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The theory of modules of separably closed fields 2

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Abstract

In Dellunde et al. (J. Symbolic Logic 67(3) (2002) 997–1015), we determined the complete theory T_e of modules of separably closed fields of characteristic p and imperfection degree e, $e \in \omega \cup \{\infty\}$. Here, for $0 \neq e \in \omega$, we describe the closed set of the Ziegler spectrum corresponding to T_e. Further, we establish a correspondence between certain submodules and n-types and we investigate several notions of dimensions and their relationships with the Lascar rank. Finally, we show that T_e has uniform p.p. elimination of imaginaries and deduce uniform weak elimination of imaginaries.

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1. Introduction

Let $SCF_{p,e}$ be the theory of separably closed fields of characteristic p and of imperfection degree e, $e \in \omega$. Let α be the Frobenius map and B a set of cardinality e.

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In [8], we considered the models of $SCF_{p,e}$ as modules over the skew polynomial ring $\mathbb{F}_p(B)[t;\alpha]$ (see [6]). It is straightforward to adapt our preceding results to a slightly more general setting, namely instead of $\mathbb{F}_p(B)[t;\alpha]$, we will consider the skew polynomial ring $R := K[t,\alpha]$, where K is any field of characteristic p and imperfection degree e. We axiomatized the theory T_e of modules over R of the models of $SCF_{p,e}$ and we showed that it admits quantifier elimination in the usual language of R-modules augmented with unary functions which are the analog in this context of the p-component functions. (In that article, we considered and treated the case of infinite imperfection degree analogously to the case of finite imperfection degree. Here, however, we will mostly restrict ourselves to the case of finite imperfection degree).

Our purpose here is two-fold, first to describe a closed subset of the Ziegler spectrum of R corresponding to the additive reduct we are considering, of the theory of the separable closure of K, second to show that in our particular setting one obtains tighter correspondence than in the field case between the pure model theoretic objects as types, forking, ranks and their algebraic counterparts.

We will first show that the theory T_e, in the case $e \neq 0$, is not superstable (as in the field case). We will give a description of the closed set $\mathbb{U}_{K^{\text{sep}}}$ of the Ziegler spectrum of R with the additional hypothesis of countability on K. We will show on one hand that there are uncountably many torsion-free points (which are all elementarily equivalent from our previous analysis) and on the other hand that there are $\aleph_0$ many non-isomorphic points with non-trivial torsion.

Then we will establish a correspondence between 1-types and certain submodules of the free R-module on countably many generators. We will prove an analogue of the Nullstellensatz (see [7, p. 154]). We will use it together with the fact that the set of separable elements in our skew polynomial ring is a right denominator set, which entails that we have a separable closure for torsion-free summands of models of T_e and so a prime model construction, to show that one can read off the U-rank of a type in the length of a chain of sub-models.

We will show that definable subgroups in the models of T_e are (definably) connected-by-finite (this has already been proven in the case of separably closed fields (see [3]), but the proof there is more involved, as one can expect) and that a definable group over the empty set is definable by a positive primitive formula (this is unknown in general).

We will compare different notions of dependence relations, one (the R-dependence relation) coming from the fact that our ring is right Ore and the others coming from stability theory and the model theory of modules.

We will observe that one obtains with the rank coming from the R-dependence relation and the U-rank similar pathologies to the ones shown in [1,3,5].

Finally, reading the results of Kucera and Prest [15] in our context, we show that T_e has uniform primitive positive (p.p.) elimination of imaginaries and from that we deduce uniform weak elimination of imaginaries, for $e \in \omega \cup \{\infty\}$. Note that in the field case to get elimination of imaginaries, when e is finite, one has to add constants for elements of a p-basis and that one does not know, when e is infinite, an appropriate language to get elimination of imaginaries (see [7]).

2. Preliminaries

Let us recall the notations and basic results obtained in [8].

Let K be a field of characteristic p and imperfection degree e (which we do not assume to be separably closed) and let R be the skew polynomial ring $K[t;\alpha]$, with the commutation rule $k.t=t.k^{\alpha}$, for all $k\in K$, where α is in this case the Frobenius map. The elements of R are of the form $q(t)=\sum_{i=0}^{n} t^i.a_i,\ a_i\in K$. The element $q(t)$, or simply q, is called *separable* if a_0 is non-zero (and *monic* if a_n is 1). This ring R is right principal (see [6, Chapter 2]), but not left Ore whenever α is not surjective. The set X of all separable elements of R is a *right denominator set*, which means that, for all $r\in R$ and $x\in X$, the set $r.X\cap x.R$ is not empty (see Section 4, Definition 4.1).

Let $B=\{b_i;0\leqslant i<e\}$ be a p-basis of K with the convention that if $e=0$, then $B=\emptyset$.

The language of right R-modules is $\mathscr{L}_R=\{+,-,0,r(\cdot);r\in R\}$, where for any $r\in R$, $r(x):=x\cdot r$ (scalar multiplication by the ring element r), where x belongs to a right R-module. Let T_R be the theory of all right R-modules in this language.

We will always deal with *right* modules, and so we will no longer specify it.

Definition 2.1. Let e be finite. Given $i\in p^e$, let m_i be the monomial $b_0^{i(0)}\cdot\ldots\cdot b_{e-1}^{i(e-1)}$.
Let $T(e)$ be the following $\mathscr{L}_R$-theory:

(1) T_R the theory of all (right) R-modules,
(2) $\forall x\exists y,\ x=y\cdot q(t)$, for any $q(t)\in X$,
(3) $(\exists x_i)_{i=1}^{d}$ (the x_i's are linearly independent over $\mathbb{F}_p\wedge\bigwedge x_i\cdot q(t)=0$) and $\neg[(\exists x_i)_{i=1}^{d+1}$ (the x_i's are linearly independent over $\mathbb{F}_p\wedge\bigwedge x_i\cdot q(t)=0)]$, for any $q(t)\in X$ monic of degree d,
(4) $\forall x\ \exists(x_i)_{i\in p^e}\ \ x=\sum_{i\in p^e} x_i\cdot t\cdot m_i$,
(5) $\forall x\forall(x_i)_{i\in p^e}\left(\sum_{i\in p^e} x_i\cdot t\cdot m_i=0\rightarrow\bigwedge_{i\in p^e} x_i=0\right)$.

It is convenient to single out different sub-theories of $T(e)$, namely, let T be the subtheory where we do not specify the torsion, so T consists of axioms schemes (1), (2), (4) and (5); let T_λ be axioms schemes (1), (4) and (5) and finally let $T_{\rm sep}$ be axioms schemes (1)–(3). Let $T^{\rm tf}$ be the theory T plus the scheme of axioms (one for each $r\in R-\{0\}$): $\forall m\ (m\cdot r=0\rightarrow m=0)$. (We will use this superscript $^{\rm tf}$ to denote such extension of a theory of R-modules.) Note that in any model M of T_λ one can define p^e unary additive functions $\lambda_i,\ i\in p^e$, as follows. By axiom (4), any element x of M can be written as $x=\sum_{i\in p^e} x_i\cdot t\cdot m_i$ and we set $\lambda_i(x)=x_i$; note that axiom (5) ensures that it is well-defined.

Each of the class of models of either T_λ, T or $T^{\rm tf}$ is closed under direct summands and direct products. Note also that if M is a model of $T^{\rm tf}$ and N be a model of $T(e)$, then $M\oplus N$ is a model of $T(e)$.

In [8] we considered only the case where $K=\mathbb{F}_p(B)$, but it is straightforward to extend our former results to the present setting. Let us rephrase them as follows. In order to get a quantifier elimination result, we considered an expansion by definition of $T(e)$, which will be denoted by T_e. We extended the usual R-module language by adding new unary additive functions (the functions λ_i's defined above). We will repeatedly use the fact that they are definable by p.p. $\mathscr{L}_R$-formulas.

Let $\mathscr{L} = \mathscr{L}_R \cup \{\lambda_i;\ i \in p^e\}$, where the λ_i's are unary functions.

Let T_e (respectively T_{free}) be the following $\mathscr{L}$-theory:

(1) $T(e)$ (respectively T),
(2) $\forall x\ x = \sum_{i \in p^e} \lambda_i(x) \cdot t \cdot m_i$.

It is straightforward to show that any model M of T can be uniquely expanded to a model of T_{free} and any pure submodule of M can be expanded to an $\mathscr{L}$-substructure of M.

In [8, Corollary 6.5], we showed that in T_{free}, any p.p. $\mathscr{L}_R$-formula is equivalent to a conjunction of atomic $\mathscr{L}$-formulas. So, conversely, any $\mathscr{L}$-substructure N of a model M of T_{free} is a pure submodule of M.

We will use several times the following fact [8, Proof of 3.5, item 3]: let N be an $\mathscr{L}$-substructure of a model M of T_{free}, let $x \in M$ be such that $x \cdot q = n \in N$ for a separable $q \in R$. Then $\lambda_i(x)$, $i \in p^e$, belongs to $N\langle x\rangle_R$, the R-submodule of M generated by N and x. More precisely, we may assume without loss of generality that $q = 1 - t.q'$, so $t.q' = \sum_i \sqrt[p]{q'_{(i)}}.t.m_i$, and then $\lambda_i(x) = x \cdot \sqrt[p]{q'_{(i)}} + \lambda_i(n)$.

Together with the relative quantifier elimination result mentioned above, this implies that the torsion submodule M_{tor} of any model M of T_{free} is an $\mathscr{L}$-substructure and so a pure submodule (see [8, Corollary 6.6]).

The index of two p.p. definable subgroups with one included in the other is either 1 or infinite simultaneously in all torsion-free summands of a model of T_e (see [8, Lemma 6.8]). Note that it implies that T^{tf} is complete.

Furthermore, it entails that T_e (respectively $T^{\mathrm{tf}}_{\mathrm{free}}$) admits quantifier elimination, even positive quantifier elimination, i.e. any positive primitive formula is equivalent to a conjunction of atomic formulas [8, 6.5, Remark 6]. Moreover, using axiom 5 of $T(e)$, and iterates of the λ functions (see [8, Notation 3.3]), it follows that a conjunction of atomic formulas is equivalent in T_{free} to one atomic formula:

$$\bigwedge_{k < p^{m.e}} t_k(\bar{x}) = 0 \leftrightarrow \sum_{k \in p^{m.e}} t_k(\bar{x}) \cdot t^m \cdot m_k = 0.$$

We showed that in a model of T_e the torsion submodule is isomorphic to the separable closure K^{sep} of the field K and in addition that any $\mathscr{L}$-substructure M is contained in a minimal submodel M^{sep} of T_e, the *separable closure* of M [8, Corollary 5.9].

Therefore, any pure-injective model of T_e can be decomposed as a direct sum of the pure-injective hull of K^{sep} and a torsion-free summand. It follows that T_e is a complete reduct of the theory of the separably closed fields of characteristic p and imperfection degree e and so is decidable. This entails that the $\mathscr{L}_R$-reduct $T(e)$ of T_e is complete and decidable as well.

Considering now two models N and N' of T such that $N_{\mathrm{tor}} \cong N'_{\mathrm{tor}}$, note that $N \equiv N'$. Indeed, by the same argument as above but now applied to N_{tor} (instead of K^{sep}), we have that N_{tor} is pure in N and so if we denote by N^* a non-principal ultrapower of N, we get on one hand that $N^* = N^*_{\mathrm{tor}} \oplus N_0$ and on the other hand that $N'^* = N'^*_{\mathrm{tor}} \oplus N_1$, where $N_0 \subseteq N^*$, $N_1 \subseteq N'^*$ and $N_0,\ N_1 \models T^{\mathrm{tf}}$. Since $N_0 \equiv N_1$, we obtain $N' \equiv N$.

For the axiomatization in the infinite imperfection degree case (namely when e is infinite), see [8, Sections 4 and 7]. We got analogous results to the ones in the finite imperfection degree case. Recall that we have denoted the corresponding theories by T_∞. These last theories will come up in the last section when we discuss the elimination of imaginaries.

3. Ziegler space of indecomposables

Let Zg_R be the Ziegler spectrum associated with the ring R. It is a topological space whose points are the indecomposable pure-injective R-modules and whose basic open sets are of the form (ϕ/ψ) where ϕ, ψ are two p.p. formulas in one variable with ψ implying ϕ in T_R, and (ϕ/ψ) consists of all indecomposable pure-injective R-modules where the index of ψ in ϕ is strictly greater than 1 (see [22]).

Definition 3.1 (see [22, Section 8 and Corollary 4.10]). Let M be an R-module. Let $\mathbb{U}_M$ be the class of all indecomposable pure-injective modules which are direct summands in modules elementarily equivalent to M. Such subsets are the closed subsets of Zg_R.

Since $T(e)$ is complete, any model of $T(e)$ is elementarily equivalent to K^{sep}. Therefore, the pure-injective indecomposable R-modules occurring as direct summands of models of T in Zg_R belong to $\mathbb{U}_{K^{\mathrm{sep}}}$.

