in any non zero R-submodule M_0 of M containing $\langle \bar{a} \rangle^{sep}$.

Proof. We proceed by induction on m. Write the matrix A as $(\bar{a}, A_1)$, where $\bar{a}$ is the first column of A. We associate pairwise the first equation $\sum_i x_i \cdot a_i = u_1$ of Σ and the j^{th} inequation of Γ: $\sum_i x_i \cdot b_{ij} \neq t_j(\bar{v})$, where $t_j(\bar{v})$ is an $\mathscr{L}_R$-term. Since R is right Ore, for each j such that $b_{1j} \neq 0$, there exist $c_j, b_j \in R - \{0\}$ such that $a_1.c_j = b_{1j}.b_j$. The system:

$$\begin{cases} \sum_i x_i \cdot a_i \cdot c_j = u_1 \cdot c_j \\ \sum_i x_i \cdot b_{ij} \cdot b_j \neq t_j(\bar{v}) \cdot b_j \end{cases}$$

is equivalent in any torsion-free module to

$$\begin{cases} \sum_i x_i \cdot a_i = u_1 \\ \sum_{i>1} x_i \cdot (b_{ij}.b_j - a_i.c_j) \neq t_j(\bar{v}) \cdot b_j - u_1 \cdot c_j. \end{cases}$$

Let us denote by $\Gamma''(x_2, \dots, x_m, \bar{u}, \bar{v})$ the set of inequations so obtained together with all the inequations of Γ where the coefficient of x_1 is equal to 0. Either $k = 1$ and so $\Gamma' := \Gamma''$ is the required set of inequations, or $k \geq 2$, so we apply the induction hypothesis. The system $(x_2, \dots, x_m) \cdot A_1 = (u_2, \dots, u_{m-\ell}, \bar{0}) \wedge \Gamma''(x_2, \dots, x_m, \bar{u}, \bar{v})$ is equivalent to $(x_2, \dots, x_m) \cdot A_1 = (u_2, \dots, u_{m-\ell}, \bar{0}) \wedge \Gamma'(x_{m-\ell+1}, \dots, x_m, \bar{u}, \bar{v})$, for a system Γ' of inequations. Finally we obtain that the system $\Sigma \wedge \Gamma$ is equivalent to $\Sigma \wedge \Gamma'$.

Let $\Gamma'_1(\bar{u}, \bar{v})$ consist of all the trivial inequations in Γ' and $\Gamma'_0(x_{m-\ell+1}, \dots, x_m, \bar{u}, \bar{v})$ of the other ones.

Let M be a model of T, let $\bar{a}$, $\bar{c}$ be in M and assume that $\Gamma'_1(\bar{a}, \bar{c})$ holds. We will construct a solution $\bar{x}$ in M of $\Sigma \wedge \Gamma$. Let us index the elements of Γ'_0 by J and let us rewrite the j^{th} inequation as $\sum_{i \geq m-\ell+1} x_i \cdot d_{ij} \neq t_j(\bar{u}, \bar{v})$, where $d_{ij} \in R$ and t_j is an $\mathscr{L}_R$-term, and let i_j be the first i such that $d_{ij} \neq 0$. Let m_1 be the maximal such i_j, $j \in J$, and let J_1 be the subset $\{j \in J : i_j = m_1\}$ of J. If the set $\{i : m_1 < i \leq m\}$ is not empty, choose arbitrarily in M_0 the components $x_{m_1+1}, \dots, x_m$ of $\bar{x}$. Then, choose $x_{m_1} \in M_0$ satisfying the following inequations: $\bigwedge_{j \in J_1} x_{m_1} \cdot d_{m_1 j} \neq t_j(\bar{u}, \bar{v}) - \sum_{i > m_1} x_i \cdot d_{ij}$ (there exists such an x_{m_1} in M_0, since there are only finitely many forbidden choices and M_0 is infinite). Then, let m_2 be the maximal i_j, $j \in J - J_1$, and let J_2 be the subset $\{j \in J - J_1 : i_j = m_2\}$ of $J - J_1$. If the set $\{i : m_2 < i < m_1\}$ is not empty, choose arbitrarily in M_0 the components $x_{m_2+1}, \dots, x_{m_1-1}$ of $\bar{x}$. Then, choose $x_{m_2} \in M_0$ satisfying the following inequations: $\bigwedge_{j \in J_2} x_{m_2} \cdot d_{m_2 j} \neq t_j(\bar{u}, \bar{v}) - \sum_{i > m_2} x_i \cdot d_{ij}$. Proceed similarly up to the $(m+1-\ell)^{th}$-component of $\bar{x}$, then use the system of equations $\bar{x} \cdot A = \bar{a}$ to find successively, using axiom scheme 2, the first $m - \ell$ components of $\bar{x}$ in M_0. It is at this point that we use the fact that M_0 contains $\langle \bar{a} \rangle^{sep}$. ⊣

