On Reduced Semantics for Fuzzy Predicate Logics

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Abstract— Our work is a contribution to the model-theoretic study of equality-free fuzzy predicate logics. We present a *reduced semantics* and we prove a completeness theorem of the logics with respect to this semantics. The main concepts being studied are the *Leibniz congruence* and the *relative relation*. On the one hand, the Leibniz congruence of a model identifies the elements that are indistinguishable using equality-free atomic formulas and parameters from the model, a reduced structure is the quotient of a model modulo this congruence. On the other hand, the relative relation between two structures plays the same role that the isomorphism relation plays in classical predicate languages with equality.

Keywords— Equality-free Language, Fuzzy Predicate Logic, Model Theory, Reduced Structure, Relative Relation.

1 Introduction

This work is a contribution to the model-theoretic study of equality-free fuzzy predicate logics. Model theory is the branch of mathematical logic that studies the construction and classification of structures. Construction means building structures or families of structures, which have some feature that interest us. In our case we devote our investigation to the class of reduced structures, that help us to shed light to the characteristic role played by equality in predicate fuzzy logics.

Classifying a class of structures means grouping the structures into subclasses in a useful way, and then proving that every structure in the collection does belong in just one of the subclasses. The most basic classification in classical model theory is given by the relations of elementary equivalence and isomorphism. Our purpose in the present article is to investigate and characterize the structure-preserving maps between structures in a fuzzy setting. In classical predicate logics with equality, homomorphisms are structure-preserving, but if they are not isomorphisms, they don't necessarily preserve all the formulas of the language. On the contrary, in equality-free fuzzy predicate logics, σ-homomorphisms preserve all the formulas, but unlike isomorphisms, the relation between structures of one being the σ-homomorphic image of another is not an equivalence relation in the class of structures.

The main concepts studied in this work are the *Leibniz congruence* and the *relative relation*. The notion of Leibniz congruence arises in a very natural way. It is said that two elements of a model are related by this congruence when they satisfy exactly the same equality-free atomic formulas with parameters in the model. This congruence always exists, and it turns out to be the greatest congruence of the model. This idea has its origin in the Principle of the identity of the indiscernibles of G. W. Leibniz.

Given a model $(\mathbf{M}, \mathbf{B})$ the quotient structure modulo the Leibniz congruence $\Omega(\mathbf{M}, \mathbf{B})$ is denoted by $(\mathbf{M}, \mathbf{B})^r$ and is called its *reduction*. When we make the quotient modulo the Leibniz congruence, we identify the elements that are indistinguishable using equality-free atomic formulas and parameters in the model, thus the Leibniz congruence of the reduction of a model is always the identity. An structure with the property that its Leibniz congruence is the identity is said to be a *reduced* structure. The importance of reduced structures in equality-free logic comes from the fact that the reduction of a model is a σ- homomorphic image of the model and therefore, the model and its reduction satisfy exactly the same equality-free sentences.

The other main concept analysed is the relative relation. It is said that two structures are *relatives* when they have isomorphic reductions. Along this work we will give different characterizations of this relation. Our aim is to point out that it plays the same role in equality-free logic that the isomorphism relation plays in logic with equality. The actual interest of the Leibniz congruence and the relative relation comes from the work of W. Blok and D. Pigozzi. They introduced the concept of relative relation for the special case of logical matrices in [1], and in [2] they made an extensive use of what they named the Leibniz congruence.

Different definitions have been introduced so far for basic model-theoretic operations on structures. For instance, the notion of *elementary submodel, morphism* and *congruence of a fuzzy model* of [3], elementary embeddings and submodels of [4], *fuzzy submodel, elementary fuzzy submodel* and *isomorphism* of structures of first-order fuzzy logic with graded syntax of [5], *complete morphism and congruence* in languages with a similarity predicate of [6] and the notion of σ-embedding of [7]. Being our starting point all these works, in the Preliminaries section we introduce the notions of homomorphism and congruence of a model, trying both to encompass the most commonly used definitions in the literature and to extend the corresponding notions of classical predicate logics.

