A first approach to the Deduction-Detachment Theorem in logics preserving degrees of truth

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Abstract

This paper studies the Deduction-Detachment Theorem (DDT) in the realm of logics associated with bounded, commutative and integral residuated lattices whose consequence relation preserves degrees of truth (strictly speaking, it preserves the lower bounds of truth values of the premises). It is given some necessary conditions that must enjoy the varieties with a logic having the DDT. In two particular cases these conditions are indeed sufficient to characterize the DDT. In the paper it is also considered the case where the Delta operator is added to the language, and the case of a kind of local version of the DDT.

Keywords: Deduction theorem, Deduction-detachment theorem, Local deduction theorem, substructural logic, fuzzy logic, degrees of truth.

1 Introduction

The **Deduction-Detachment Theorem (DDT)** [6, 2, 8] is the metalogical property, which a sentential logic L may or may not have, that there exists a set of formulas $\Sigma(p,q)$ with two variables such that for all formulas φ, ψ and all sets of formulas Γ,

$$\Gamma, \varphi \vdash_L \psi \quad \text{iff} \quad \Gamma \vdash_L \Sigma(\varphi, \psi), \qquad \text{(DDT)}$$

where $\vdash_L$ is the consequence relation of the logic L. The implication from right to left is equivalent to the Hilbert-style rule of Modus Ponens or "**Detachment**"

$$\varphi, \Sigma(\varphi, \psi) \vdash_L \psi, \qquad \text{(MP)}$$

while the other implication is equivalent to what in this paper will be called, strictly speaking, the "**Deduction Theorem**"

$$\text{if} \quad \Gamma, \varphi \vdash_L \psi \quad \text{then} \quad \Gamma \vdash_L \Sigma(\varphi, \psi). \quad \text{(DT)}$$

We point out that very often the DDT has been called in the literature the Deduction Theorem, but we prefer to keep the previous distinction.

An easy consequence of the previous definition is that the set $\Sigma(p,q)$ is unique up to equivalence, i.e., if $\Sigma(p,q)$ and $\Sigma'(p,q)$ satisfy the (DDT) condition then

$$\Sigma(p,q) \dashv\vdash_L \Sigma'(p,q).$$

This explains why the stress must be on the existence of the DDT more than in the particular set $\Sigma(p,q)$ involved.

Another easy consequence of this definition is that if L is a finitary logic then we can always assume that the set $\Sigma(p,q)$ is a finite set. If moreover the logic is conjunctive [15] then without loss of generality we can assume that $\Sigma(p,q)$ is a singleton $\{\sigma(p,q)\}$.

The DDT is one of the main metalogical properties of classical and intuitionistic logic (being the role of $\Sigma(p,q)$ played by the implication $p \to q$). For instance, it is well known that the DDT characterizes the implication

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fragment of intuitionistic logic, i.e., this fragment is the smallest logic satisfying the DDT for the implication.

The study of the DDT (and several of its generalizations) has played an important role in the field of Abstract Algebraic Logic (AAL). The birth of AAL in the eighties [1, 2, 6] is closely connected with the following theorem characterizing the DDT.

Theorem 1.1. *[10, Theorem 3.10] A finitary and finitely algebraizable logic L has the DDT if and only if the principal relative congruences of algebras in the class Alg^*L*[1] *are equationally definable.*

The aim of this paper is to study the DDT for the logics preserving degrees of truth associated with subvarieties K of the variety RL of bounded, commutative and integral residuated lattices; for the sake of brevity we will call residuated lattices to the members of the previous class RL. These logics have been studied in [4] (see also [3]). Although the approach of [4] is based on the general theory of AAL, no study of DDT has been undertaken there. The present paper wants to fill this gap, and it can be considered as an extension of the work previously done in [4]. From now on, we will assume that the reader is familiar with both the content and the notation of [4].