LEMMA 6.8. *Let ϕ, ψ be two p.p. $\mathscr{L}_R$-formulas defining two subgroups with ψ implying ϕ in T_R. Suppose that there exists a torsion-free $\mathscr{L}$-structure M modelling T_{free} with $[\phi(M) : \psi(M)] > 1$. Then, for any non-trivial torsion-free $\mathscr{L}$-structure N modelling T_{free}, $[\phi(N) : \psi(N)] = \infty$.*

PROOF. Let $a \in M$ be such that $(\phi(a) \wedge \neg\psi(a))$. By Corollary 6.5, there exist finitely many $\mathscr{L}$-terms $t_k(x)$ (respectively $s_l(x)$) such that $\phi(x)$ is equivalent modulo T_{free} to $\bigwedge_k t_k(x) = 0$ (respectively $\psi(x)$ is equivalent (modulo T_{free}) to $\bigwedge_{l \in L} s_l(x) = 0$). So, $\exists x \in M(\phi(x) \wedge \neg\psi(x))$ is equivalent to $\bigvee_{l \in L} \exists x \in M\, (\bigwedge_k t_k(x) = 0 \wedge s_l(x) \neq 0)$. There is some $m \geq 1$ such that the terms $t_k(x)$ and $s_l(x)$ are modulo T_{free} of the form $\sum_{j \in p^{me}} \lambda_j(x) \cdot d_j$ with $d_j \in R$ (see Corollary 3.3). Let us use new variables y_j for the $\lambda_j(x)$'s. Identify p^{me} with $\{0, \dots, p^{me} - 1\}$ and set $\bar{y} = (y_0, \dots, y_{p^{me}-1})$, and let $t'_k(\bar{y}) = \sum_j y_j \cdot r_{kj}$ (respectively $s'_l(\bar{y}) = \sum_j y_j \cdot s_{lj}$) be the terms obtained by substituting y_j for $\lambda_j(x)$, from $t_k(x)$ (respectively $s_l(x)$).

The system $\bigwedge_k t'_k(\bar{y}) = 0$ can be written as $\bar{y} \cdot A = 0$ and by Lemma 6.4, it is equivalent (modulo T_{free}) to the system $\bar{y} \cdot P \cdot \tilde{A} = 0$, where the matrix P is a permutation matrix and $\tilde{A}$ is an l.t.s. matrix of non zero co-rank ℓ since $\phi/\psi(M) > 1$. For ease of notation, set $n = p^{me}$, $\bar{z} = \bar{y} \cdot P$, and $\sum_{0 \leq j < n} y_j \cdot s_{lj} = \sum_{0 \leq j < n} z_j \cdot \tilde{s}_{lj}$, for some $\tilde{s}_{lj}$'s in R.