In section 3 we introduce the notion of reduced structure and some basic model-theoretic properties of this kind of structures. In section 4 we characterize when two structures are relative and we prove that the relative relation is the transitivization of the relation of being a σ-homomorphic image. Finally, section 5 is devoted to future work.

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2 Preliminaries

Our study of the model theory of fuzzy predicate logics is focused on the basic fuzzy predicate logic MTL∀ and stronger t-norm based predicate calculi, the so-called *core fuzzy logics*. We start by introducing the notion of core fuzzy logic in the propositional case.

Definition 1 *A propositional logic L is a* core fuzzy logic *iff L satisfies:*

1. *For all formulas ϕ, φ, α, $\varphi \equiv \phi \vdash \alpha(\varphi) \equiv \alpha(\phi)$.*
2. (LDT) Local Deduction Theorem: *for each theory and formulas ϕ, φ:*

$$T, \varphi \vdash \phi \text{ iff for some natural number } n,\ T \vdash \varphi^n \to \phi.$$

3. *L expands* MTL.

For a thorough treatment of core fuzzy logics we refer to [4], [8] and [7]. A *predicate language* Γ is a triple (P,F,A) where P is a non-empty set of predicate symbols, F is a set of function symbols and A is a function assigning to each predicate and function symbol a natural number called the *arity of the symbol*. Functions f for which $\mathrm{A}(f) = 0$ are called *object constants*. Formulas of the predicate language Γ are built up from the symbols in (P,F,A) together with logical symbols ($\forall, \exists, \&, \to, \overline{0}, \overline{1}$), variables and punctuation. Throughout the paper we consider the equality symbol as a binary predicate symbol not as a logical symbol, we work in equality-free fuzzy predicate logics. That is, the equality symbol is not necessarily present in all the languages and its interpretation is not fixed.

Let L be a fixed propositional core fuzzy logic and $\mathbf{B}$ an L-algebra, we introduce now the semantics for the fuzzy predicate logic $L\forall$. A $\mathbf{B}$-structure for predicate language Γ is a tuple $\mathbf{M} = (M, (P_{\mathbf{M}})_{P\in\Gamma}, (F_{\mathbf{M}})_{F\in\Gamma}, (c_{\mathbf{M}})_{c\in\Gamma})$ where M is a non-empty set and

1. For each n-ary predicate $P \in \Gamma$, $P_{\mathbf{M}}$ is a $\mathbf{B}$-fuzzy relation $P_{\mathbf{M}} : M^n \to \mathbf{B}$.
2. For each n-ary function symbol $F \in \Gamma$, $F_{\mathbf{M}} : M^n \to M$.
3. For each constant symbol $c \in \Gamma$, $c_{\mathbf{M}} \in M$.

Let $\mathbf{M}$ be a $\mathbf{B}$-structure, an $\mathbf{M}$-*evaluation* of the variables is a mapping v which assigns to each variable an element from M. By $\phi(x_1, \ldots, x_k)$ we mean that all the free variables of ϕ are among $x_1, \ldots, x_k$. If v is an evaluation such that for each $0 < i \leq n$, $v(x_i) = d_i$, and λ is either a Γ-term or a Γ-formula, we abbreviate by $\|\lambda(d_1, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}$ the expression $\|\lambda(x_1, \ldots, x_n)\|^{\mathbf{B}}_{\mathbf{M},v}$. Let ϕ be a Γ-sentence, given a $\mathbf{B}$-*structure* $\mathbf{M}$, it is said that $\mathbf{M}$ is a *model* of ϕ iff $\|\phi\|^{\mathbf{B}}_{\mathbf{M}} = 1$.