For every variety $\mathsf{K} \subseteq \mathsf{RL}$, we know that we can consider two finitary and conjunctive logics: the logic $\vdash_{\mathsf{K}}$ **preserving the truth**, and the logic $\models^{\leqslant}_{\mathsf{K}}$ **preserving the degrees of truth**. We remind that they can be characterized (see [4]) as the finitary logics introduced by the schemes

sch1: $\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathsf{K}} \psi$ iff it holds that $\mathsf{K} \models \varphi_0 \approx 1 \;\&\; \ldots \;\&\; \varphi_{n-1} \approx 1 \Rightarrow \psi \approx 1$.

sch2: $\varphi_0, \ldots, \varphi_{n-1} \models^{\leqslant}_{\mathsf{K}} \psi$ iff it holds that $\mathsf{K} \models \varphi_0 \wedge \cdots \wedge \varphi_{n-1} \preccurlyeq \psi$.

It is well known that $\vdash_{\mathsf{K}}$ is always finitely algebraizable, while in general $\models^{\leqslant}_{\mathsf{K}}$ is not algebraizable (even not protoalgebraic). Therefore, by [11, Proposition 11.2] together with Theorem 1.1 it is well known which are the logics $\vdash_{\mathsf{K}}$ having DDT.

Theorem 1.2. *The logic $\vdash_{\mathsf{K}}$ has DDT iff exists $n \in \omega$ such that all algebras in K are n-contractive (i.e., $\mathsf{K} \subseteq \mathsf{E_n}$)*[2]*. In such a case, a formula defining the DDT is $\sigma(p, q) := p^n \to q$.*

Note that since the logic $\models^{\leqslant}_{\mathsf{K}}$ is in general not algebraizable we do not have any general method to characterize the logics $\models^{\leqslant}_{\mathsf{K}}$ that have DDT.

In Section 2 we will give some necessary conditions that must enjoy all K's such that the logic $\models^{\leqslant}_{\mathsf{K}}$ has the DDT. We will also show two cases where these conditions are indeed sufficient, and so in these two cases we know how to characterize the existence of DDT. The first of them is when K is a variety of MTL algebras, and the second one is when the variety K is generated by a finite algebra (indeed something weaker is sufficient).

In Section 3 we will consider the expansions by the Δ operator (see [14]) of the logics $\models^{\leqslant}_{\mathsf{K}}$. As it happens for the case of the logics preserving the truth we will see that the addition of Δ turns all logics $\models^{\leqslant}_{\mathsf{K}}$ into logics having the DDT.

Section 4 deals with a kind of "Local Deduction-Detachment Theorem" that holds for all logics $\models^{\leqslant}_{\mathsf{K}}$ where K is a variety generated by a family of continuous t-norms.

Remark 1.3. *Due to size limitations the proofs of the results stated in this paper are not included. They will be included in a extended version of this paper under preparation.*

2 The DDT for the logics $\models^{\leqslant}_{\mathsf{K}}$

In this section we focus our attention on the logics $\models^{\leqslant}_{\mathsf{K}}$ where K is a variety of residuated lattices. Since $\models^{\leqslant}_{\mathsf{K}}$ is always finitary and conjunctive we can assume that $\Sigma(p, q)$ is a singleton $\{\sigma(p, q)\}$. And by the uniqueness it is

[1] Remind that Alg^*L is the class of algebras $\boldsymbol{A}$ such that there is a set $F \subseteq A$ satisfying that $\langle \boldsymbol{A}, F\rangle$ is a model of the logic L and that the identity relation is the only congruence on $\boldsymbol{A}$ not relating elements inside F with elements outside F.

[2] The class $\mathsf{E_n}$ of n-contractive algebras is the subvariety of RL defined by the equation $x^n \approx x^{n+1}$.

clear that if K has the DDT both for a formula $\sigma(p,q)$ and a formula $\sigma'(p,q)$ then

$$\mathsf{K} \models \sigma(p,q) \approx \sigma'(p,q).$$

By the semantic definition of $\models_{\mathsf{K}}^{\leqslant}$ it is easy to prove the following result.

Lemma 2.1. *Given a formula $\sigma(p,q)$ the following conditions are equivalent:*

1. $\models_{\mathsf{K}}^{\leqslant}$ *has the DDT for the formula $\sigma(p,q)$.*

2. $\mathsf{K} \models \forall x \forall y \forall z (\, x \wedge z \preccurlyeq y \iff z \preccurlyeq \sigma(x,y)\,)$.