Now, the system $S_l(\bar{z})$ below, which is assumed to be consistent for each $l \in L$:

$$\begin{cases} \bar{z} \cdot \tilde{A} = 0 \\ \sum_{0 \leq j < n} z_j \cdot \tilde{s}_{lj} \neq 0, \end{cases}$$

is equivalent by Lemma 6.7 to a system of the form:

$$\begin{cases} \bar{z} \cdot \tilde{A} = 0 \\ \sum_{n-\ell<j<n} z_j \cdot s'_{lj} \neq 0, \end{cases}$$

where the inequation is non-trivial by consistency of $\phi \wedge \neg\psi$.

Let now N be a torsion-free $\mathscr{L}$-structure, modelling T_{free}. Suppose we have constructed d elements $x(1), \dots, x(d)$ satisfying ϕ and such that all $x(i)$, $x(i) - x(j)$, $i \neq j$, do not satisfy ψ. Denote by $v_j(i)$, $1 \leq i \leq d$, the λ_j-components, $0 \leq j \leq n-1$, of $x(i)$. We are looking for an element $x(d+1)$ satisfying ϕ and such that $x(d+1)$ and $x(d+1) - x(i)$, $1 \leq i \leq d$ do not satisfy ψ, i.e. $x(d+1) = \sum_{0\leq j<n} y_j \cdot t^m \cdot m_j$ such that $\bar{z} := (y_j)_{0\leq j<n} \cdot P$ satisfies one of the following systems $S_{l,f}(\bar{z}, \bar{v}(1), \dots, \bar{v}(d)) := \{\bar{z} \cdot \tilde{A} = 0\} \cup \Gamma_{l,f}(\bar{z}, \bar{v}(1), \dots, \bar{v}(d))$, where $l \in L$, $f = (f(1), \dots, f(d)) \in L^d$ and $\Gamma_{l,f}(\bar{z}, \bar{v}(1), \dots, \bar{v}(d))$ consists of:

$$\begin{cases} \sum_{n-\ell<j<n} z_j \cdot s'_{lj} \neq 0 \\ \bigwedge_{1\leq i\leq d} \sum_{n-\ell<j<n} (z_j - v_j(i)) \cdot s'_{f(i)j} \neq 0. \end{cases}$$

It suffices to apply Lemma 6.7 with $\bar{a} = \bar{0}$ and $\bar{c} = (\bar{v}(1), \dots, \bar{v}(d))$; note that in this case Γ'_1 is empty. ⊣

Proposition 6.9. *The theory T_e admits quantifier elimination and it is complete.*

Proof. First, any $\mathscr{L}$-formula is equivalent to an $\mathscr{L}_R$-formula in T_λ. Second, any theory of modules admits p.p. elimination i.e., every formula is equivalent to a boolean combination of positive primitive formulas modulo invariant sentences. An invariant sentence is one which specifies the index for one p.p. definable subgroup included in another (see [10], Chapter 2). By Corollary 6.5, it suffices to prove that the index (in $\omega \cup \{\infty\}$) of two p.p. definable subgroups (with one included in the other) is the same in any model of T_e. Note that this last property implies that T_e is complete. We may always work in an $\aleph_1$-saturated model M of T_e. By Proposition 5.5, the torsion submodel M_{tor} is isomorphic to $\mathbb{F}_p(B)^{sep}$ and in a nonprincipal ultrapower of M, say M^*, the corresponding ultrapower M^*_{tor} of M_{tor} is a direct summand as a pure-injective pure submodule (by Corollary 6.6) and $M^*/M^*{}_{tor}$ is a torsion-free non zero model of T_{free}. But by the Lemma above, the index of two p.p. definable subgroups (with one included in the other) is the same in any non-trivial torsion-free R-module purely embedded in a model of T_e. This suffices by [10] Lemma 2.23. ⊣

Remark 6.

1. In fact, we proved that not only our theory T_e admits quantifier elimination but that any positive primitive $\mathscr{L}$-formula is equivalent to a conjunction of atomic $\mathscr{L}$-formulas. This last property appears in the theory of modules and is denoted by "*elim* $-Q^+$" (see [10] p. 319). In particular, it means that the p.p. definable functions are in fact definable by a conjunction of atomic formulas.