Since the theory $T(e)$, $e \in \omega$ is decidable, this closed subset $\mathbb{U}_{K^{\mathrm{sep}}}$ of Zg_R should be well understood (see [22, Remark after Theorem 9.4]). Note that a pure-injective indecomposable module which is not in $\mathbb{U}_{K^{\mathrm{sep}}}$ can be obtained as follows. Consider $\hat{R}$ the completion of R, i.e. it consists of the power series of the form $\sum_{i\in\mathbb{N}} t^i.a_i$, then consider the right denominator set $\{t^i; i \in \mathbb{N}\}$ and take the field of fractions of $\hat{R}$ with respect to that subset. One obtains a skew field (see [6, Section 2.4]) whose elements are of the form $\sum_{i\in\mathbb{N}} t^i.a_i.t^{-n}$, $n \in \mathbb{N}$. It is not a model of $T(e)$ because it does not satisfy axiom 5. (Take $a \in K - K^p$, then $a = a.t^{-1}.t = \sum a_i^p.t^{-1}.t.m_i$.)

First, we prove, as in the case of separably closed fields (see [21]), that the theory $T(e)$ is not superstable, for $e \in (\omega - \{0\}) \cup \{\infty\}$.

This follows from the following more general result. In the next proposition and lemma, we place ourselves in the following setting, namely in a class of modules over a skew polynomial ring $S := F[t; \alpha]$, where F is an infinite field, α an endomorphism of F, and where multiplication by t is injective.

Proposition 3.2. *Let M be an S-module. Assume in addition that α is a non-surjective endomorphism of F and that in M multiplication by t is not surjective. Then, the theory of M is stable but not superstable.*

Proof. First, any complete theory of modules is stable (see [18, Theorem 3.1]). Second, we apply Theorem 2.1 in [22]. Let us show that in the following chain of p.p. definable subgroups $M \supseteq M \cdot t \supseteq M \cdot t^2 \supseteq \cdots$, each one is of infinite index in its predecessor. By the way of contradiction, suppose that $[M : M\cdot t]=\ell$ (the argument is the same for any consecutive pair in the chain); let $f_0,\dots,f_\ell$ be distinct elements of $F-\{0\}$ and let $m\in M-M\cdot t$. Then, the elements $m\cdot f_0^\alpha$, $m\cdot f_1^\alpha,\dots,m\cdot f_\ell^\alpha$ are in different classes modulo $M\cdot t$. For if not, then $m\cdot(f_i^\alpha-f_j^\alpha)\in M\cdot t$, where $0\leqslant i<j\leqslant\ell$, i.e. $m\cdot(f_i^\alpha-f_j^\alpha)=n\cdot t$, for some $n\in M$. So, $m=n\cdot t.((f_i-f_j)^\alpha)^{-1}=n\cdot(f_i-f_j)^{-1}.t$ by the commutation rule and so $m\in M\cdot t$, a contradiction. □

To apply it to $T(e)$, it suffices to take $M:=K^{\text{sep}}$.

Then, we note the following property of models of T (which we prove in a slightly more general setting).

Lemma 3.3. *Let N be an S-module where multiplication by t is injective. Then N is a fully faithful S-module.*

Proof. Recall that N is *fully faithful* if for any non-zero S-submodule M of N, $ann(M):=\{r\in S;\ \forall m\in M,\ m\cdot r=0\}$ is equal to the zero ideal (see [11, p. 31]).

First, note that it is a two-sided ideal of S. Since the right ideals are principal, $ann(M)$ is of the form $f.S$ with in this case the additional property that $S.f\subseteq f.S$. Whenever f is non-zero, such an element is called *right invariant* (see [6, p. 57]) and so f is of the form $t^n.c$, where $c\in F-\{0\}$ (see [6, Proposition 2.2.8]). Therefore, if M is a non-zero module, $ann(M)=\{0\}$. □

In the following, assuming that $e\in\omega-\{0\}$ and that K is countable, we will show that if M is a model of T^{tf}, then there are $2^{\aleph_0}$ points in $\mathbb{U}_M\subseteq\mathbb{U}_{K^{\text{sep}}}$ We will need the following technical lemma (which holds whatever the cardinality of K is).

Lemma 3.4. *Let $e\in\omega-\{0\}$ and ϕ, ψ be a pair of p.p. $\mathscr{L}_R$-definable subgroups of M with $T_R\models\psi\to\phi$ and $T^{\text{tf}}\models[\phi:\psi]>1$. Then, there exists a p.p. $\mathscr{L}_R$-formula γ with $T_\lambda\models\psi\to\gamma\to\phi$ such that the indices of ψ in γ and of γ in ϕ are infinite in any model of T^{tf}. Further, there is another p.p. $\mathscr{L}_R$-formula γ' with the same properties and such that, modulo T^{tf}, γ and γ' are incomparable in the lattice of p.p. $\mathscr{L}_R$-definable subgroups.*

This means that, if we consider the pair (ϕ,ψ) as a sublattice of the lattice of p.p. definable subgroups in models of any completion of T, its m-dimension and its breadth, or its width, in the sense of [18, p. 205], are undefined (for the fact that the breadth of a pair is undefined iff its width is undefined, see [18, Lemma 10.7]).

Proof. Recall that any model of T can be expanded to an $\mathscr{L}$-structure and so becomes a model of T_{free}. Since T_{free} admits positive quantifier elimination, $\phi(x)$ and $\psi(x)$ are equivalent (in T_{free}) to a conjunction of atomic $\mathscr{L}$-formulas. Using Corollary 3.3 in [8] on normal form for terms, as in Lemma 6.8 in [8], substituting new variables x_i for the unary functions $\lambda_i(x)$, we get that $\phi(x)$ (respectively $\psi(x)$) is equivalent in T to an $\mathscr{L}_R$-formula of the form $\exists\bar{x}\ \ \theta(x,\bar{x})$ (respectively $\exists\bar{x}\ \ \theta'(x,\bar{x})$), where $\theta(x,\bar{x})$ is the formula $x=\sum_{i\in p^{en}}x_i\cdot t.m_i\wedge\bar{x}\cdot P.C=\bar{0}$, (respectively $\theta'(x,\bar{x})$ is the formula

$x=\sum_{i\in p^{en}} x_i \cdot t.m_i \wedge \bar{x}\cdot B=\bar{0})$, where $n\geqslant 0$, P is a permutation matrix and C is a lower triangular $p^{en}\times(j-1)$-matrix with coefficients in R, and with only separable coefficients on its diagonal. Since $|\phi(x)|$ is infinite, one has $j\leqslant p^{en}$. Set $\bar{z}=\bar{x}\cdot P$. Then for any solution $(a,\bar{a})$ of θ in a model of T, the tuple $\bar{a}':=\bar{a}\cdot P$ has the property that $(a'_0,\dots,a'_{j-2})\in\langle a'_{j-1},\dots,a'_{p^{en}-1}\rangle_R^{\text{sep}}$, with the convention that if $j=1$ (namely if $\phi(x)$ is equivalent to $x=x$) the tuple $(a'_0,\dots,a'_{j-2})$ is empty.

Since $\phi\wedge\psi$ is equivalent to ψ in T_R, using Euclidean division in R and Gauss elimination as in [8, 6.1] (first between $\bar{z}\cdot C=\bar{0}$ and $\bar{z}\cdot P^{-1}.B=\bar{0}$ and then inside what is, after having done that, the block of last columns of the same breadth as B), we obtain a permutation σ on p^{en} which is the identity on $\{0,\dots,j-2\}$ if $j\neq 1$, and an invertible matrix Q with coefficients in R such that, if P' is the matrix of σ, then $P'.P^{-1}.B.Q=:B'=:(b'_{i,h})$ is lower triangular and has the property that the coefficients $b'_{i,i}$, $0\leqslant i\leqslant j-1$, are non-zero. So, $\bar{z}\cdot P^{-1}.B=\bar{0}$ is equivalent to the system $\bar{z}\cdot P'^{-1}.B'=\bar{0}$; set $\bar{z}'=\bar{z}\cdot P'^{-1}=(z_0,\dots,z_{j-2},z_{\sigma(j-1)},\dots,z_{\sigma(p^{en}-1)})$, with the convention that if $j=1$ then the first part of the tuple is empty. Moreover, since there exists x in a model of T^{tf} such that $\phi(x)\wedge\neg\psi(x)$, we can take σ such that the coefficient $b'_{j,j}$ in the matrix B' is non-zero. Consider the equation $\sum_{k=j-1}^{p^{en}-1} z'_k\cdot b'_{k+1,j}=0$. Applying Lemma 3.4 in [8], we may always assume that at least one coefficient $b'_{k_0,j}$, for some $k_0\in\{j,\dots,p^{en}-1\}$, is separable. (Otherwise, we replace the equation by the system $\bigwedge_{i\in p^e}\sum_{k=j-1}^{p^{en}-1} z'_k\cdot\sqrt[p]{b''_{k+1,j(i)}}=0$, where each $b'_{k+1,j}=t.b''_{k+1,j}$.)

The equation $\sum_{k=j-1}^{p^{en}-1} z'_k\cdot b'_{k+1,j}=0$ is equivalent to the conjunction $\bigwedge_{i\in p^e}\lambda_i(\sum_{k=j-1}^{p^{en}-1} z'_k\cdot b'_{k+1,j})=0$. Now, the basic idea for defining the intermediate subgroup is to intersect ϕ with one of the elements of the above conjunct. Since the functions λ_i are additive, it is equivalent to $\bigwedge_{i\in p^e}\sum_{k=j-1}^{p^{en}-1}\lambda_i(z'_k\cdot b'_{k+1,j})=0$. We write $b'_{k_0+1,j}$ as $\alpha+t.q$, where α belongs to $K-\{0\}$ and q to R. We may assume without loss of generality that $\alpha=1$. We have $\lambda_i(z'_{k_0}\cdot b'_{k_0+1,j})=\lambda_i(z'_{k_0})+\lambda_i(z'_{k_0}\cdot t.q)$ and, by Remark 2, item 3 in [8], we have $\lambda_i(z'_{k_0}\cdot t.q)=z'_{k_0}\cdot\sqrt[p]{q_{(i)}}$. Replacing in that last term, z'_{k_0} by $\sum_{h\in p^e}\lambda_h(z'_{k_0})\cdot t.m_h$, we get a system of equations $\bigwedge_{i\in p^e}\gamma_i(\overline{\lambda(z'_k)})$ where the formula $\gamma_i(\overline{\lambda(z'_k)})$ is

$$\lambda_i(z'_{k_0})\cdot(1+t.m_i.\sqrt[p]{q_{(i)}})+\sum_{\substack{h\in p^e\\ h\neq i}}\lambda_h(z'_{k_0})\cdot t.m_h.\sqrt[p]{q_{(i)}}+\sum_{\substack{k=j-1\\ k\neq k_0}}^{p^{en}-1}\lambda_i(z'_k\cdot b'_{k+1,j})=0.$$

For $i\neq h$, the coefficients of $\lambda_i(z'_{k_0})$ in γ_i, respectively in γ_h, are distinct, namely that in γ_i is separable and that in γ_h is not. To see now that $\gamma_i\wedge\gamma_h\wedge\phi(x)$ is strictly smaller than say $\gamma_i\wedge\phi(x)$ (the argument for γ_h is similar), we proceed as follows. In order to simplify the notation, set $v_1:=\lambda_i(z'_{k_0})$ and $v_2:=\lambda_h(z'_{k_0})$ and write γ_i as $v_1\cdot q_1+v_2\cdot p_1+u=0$, where $q_1\in X$ (the set of separable elements of R), $p_1\in R-X$ and u is a term in the remaining variables; similarly write γ_h as $v_1\cdot p_2+v_2\cdot q_2+w=0$, where $q_2\in X$, $p_2\in R-X$ and w is a term in the other variables.

Now, X is a right denominator set and so $\exists t_1,\ t_2\in R$ with $q_1.t_1=p_2.t_2$ and $t_2\in X$. So, the system consisting of the two equations $\gamma_i\wedge\gamma_h$ is equivalent in T^{tf} to the system

$$\begin{cases} v_1 \cdot q_1 + v_2 \cdot p_1 + u = 0, \\ v_2 \cdot (p_1.t_1 - q_2.t_2) + u \cdot t_1 - w \cdot t_2 = 0 \end{cases}$$

with $(p_1.t_1 - q_2.t_2) \in X$. Now, take u and w arbitrarily, choose v_2 such that it does not satisfy the second equation of the system (this is possible since there is only one such element whenever u and w are fixed) and then choose v_1 in order to satisfy γ_i: this is always possible since its coefficient is separable in that equation. Then choose z'_k for $k < (j-1)$ in order to satisfy ϕ, this is always possible since we are in a model of T. We finish the proof by applying Lemma 6.8 of [8]. □

Corollary 3.5. *Suppose that* $e \in \omega - \{0\}$ *and* K *is countable and let* M *be a model of* T^{tf}. *Then there are* $2^{\aleph_0}$ *points in* $\mathbb{U}_M$ *(and therefore in* $\mathbb{U}_{K^{\mathrm{sep}}}$*).*

Proof. We apply Lemma 8.3 in [22] (it is there where the countability hypothesis is needed). □

Corollary 3.6. *Suppose that* $e \in \omega - \{0\}$ *and* K *is countable. Then, there exists a model* N *of* T^{tf} *which has no indecomposable direct summand.*

Proof. A complete theory of modules over a countable ring has a continuous part zero iff its width (or equivalently its breadth) is defined (M. Ziegler uses the term *bounded*) [22, Theorem 7.1]. □

Corollary 3.7. *Suppose that* $e \in \omega - \{0\}$ *and* K *is countable. Then, the Cantor–Bendixon rank of the closed subset corresponding to* T^{tf} *of the Ziegler spectrum of* R *is undefined.*

Definition 3.8 (see [22, before Theorem 3.6]). Let A be a subset of a pure-injective R-module M. The *hull* $H(A)$ of A (in M) is a maximal *small* extension of A [22, Corollary 3.10], i.e. every partial homomorphism from A to an R-module whose restriction to A is a *partial isomorphism* (recall that a partial isomorphism is a partial mapping that preserves p.p. formulas and negations of p.p. formulas), and it is defined by the following two properties:

(i) $H(A)$ is a pure-injective module such that $A \subseteq H(A) \subseteq M$, and $H(A)$ is pure in M.
(ii) for any pure-injective module B with $A \subseteq B \subseteq H(A)$ and B pure in M, then $B = H(A)$.