From now on, we say that $(\mathbf{M}, \mathbf{B})$ is a Γ-structure instead of saying that $\mathbf{M}$ is a $\mathbf{B}$-structure in the language Γ. We say that a structure is *safe*, if a truth value is defined for each formula and evaluation. We assume that all our structures are safe. It is denoted by $(\mathbf{M}, \mathbf{B}) \equiv (\mathbf{N}, \mathbf{A})$ when these two structures are elementarily equivalent. In this section we have presented only a few definitions and notation, a detailed introduction to the syntax and semantics of fuzzy predicate logics can be found in [9].

Definition 2 *Let $(\mathbf{M}_1, \mathbf{B}_1)$ be a Γ_1-structure and $(\mathbf{M}_2, \mathbf{B}_2)$ be a Γ_2-structure with $\Gamma_1 \subseteq \Gamma_2$. We say that the pair (f, g) is a* homomorphism *of $(\mathbf{M}_1, \mathbf{B}_1)$ into $(\mathbf{M}_2, \mathbf{B}_2)$ iff*

1. *$g : \mathbf{B}_1 \to \mathbf{B}_2$ is a L-algebra homomorphism of $\mathbf{B}_1$ into $\mathbf{B}_2$.*
2. *$f : M_1 \to M_2$ is a mapping of M_1 into M_2.*
3. *For each constant symbol $c \in \Gamma_1$, $f(c_{\mathbf{M}_1}) = c_{\mathbf{M}_2}$.*
4. *For each n-ary function symbol $F \in \Gamma_1$ and elements $d_1, \ldots, d_n \in M_1$,*

$$f(F_{\mathbf{M}_1}(d_1, \ldots, d_n)) = F_{\mathbf{M}_2}(f(d_1), \ldots, f(d_n))$$

5. *For each n-ary predicate $P \in \Gamma_1$ and elements $d_1, \ldots, d_n \in M_1$,*

$$g(P_{\mathbf{M}_1}(d_1, \ldots, d_n)) = P_{\mathbf{M}_2}(f(d_1), \ldots, f(d_n))$$

*We say that (f, g) is a σ-*homomorphism *if g preserves the existing infima and suprema (that is, if I is a non-empty set and $\sup_{i\in I} a_i$ and $\sup_{i\in I} g(a_i)$ exist, then $g(\sup_{i\in I} a_i) = \sup_{i\in I} g(a_i)$ and analogously for the infima).*

It is denoted by $(\mathbf{M}, \mathbf{B}) \cong (\mathbf{N}, \mathbf{A})$ when these two structures are isomorphic (that is, there is a homomorphism (f, g) from $(\mathbf{M}, \mathbf{B})$ into $(\mathbf{N}, \mathbf{A})$ with f and g onto and one-to-one). It is easy to check, by induction on the complexity of the formulas, that homomorphisms preserve quantifier-free formulas. Note that, by definition, homomorphisms are not always σ-complete, as are in [3] or [6], and unlike [6] homomorphisms are crisp on the algebraic reduct of the first-order structure. If (f, g) is a σ-homomorphism of $(\mathbf{M}_1, \mathbf{B}_1)$ into $(\mathbf{M}_2, \mathbf{B}_2)$ such that f is onto, then for each formula $\phi(x_1, \ldots, x_n) \in \Gamma_1$ and elements $d_1, \ldots, d_n \in M_1$,

$$g(\|\phi(d_1, \ldots, d_n)\|^{\mathbf{B}_1}_{\mathbf{M}_1}) = \|\phi(f(d_1), \ldots, f(d_n))\|^{\mathbf{B}_2}_{\mathbf{M}_2} \quad (1)$$

The proof can be found in [3] (Propositions 6.1 and 6.2). We will refer to homomorphisms satisfying condition (1) as *elementary homomorphisms*.