3. K *satisfies the following three equations*

$$\begin{gathered} x \wedge \sigma(x,y) \preccurlyeq y \\ y \preccurlyeq \sigma(x, x \wedge y) \\ \sigma(x,y) \preccurlyeq \sigma(x, y \vee z). \end{gathered}$$

4. *For every $\boldsymbol{A} \in \mathsf{K}$, the function $\sigma^{\boldsymbol{A}}$ together with the meet operation forms a residuated pair (i.e., $\sigma^{\boldsymbol{A}}(a,b) = \min\{c \in A : a \wedge c \leqslant b\}$).*

A trivial consequence of the previous lemma is that the existence of a DDT for $\models_{\mathsf{K}}^{\leqslant}$ only depends on the free algebra generated by three elements of the variety K (i.e., on equations using three variables). Up to now it is still open whether the "three" appearing here could be improved or not.

Next we state some necessary conditions that must hold in case that the logic $\models_{\mathsf{K}}^{\leqslant}$ has the DDT.

Lemma 2.2. *Let us assume that $\models_{\mathsf{K}}^{\leqslant}$ has the DDT for the formula $\sigma(p,q)$. Then,*

1. *The logic $\models_{\mathsf{K}}^{\leqslant}$ is protoalgebraic.*

2. *If $\boldsymbol{A} \in \mathsf{K}$, then every subalgebra of $\boldsymbol{A}$ is closed under the binary operation $\sigma^{\boldsymbol{A}}$.*

3. *For every $\boldsymbol{A} \in \mathsf{K}$, the algebra $\langle A, \wedge, \vee, \sigma^{\boldsymbol{A}}, 0, 1\rangle$ is a Heyting algebra (where $\sigma^{\boldsymbol{A}}$ corresponds to the interpretation of the implication connective).*[3]

[3] In particular this implies that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice, that $\sigma^{\boldsymbol{A}}(a, \sigma^{\boldsymbol{A}}(b,c)) = \sigma^{\boldsymbol{A}}(a \wedge b, c))$, that $\sigma^{\boldsymbol{A}}$ is anti-monotone in the first variable and monotone in the second one, etc.

We remind the reader that all protoalgebraic logics $\models_{\mathsf{K}}^{\leqslant}$ are also finitely equivalential (see [4, 3]). By [4, Theorem 3.4], the first of the previous items tells us that there is an $n \in \omega$ such that K satisfies the equation

$$x \wedge \big((x \to y)^n \star (y \to x)^n\big) \preccurlyeq y, \qquad (\mathrm{Prot}_n)$$

i.e., $\mathsf{K} \subseteq \mathsf{Prot_n}$. Using [4, Theorem 3.3] it is not hard to prove the following statement.

Lemma 2.3. *Let us assume that $\models_{\mathsf{K}}^{\leqslant}$ has the DDT for the formula $\sigma(p,q)$ and that $n \in \omega$ is such that $\mathsf{K} \subseteq \mathsf{Prot_n}$. Then,*

1. $\mathsf{K} \models \sigma(x^n, y) \approx x^n \to y$.

2. $\mathsf{K} \models \alpha(x,y,z) \preccurlyeq \sigma(x,y)$ *where* $\boldsymbol{\alpha(x,y,z)}$ *is the formula* $z \wedge ((x \wedge z) \to y)^n$.

3. $\mathsf{K} \models \alpha(x,y,\sigma(x,y)) \approx \sigma(x,y)$.