The *injective hull* $E(A)$ of a subset A in an injective R-module M is defined similarly as above replacing "pure-injective" by "injective" and "pure" by "included".

Notation 3.9 (see [22, Notation after Theorem 3.6]). Let M be a pure-injective R-module, a an element of M and p its p.p.-complete type over $\emptyset$ (i.e. the set of p.p. formulas and negations of such satisfied by a in M). A consequence of the Baur–Monk quantifier elimination result is that, in any theory of modules, the type of an element is axiomatized by its p.p.-complete type. The hull $H(a)$ of a in M is up to isomorphism determined by its p.p.-complete type p and so we will write indifferently $H(p)$ or $H(a)$.

Existence and uniqueness of the hull was proven by Fisher and may be found in [22, Theorem 3.6] or in [18, Chapter 4.1]. In particular, let $A \subseteq M$, M', where M, M' are pure-injective modules and the p.p. type of A is the same in M and in M'. Then, $H_M(A) \cong H_{M'}(A)$. The hull of any model M of $T(e)$ (respectively T^{tf}) can be expanded to an $\mathscr{L}$-structure (the functions λ_i are p.p. definable) and this expansion is a model of T_e (respectively $T^{\mathrm{tf}}_{\mathrm{free}}$) (both theories are model-complete). Note that hull and *pure* hull coincide in this case (for a definition of pure hull, see [22, p. 162]).

Further, suppose that N is an R-module included in a model M of T_e. Then, $H(N)$ contains the $\mathscr{L}$-substructure of M generated by N, in particular if N is a pure-injective R-submodule, it can be expanded to an $\mathscr{L}$-structure. Indeed, the functions λ_i are p.p. definable and so for $a \in N$, the set $\Lambda := \{\lambda_i(a)\colon\ i \in p^e\}$ is small over N and therefore is included in $H(N)$.

Corollary 3.10. *Assume K countable and that $e \notin \{0, \infty\}$, and let $S_1(\emptyset)$ be the set of complete 1-types over $\emptyset$ of T^{tf}. Then there are $2^{\aleph_0}$-points in $S_1(\emptyset)$.* □

Now, we will consider the points N in $\mathbb{U}_{K^{\mathrm{sep}}}$ for which the torsion submodule N_{tor} is non-trivial. We already know that $N_{\mathrm{tor}} \subseteq K^{\mathrm{sep}}$ (see [8]) and so N is included in $H(K^{\mathrm{sep}})$.

Recall that $q \in R$ is *prime separable* if it is separable, non-invertible, and cannot be written as a product of two non-zero invertible elements of R.

Lemma 3.11. *Let N be a model of T which is pure-injective and indecomposable and suppose that N_{tor} contains a non-zero element a annihilating some prime separable element p_0 of R. Let N' be another model of T which is pure-injective and indecomposable and suppose that $ann(p_0) \cap N'$ is non-trivial, i.e. of cardinality strictly bigger than 1. Then $N \cong N'$.*

Proof. First, let us note that the type of a is determined by the fact that $a \cdot p_0 = 0 \wedge a \neq 0$. First, since the element p_0 is prime, it is of minimal degree such that it annihilates a and so any polynomial q annihilating a is right divisible by p_0. Then, any p.p. $\mathscr{L}_R$-formula is equivalent to an atomic $\mathscr{L}$-formula [8, Corollary 6.5] and the $\mathscr{L}$-substructure generated by a coincides with the R-submodule generated by a. So any formula satisfied by a is equivalent to a boolean combination of atomic formulas of the form $a \cdot q = 0$, for some $q \in R$ and whether the later holds or not only depends on whether p_0 divides q. □

The preceding Lemma shows that there are at most $\max\{|K|, \aleph_0\}$ non-isomorphic indecomposable direct summands in $H(K^{\mathrm{sep}})$. When K is countable, we will show that there are exactly $\aleph_0$ of them.

First, we need to investigate in more detail the structure of those elements of $\mathbb{U}_{K^{sep}}$ with non-trivial torsion.

We begin by proving a property of pure-injective indecomposable models of $T_{\rm free}$, which follows from the fact that $T_{\rm free}$ has positive quantifier-free elimination. We need the following notion which appears in the setting of R-modules (see for instance [11, p. 71]).

Definition 3.12. An $\mathscr{L}$-structure is *trivial* if its domain is equal to $\{0\}$. An $\mathscr{L}$-structure is *uniform* if the intersection of any two non-trivial $\mathscr{L}$-substructures is non-trivial.

Lemma 3.13. *Let N be a model of $T_{\rm free}$ which is pure-injective and indecomposable. Then N is uniform.*

Proof. In a pure-injective indecomposable R-module, any two non-zero elements a, b are linked (see [18, Corollary 4.11]), i.e. there is a p.p. formula $\phi(x, y)$ such that $\phi(a, b)$ and $\neg\phi(a, 0)$. Then, as in Corollary 16.7 in [18], one applies the fact that $T_{\rm free}$ has positive quantifier elimination. □

Lemma 3.14. *Let N be a model of T which is pure-injective and indecomposable and suppose that $N_{\rm tor}$ contains a non-zero element a annihilating some prime separable element p_0 of R. Then, for any other element b of $N_{\rm tor} - \{0\}$, there exists $s \in R$ with $a = b \cdot s$. Moreover, a prime separable element q of R is annihilated by an element of $N - \{0\}$ iff it has the same degree as p_0 and if there exist q_1, $q_2 \in R$ with $deg(q_1)$, $deg(q_2) < deg(p_0)$ such that $q.q_1 = q_2.p_0$. In addition, the annihilators of two prime separable elements of R, whenever they are non-trivial, are isomorphic as $\mathbb{F}_p$-vector spaces.*

Proof. We first expand N into an $\mathscr{L}$-structure, this expansion is a model of $T_{\rm free}$ and so by the preceding lemma, there exist $\mathscr{L}$-terms σ, τ such that $\sigma(a) = \tau(b) \neq 0$. Since a, b both belong to $N_{\rm tor}$, we have that, for $i \in p^e$, $\lambda_i(a)$ (respectively $\lambda_i(b)$) belongs to the R-submodule generated by a (respectively by b). So we have a relation of the form (*) $a \cdot r + b \cdot s = 0$, with $r, s \in R$ and $a \cdot r \neq 0$. Assume now that r is of minimal degree but does not belong to K, w.l.o.g., $deg(r) < deg(p_0)$. Then, applying the right Euclidean algorithm in R, we get: $p_0 = r.r_1 + r_2$ with $deg(r_2) < deg(r)$ and $r_2 \neq 0$ (p_0 is prime), hence $a \cdot r_2 + b \cdot s.r_1 = 0$, a contradiction. So, we may assume that the relation (*) is of the form $a = b \cdot s'$, with $deg(s')$ less than $deg(q)$ where q is of minimal degree such that $b \cdot q = 0$. (Note that such q is necessarily separable.)

Now, assume furthermore that q is a prime polynomial. Since $b \cdot s'.p_0 = 0$, there exists q_1 in R such that $q.q_1 = s'.p_0$. Note that since $deg(q) > deg(s')$, s' and q are relatively prime. Now we apply Theorem 11 in [16], so p_0 is divisible by $s'^{-1}.lcm(s', q)$ and $deg(s'^{-1}.lcm(s', q)) = deg(q)$. But, p_0 is prime, so $p_0 = s'^{-1}.lcm(s', q)$ and $deg(p_0) = deg(q)$. We also have that whenever $v \in ker(q)$, $v \cdot s' \in ker(p_0)$, and since $deg(s') < deg(q)$, $ker(s') = \{0\}$. So, there is an injection from $ann_N(q)$ into $ann_N(p_0)$, both finite. But in this reasoning the roles of a and b can be reversed and so there is a bijection between the two annihilators. The condition on the prime separable q is that it has the same degree as p_0 and that there exist q_1, q_2 with $deg(q_1)$, $deg(q_2) < deg(p_0)$ such that $q.q_1 = q_2.p_0$. □

Proposition 3.15. *Assume K is countable. Then, there are exactly $\aleph_0$ non-elementarily equivalent indecomposable pure-injective direct summands in $H(K^{\text{sep}})$.*

Proof. Let $\{p_n : n \in \mathbb{N}\}$ be an enumeration of the prime separable elements of R. Given n, let p_{n+k} be the first element on the list such that there is in K^{sep} a zero of p_{n+k} which does not belong to the $\mathbb{F}_p$-subspace generated by the annihilators of the p_i, $1 \leqslant i \leqslant n$. Consider the set of formulas $q_n(x) = \{\neg(\exists x_1, \ldots, x_n \ \ (x = \sum_{i=1}^{n} x_i \wedge \bigwedge_{i=1}^{n} x_i.p_i = 0)) \wedge x.p_{n+k} = 0\}$. Then $q_n(x)$ satisfies the hypothesis of Lemma 4.7 in [22]. So, there is an extension t_n of q_n which is consistent with K^{sep}, complete and indecomposable, in the sense that $H(t_n)$ is indecomposable. We get that $H(t_n)$ embeds in $H(K^{\text{sep}})$. This provides at least $\aleph_0$ non-elementary equivalent indecomposable direct summands in $H(K^{\text{sep}})$. By Lemma 3.11, we have at most $\aleph_0$ non-isomorphic pure-injective indecomposable direct summands in $H(K^{\text{sep}})$. □

Question: Does the conclusion of the above proposition holds without the hypothesis of countability?

In [8], we pointed out that any model of T_λ can be viewed as a module over the ring of endomorphisms $R' := R[\lambda_i; i \in p^e]$. We will prove in 5.3 that R' is the ring of all definable scalars $R_{\mathbb{U}_{K^{\text{sep}}}}$ (see [4, p. 189]). It is classical to consider modules as modules over their ring of definable endomorphisms and Blossier [1] has developed this point of view for separably closed fields. He has also given an abstract description of R'.

4. Separable closure

In this section, we will show that we can embed any torsion-free R-module which is a model of T_λ into a model of T. Further, we will prove that any model of T^{tf} can be viewed as a module over a non-commutative $\mathbb{Z}$-valued valuation ring (see [6, p. 83, 2.6]), namely the ring of fractions $R.X^{-1}$ of R with respect to the set X of separable elements of R.

In the next section, we will use this construction in order to establish the correspondence between certain submodules of the free countably generated R-module and $\mathscr{L}_R$-types over $T(e)$.

Definition 4.1. A multiplicative subset X of a domain S is a *right denominator set* if $\forall r \in S, \forall x \in X,\ r.X \cap x.S \neq \emptyset$.

In such a situation, there exists a ring of fractions $S.X^{-1}$ for S with respect to X (we use here that two elements in a right denominator set have a right common multiple belonging to this set, see [11, Lemma 9.6, Theorem 9.7]). Then any right S-module A has a module of fractions $A \cdot X^{-1}$ with respect to X, which carries a unique right

$S.X^{-1}$-module structure [11, Theorem 9.13, 9.12(b)], in fact $A\cdot X^{-1}\simeq A\otimes_S(S.X^{-1})$ and any element m of $A\otimes S.X^{-1}$ may be written as $a\otimes x^{-1}$ with $a\in A$ and $x\in X$ (see [11, Proposition 9.14 and its proof]).

Note that A embeds in its module of fractions whenever A has no X-torsion, i.e. $\{a\in A: \exists x\in X\ a\cdot x=0\}=\{0\}$.

Lemma 4.2. *The set X of separable elements of the skew polynomial ring R is a right denominator set and R embeds canonically in $R.X^{-1}$.* □

This is Lemma 5.7 in [8] and the fact that X contains 1 and that R has no X-torsion.

Proposition 4.3. *Let A be a torsion-free right R-module. Then,*

(1) *A embeds in $A\cdot X^{-1}\oplus K^{\rm sep}$ which is a model of $T_{\rm sep}$.*
(2) *If in addition A is a model of T_λ, then $A\cdot X^{-1}$ is a model of $T^{\rm tf}$.*
Therefore, if A is a model of $T^{\rm tf}$, then A can uniquely be endowed with a structure of right $R.X^{-1}$-module (compatible with its right R-module structure).

So, $T^{\rm tf}_{\rm free}$ is a model-companion of the expansion by definition to $\mathscr{L}$ of $T_\lambda{}^{\rm tf}$.

Proof. (1) By the above Lemma, X is a right denominator set. Since A is torsion-free, A embeds in $A\cdot X^{-1}$. Consider now the R-module $A\cdot X^{-1}\oplus K^{\rm sep}$. It satisfies axiom scheme 2 since $A\cdot X^{-1}$ is X-divisible and it satisfies axiom scheme 3 since $A\cdot X^{-1}$ is X-torsion-free (see [11, Proposition 9.12(a)]).