Definition 3 *A* congruence *on a Γ-structure $(\mathbf{M}, \mathbf{B})$ is a pair (E, θ) where:*

1. *θ is an L-congruence on the algebra $\mathbf{B}$.*
2. *E is an equivalence relation $E \subseteq M \times M$ such that:*
   - *For each n-ary function symbol $F \in \Gamma$ and elements $d_1, \ldots, d_n, e_1, \ldots, e_n \in M$, if for each $0 < i \leq n$, $(d_i, e_i) \in E$, then*

$$(F_{\mathbf{M}}(d_1, \ldots, d_n), F_{\mathbf{M}}(e_1, \ldots, e_n)) \in E$$

   - *For each n-ary predicate $P \in \Gamma$ and elements $d_1, \ldots, d_n, e_1, \ldots, e_n \in M$, if for each $0 < i \leq n$, $(d_i, e_i) \in E$, then*

$$(P_{\mathbf{M}}(d_1, \ldots, d_n), P_{\mathbf{M}}(e_1, \ldots, e_n)) \in \theta$$

Now, given a congruence (E, θ) on $(\mathbf{M}, \mathbf{B})$ we define the *quotient structure* $(\mathbf{M}/E, \mathbf{B}/\theta)$ by:

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- For each constant symbol $c \in \Gamma$, $c_{(\mathbf{M}/E,\mathbf{B}/\theta)} = [c_{(\mathbf{M},\mathbf{B})}]_E$.
- For each n-ary function symbol $F \in \Gamma$ and elements $d_1, \ldots, d_n \in M$,

$$F_{\mathbf{M}/E}([d_1]_E, \ldots, [d_n]_E) = [F_{\mathbf{M}}(d_1, \ldots, d_n)]_E$$

- For each n-ary predicate $P \in \Gamma$ and elements $d_1, \ldots, d_n \in M$,

$$P_{\mathbf{M}/E}([d_1]_E, \ldots, [d_n]_E) = [P_{\mathbf{M}}(d_1, \ldots, d_n)]_\theta$$

where, given an element $d \in M$ and $b \in B$, $[d]_E$ and $[b]_\theta$ denote respectively the equivalence classes of d modulo E and of b modulo θ. We will say that a (E, θ) is an *elementary* congruence (σ-congruence, respectively) if its canonical mapping (f_E, g_θ) is an elementary homomorphism (σ-homomorphism, respectively).

3 Reduced Structures

In this section we introduce the notions of Leibniz congruence and of reduced structure and we establish some basic model-theoretic properties of this kind of models, giving some examples of first-order theories with reduced models. The study of reduced structures and Leibniz congruences for classical predicate logics was done in [10]. In the context of fuzzy predicate logics, X. Caicedo introduced this notion in [11] for the particular case of models of first-order Rational Pavelka's logic in a language with the $\approx$ symbol. At the end of this section we study similarities on reduced structures.

Definition 4 *Let $(\mathbf{M}, \mathbf{B})$ be a Γ-structure and θ an L-congruence on $\mathbf{B}$. We define the relation $\Omega(\mathbf{M}, \mathbf{B}, \theta) \subseteq M \times M$ as follows: for every $d, e \in M$, $(d, e) \in \Omega(\mathbf{M}, \mathbf{B}, \theta)$ iff for every atomic formula, $\phi(y, x_1, \ldots, x_n) \in \Gamma$ and elements $d_1, \ldots, d_n \in M$,*

$$(\|\phi(d, d_1, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}, \|\phi(e, d_1, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}) \in \theta$$

Fixed an L-congruence θ on the L-algebra $\mathbf{B}$, next lemma shows that $\Omega(\mathbf{M}, \mathbf{B}, \theta)$ is the greatest E such that (E, θ) is a congruence on the model $(\mathbf{M}, \mathbf{B})$.