4. *For every $\boldsymbol{A} \in \mathsf{K}$ and $a,b \in A$, the element $\sigma^{\boldsymbol{A}}(a,b)$ coincides with $\max\{\alpha^{\boldsymbol{A}}(a,b,c) : c \in A\}$.*

5. *For every $\boldsymbol{A} \in \mathsf{K}$ and $a,b \in A$, the element $\sigma^{\boldsymbol{A}}(a,b)$ is the greatest fixed point of the unary map $\alpha^{\boldsymbol{A}}(a,b,\bullet)$.*

Using the trivial validities $\alpha(x,y,y) \approx y$ and $\alpha(x,y,1) \approx (x \to y)^n$, by the second item of the previous lemma we know that

$$\mathsf{K} \models y \vee (x \to y)^n \preccurlyeq \sigma(x,y).$$

In general, under the assumptions of Lemma 2.3 we will see at the end of this section that it is false that $y \vee (x \to y)^n$ is a formula defining the DDT, i.e., it is false that $\mathsf{K} \models y \vee (x \to y)^n \approx \sigma(x,y)$. But first of all we are going to see that for the particular case of MTL algebras if there is a formula defining the DDT then the formula $y \vee (x \to y)^n$ is also defining the DDT. The proof of this result, Theorem 2.5, is an easy consequence of the next lemma (together with some results stated in [4]).

Lemma 2.4. *Let $n \in \omega$. The equation* (Prot_n) *is equivalent, in any MTL algebra, to the quasi equation(s)*

$$x \wedge z \preccurlyeq y \iff z \preccurlyeq y \vee (x \to y)^n.$$

Theorem 2.5. *Let us assume that* K *is a variety of MTL algebras. The following conditions are equivalent.*

1. $\models_{\mathsf{K}}^{\leqslant}$ *has the DDT.*

2. $\models_{\mathsf{K}}^{\leqslant}$ *is protoalgebraic.*

3. $\models_{\mathsf{K}}^{\leqslant}$ *is finitely equivalential.*

4. *There exists* $n \in \omega$ *such that all chains in* K *are ordinal sums (as semihoops) of simple* n*-contractive MTL chains.*

5. *There exists* $n \in \omega$ *such that* $\models_{\mathsf{K}}^{\leqslant}$ *has the DDT for the formula* $\sigma(p,q) := q \vee (p \to q)^n$.

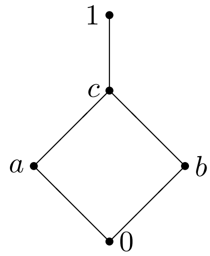
Next we state another case where we know how to characterize the existence of the DDT. However, the framework is not so simple as in the case of MTL algebras where we have been able to describe the formula defining the DDT.

Theorem 2.6. *Let us assume that* $\boldsymbol{A}$ *is a finite residuated lattice satisfying the same equations with three variables than the variety* K*. The following conditions are equivalent.*

1. $\models_{\mathsf{K}}^{\leqslant}$ *has the DDT.*

2. *It holds that (i)* $\mathsf{K} \subseteq \mathsf{Prot_n}$ *for certain* $n \in \omega$*, (ii)* $\mathsf{K} \models x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$*, and (iii) Every subalgebra of* $\boldsymbol{A}$ *is closed under the binary map defined by the rule* $(a,b) \longmapsto \max\{c \in A : a \wedge c \leqslant b\}$.

We point out that no one of the three properties stated in the previous characterization can be deleted. The variety of MTL algebras generated by the nilpotent minimum algebra $\boldsymbol{N_4}$ with four elements (see [9, 13]) shows that the condition (i) cannot be deleted. The condition (ii) (i.e., the distributive law) is the one that guarantees the existence of the map $(a,b) \longmapsto \max\{c \in A : a \wedge c \leqslant b\}$ considered in the condition (iii). Finally, a variety showing that (iii) cannot be deleted is the one generated by the simple 2-contractive algebra whose underlying lattice is obtained by adding one point above the Boolean algebra of four elements (i.e., the lattice depicted in Figure 1).



$\star$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	b	b	b
c	0	0	b	b	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	c	1	1
b	a	a	1	1	1
c	a	a	c	1	1
1	0	a	b	c	1

Figure 1: An interesting residuated lattice

In the case of a variety K of MTL algebras inside $\mathsf{Prot_n}$ it is clear, by our previous results, that if $\models_{\mathsf{K}}^{\leqslant}$ has the DDT then $\models_{\mathsf{K}}^{\leqslant}$ has the DDT for the formula $\sigma(p,q) := q \vee (p \to q)^n$. Thus, for this case we know how the formula defining the DDT is. On the other hand, for the case of a variety K of residuated lattices inside $\mathsf{Prot_n}$ that is generated by a finite algebra, it is unknown whether we can fix the formula defining the DDT or not. This is in author's opinion the main open problem concerning this subject.