(2) Assume now in addition that A is a model of T_λ. We will show that $A\cdot X^{-1}$ can be expanded to an $\mathscr{L}$-structure. By definition, for any element b of $A\cdot X^{-1}$, there exists $q\in X$ such that $b\cdot q\in A$. Suppose that the degree of q is minimal such and denote by a the element $b\cdot q$. Then, write $q=\alpha-t.q_0$, where $\alpha\in K-\{0\}$. W.l.o.g., we may assume that $\alpha=1$. We have $b=a+b\cdot t.q_0$. So, we define $\lambda_i(b):=\lambda_i(a)+b\cdot\sqrt[p]{(q_0)_{(i)}}$, hence $b=\sum_i\lambda_i(b)\cdot t.m_i$ (see Notation 3.2, Remark 2 and Proposition 3.5 (Proof of item 3) in [8]). It remains to show that if $\sum_i b_i\cdot t.m_i=0$, then $\bigwedge_i b_i=0$. Using the isomorphism between $A\cdot X^{-1}$ and $A\otimes R\cdot X^{-1}$, we further may write each b_i as $a_i\otimes x_i^{-1}$, with $x_i\in X$ and $a_i\in A$. Since X is a right denominator set of R, there exists $r_i\in R-\{0\}$ and $x\in X$ such that $r_i.x^{-1}=x_i^{-1}$. So, we have $\sum_i a_i\cdot r_i.x^{-1}.t.m_i=0$. Now, we view this sum as an element of the tensor product of the right R-module $A\cdot X_R^{-1}$ with the left R-module ${}_RR$. We have the isomorphism $A\cdot X^{-1}\otimes_R R\cong A\cdot X^{-1}$. We consider the elements $t.m_i$'s as a set of left generators of ${}_Rt.R$. There exist elements u_j, $j\in J$, of $A\cdot X^{-1}$ and elements a_{ij} in R such that $(a_{ij}).(t.m_i)=0$ and $(a_i\cdot r_i.x^{-1})=(u_j)\cdot(a_{ij})$ (see [19, Proposition 8.8, Chapter 1]). But the ring R is a domain and so the matrix (a_{ij}) is the zero matrix. This implies that $a_i\cdot r_i.x^{-1}=0$ for each i and, A being torsion-free, this entails that each a_i and so each b_i is equal to 0. The last part of the statement is clear. □

Remark 4.4. As in the commutative case (see [22, Lemma 5.3]), we have the following. Given M,N two $R.X^{-1}$-modules and M_R, N_R their reducts in the language of

R-modules, then, $M \equiv N$ iff $M_R \equiv N_R$. (The verification is straightforward, using simply that, given n elements $r_1 x_1^{-1}, \ldots, r_n x_n^{-1} \in R.X^{-1}$ with $r_1, \ldots, r_n \in R$, $x_1, \ldots, x_n \in X$, there exist $x \in X$, $s_1, \ldots, s_n \in R$ such that $r_1 x_1^{-1} = s_1 x^{-1}, \ldots, r_n x_n^{-1} = s_n x^{-1}$. Hence for ϕ a p.p. $\mathscr{L}_{R.X^{-1}}$-formula, $\phi(\bar{z}) := (\exists \bar{y} \quad \bar{y} \cdot A = \bar{z})$, there is some $x \in X$ such that $\phi \cdot x := (\exists \bar{y} \quad \bar{y} \cdot A.x = \bar{z})$ is a p.p. $\mathscr{L}_R$-formula; now ϕ and $\phi \cdot x$ are equivalent in any X-torsion-free and X-divisible R-module.)

Moreover, as in the commutative case, to describe the pure-injective indecomposable modules over $R.X^{-1}$ is equivalent to describing the pure-injective indecomposable X-torsion-free modules over R.

From the preceding proposition and the fact that a direct sum of a model of $T(e)$ with a model of T^{tf} is a model of $T(e)$, we deduce the following result.

Corollary 4.5. *Let A be a model of T_λ^{tf}. Then A can be embedded in a model of $T(e)$.* $\square$

In Proposition 5.8 of [8], given a model M of T_{sep} and an R-submodule A of M, we constructed the separable closure of A in M, which we denoted by A^{sep}. Moreover, in case M is a model of T_e, A^{sep} is a model of T_e. Let us see the relationship between A^{sep} and the module of fractions $A \cdot X^{-1}$. Assume that $K^{\mathrm{sep}} \subseteq A$, then $A^{\mathrm{sep}}/K^{\mathrm{sep}}$ is a module of fractions for A/K^{sep} and therefore by uniqueness of the $R.X^{-1}$ module structure on A/K^{sep}, we have that $A^{\mathrm{sep}}/K^{\mathrm{sep}} = (A/K^{\mathrm{sep}}) \cdot X^{-1}$. Observe that $A^{\mathrm{sep}}/K^{\mathrm{sep}}$ is also a module of fractions for A, so we get that

$$A \cdot X^{-1} \cong A^{\mathrm{sep}}/K^{\mathrm{sep}} \cong (A/K^{\mathrm{sep}}) \cdot X^{-1}.$$

Let us denote by $\mathscr{M}$ the monster model of $T(e)$ and by $\mathscr{M}_{\mathscr{L}}$ the expansion of $\mathscr{M}$ to a model of T_e. Given any subset A of $\mathscr{M}$, we denoted by $(A)_{\mathscr{L}}$ the $\mathscr{L}$-substructure of $\mathscr{M}_{\mathscr{L}}$ generated by A and $\langle\langle A \rangle\rangle := ((A)_{\mathscr{L}})^{\mathrm{sep}}$ is the model-theoretic algebraic closure of A (see [8]), it is also the unique minimal prime model of T_e over A. We give now a characterization of the definable closure of A, which we denote by $(A)_{\mathrm{def}}$.

Lemma 4.6. *Let A be a subset of $\mathscr{M}$, then $(A)_{\mathscr{L}} = (A)_{\mathrm{def}}$ and thus every element of $(A)_{\mathrm{def}}$ is equal to an $\mathscr{L}$-term in elements of A.*

Proof. Clearly $(A)_{\mathscr{L}} \subseteq (A)_{\mathrm{def}}$. Suppose that there is an element $x \in (A)_{\mathrm{def}} - (A)_{\mathscr{L}}$. Since $((A)_{\mathscr{L}})^{\mathrm{sep}}$ is a model of T_e and T_e model-complete (see [8, Corollary 5.10]), $x \in ((A)_{\mathscr{L}})^{\mathrm{sep}}$. So there is $q \in R$ separable and of minimal degree with the property that $x \cdot q \in (A)_{\mathscr{L}}$. Among all such elements x, take one with the degree of the corresponding polynomial q minimal. Note then that such q is prime. Let $a \in (A)_{\mathscr{L}}$ be such that $x \cdot q = a$. We claim that $q \in K$ and so we get a contradiction. Assume, searching for a contradiction, that $\deg(q) \geqslant 1$, so there is $y \in \mathscr{M}$, $y \neq x$ such that $y \cdot q = a$. Since q is prime, the map $\phi : \langle x \rangle_R + (A)_{\mathscr{L}} \to \langle y \rangle_R + (A)_{\mathscr{L}}$ defined by $x \cdot r + c \mapsto y \cdot r + c$ is well defined and it is an isomorphism of R-modules. Since $\langle x \rangle_R + (A)_{\mathscr{L}}$ and $\langle y \rangle_R + (A)_{\mathscr{L}}$ are closed under the λ_i (see [8, Proof of Proposition 3.6]), they are isomorphic $\mathscr{L}$-

structures over A. Therefore, x and y have same type over A, which is absurd because $x\in(A)_{\mathrm{def}}$. We can conclude that $q\in K$ and thus that $x\in(A)_{\mathscr{L}}$, a contradiction. □

5. Description of types. Finite imperfection degree

In this section, we are going to establish a correspondence between p.p. complete $\mathscr{L}_R$-types of $T(e)$ and certain submodules of the free right R-module generated by countably many variables.

Notation 5.1. Let x_∞ be the ω-tuple of variables $(x_i : i\in(p^e)^{<\omega})=(x,x_0,\dots,x_{p^e-1},x_{00},\dots)$ and S any ring extension of R. We denote by $\langle x_\infty\rangle_S$ the free right S-module generated by x_∞ and by I^0_S be the right S-submodule of $\langle x_\infty\rangle_S$ generated by all the expressions of the form $x_j-\sum_{i\in p^e}x_{(j,i)}\cdot t\cdot m_i$, for any indeterminate x_j. When the context is clear, we will allow ourselves to drop the subscript S.

Lemma 5.2. *One can define unary functions λ_i on $\langle x_\infty\rangle_R/I^0$ in such a way that it becomes a model of T_λ. This $\mathscr{L}_R$-structure can in turn be embedded in a model of T^{tf} which is of the form $\langle x_\infty\rangle_{R.X^{-1}}/I^0_{R.X^{-1}}$, where X is the set of separable elements of R.*

Proof. A typical element of $\langle x_\infty\rangle_R$ is of the form $\sum_i x_i\cdot r_i$, $r_i\in R$. We define $\lambda_i(x_j)$ as x_{ji} and we extend λ_i on $\langle x_\infty\rangle_R$ proceeding as in Lemma 3.2 in [8]. So, $\langle x_\infty\rangle_R/I^0$ becomes an $\mathscr{L}$-structure. Note that it is torsion-free: suppose that $t(x_\infty)\cdot r\in I^0$, using the generators of I^0, we may assume that any x_j occurring in $t(x_\infty)$ has the property that j is of fixed length ℓ, then we use that $\langle x_\infty\rangle_R$ is torsion free and that $I^0\cap\langle(x_i)_{i\in(p^e)^\ell}\rangle_R=0$. Then to embed it in a model of T^{tf}, we apply Proposition 4.3(2).

Let $(I^0)^e:=\{n\cdot z^{-1} : n\in I^0,\ z\in X\}=I^0_{R.X^{-1}}$ be the extension of I^0 in $\langle x_\infty\rangle_R\cdot X^{-1}$. Since $\langle x_\infty\rangle_R/I^0$ is X-torsion-free, $((I^0)^e)^{\mathrm{c}}=I^0$, where $(\cdot)^{\mathrm{c}}$ denotes the contraction i.e. if J is a right $R.X^{-1}$-submodule of $\langle x_\infty\rangle_R\cdot X^{-1}$ then $J^{\mathrm{c}}:=\{m\in\langle x_\infty\rangle_R : m\cdot 1^{-1}\in J\}$ (see [11, Theorem 9.17]). We have, by uniqueness of the $R.X^{-1}$-structure (see [11, Proposition 9.12]), that $\langle x_\infty\rangle_R\cdot X^{-1}/I^0_R\cdot X^{-1}\cong\langle x_\infty\rangle_{R.X^{-1}}/I^0_{R.X^{-1}}$. □

By the previous lemma, $\langle x_\infty\rangle_R/I^0$ is equipped with a ring structure, multiplication by x and elements of K being given by the R-module structure and each x_i acting like λ_i. This ring acts naturally on each model of $T(e)$.

Proposition 5.3. *$\langle x_\infty\rangle_R/I^0$ is the ring of definable endomorphisms of any model of $T(e)$.*

Proof. By 4.6 any definable endomorphism is of the form $x\mapsto\tau_x(x)$ for terms τ_x from $\langle x_\infty\rangle_R$. By a compactness argument τ_x does not depend on x. Now a term $\tau\in\langle x_\infty\rangle_R$ vanishes identically on some (any) model of $T(e)$ iff it belongs to I_0 (for more details on these two last assertions, see [1, 3.3.1]). □

The following definitions are made to reflect what happens in the field case (see [7, pp. 148–149]).

Definition 5.4. Let M be an R-module. An R-submodule I of M is called *separable* if whenever it contains an element of the form $\sum_{i\in p^e} y_i \cdot t \cdot m_i$ with $y_i \in M$, it contains every element y_i for $i \in p^e$.

So, if M is in addition a model of T_λ, any separable R-submodule of M is closed under the λ-functions.

Lemma 5.5. (1) *Let I be a separable R-submodule of an R-module M. If $a\cdot q$ belongs to I for some $a\in M$ and $q\in R-\{0\}$, then $a\cdot q'$ belongs to I for some $q'\in X$.*

(2) *Assume furthermore that M is a model of $T(e)$, then the separable closure I^{sep} of I is again separable.*

(3) *Let I be a non-zero right ideal of R and suppose it is separable. Then it is generated by a separable polynomial.*

Proof. (1) If $q\notin X$, decompose it as in Lemma 3.2 in [8]: $q=t.q'=\sum_i \sqrt[p]{q'_{(i)}}.t.m_i$. We have that $a\cdot t\cdot q'=\sum_i a\cdot \sqrt[p]{q'_{(i)}}.t.m_i$, so by separability of I, we have that for each i, $a\cdot \sqrt[p]{q'_{(i)}}$ belongs to I. In case of q' separable, for some i, we have that $\sqrt[p]{q'_{(i)}}$ is separable, if not we iterate the procedure.

(2) Let a belong to I^{sep}, then there exists $q\in X$ such that $a\cdot q\in I$. So as in the proof of Proposition 3.6(3) in [8], $\lambda_i(a)$ belongs to $\langle I,a\rangle_R$.