Lemma 5 *Let $(\mathbf{M}, \mathbf{B})$ be a Γ-structure and θ an L-congruence on $\mathbf{B}$, then*

1. *$(\Omega(\mathbf{M}, \mathbf{B}, \theta), \theta)$ is a congruence on $(\mathbf{M}, \mathbf{B})$.*
2. *For every (E, θ) congruence on $(\mathbf{M}, \mathbf{B})$, $E \subseteq \Omega(\mathbf{M}, \mathbf{B}, \theta)$.*

Proof: 1. Since θ is an equivalence relation, by definition, $\Omega(\mathbf{M}, \mathbf{B}, \theta)$ is also an equivalence relation. For each n-ary predicate $P \in \Gamma$ and elements $d_1, \ldots, d_n, e_1, \ldots, e_n \in M$, if for each $0 < i \leq n$, $(d_i, e_i) \in \Omega(\mathbf{M}, \mathbf{B}, \theta)$, then by using the definition of $\Omega(\mathbf{M}, \mathbf{B}, \theta)$, for every $0 < i \leq n$, we have the following chain:

$$(\|P(d_1, d_2, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}, \|P(e_1, d_2, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}) \in \theta$$

$$(\|P(e_1, d_2, d_3, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}, \|P(e_1, e_2, d_3, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}) \in \theta$$

$$\vdots$$

$$(\|P(e_1, e_2, \ldots, e_{n-1}, d_n)\|^{\mathbf{B}}_{\mathbf{M}}, \|P(e_1, \ldots, e_n)\|^{\mathbf{B}}_{\mathbf{M}}) \in \theta$$

Assume now that $F \in \Gamma$ is an n-ary function symbol and $d_1, \ldots, d_n, e_1, \ldots, e_n \in M$ such that for each $0 < i \leq n$, $(d_i, e_i) \in \Omega(\mathbf{M}, \mathbf{B}, \theta)$. Let $\overline{k} = k_1, \ldots, k_s \in M$, $\phi(y, x_1, \ldots, x_s) \in \Gamma$ an atomic formula and ϕ' the formula obtained from ϕ by substitution of the variable y for the term $F(z_1, \ldots, z_n)$ (where $z_1, \ldots, z_n$ are new variables not occurring in ϕ). By definition of $\Omega(\mathbf{M}, \mathbf{B}, \theta)$, since ϕ' is also atomic, we can build a chain similar to the one defined in the predicate case and then, we obtain $(\|\phi'(d_1, \ldots, d_n, \overline{k})\|^{\mathbf{B}}_{\mathbf{M}}, \|\phi'(e_1, \ldots, e_n, \overline{k})\|^{\mathbf{B}}_{\mathbf{M}}) \in \theta$, consequently,

$$(\|\phi(F_{\mathbf{M}}(d_1, \ldots, d_n), \overline{k})\|^{\mathbf{B}}_{\mathbf{M}}, \|\phi(F_{\mathbf{M}}(e_1, \ldots, e_n), \overline{k})\|^{\mathbf{B}}_{\mathbf{M}}) \in \theta$$

and then $(F_{\mathbf{M}}(d_1, \ldots, d_n), F_{\mathbf{M}}(e_1, \ldots, e_n)) \in \Omega(\mathbf{M}, \mathbf{B}, \theta)$.

2. By definition of $\Omega(\mathbf{M}, \mathbf{B}, \theta)$, because (E, θ) is a congruence. □

Definition 6 *A Γ-structure $(\mathbf{M}, \mathbf{B})$ is* reduced *iff $\Omega(\mathbf{M}, \mathbf{B}, Id_{\mathbf{B}})$ is the identity relation.*

From now on we denote $(\Omega(\mathbf{M}, \mathbf{B}, Id_{\mathbf{B}}), Id_{\mathbf{B}})$ simply by $\Omega(\mathbf{M}, \mathbf{B})$ and we call it the *Leibniz congruence* of $(\mathbf{M}, \mathbf{B})$. Since the identity map clearly preserves infima and suprema, $\Omega(\mathbf{M}, \mathbf{B})$ is always a σ-congruence, therefore for every $(d, e) \in \Omega(\mathbf{M}, \mathbf{B})$, every formula, $\phi(y, x_1, \ldots, x_n) \in \Gamma$ and elements $d_1, \ldots, d_n \in M$,

$$\|\phi(d, d_1, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}} = \|\phi(e, d_1, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}$$

We will denote by $(\mathbf{M}, \mathbf{B})^r$ the quotient structure modulo the Leibniz congruence $\Omega(\mathbf{M}, \mathbf{B})$ and call it the *reduction* of $(\mathbf{M}, \mathbf{B})$.