2. The same proof also gives that the theory of the class of torsion-free models of T_{free} admits q.e. in $\mathscr{L}$ as well as any extension of T_{free} which specifies for each separable polynomial the finite number of elements it annihilates.

Corollary 6.10. *The $\mathscr{L}_R$-reduct of T_e is model-complete. It is axiomatized by T_{sep} and $\forall x\ \exists!(x_i)_{i\in p^e}\ x = \sum x_i \cdot t \cdot m_i$.*

PROOF. The functions λ_i's are $\mathscr{L}_R$-existentially definable in any model of T_e and T_e admits quantifier elimination in the language $\mathscr{L}$ by the above proposition. ⊣

COROLLARY 6.11. *T_e has a prime model.*

PROOF. Apply Propositions 5.4 and 6.9. ⊣

COROLLARY 6.12. *T_e is decidable.*

PROOF. T_e is complete and recursively axiomatizable (a "natural" enumeration of R provides a recursively enumerable axiomatisation of T_e). The result may also be deduced from the fact that T_e is the $\mathscr{L}$-reduct of the theory of the separably closed field $\mathbb{F}_p(B)^{sep}$ which is decidable. ⊣

§7. Quantifier elimination in the case of infinite imperfection degree.

As in the finite imperfection degree case, the torsion submodule of any model of T_∞ is isomorphic to $\mathbb{F}_p(B)^{sep}$.

Any $\mathscr{L}_\infty$-formula is equivalent to an $\mathscr{L}_R$-formula in T_∞ since the $\lambda_i^{\bar{b}}$ functions are positively existentially $\mathscr{L}_R$-definable.

LEMMA 7.1. *Let A be a $h \times k$-matrix with coefficients in $\mathbb{F}_p(\bar{b})[t;\alpha]$, for some tuple $\bar{b} \in B^n$. Then, there are $\mathscr{L}_\infty$-terms $(\tau_l(.))_{l=1}^{s}$ of the form $\tau_l(.) = \lambda_{i_{l1}}^{\bar{b}} \circ \cdots \circ \lambda_{i_{lr_l}}^{\bar{b}}(.)$, with $i_{l,j} \in p^{nm_{l,j}}$ for all $j = 1, \ldots, r_l$, a k'-tuple of $\mathscr{L}_R$-terms $\bar{t}$, a permutation matrix P, an l.t.s. $h \times k'$-matrix B, and integers g_l, $1 \le g_l \le k$, such that the system of equations $\bar{y} \cdot A = \bar{u}$ is equivalent, modulo $T_{free,\infty}$, to*

$$\bar{y} \cdot P^{-1} \cdot B = \bar{t}\left(\left(\lambda_{i_{l,1}}^{\bar{b}} \circ \cdots \circ \lambda_{i_{l,r_l}}^{\bar{b}}(u_{g_l})\right)_{l=1}^{s}\right)$$
$$\wedge \bigwedge_{\substack{1 \le l \le s \\ 1 \le j \le r_l}} \lambda_{i_{l,j+1}}^{\bar{b}} \circ \cdots \circ \lambda_{i_{l,r_l}}^{\bar{b}}(u_{g_l}) \text{ has the } (\bar{b}, m_{l,j})\text{-t.p.}$$

PROOF. The proof is analogous to the proof of Lemma 6.4. Note that at each step, we look at a coefficient of minimal valuation and if $\bar{v} \cdot Q = \bar{w}$, where Q is an invertible matrix with coefficients in $\mathbb{F}_p(\bar{b})[t;\alpha]$, and if each component w_i of $\bar{w}$ has the $(\bar{b}, m)$-tp property for some m, then this also holds for each component v_i of $\bar{v}$. ⊣

PROPOSITION 7.2. *Every primitive positive $\mathscr{L}_R$-formula is equivalent, modulo $T_{free,\infty}$, to a positive quantifier-free $\mathscr{L}_\infty$-formula.*