(3) The ring R is right principal and so I is generated by a polynomial $q\neq 0$. Suppose that $q=t.q'$, then by Lemma 3.2 in [8], $q=t.q'=\sum_i \sqrt[p]{q'_{(i)}}t.m_i$. So, since I is separable, each $\sqrt[p]{q'_{(i)}}$ belongs to I, but they are of degree smaller than q, a contradiction. □

Let M be a substructure of $\mathscr{M}$ and assume that it is a model of T_λ. We will denote by $M_{\mathscr{L}}$ the expansion of M to an $\mathscr{L}$-substructure of $\mathscr{M}_{\mathscr{L}}$. Now we show that there is a bijection between 1-types over M and some right R-submodules of $M\langle x_\infty\rangle := \langle x_\infty\rangle_R \oplus M$.

Notation 5.6. Let $a\in\mathscr{M}$, set $a_\infty := (\lambda_j(a) : j\in (p^e)^{<\omega})$, let $p(a,M)$ be the set of all $\mathscr{L}_R$-formulas satisfied by a in $\mathscr{M}$ over M; it is axiomatized by the p.p. complete $\mathscr{L}_R$-type $p_+(a,M)$ of a over M, or equivalently—using the positive quantifier elimination for T_e (see Section 2)—by the set $p_+(a,M_{\mathscr{L}})$ of all atomic $\mathscr{L}$-formulas satisfied by a over M in the expansion $\mathscr{M}_{\mathscr{L}}$.

We will denote by $S_1(M)$ the set of complete 1-$\mathscr{L}_R$ types with parameters in M.

To the $\mathscr{L}$-type $p_+(a,M_{\mathscr{L}})$, we associate:

- the (a priori partial) $\mathscr{L}_R$-type $p_+(a_\infty,M)$ in *countably* many variables, obtained by substituting x_j for $\lambda_j(x)$ in $p_+(a,M_{\mathscr{L}})$,

- the R-submodule of $M\langle x_\infty\rangle$

$$I(a_\infty,M) := \{t(x_\infty)+m \in M\langle x_\infty\rangle;\ \mathscr{M} \models t(a_\infty)+m=0\}$$

(observe that, if $e>0$, independently of M, $I(a_\infty,M)$ has the cardinality of R),
- and the R-submodule of $R\langle x_\infty\rangle$

$$I_R(a_\infty,M) := \{t(x_\infty) \in \langle x_\infty\rangle_R; t(a_\infty) \in M\}.$$

For $a\in\mathscr{M}$ and $q := tp(a,M)$, we write $I(q)$ and $I_R(q)$ for $I(a_\infty,M)$ and $I_R(a_\infty,M)$.

Lemma 5.7. *Let a, $b\in\mathscr{M}$, the following are equivalent*:

(1) $tp(a,M)=tp(b,M)$,
(2) $p_+(a,M_{\mathscr{L}})=p_+(b,M_{\mathscr{L}})$,
(3) $I(a_\infty,M)=I(b_\infty,M)$.

Proof. (1)⇒(3) is clear.

(3)⇒(2) Observe that, by Corollary 3.4 in [8], any atomic formula in $p_+(a,M_{\mathscr{L}})$ is equivalent to one of the following form $\sum_{i\in p^{en}} \lambda_i(x)\cdot d_i + m=0$, for some $n\geqslant 1$, $d_i\in R$ and $m\in M$.

(2)⇒(1) by the positive quantifier elimination. □

Proposition 5.8. (1) *If M and N be models of $T(e)$, $M\preceq N\preceq\mathscr{M}$, and $a\in\mathscr{M}$. Let $t(a,N)\in S_1(M)$. Then, $t(a,N)$ does not fork over M iff $I(a_\infty,M)=I(a_\infty,N)$ iff $I_R(a_\infty,M)=I_R(a_\infty,N)$.*

(2) *If M and N are $\mathscr{L}$-substructures of $\mathscr{M}_{\mathscr{L}}$, $M\subseteq N$, and $a\in\mathscr{M}$, then $t(a,N)$ does not fork over M iff any element in $I(a_\infty,N)$ has a multiple in $I(a_\infty,M)$ iff any element in $I_R(a_\infty,N)$ has a multiple in $I_R(a_\infty,M)$.*

For a definition of forking, see [2, p. 3; 18, p. 111].

Proof. (1) First, $I(a_\infty,M)=I(a_\infty,N)$ and $I_R(a_\infty,M)=I_R(a_\infty,N)$ are trivially equivalent when $M\subseteq N$. Now, if $t(a,N)$ does not fork over M then $t(a,M)$ should not represent more formulas than $t(a,N)$; hence, $I_R(a_\infty,M)=I_R(a_\infty,N)$. The converse follows by 5.7 and uniqueness of non-forking extensions over models.

(2) Follows from (1) and the following facts and their analogues for the I_R's.

- If M and N are models, "any element in $I(a_\infty,N)$ has a multiple in $I(a_\infty,M)$" is equivalent to "$I(a_\infty,M)=I(a_\infty,N)$".
- For $N=\langle\langle M\rangle\rangle$, since M is an $\mathscr{L}$-structure, $N=M^{\mathrm{sep}}$ and any element in $I(a_\infty,N)$ has a multiple in $I(a_\infty,M)$.
- Since the prime model over some set of parameters is algebraic over these parameters, $t(a,N)$ does not fork on M iff $t(a,\langle\langle N\rangle\rangle)$ does not fork on $\langle\langle M\rangle\rangle$. □

Definition 5.9. (1) We call *standard* an R-submodule of $M\langle x_\infty\rangle$ which includes I^0 (see Notation 5.1).

(2) Now, assume that M contains K^{sep}. Then, an R-submodule I of $M\langle x_\infty\rangle$ is called *separably closed* if whenever $v \cdot q \in I$, for some $q \in X$ and $v \in M\langle x_\infty\rangle$, then $v - c \in I$, for some $c \in K^{\text{sep}}$.

Now we will further assume that M contains K^{sep}, in other words all the torsion elements. We will characterize submodules of the form $I(a_\infty, M)$ amongst submodules of $M\langle x_\infty\rangle$.

Proposition 5.10. *Let M be an R-submodule of $\mathscr{M}$. Suppose that M is a model of T_λ containing K^{sep}, and let $I \subseteq M\langle x_\infty\rangle$ be a right R-submodule. Then there is $a \in \mathscr{M}$ such that $I = I(a_\infty, M)$ iff I is standard, separable, separably closed and such that $I \cap M = \{0\}$.*

Proof. That $I(a_\infty, M) \cap M = \{0\}$ is formally clear. Now, $I(a_\infty, M)$ is standard because $\mathscr{M}$ is a model of axiom (4), separable because $\mathscr{M}$ is a model of axiom (5), and separably closed because $\mathscr{M}$ is a model of axiom scheme (3).

Conversely let I be as stated. Let M^{sep} be the separable closure of M in $\mathscr{M}$, $S := R.X^{-1}$ be the ring of fractions of R with respect to the set X of separable elements of R and $\tilde{M}$ be the module $M^{\text{sep}} \oplus \langle x_\infty\rangle_S / I_S^0$. It is a model of $T(e)$ since M^{sep} is a model of $T(e)$ and $\langle x_\infty\rangle_S / I_S^0$ is a model of T^{tf} (see the proof of Lemma 5.2). Consider the quotient $N = \tilde{M}/(I/I_S^0)$. As $M \cap I = \{0\}$, M embeds in N. Since I is standard and separable, N is a model of T_λ (and so the torsion is equal to the separable torsion). Since I is separably closed, N_{tor} is isomorphic to K^{sep} and can be identified with the copy of K^{sep} which is in M. Since $\tilde{M}$ satisfies axiom scheme 2, a fortiori N satisfies it. So N is a model of $T(e)$ containing M and which contains an element $x + I$ with the property that $I((x + I)_\infty, M) = I$. The result follows by $|R|^+$-saturation of $\mathscr{M}$. $\square$

Corollary 5.11. *Let M and a be as above. The map $a \mapsto I(a_\infty, M)$ defines a bijection between 1-types over M and right R-submodules I of $M\langle x_\infty\rangle$ standard, separable and separably closed with $I \cap M = \{0\}$.* $\square$

Remark. Note that, for M as above and $q \in S_1(M)$, the proof of 5.10 describes the prime model $M\langle\langle q\rangle\rangle$ over M and q.

Corollary 5.12. *Let M and N be models of $T(e)$, $M \preceq N \preceq \mathscr{M}$, and $a \in \mathscr{M}$. Then, $t(a, N)$ does not fork over M iff $N\langle\langle a\rangle\rangle/N = M\langle\langle a\rangle\rangle/M$ for the canonical inclusion.*

Proof. Clear from 5.8 and proof of 5.10. $\square$

We describe now types over R-submodules M of models of $T(e)$, with M a model of T_λ, without assuming that M contains K^{sep}.

Definition 5.13. (1) An R-submodule I of $M\langle x_\infty\rangle$ is called *compatible* if, for each $q \in X$, there are at most $p^{deg(q)}$ elements $t(x_\infty) + m \in M\langle x_\infty\rangle$ different modulo I, such that $(t(x_\infty)+m)\cdot q\in I$. If $K^{\rm sep}\subseteq M$ and $I\cap M=\{0\}$, it is equivalent to be separably closed.

(2) An R-submodule I of $\langle x_\infty\rangle_R$ is called *X-saturated* if whenever $v\cdot q\in I$, for some $q\in X$ and $v\in\langle x_\infty\rangle_R$, then $v\in I$ (see [19, p. 62]). Note that if in addition I is separable, then the properties of being X-saturated and $(R-\{0\})$-saturated are equivalent (see Lemma 5.5(1)).

Proposition 5.14. *Let M be a $\mathscr{L}$-substructure of $\mathscr{M}_{\mathscr{L}}$ and $a\in\mathscr{M}$. The map $a\mapsto I(a_\infty,M)$ defines a bijection between 1-types over M and right R-submodules I of $M\langle x_\infty\rangle$ which are standard, separable, compatible and satisfy $I\cap M=\{0\}$.*

Proof. It is clear that a module of the form $I(a_\infty,M)$ has the properties above. Conversely let I be such.

Claim. *An R-module $N\supseteq M$ and* of small torsion, *i.e. containing, for each separable $q\in R$, at most $p^{deg(q)}$ elements x satisfying $x\cdot q=0$, embeds over M in a model of $T_{\rm sep}$, which can be taken minimal over N.*

Proof of the Claim. After the techniques from [8, 5.3, 5.4 and 5.8], we decompose $N=\bigcup_{i\in\omega+1}N_i$ where $N_0=M$, the union is increasing and continuous, and each N_i contains all elements $m\in N$ satisfying $m\cdot q=0$ for some separable $q\in R$ of degree $<i$, and we embed inductively each N_i in any $|N|^+$-saturated model of $T_{\rm sep}$. By [8, 5.8], there is a minimal model $N^{\rm sep}\models T_{\rm sep}$ containing N, which can be taken minimal over N.

Now, $N:=M\langle x_\infty\rangle/I$ has small torsion and is provided with an $\mathscr{L}$-structure extending M. By the claim it embeds in $N^{\rm sep}\models T_{\rm sep}$, by [8, 5.9] $N^{\rm sep}$ has a unique $\mathscr{L}$-structure extending N, and, as such, $N^{\rm sep}\models T_e$. Clearly $I(x_\infty+I,M)=I$. □

Proposition 5.15. *Let M be a model of $T(e)$. Then, a submodule I of $\langle x_\infty\rangle_R$ is of the form $I_R(a_\infty,M)$ for some $a\in\mathscr{M}$ iff it is standard, separable and X-saturated iff it is of the form $I_R(a_\infty,N)$ for some $a\in\mathscr{M}$ and $N_{\mathscr{L}}\preceq\mathscr{M}_{\mathscr{L}}$.*

Proof. The second equivalence follows from the first one, which we get by the previous proposition, with $M=0$. □

Notation 5.16. By Lemma 5.7, the submodule $I(a_\infty,M)$ (respectively $I_R(a_\infty,M)$) of $M\langle x_\infty\rangle$ (respectively $\langle x_\infty\rangle_R$), $a\in\mathscr{M}$, only depends on the type $q\in S_1(M)$ of a over M. We will denote it by $I(q)$ (respectively $I_R(q)$). By Proposition 5.14 (respectively 5.15), any submodule I of $M\langle x_\infty\rangle$ (respectively $\langle x_\infty\rangle_R$) which is standard, separable, compatible (respectively X-saturated) and satisfies $I\cap M=\{0\}$ is of that form.

We will call *type submodule* any such submodule of $\langle x_\infty\rangle_R$ of the form $I_R(q)$.

If we consider the theory $T^{\rm tf}$ instead of $T(e)$, using analogous proofs to the ones in this section, we can also obtain a correspondence between 1-types over a model M and standard, separable and X-saturated submodules I of $M\langle x_\infty\rangle$ satisfying $I\cap M=\{0\}$.

6. Types and infinitely definable subgroups

Classically, one associates with a type q with parameters over a subset A, an infinitely definable group without parameters and its connected component (see [18, Chapter 2, pp. 25–26]). Whenever, q is a type over a model, those two infinitely definable groups coincide (see [18, Lemma 2.6]) and in that case the free part of q (see [18, pp. 134, 135]) is the generic type of the associated subgroup.

We will first show that the connected component of a definable subgroup of $(M,+,0)$, where $M \models T(e)$ is still definable.