Lemma 7 *For every Γ-structure $(\mathbf{M}, \mathbf{B})$ we have:*

1. *$(\mathbf{M}, \mathbf{B}) \equiv (\mathbf{M}, \mathbf{B})^r$.*
2. *$(\mathbf{M}, \mathbf{B})^r$ is a reduced structure.*
3. *$((\mathbf{M}, \mathbf{B})^r)^r \cong (\mathbf{M}, \mathbf{B})^r$.*
4. *If there exists a σ-congruence (E, θ) on $(\mathbf{M}, \mathbf{B})$, then*

$$(\mathbf{M}, \mathbf{B}) \equiv (\mathbf{M}/\Omega(\mathbf{M}, \mathbf{B}, \theta), \mathbf{B}/\theta)$$

Proof: 1. holds because $(\mathbf{M}, \mathbf{B})^r$ is a σ-homomorphic image of $(\mathbf{M}, \mathbf{B})$, 2. and 3. by definition of the Leibniz congruence and of quotient structure. To prove 4. use the fact that for every σ-congruence (E, θ), $(\Omega(\mathbf{M}, \mathbf{B}, \theta), \theta)$ is also a σ-congruence. □

Corollary 8 [Completeness Theorem] *Let $L\forall$ be a fuzzy predicate logic and $\mathbb{K}$ a class of structures such that $L\forall$ is $\mathbb{K}$-complete (strong or finite strong $\mathbb{K}$-complete, respectively), then $L\forall$ is $\mathbb{K}^r$-complete (strong or finite strong $\mathbb{K}^r$-complete, respectively), where $\mathbb{K}^r$ is the class of reductions of the structures in $\mathbb{K}$.*

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Reduced structures are of common use in computer science and in mathematics. The Rado Graph (infinite random graph) and fuzzy linear orders are examples of reduced structures. However, we can see a more developed example of this process of reduction in the well-known case of similarities. In Section 5 of [9], P. Hájek studies similarities and applies the obtained results to the analysis of fuzzy control in Chapter 7 of of [9]. For a reference about model-theoretic properties of algebras with fuzzy equalities see [8] and [6]. Now (and only for the rest of this section) we assume that our predicate language Γ contains a binary predicate symbol $\approx$. Similarity is understood as fuzzified equality (for a reference see [12] or [13]). Given a core fuzzy logic L, let our axiomatic system for $L\forall$ contain also the following axioms:

1. (Reflexivity) $\forall x\, x \approx x$
2. (Symmetry) $\forall x \forall y (x \approx y \to y \approx x)$
3. (Transitivity) $\forall x \forall y \forall z ((x \approx y \& y \approx z) \to x \approx z)$
4. For each n-ary function symbol $F \in \Gamma$, $\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n ((x_1 \approx y_1 \& \dots \& x_n \approx y_n) \to (F(x_1, \dots, x_n) \approx F(y_1, \dots, y_n))$
5. For each n-ary predicate $P \in \Gamma$, $\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n ((x_1 \approx y_1 \& \dots \& x_n \approx y_n) \to (P(x_1, \dots, x_n) \leftrightarrow P(y_1, \dots, y_n))$

Axioms 1-3 are called *Similarity Axioms* (Sim) and axioms 4-5 *Congruence Axioms* (Cong).