What it is known about this open problem is that in general the formula $q \vee (p \to q)^n$ does not work. For instance, the variety generated by the algebra stated in Figure 1 has the DDT for the formula (where $n \geqslant 2$)

$$q \vee (p \to q)^n \vee \alpha(p, q, p \to q), \qquad (1)$$

while it fails for the formula $q \vee (p \to q)^n$ (because $b \wedge a \leqslant 0$ while $a \nleqslant b \vee (a \to b)^n$). Is (1) working in general for varieties inside $\mathsf{Prot_n}$? Unfortunately not, for instance there is a residuated lattice in $\mathsf{Prot_n}$ with seven points that has the DDT but does not have the DDT for the formula (1). Can we extend this process again and again ad infinitum or not? This is what the open problem stated above is inquiring.

3 The DDT for the logics $\models_{\mathsf{K}}^{\leqslant}$ expanded with Δ

In the realm of fuzzy logics it is common to consider the expansions by the Δ operator of the logics preserving the truth (see [14]). In the context of the logics preserving the degrees of truth their expansions by the Δ operator have not yet been studied. In this section we will introduce them and we will see that all of them have the DDT.

In this section we will assume that K is a variety of MTL-Δ algebras. We remind the reader that this means that K is a variety in the language $\langle \wedge, \vee, \star, \rightarrow, 1, 0, \Delta\rangle$ generated by a family of algebras $\{\boldsymbol{A_i} : i \in I\}$ such that for every $i \in I$,

- the reduct without Δ of $\boldsymbol{A_i}$ is an MTL chain.
- the $\Delta^{\boldsymbol{A_i}}$ operator satisfies that (i) $\Delta^{\boldsymbol{A_i}} 1 = 1$, and that (ii) $\Delta^{\boldsymbol{A_i}} x = 0$ if $x \neq 1$.

Thus, we can introduce, in the language $\langle \wedge, \vee, \star, \rightarrow, 1, 0, \Delta\rangle$, two different logics $\vdash_{\mathsf{K}}$ and $\models_{\mathsf{K}}^{\leqslant}$ just using, respectively, the schemes **sch1** and **sch2**. We are using the same notation than in the other (previous and future) sections, but there is no confusion because the similarity type of the class K used here is different. We notice that, by definition, these logics are conservative expansions of their corresponding logics associated with the class of $\langle \wedge, \vee, \star, \rightarrow, 1, 0\rangle$-reducts of algebras in K.

As we have pointed out above the logic $\vdash_{\mathsf{K}}$ has been widely studied in the literature. It is well known that $\vdash_{\mathsf{K}}$ is always finitely algebraizable and that it has the DDT for the formula $\sigma(p, q) := \Delta p \rightarrow \Delta q$, i.e.,

$$\Gamma, \varphi \vdash_{\mathsf{K}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathsf{K}} \Delta\varphi \rightarrow \Delta\psi.$$

Next, we state a result showing that the behaviour of the logics $\models_{\mathsf{K}}^{\leqslant}$ is very different from the behaviour of their fragments without Δ.

Theorem 3.1. *Let us assume that K is a variety of MTL-Δ algebras. Then,*

1. *$\models_{\mathsf{K}}^{\leqslant}$ has the DDT for the formula $\sigma(p, q) := q \vee \Delta(p \rightarrow q)$.*

2. *$\models_{\mathsf{K}}^{\leqslant}$ is finitely equivalential (in particular, protoalgebraic) being the equivalential set $E(p, q) := \{\Delta(p \leftrightarrow q)\}$.*

3. *$\models_{\mathsf{K}}^{\leqslant}$ is algebraizable iff the class of reducts without Δ of algebras in K is the class of Boolean algebras.*

To finish the section we show an axiomatization for the logic $\models_{\mathsf{K}}^{\leqslant}$. An easy consequence of the definition of logics $\vdash_{\mathsf{K}}$ and $\models_{\mathsf{K}}^{\leqslant}$ is the equivalence between

- $\varphi_0, \ldots, \varphi_{n-1} \models_{\mathsf{K}}^{\leqslant} \psi$, and
- $\emptyset \vdash_{\mathsf{K}} \varphi_0 \wedge \cdots \wedge \varphi_{n-1} \rightarrow \psi$.