Then, we will prove an analog of the Nullstellensatz in our context using on one hand the description of type submodules given in the previous section and on the other hand the generic types.

Finally, thanks to the fact that we do have prime models, we will show a lattice isomorphism between type submodules and models.

All the groups G we will be considering are subgroups of $(M,+,0)$, where M is a model of $T(e)$. To stress that fact we will use the phrase *additive group*, also we will say that a group is $\emptyset$-definable when it is definable without parameters. Let us first recall some terminology.

A subgroup H is *relatively definable* in a group G if it is the intersection of a definable subset with G.

An infinitely definable group is *connected* iff it does not contain any relatively definable proper subgroup of finite index. The *connected component*, usually denoted by G^0, of an infinitely definable group G is the intersection of all relatively definable subgroups of finite index. In a stable theory T, it is infinitely definable using the same parameters as the ones used for defining G and has index at most $2^{|T|}$ [20, p. 61], it is the intersection of all relatively definable subgroups of bounded index. We prove in the following proposition that we have more in our context: in case G is definable, G^0 has finite index in G, i.e. G is *connected-by-finite*. This implies that, for a definable G, G^0 itself is definable, and not only infinitely definable.

Proposition 6.1. *Any additive subgroup definable in $T(e)$ is connected-by-finite.*

Proof. We can deduce this from the corresponding result for separably closed fields (see [3, Corollary 4.6]), but we give here a direct proof.

First, we note that in any theory of modules, to show that a definable group is connected-by-finite, it suffices to prove it for p.p. definable groups. Indeed, one observes using B.H. Neumann's Lemma [13, p. 140] that any definable group has a p.p. definable subgroup of finite index.

Using positive elimination for T_e in $\mathscr{L}$ [8, 6.9, Remark 6], the fact that, for each integer n and $M_{\mathscr{L}} \models T_e$, $(\lambda_i)_{i\in p^{en}}$ defines an isomorphism between $(M,+)$ and $(M^{p^{en}},+)$, and the triangularization of matrices [8, 6.4], any subgroup of some Cartesian power of $(M,+)$ is definably isomorphic to a group G defined by a system $\bar{x}\cdot A=\bar{0}$, where A is a lower triangular $(m\times k)$-matrix with only separable, hence non-zero, elements on its diagonal. Let $r_0,\dots,r_{k-1}$ be these coefficients. Let us prove now, by induction on k independently of m, that G is connected-by-finite, with $[G:G^0]\leqslant \sum_{i\in k} p^{deg(r_i)}$.

It is clear for $k=0$ as, in this case, G is the additive group M^m. For $k\geqslant 1$, consider the definable additive homomorphism $f:(x_i)_{i\in m}\mapsto(x_0\cdot r_0,x_1,\ldots,x_{m-1})$. The image of G under f lies in the group $H:=\{\bar{x}\in M^m;\bar{x}\cdot B=\bar{0}\}$, where B is the matrix obtained from A by replacing r_0 by 1. Because r_0 is separable and $M_{\mathscr{L}}$ models T_e, f is surjective on H and its kernel is finite, of cardinality $p^{deg(r_0)}$. Now H is isomorphic to $M\times G'$ where $G':=\{\bar{x}\in M^{m-1};\bar{x}\cdot A'=\bar{0}\}$, for A' the $(m-1)\times(k-1)$-matrix obtained from A by removing the first line and the first column. By induction ($[H:H^0]=$) $[G':G'^0]\leqslant\sum_{i=1}^{k-1}p^{deg(r_i)}$ and G is finite-by-(connected-by-finite). As it is stable, it implies that it is connected-by-finite (we can also prove it directly). Indeed, $f(G^0)$ is a connected subgroup of H of bounded index, hence $f(G^0)=H^0$, which also proves $[G:G^0]\leqslant[H:H^0]+|ker\, f|\leqslant\sum_{i\in k}p^{deg(r_i)}$. □

Note that a module is connected iff it does not contain any proper p.p. definable subgroup of finite index (see [12, Lemma 1, paragraph 6, part 2, p. 155].)

In an arbitrary module, a connected definable subgroup is p.p. definable (apply Neumann's Lemma) and so $\emptyset$-definable; here, we will prove that any additive subgroup $\emptyset$-definable in $T(e)$ is p.p. definable. However, it is not known whether this holds in general (see [18, exercise 2, p. 118]).

Proposition 6.2. *Any additive subgroup $\emptyset$-definable in $T(e)$ is p.p. definable.*

Proof. Let G be an additive definable subgroup. We argue in the prime model K^{sep}. As we just have seen G is connected-by-finite, hence (p.p. definable)-by-finite. Since all groups are here vector spaces over $\mathbb{F}_p$, G is the product of G^0 by a finite group F, which is a finite additive subgroup of K^{sep}. Consider the group $F'=\Sigma\{\sigma F;\sigma\in Gal(K^{\mathrm{sep}};K)\}$. Since G is $\emptyset$-definable without parameters, $G=G^0+F'$. Now F', as a finite additive subgroup of K^{sep}, is the zero set of some additive polynomial over K^{sep}, and in fact over K since it is invariant under $Gal(K^{\mathrm{sep}};K)$, hence the r-torsion for some $r\in R$. □

After these preliminaries, we relate the description of types in $T(e)$ we have given in Section 5 to the classical description of types in modules.

Definition 6.3. We define (see [18, pp. 24–26]).

$$\mathscr{G}:=\{\text{definable additive subgroups of } \mathscr{M}\},$$

$$\mathscr{G}_0:=\{\text{connected definable additive subgroups of } \mathscr{M}\},$$

$$\bar{\mathscr{G}}_0:=\{\text{connected infinitely definable additive subgroups of } \mathscr{M}\},$$

$$\mathscr{T}:=\{\text{types submodules of } \langle x_\infty\rangle_R\}$$

and for $M \preceq \mathscr{M}$ and $q \in S_1(M)$

$$\mathscr{G}(q) := \{G \in \mathscr{G};\ q \vdash x - a \in G \text{ for some } a \in M\},$$

$$\mathscr{G}_0(q) := \{G \in \mathscr{G}_0;\ q \vdash x - a \in G \text{ for some } a \in M\},$$

$$\mathscr{T}(q) := \{N \in \mathscr{T}; I_R(q) \subseteq N\},$$

$$G(q) := \cap\{G; G \in \mathscr{G}(q)\}.$$

Equipped with the operations connected component of the intersection, and of the sum, $\bar{\mathscr{G}}_0$ is a complete lattice. It happens here that $\mathscr{G}_0$ is a sublattice. Indeed, the intersection and the sum of definable groups are definable, and their connected components too, by 6.1. $\bar{\mathscr{G}}_0$ is in fact the completion of $\mathscr{G}_0$.

As a consequence, because we are considering types over a model M where all classes of G/G^0 exist, we have that $G \in \mathscr{G}(q)$ iff $G^0 \in \mathscr{G}_0(q)$, and $G(q)$ is also the intersection of all groups in $\mathscr{G}_0(q)$. Note that, if $tp(x,M)$ is the generic of some group $G \in \bar{\mathscr{G}}_0$, then $G = G(x,M)$ (we use here the notation $G(x,M)$ for $G(tp(x,M))$). Indeed $G(x,M) \leqslant G$ by definition of $G(x,M)$, and q is stable under (over M) independent addition because it is a principal generic [17, p. 163], hence $\mathscr{G}(q) = \{G \in \mathscr{G}; q \vdash x \in G\}$; now x is in no proper definable subgroup of G. Hence $\bar{\mathscr{G}}_0 = \{G(q); q$ a type over some model of $T(e)\}$, (see also [18], where $\bar{\mathscr{G}}_0$ appears as $\mathrm{PP}_0^{(1)}$, in particular Lemma 5.10). Our description of types is related to the classical one in the following way: for $q \in S_1(M)$ and $t \in \langle x_\infty \rangle_R$, $t \in I_R(q)$ iff $t(a_\infty) \in M$ for any a realizing q, iff $\{y; t(y_\infty) = 0\} \in \mathscr{G}(q)$.

Proposition 6.4. *Suppose M is a $|K|^+$-saturated model of $T(e)$. Then the maps*

$$\bar{\mathscr{G}}_0 \to \mathscr{T} : G \to I_R(G) := \{t \in \langle x_\infty \rangle_R \ ;\ \forall g \in G \ \ t(g_\infty) = 0\},$$

$$\mathscr{T} \to \bar{\mathscr{G}}_0 : N \to V(N) := \{x \in M \ ;\ \forall t \in N \ \ t(x_\infty) = 0\}$$

are inverse bijections between $\bar{\mathscr{G}}_0$ and $\mathscr{T}$. Their restrictions are inverse bijections between $\{G \in \bar{\mathscr{G}}_0; G \leqslant G(q)\}$ and $\mathscr{T}(q)$. For $G \in \bar{\mathscr{G}}_0$, the generic type of G is the unique type q over M satisfying $I_R(G) = I(q)$.

Proof. For $N \in \mathscr{T}$, $H := \langle x_\infty \rangle_R / N$ is a torsion-free $\mathscr{L}$-structure which embeds in any $|K|^+$-saturated model of T_e, is as such a subset of $V(N)$ and satisfies $I_R(H) = N$. Hence $I_R(V(N)) = N$, this is the analogue in the module framework of the Nullstellensatz for separably closed fields. It is clear that $V(N)$ is an infinitely definable subgroup of M, let us prove now it is connected. Since N is X-saturated the condition $N = I(q)$ defines a unique type $q \in S_1(M)$. As we already noticed, for $t \in \langle x_\infty \rangle_R$, $t \in I_R(q)$ iff $\{y; t(y_\infty) = 0\} \in \mathscr{G}(q)$. Hence $G(q) = V(N)$, but we know that $G(q)$ is connected. Now q is clearly contained in $V(N)$ and stable under addition of any element of $M \cap V(N)$, hence it is the generic of this group.

Conversely, by quantifier elimination in the language $\mathscr{L}$, any $G \in \mathscr{G}_0$ is the zero set of terms of $\langle x_\infty \rangle_R$ and $I_R(G)$ is clearly standard and separable. We show now it is X-saturated if G is connected. Let $t(G)\cdot g=0$ for some $t\in\langle x_\infty\rangle_R$ and a separable polynomial g (of degree d). Then the condition "$t(y_\infty)=0\wedge y\in G$" defines a subgroup of finite index ($\leqslant d$) of G, hence by connexity G itself, hence $t\in I_R(G)$. □

For $q\in S_1(M)$, let q_G the generic type over M of $G(q)$ (as q is a type over a model, q_G is the "free part" q_* of q in [18, Section 6.1]), hence $I_R(q)=I(q_G)$ and $\mathscr{G}(q)=\mathscr{G}(q_G)$. The construction of the minimal model realizing some type allows us to understand an interesting connection between q and q_G.

Proposition 6.5. *Let M be a model of $T(e)$. Then, the models $M\langle\langle q_G\rangle\rangle$ and $M\langle\langle q\rangle\rangle$ are isomorphic over M iff M is a direct summand in $M\langle\langle q\rangle\rangle$. This is in particular the case when in addition M is pure-injective.*

Proof. By the construction of $M\langle\langle q\rangle\rangle$ in 5.10, if $I(q)=I_R(q)$ then $M\langle\langle q\rangle\rangle$ is equal to $M\oplus\langle x_\infty\rangle_{R.X^{-1}}/I_R(q).X^{-1}$ and so M is a direct summand in $M\langle\langle q\rangle\rangle$.

In particular, note that M is always a direct summand in $M\langle\langle q_G\rangle\rangle$.

In the other direction, $M\langle\langle q_G\rangle\rangle/M=M\langle\langle q\rangle\rangle/M$ and M is a direct summand in $M\langle\langle q\rangle\rangle$ exactly when $M\langle\langle q_G\rangle\rangle$ and $M\langle\langle q\rangle\rangle$ are M-isomorphic over M. □

Given two models $M\preceq N\preceq\mathscr{M}$ such that M is direct summand in N, then the set of models P, $M\preceq P\preceq N$, equipped with sum and intersection, is a complete lattice, as is $\mathscr{T}(q)$.

Proposition 6.6. *The lattice of models between M and $M\langle\langle q_G\rangle\rangle$ and $\mathscr{T}(q)$ are isomorphic.*

Proof. Define

$$\mathscr{S}^{\text{tf}}(q):=\{N;N \text{ is a model of } T^{\text{tf}},\ N\preceq\langle x_\infty\rangle_S/(I_R(q)\cdot X^{-1})\},$$

$$\mathscr{S}(q_G):=\{N;N \text{ is a model of } T(e),\ M\preceq N\preceq M\langle\langle q_G\rangle\rangle\}$$

and consider the maps

$$\alpha\colon\ N\in\mathscr{T}(q)\mapsto(N\cdot X^{-1})/(I_R(q)\cdot X^{-1})\in\mathscr{S}^{\text{tf}}(q),$$

$$\beta\colon\ N\in\mathscr{S}^{\text{tf}}(q)\mapsto M\oplus N\in\mathscr{S}(q_G),$$

$$\gamma\colon\ N\in\mathscr{S}(q_G)\mapsto(N/M)\cap(\langle x_\infty\rangle_R/I_R(q)),$$

this intersection being taken in $\langle x_\infty\rangle_S/I_R(q)\cdot X^{-1}$, where N/M canonically embeds. Since modules in $\mathscr{T}(q)$ are X-saturated, one has $\gamma\circ\beta\circ\alpha=Id$. □

7. Dependence relations

In this section, we are going to consider several notions of dependence and investigate the relationships among them. The first ones are algebraically defined and use either the fact that the ring is right Ore or that X is a right denominator set. Then we will consider relations which are usually defined in the context of modules using the p.p. elimination. In general, those are not transitive (nor reflexive) unless if we restrict to particular subsets. And finally the relation of forking which is again in general non-transitive. Using our prime model construction, we will be able to obtain tighter connections than usual.