Definition 9 *A Γ-structure $(\mathbf{M}, \mathbf{B})$ has the* equality property (EQP) *if the following condition holds: for every $d, e \in M$, $\|d \approx e\|^{\mathbf{B}}_{\mathbf{M}} = 1$ iff $d = e$.*

Lemma 10 *Given a set Σ of Γ-sentences, $\Sigma \cup Sim \cup Cong$ is satisfiable iff Σ has a model $(\mathbf{M}, \mathbf{B})$ that has* EQP.

Proof: Let $(\mathbf{M}, \mathbf{B})$ be a model of $\Sigma \cup Sim \cup Cong$. If we define $E = \{(a, b) \in M \times M : \|a \approx b\|^{\mathbf{B}}_{\mathbf{M}} = 1\}$, then $(E, Id_{\mathbf{B}})$ is a congruence. Thus, in the quotient structure $(\mathbf{M}/E, \mathbf{B})$, $\|x \approx y\|^{\mathbf{B}}_{\mathbf{M}/E} = 1$ iff $x = y$. □

In [11] X. Caicedo called *reduced structure* to a model of first-order Rational Pavelka's logic with the property EQP. Next lemma shows that Caicedo's notion coincides with ours when we consider axiomatic systems including axioms $Sim \cup Cong$.

Lemma 11 $(\mathbf{M}, \mathbf{B})$ *is a reduced structure iff* $(\mathbf{M}, \mathbf{B})$ *has property* EQP.

Proof: Assume that $(\mathbf{M}, \mathbf{B})$ is a reduced structure, by definition, the Leibniz congruence $\Omega(\mathbf{M}, \mathbf{B}, Id_{\mathbf{B}})$ is the identity on M. Then, if we define $(E, Id_{\mathbf{B}})$ as in the previous proof, by Lemma 5, $E \subseteq \Omega(\mathbf{M}, \mathbf{B}, Id_{\mathbf{B}})$, consequently E is also the identity and thus $(\mathbf{M}, \mathbf{B})$ has the EQP. Conversely, assume that $(\mathbf{M}, \mathbf{B})$ has the EQP. If $d, e \in M$ and $(d, e) \in \Omega(\mathbf{M}, \mathbf{B}, Id_{\mathbf{B}})$, since $x \approx y$ is an atomic formula and $\|d \approx d\|^{\mathbf{B}}_{\mathbf{M}} = 1$, by definition of the Leibniz congruence, $\|d \approx e\|^{\mathbf{B}}_{\mathbf{M}} = 1$. Then, by EQP, we have that $d = e$, therefore $(\mathbf{M}, \mathbf{B})$ is a reduced structure. □

Note that the interpretation of the $\approx$ symbol in a reduced structure is not necessarily crisp. Adding a new axiom it is possible to obtain crisp interpretations: Crispness Axiom (Crisp) $\forall x \forall y (x \approx y \vee \neg(x \approx y))$ (for the details about this axiom see Chapter 5 of [8]).

Corollary 12 *Let T be a set of Γ-sentences containing axioms $Sim \cup Cong \cup Crisp$. Then, for every formula $\phi(x_1, \dots, x_n) \in \Gamma$, the following holds:*
$T \vdash \forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n ((x_1 \approx y_1 \& \dots \& x_n \approx y_n) \to (\phi(x_1, \dots, x_n) \leftrightarrow \phi(y_1, \dots, y_n))$

Proof: Since axiom Crisp holds, the interpretation of the $\approx$ symbol in a reduced structure is the identity. Therefore, since the logic is complete with respect to its reduced models, we obtain the desired result. □

4 The relative relation

We now present the notion of relative relation, a relation between structures that will play in fuzzy predicate languages the same role that the isomorphism relation plays in classical predicate languages with equality. This notion was introduced by G. Zubieta in [14], but only for relational structures, and independently by W. Blok and D. Pigozzi in [1], for the special case of logical matrices. A characterization of the relative relation for classical first-order logics can be found in [10].