Using this fact it can be proved (cf. [4]) that $\models_{\mathsf{K}}^{\leqslant}$ is axiomatized by the set of tautologies $TAUT(\mathsf{K})$ of $\vdash_{\mathsf{K}}$ as axioms together with the following rules

(Adj-$\wedge$)	$\{\langle\{\varphi, \psi\}, \varphi \wedge \psi\rangle : \varphi, \psi \in Fm\}$.
(MP-r)	$\{\langle\{\varphi, \varphi \rightarrow \psi\}, \psi\rangle : \varphi, \psi \in Fm$ and $\varphi \rightarrow \psi \in TAUT(\mathsf{K})\}$.
(Δ-r)	$\{\langle\{\varphi\}, \Delta\varphi\rangle : \varphi \in TAUT(\mathsf{K})\}$.

4 Some additional remarks

In this section we consider again K as an arbitrary variety of residuated lattices (without Δ). The aim is to discuss some conditions that resemble the Local Deduction-Detachment Theorem for the logics preserving the truth (see [7, 12]).

One result that helps to understand when the DDT holds for the logics $\vdash_{\mathsf{K}}$ (see Theorem 1.2) is the Local Deduction-Detachment Theorem. This theorem states that the following items are equivalent:

- $\Gamma, \varphi \vdash_{\mathsf{K}} \psi$,
- $(\exists n \in \omega)\ (\forall m \geqslant n)\ \Gamma \vdash_{\mathsf{K}} \varphi^m \rightarrow \psi$.

Thus, looking at Theorem 2.5 it seems natural to wonder whether the statements

st1: $\Gamma, \varphi \models^{\leqslant}_{\mathsf{K}} \psi$,

st2: $(\exists n \in \omega)\,(\forall m \geqslant n)\,\Gamma \models^{\leqslant}_{\mathsf{K}} \psi \vee (\varphi \to \psi)^m$.

are also equivalent or not. Before answering this question we notice that **st2** is equivalent to the statement

st3: $(\forall n \in \omega)\,\Gamma \models^{\leqslant}_{\mathsf{K}} \psi \vee (\varphi \to \psi)^n$.

Hence, what we will discuss is the relationship between **st1** and **st3**. This relationship is summarized in the next proposition.

Proposition 4.1.

1. **st1⇒st3** *fails (even with Γ finite) in general.*
2. **st1⇒st3** *holds (even with Γ infinite) for varieties* $\mathsf{K} \subseteq \mathsf{MTL}$.
3. **st3⇒st1** *fails (even with Γ finite) for the variety* K *of MTL algebras.*
4. **st3⇒st1** *fails (even with Γ finite) for the variety* K *of MV algebras generated by Chang's algebra (see [5, p. 481]).*
5. **st3⇒st1** *fails (with Γ infinite) for any variety* K *containing Chang's algebra.*
6. **st1⇔st3** *holds for Γ finite and any variety* K *of MTL algebras generated by chains that are ordinal sums (as semihoops) of Gödel algebras and Archimedean[4] algebras.*

Notice that all varieties generated by a family of continuous t-norms (this includes very famous varieties like BL, MV, G and Π) satisfy the assumption stated in the last item. Hence, the last proposition tells us that for $\mathsf{K} \in \{\mathsf{BL}, \mathsf{MV}, \mathsf{G}, \Pi\}$ the following statements are equivalent:

- $\gamma, \varphi \models^{\leqslant}_{\mathsf{K}} \psi$,
- $\gamma \models^{\leqslant}_{\mathsf{K}} \psi \vee (\varphi \to \psi)^n$ for every $n \in \omega$.

[4] This means that for every $a \neq 1$, it holds that $0 = \inf\{a^n : n \in \omega\}$.

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