We call here *dependence relation* over some abstract structure $\mathcal{N}$ a relation of the form $b \prec A$, for $b \in \mathcal{N}$ and $A \subseteq \mathcal{N}$, satisfying the following axioms:

- If $b \in A$, then $b \prec A$ (reflexivity);
- $b \prec A$ iff $b \prec A_0$ for some finite $A_0 \subseteq A$ (finite character);
- if $b \prec A \cup \{c\}$ and $c \prec A$ then $b \prec A$ (transitivity);
- if $b \prec A \cup \{c\}$ and $b \not\prec A$ then $c \prec A \cup \{b\}$ (symmetry, or exchange property).

These axioms allow us to extend coherently the dependence relation to subsets

$$\{b_1, \ldots, b_k\} \prec A \text{ iff, for some } i = 1, \ldots, k-1, b_{i+1} \prec A \cup \{b_1, \ldots, b_i\}$$

and

$$B \prec A \text{ iff for any finite } B_0 \subseteq B, B_0 \prec A$$

and to localize to a set of parameters: $x \prec_A B$: iff $x \prec B \cup A$. The corresponding *closure* is defined as follows: $\mathrm{cl}(A) = \{x \in \mathcal{N};\ x \prec A\}$, and it characterizes the relation $\prec$.

To a dependence relation is naturally associated a notion of *dimension* of a set as the cardinality of a *basis*, i.e. a maximally independent subset. This is well defined since all bases have the same cardinality [14, p. 122].

$\mathcal{N}$ is now a model of $T(e)$ or T^{tf} and let $x \in \mathcal{N}$.

Definition 7.1. (1) x is *R-dependent* on A iff there exist $a_1, \ldots, a_n$ in A and nonzero elements $r_0, r_1, \ldots, r_n$ in R, such that $x \cdot r_0 = \sum_{i \geqslant 1} a_i \cdot r_i$.

(2) x is *separably R-dependent* on A, $x \prec A$, iff there exist $a_1, \ldots, a_n$ in A and separable elements $r_0, r_1, \ldots, r_n$ in R, such that $x \cdot r_0 = \sum_{i \geqslant 1} a_i \cdot r_i$.

Proposition 7.2. (1) *The R-dependence is a dependence relation.*

(2) *The separable R-dependence is a dependence relation. The corresponding closure of A is* $\langle A \rangle_R^{\mathrm{sep}}$.

(3) *If A is an* $\mathcal{L}$*-substructure, then R-dependence and separable R-dependence on A coincide.*

Proof. Reflexivity, finite character and symmetry are clear for both relations. Transitivity of the first one follows from the right Ore property of R: two non-zero elements have a non-zero common multiple. Transitivity of the second one follows from the fact that X is a right denominator set of R: two separable elements in R have a separable common multiple. The description of both closures is clear, and also the fact that the separable dependence is stronger. The converse in case A is closed under the functions λ follows from Lemma 3.2, item 3 in [8]. □

Amongst relations arising from stability theory, the relation "$t(x, A)$ forks over $\emptyset$" ("$x \not\downarrow A$") is in general not transitive, and the relation "x is model theoretically algebraic over A" not symmetric. But they are when restricted to minimal types, in which case they coincide and give rise to a dependence relation. This is a basic fact in stability. Existence and description of prime models give us a bit more in our case (recall that $(A)_{\mathscr{L}}$ denotes the $\mathscr{L}$-substructure generated by A). In fact one has to be careful since, for an algebraic x, one has $x \downarrow x$, which means that $\not\downarrow$ is not reflexive. Classically this fact is handled by identifying x to its algebraic closure. We proceed here slightly differently and consider the relation "x is algebraic or $x \not\downarrow A$".

Proposition 7.3. *Let $x \in \mathscr{M}$ and $A, B \subseteq \mathscr{M}$ be such that $RU(x, A)$ and $RU(b, A)$ are at most* 1, *for each $b \in B$. Then the relation "$RU(x, A) = 0$ or $x \not\downarrow_A B$" is a dependence relation, the corresponding closure of B is the model theoretical algebraic closure of $A \cup B$, which is also the prime model $\langle\langle A \cup B \rangle\rangle$ over $A \cup B$.*

Without restricting ourselves to minimal types there are some classical algebraic characterizations of forking, which we can in our case make more precise or more general. For example let us consider the independence relation introduced by Garavaglia, which works over any ring and which we will here call the *G-independence* (see for instance [22, p. 177]).

Definition 7.4. Two subsets A, B of a pure injective R-module $\mathscr{N}$ are *G-independent* if for any p.p. formula $\phi(\bar{x}, \bar{y})$ such that $\phi(\bar{a}, \bar{b})$ holds in $\mathscr{N}$ with $\bar{a} \in A$, $\bar{b} \in B$, then $\phi(\bar{0}, \bar{b})$ holds, or, equivalently, if there is a decomposition $\mathscr{N} = N_1 \oplus N_2$, with $A \subseteq N_1$ and $B \subseteq N_2$.

Note that if two subsets of parameters A and B included in a pure-injective model $\mathscr{N}$ are *G-independent*, then they remain *G-independent* in any pure-injective extension of $\mathscr{N}$ in which $\mathscr{N}$ is pure. So, in the case when $\mathscr{N}$ is a model of $T(e)$ or T^{tf}, this property is invariant under taking $\mathscr{L}$-extensions.

Proposition 7.5. *In T^{tf}, we have the following equivalence: $A \downarrow_{\emptyset} B$ iff A and B are G-independent.*

Proof. Forking and G-independence are linked in any theory of modules closed under direct sums, by the above equivalence (see [22, Corollary 11.2]). □

Lemma 7.6. *Let A, B be included in $\mathscr{M}_{\mathscr{L}}$. Then, $(A)_{\mathscr{L}} \cap (B)_{\mathscr{L}} = \{0\}$ (i.e. $(A)_{\mathscr{L}}$ and $(B)_{\mathscr{L}}$ are R-independent) iff A and B are G-independent.*

Proof. ($\rightarrow$) Let $\phi(\bar{x}, \bar{y})$ be a p.p. formula such that $\phi(\bar{a}, \bar{b})$ holds with $\bar{a} \in A$, $\bar{b} \in B$. Since T_e admits positive quantifier elimination $\phi(\bar{x}, \bar{y})$ is equivalent to a conjunction of $\mathscr{L}$-terms $t_i(\bar{x}, \bar{y}) = 0$. By Lemma 3.3 and Corollary 3.4 in [8], we have that $t_i(\bar{x}, \bar{y}) = t_{i1}((\bar{x}_j)_{j \in p^{en}}) + t_{i2}((\bar{y}_j)_{j \in p^{en}})$ for some integer n and t_{i1}, t_{i2} two $\mathscr{L}_R$-terms. By hypothesis, we get that $t_{i1}((\bar{a}_j)_{j \in p^{en}}) = 0$ and since $t_{i2}((\bar{0}_j)_{j \in p^{en}}) = 0$, we have that $\phi(\bar{a}, \bar{0})$ holds.

($\leftarrow$) Conversely, suppose that A and B are G-independent and suppose that $a=b$, with $a\in(A)_{\mathscr{L}}$ and $b\in(B)_{\mathscr{L}}$. Since a and b are $\mathscr{L}$-terms in respectively A and B, we get that some p.p. formula in $\mathscr{L}_R$ holds for $\bar{a}$ and $\bar{b}$ in respectively A and B, then it should hold for $\bar{a}$ and $\bar{0}$, hence $a=0$. $\square$

Lemma 7.7. *Let A, B and C be included in $\mathscr{M}_{\mathscr{L}}$. Then, $((B)_{\mathscr{L}}+\langle\langle A\rangle\rangle)\cap((C)_{\mathscr{L}}+\langle\langle A\rangle\rangle)=\langle\langle A\rangle\rangle$ iff $B+\langle\langle A\rangle\rangle$ and $C+\langle\langle A\rangle\rangle$ are G-independent in $T^{\mathrm{tf}}_{\mathrm{free}}$.*

Proof. The proof is the same for $(B)_{\mathscr{L}}+\langle\langle A\rangle\rangle$ and $(C)_{\mathscr{L}}+\langle\langle A\rangle\rangle$ in $T^{\mathrm{tf}}_{\mathrm{free}}$ as for the previous proposition, since $T^{\mathrm{tf}}_{\mathrm{free}}$ admits also positive quantifier elimination in $\mathscr{L}$. $\square$

Proposition 7.8. *In $T(e)$, $B\downarrow_A C$ iff $(B)_{\mathscr{L}}$ and $(C)_{\mathscr{L}}$ are independent in the sense of $\prec_A$ iff $(B)_{\mathscr{L}}+\langle\langle A\rangle\rangle\cap(C)_{\mathscr{L}}+\langle\langle A\rangle\rangle=\langle\langle A\rangle\rangle$ iff $(B)_{\mathscr{L}}+\langle\langle A\rangle\rangle$ and $(C)_{\mathscr{L}}+\langle\langle A\rangle\rangle$ are G-independent in $\mathscr{M}_{\mathscr{L}}/\langle\langle A\rangle\rangle$.*

Proof. The first equivalence is trivial since any point of $(B)_{\mathscr{L}}$ is definable over B, the last one is the previous lemma, let us prove the middle one. We use the following criterion, true in any complete theory of modules (see [18, p. 113]): $\bar{b}\downarrow_{\bar{a}}\bar{c}$ iff for any p.p. formula ψ such that $\mathscr{M}\models\psi(\bar{b},\bar{c},\bar{a})$ there exists a p.p. formula ϕ and a tuple of elements $\bar{a}'\subseteq\bar{a}$ such that $\mathscr{M}\models\phi(\bar{b},\bar{a}')$ and the index $[\phi(\bar{v},\bar{0}):\phi(\bar{v},\bar{0})\wedge\psi(\bar{v},\bar{0},\bar{0})]$ is finite.

Set $\mathscr{M}_0:=\mathscr{M}_{\mathscr{L}}/\langle\langle\bar{a}\rangle\rangle$, note that it is now a model of $T^{\mathrm{tf}}_{\mathrm{free}}$. Let us denote the class in $\mathscr{M}_0$ of any element $d\in\mathscr{M}$ by $d_{\equiv}:=d+\langle\langle\bar{a}\rangle\rangle$.

($\longrightarrow$) Suppose that $\mathscr{M}_0\models\chi(\bar{b}_{\equiv},\bar{c}_{\equiv})$, where χ is a p.p. formula. By positive quantifier elimination in $T^{\mathrm{tf}}_{\mathrm{free}}$, there exist $\mathscr{L}$-terms t_i, s_i such that $\chi(\bar{u},\bar{v})\longleftrightarrow\bigwedge_i t_i(\bar{u})+s_i(\bar{v})=0$. So, there exist $\mathscr{L}$-terms r_i and separable elements $q_i\in X$ such that $\mathscr{M}_{\mathscr{L}}\models\bigwedge_i(t_i(\bar{b})+s_i(\bar{c}))\cdot q_i=r_i(\bar{a})$. The $\mathscr{L}$-formula $\bigwedge_i(t_i(\bar{v})+s_i(\bar{w}))\cdot q_i=r_i(\bar{u})$ is equivalent mod T_λ to a p.p. $\mathscr{L}_R$-formula $\psi(\bar{v},\bar{w},\bar{u})$. By the criterion, there exists a p.p. formula ϕ and $\bar{a}'\subseteq\bar{a}$ such that $\mathscr{M}\models\phi(\bar{b},\bar{a}')$ and the index $[\phi(\bar{v},\bar{0}):\phi(\bar{v},\bar{0})\wedge\psi(\bar{v},\bar{0},\bar{0})]$ is finite. So, $\mathscr{M}_0\models\phi(\bar{b}_{\equiv},\bar{0}_{\equiv})$. But in $\mathscr{M}_0$, if the index of two p.p. definable subgroups is finite, it is equal to 1.

So, $\mathscr{M}_0\models\psi(\bar{b}_{\equiv},\bar{0}_{\equiv},\bar{0}_{\equiv})$. So, $\mathscr{M}_0\models\bigwedge_i(t_i(\bar{b})+s_i(\bar{0}))\cdot q_i=r_i(\bar{0})=0$. But $\mathscr{M}_0$ is torsion-free and so $\mathscr{M}_0\models\chi(\bar{b}_{\equiv},\bar{0}_{\equiv})$.