PROOF. The proof is analogous to the proof of Corollary 6.5 with the additional information that the obtained terms occurring in the conjunction of atomic formulas equivalent to a given p.p. formula have the $(\bar{b}, m)$-tp.-property, for some m and tuple $\bar{b} \in B^n$, for some n. Note that this last property is expressible by a positive quantifier-free $\mathscr{L}_\infty$-formula (see Definition 5). ⊣

LEMMA 7.3. *Let ϕ, ψ be two p.p. $\mathscr{L}_R$-formulas respectively defining two subgroups with ψ implying ϕ in T_R. Suppose that there exists a torsion-free $\mathscr{L}_\infty$-structure M modelling $T_{free,\infty}$ with $[\phi(M) \colon \psi(M)] > 1$, then $[\phi(N) \colon \psi(N)] = \infty$, for any non-trivial torsion-free $\mathscr{L}_\infty$-structure N modelling $T_{free,\infty}$.*

PROOF. Again the proof of this lemma is analogous to the proof of Lemma 6.8. Let $t_k(x), s_\ell(x)$ be $\mathscr{L}_\infty$-terms such that $[\phi(x) \wedge \neg\psi(x)] \leftrightarrow \bigvee_\ell [\bigwedge_k t_k(x) = 0 \wedge s_\ell(x) \neq 0]$. Each $\mathscr{L}_\infty$-term t_k (respectively s_ℓ) is equivalent to a term of the form

$\sum_{\bar{\imath}} \lambda_{\bar{\imath}}^{\bar{b}}(x) \cdot r_{\bar{\imath}}$ (respectively $\sum_{\bar{\jmath}} \lambda_{\bar{\jmath}}^{\bar{b}}(x) \cdot s_{\bar{\jmath}}$), where $r_{\bar{\imath}}, s_{\bar{\jmath}} \in R - \{0\}$ (see Lemma 4.1). Notice that we may assume that the λ functions have the same superscript $\bar{b} \in B^n$. Also, for the clarity of the exposition, we reverted to the tuple notation for their subscripts which might not have the same length. We prove by induction on the maximal length r of $\bar{\imath}$ for functions $\lambda_{\bar{\imath}}$ occuring in ϕ and ψ that, as in 6.8, a system Σ of the form $\bigwedge_k t_k(x) = 0 \wedge s_\ell(x) \neq 0$ has infinitely many "unequivalent" solutions if it has one. In other words there is a quantifier-free $\mathscr{L}_R$-formula Θ such that Σ is equivalent to $\Theta(\lambda_{\bar{\imath}}^{\bar{b}}(x) \colon \bar{\imath} \text{ of length } \leq r)$. We know the result for $r = 0$ as Lemma 6.7 also holds for infinite e. Now, for $e > 0$, Σ is equivalent to

$$\left[\Theta\left(\lambda_{\bar{\imath}}^{\bar{b}}(x) \colon \bar{\imath} \text{ of length } \leq r\right) \wedge x = \sum \lambda_{\bar{\imath}}^{\bar{b}}(x) \cdot t \cdot m_{n,\bar{\imath}}(\bar{b})\right] \vee \Theta\left(\left(\lambda_{\bar{\imath}}^{\bar{b}}(x) \colon \bar{\imath} \text{ of length } < r\right), \bar{0}\right).$$

Using induction hypothesis, we have only to deal with the first term of the disjunction, and, by 6.7 again, if such a system has in some non zero torsion-free model N of $T_{free,\infty}$ a solution x having the $(\bar{b}, r)$-t.p., then it has one in any such structure. Therefore, we may proceed as in the finite imperfection degree case to produce unequivalent solutions. ⊣

PROPOSITION 7.4. *T_∞ admits quantifier elimination and is complete; for any countable $B_0 \subset B$, the corresponding reduct is decidable.*

PROOF. As in Corollary 6.12, we take a recursive presentation of the ring $\mathbb{F}_p(B_0)[t; \alpha]$. ⊣

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