($\longleftarrow$) Let ψ be a p.p. $\mathscr{L}_R$-formula such that $\mathscr{M}\models\psi(\bar{b},\bar{c},\bar{a})$. By positive quantifier elimination result in T_e, there exist $\mathscr{L}$-terms t_i, s_i, r_i such that $\psi(\bar{v},\bar{u},\bar{w})$ is equivalent to the $\mathscr{L}$-formula $\bigwedge_i t_i(\bar{v})+s_i(\bar{u})+r_i(\bar{w})=0$. In $\mathscr{M}_0$, we get $\mathscr{M}_0\models\bigwedge_i t_i(\bar{b}_{\equiv})+s_i(\bar{c}_{\equiv})=0_{\equiv}$, which entails that $\mathscr{M}_0\models\bigwedge_i t_i(\bar{b}_{\equiv})+s_i(\bar{0}_{\equiv})=0_{\equiv}$ and so there exist $q_i\in X$ and $\mathscr{L}$-terms r_i' such that $\mathscr{M}_{\mathscr{L}}\models\bigwedge_i(t_i(\bar{b})+s_i(\bar{0})).q_i=r_i'(\bar{a})$. This last formula is equivalent to a p.p. $\mathscr{L}_R$-formula and the index $[\bigwedge_i(t_i(\bar{x})+s_i(\bar{0}))\cdot q_i=r_i'(\bar{0})=0:\bigwedge_i t_i(\bar{x})+s_i(\bar{0})=r_i(\bar{0})=0]$ is finite. $\square$

Now using the above proposition and a result of Ziegler [22, Section 6, Lemma 6.2], we may state the weak transitivity property of $\downarrow$.

Namely, let $D\subseteq\mathscr{M}$, $b\in\mathscr{M}$, let $a_1,\ldots,a_{n+1}\in\mathscr{M}$ and assume that in the model $\mathscr{M}/\langle\langle D\rangle\rangle$ of T^{tf}, $H(a_i)/\langle\langle D\rangle\rangle$, $1\leqslant i\leqslant n+1$, are indecomposable and that $\{a_1,\ldots,a_n\}$

is $\downarrow$-independent. Then $b \not\downarrow_D \{a_1,\ldots,a_{n+1}\}$ and $a_{n+1} \not\downarrow_D \{a_1,\ldots,a_n\}$, implies that $b \not\downarrow_D \{a_1,\ldots,a_n\}$.

8. Ranks

Let M be a $|K|^+$-saturated model of $T(e)$ and $q \in S_1(M)$.

Theorem 8.1. *The foundation rank of* $G(q)$ *in* $\bar{\mathscr{G}}_0$, *the foundation rank of* $\mathscr{T}(q)$ *and of the lattice of models between* M *and* $M\langle\langle q_G\rangle\rangle$ *are all equal to* $RU(q)$.

Proof. The interpretation of $RU(q)$ as the foundation rank of $G(q)$ is true in any module, see [18, Theorem 5.12], the other characterizations follow from Propositions 6.4 and 6.6. □

We note here that we understand the modules we are considering much better than the separably closed fields. Indeed, we know that, in SCF_e, for a type q over an ω_1-saturated model M, if $RU(q)$ is an integer n, then there is a chain of models of length n between M and $M\langle\langle q\rangle\rangle$, but we are unable to say something about the converse, or to turn the length of chain of models into a rank. Also the relation between U-rank and the depth of type ideals, the analogue in SCF_e of the foundation rank of $\mathscr{T}(q)$ for T_e, is unknown.

On the other hand, for $M \models T(e)$ and q a type over M, we can define, if $R\text{-}dim$ ($R\text{-}dim_M$) denotes the dimension relative to R-dependence (with parameters from M),

$$RR(q) := R\text{-}dim_M M\langle\langle q\rangle\rangle = R\text{-}dim(M\langle\langle q\rangle\rangle/M),$$

ranging in $\omega \cup \{\infty\}$. This is also the maximal co-rank of a lower triangular separable matrix A such that

$$q(x) \vdash x = \Sigma_{i\in p^{en}} x_i \cdot t^n \cdot m_i \wedge (x_i)_{i\in p^{en}} \cdot A = m$$

for some m in some Cartesian power of M. Then it is a stability rank (see 5.12), the right analogue of transcendence rank for SCF_e, and we get immediately from the equivalent result in SCF_e, the following proposition.

Proposition 8.2. *There are in* T_e *minimal types of arbitrary* RR*-rank in* $\omega \cup \{\infty\}$.

We already mentioned that many results can be directly transferred from the theory SCF_e of separably closed fields of characteristic p and degree of imperfection e. Indeed we have the following.

Fact 8.3 ([1]). *For* K *a definably closed subfied of* $C \models SCF_e$, *any additive subgroup of* C *which is* (*infinitely*) *definable over* K *is also* (*infinitely*) *definable in the* $\mathscr{L}_R$-*reduct of* C.

Thomas Blossier also constructed in SCF_e $(e \geqslant 1)$ minimal additive subgroups of arbitrary transcendence degree $n \in \omega \cup \{\infty\}$. In $T(e)$ these groups are still infinitely definable and their U-rank can only decrease, hence it is still equal to 1. Their generics, or more exactly their trace on $\mathscr{L}_R$ have clearly RR-rank n. Other constructions of Thomas Blossier are of interest for us, as he produced several examples of 1-based groups. On such groups the structure induced by the field is exactly the $\mathscr{L}_R$-structure. As an example one can immediately deduce from his work that there are in $T(e)$ 1-types of any U-rank $< \omega^\omega$.

Let us finally remark that, as in separably closed fields, and for the same reasons, there are in $T(e)$ no non-algebraic type ranked by the Morley rank. (When K is countable, this also can be deduced from the description of the Ziegler spectrum. Indeed, let $M \models T^{\rm tf}$ and suppose that M is $\aleph_0$-saturated. Let $\mathbb{U}_M$ be the corresponding closed set in the Ziegler spectrum. By [22, 8.6] (and see also [18, 10.19]), we have that the Cantor–Bendixon rank of an element of $\mathbb{U}_M$ is equal to its m-dimension (in $\mathbb{U}_M$) which in turn is equal to the minimum of the m-dimension of the interval of p.p. formulas $[\phi \;\; \psi]$ with $\phi(M) > \psi(M)$. By Lemma 3.4, this dimension is undefined. Now the Morley rank of $q \in S_1(\emptyset)$ with $H(q) = N$ is equal to the Cantor–Bendixon rank in $S_1(M)$ of a non-forking extension $\tilde{q}$ of q over M (see [18, p. 120])). Therefore the Morley rank of q is greater than or equal to the Cantor–Bendixon rank of $H(\tilde{q})$ in $\mathbb{U}_M$. But this Cantor–Bendixon rank is undefined.

9. Elimination of imaginaries

If M is an infinite module, its theory does not admit elimination of imaginaries (e.i.); hence, weaker notions have been considered (see [9,15]), and we will show here that T_e has uniform weak e.i. and uniform p.p. elimination of imaginaries (see definitions below). As a matter of comparison let us recall that, for theories which expand a theory of fields, weak e.i. implies e.i. In the case of separably closed fields of finite imperfection degree, in order to get e.i., one needs to choose a p-basis and to add in the language constants for its elements, Lastly, in the infinite imperfection degree case, an appropriate language is still unknown. Some of our results will also hold for the infinite imperfection degree case (recall that the corresponding theory is denoted by T_∞).

Definition 9.1 (see [9, part 2]). Let T be a first-order theory, $\phi(\bar{x}, \bar{a})$ be a formula and A be the set defined by $\phi(\bar{x}, \bar{a})$. Then,

(1) $\bar{c}$ is a *canonical base* for $\phi(\bar{x}, \bar{a})$ if for all automorphisms σ of the monster model of T, σ fixes A setwise iff σ fixes $\bar{c}$ pointwise.
(2) $\bar{c}$ is a *weak canonical base* for $\phi(\bar{x}, \bar{a})$ if there are $\bar{c}_1 = \bar{c}, \ldots, \bar{c}_n$ such that, for all automorphisms σ of the monster model of T, σ fixes A setwise iff it fixes $\{\bar{c}_1, \ldots, \bar{c}_n\}$ setwise.
(3) T has (*weak*) *elimination of imaginaries* if every formula $\phi(\bar{x}, \bar{a})$ has a (weak) canonical base.

(4) T has *p.p. elimination of imaginaries* if every p.p. formula $\phi(\bar{x},\overline{a})$ has a canonical base.

Equivalently [9], $\bar{c}$ is a (weak) canonical base for $\phi(\bar{x},\overline{a})$ if there is a formula $\psi(\bar{x},\bar{y})$ such that $\forall\bar{x}\ \phi(\bar{x},\overline{a}) \leftrightarrow \psi(\bar{x},\overline{c})$ and there is exactly one tuple (finitely many tuples) $\overline{y}$ such that $\forall\bar{x}\ \phi(\bar{x},\overline{a}) \leftrightarrow \psi(\bar{x},\overline{y})$.

In both cases, we say that the elimination is *uniform* if ψ only depends on ϕ and not on the parameters $\overline{a}$.

Remark 9.2. Notice that, up to definable closure, a canonical base, when it exists, is uniquely determined. Moreover, if $\bar{c}$ is a canonical base for $\phi(\bar{x},\overline{a})$, then $\bar{c} \in dcl(\overline{a})$. Now, by Example 3.8 in [15], no theory of infinite modules has elimination of imaginaries.

Proposition 9.3. *T_e (respectively T_∞) has uniform p.p. elimination of imaginaries.*

Proof (See also [15, Lemma 3.7]). Let $\phi(\bar{x},\overline{a})$ be a p.p. formula. By the positive quantifier elimination for T_e and the additional fact that a conjunction of atomic formulas is equivalent to one atomic formula, we have that this formula is equivalent to $t_1(\bar{x})=t_2(\bar{a})$, where the t_1, t_2's are $\mathscr{L}$-terms. Set $\bar{c}=t_2(\bar{a})$. Now, suppose that we have another parameter $\bar{b}$ such that $T_e \models \phi(\bar{x},\bar{a}) \leftrightarrow \phi(\bar{x},\bar{b})$, then $t_2(\bar{a})=t_2(\bar{b})$, hence $\bar{c}$ is a canonical parameter for $\phi(\bar{x},\bar{a})$. This elimination is uniform since it only depends on the formula (and not on the parameters) (see [13, p. 157]). □

Corollary 9.4. *T_e (respectively T_∞) has uniform weak elimination of imaginaries.*

Proof. Any definable A can be finitely written as

$$A = \bigcup_{i=1}^{n_0} \left(A_i \Bigg\backslash \bigcup_{j=1}^{n_i} \left(A_{ij} \Bigg\backslash \bigcup_{k=1}^{n_{ij}} (A_{ijk} \backslash \ldots) \right) \right),$$

where any A_v occurring is a p.p. definable coset (with parameters). This decomposition is unique when some additional conditions are required: if $N := \{v \in \omega^{<\omega}; A_v \text{ exists}\}$, A_v should be connected for $v \in N$ (or rather the corresponding subgroup—we use here that a definable group is connected-by-finite, see Proposition 6.1), for each $i \in \omega$ such that furthermore $(v,i) \in N$, $A_{(v,i)}$ should be strictly included in A_v and finally, for fixed v, the $A_{(v,i)}$'s should be disjoint. In this situation, an automorphism of the monster model fixes A setwise iff it fixes setwise the set of A_i's, $i \in n_0$. It is clear, by induction on the maximal length of $v \in N$, that each A_v has a canonical base.

Suppose now that A is defined by the formula $\phi(x,a)$ and let us introduce formulas ϕ_v defining the A_v's. The ϕ_v's are given by the Baur–Monk p.p. quantifier elimination result in modules and are therefore uniform in a. By positive quantifier elimination in our structures, ϕ_v has the form $t_v(x)=a_v$, where t_v is a term without parameters, not depending on a, and the formula $t_v(x)=0$ defines a connected subgroup G_v, strictly containing the group $G_{(v,i)}$ (by hypothesis on the decomposition), for any $(v,i) \in N$. Hence the formula

$$\bigvee_{i=1}^{n_0} \left(\phi_i \wedge \neg \bigvee_{j=1}^{n_i} \left(\phi_{ij} \wedge \neg \bigvee_{k=1}^{n_{ij}} (\phi_{ijk} \wedge \neg \cdots) \right) \right)$$

provides a definition of A for only a finite choice of its parameters. □

In the finite imperfection case, we could have deduce weak elimination of imaginaries from our description of types, but a priori not the uniformity. Indeed, let us call *canonical base* of a type q over a substructure of a model M of T a definably closed set $D \subseteq M$ such that, for every automorphism σ of the monster model of T, σ fixes q iff σ fixes D pointwise. As we see in the next lemma, in our theory, types do have canonical bases. Now Evans, Pillay and Poizat proved that a stable theory, where each n-type over every model has a canonical base, has weak elimination of imaginaries. (In fact, only a weaker property is required, namely to have a *weak canonical base*, whose definition we will not recall since in our case a stronger property holds.)

Lemma 9.5. *Let A be an $\mathscr{L}$-substructure of a model of T_e. Then every type $q \in \mathrm{S}(A)$ has a canonical base D.*

Proof. The map $(x_i)_{i \in p^{ek}} \mapsto \sum_{i \in p^{ek}} x_i \cdot t^i \cdot m_i$ embeds $S_{p^{ek}}(A)$ in $S_1(A)$, so let us consider the case of 1-types. Let $a \in \mathscr{M}$ be a realization of q and let

$$D := \{ m \in A; x_{j_0} \cdot r_0 + \cdots + x_{j_s} \cdot r_s + m \in I(a_\infty, A) \}.$$

Note that D is a substructure of A. By Lemma 4.6, D is definably closed and by Lemma 5.7, it is clear that for every automorphism σ of $\mathscr{M}$, σ fixes q iff σ fixes D pointwise. □

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