Definition 13 *Let $(\mathbf{M}_1, \mathbf{B}_1)$ and $(\mathbf{M}_2, \mathbf{B}_2)$ be two Γ-structures, we say that the pair (R, T) is a* relative relation *between $(\mathbf{M}_1, \mathbf{B}_1)$ and $(\mathbf{M}_2, \mathbf{B}_2)$ iff*

1. *$T \subseteq B_1 \times B_2$ is a relation such that $dom(T) = B_1$, $rg(T) = B_2$ and*
   (a) *for every connective $\delta \in L$, if $(a_i, b_i) \in T$, then*
   $$(\delta_{\mathbf{B}_1}(a_1, \dots, a_n), \delta_{\mathbf{B}_2}(b_1, \dots, b_n)) \in T$$
   (b) *for every $a \in B_1$, if $b, b' \in rg(a)$, then $dom(b) = dom(b')$.*
   (c) *for every $b \in B_2$, if $a, a' \in dom(b)$, then $rg(a) = rg(a')$.*
2. *$R \subseteq M_1 \times M_2$ is a relation such that $dom(R) = M_1$, $rg(R) = M_2$ and*
   (a) *For each constant symbol $c \in \Gamma$, $(c_{\mathbf{M}_1}, c_{\mathbf{M}_2}) \in R$.*
   (b) *For each n-ary function symbol $F \in \Gamma$, if for every $0 < i \leq n$, $(a_i, b_i) \in R$,*
   $$(F_{M_1}(a_1, \dots, a_n), F_{M_2}(b_1, \dots, b_n)) \in R$$
   (c) *For each n-ary predicate $P \in \Gamma$, if for every $0 < i \leq n$, $(d_i, e_i) \in R$,*
   $$(\|P(d_1, \dots, d_n)\|^{\mathbf{B}_1}_{\mathbf{M}_1}, \|P(e_1, \dots, e_n))\|^{\mathbf{B}_2}_{\mathbf{M}_2}) \in T$$

where for every $a \in B_1$, $rg(a) = \{b \in B_2 : (a, b) \in T\}$ and for every $b \in B_2$, $dom(b) = \{a \in B_1 : (a, b) \in T\}$. We denote by $(R, T) : (\mathbf{M}_1, \mathbf{B}_1) \sim (\mathbf{M}_2, \mathbf{B}_2)$ when (R, T) is a relative relation (or simply by $(\mathbf{M}_1, \mathbf{B}_1) \sim (\mathbf{M}_2, \mathbf{B}_2)$ when there is a relative relation between them).

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Theorem 14 *Let* $(\mathbf{M}_1, \mathbf{B}_1)$ *and* $(\mathbf{M}_2, \mathbf{B}_2)$ *be two* Γ*-structures. The following are equivalent:*

1. *There is a relative relation* $(R, T) : (\mathbf{M}_1, \mathbf{B}_1) \sim (\mathbf{M}_2, \mathbf{B}_2)$
2. *There are congruences* (E_1, θ_1) *and* (E_2, θ_2) *such that*

$$(\mathbf{M}_1/E_1, \mathbf{B}_1/\theta_1) \cong (\mathbf{M}_2/E_2, \mathbf{B}_2/\theta_2)$$

Proof: 1. $\Rightarrow$ 2. Assume that (R, T) is a relative relation. We define $\theta_1 = \{(a, a') \in B_1 \times B_1 : rg(a) = rg(a')\}$, $\theta_2 = \{(b, b') \in B_2 \times B_2 : dom(b) = dom(b')\}$, $E_1 = \Omega(\mathbf{M_1}, \mathbf{B_1}, \theta_1)$ and $E_2 = \Omega(\mathbf{M_2}, \mathbf{B_2}, \theta_2)$.

It is clear by definition that θ_1 and θ_2 are equivalence relations. Now we show that they are L-congruences, we prove that, for every connective $\delta \in L$, for every $0 < i \leq n$ and $a_i, a'_i \in B_1$, if $rg(a_i) = rg(a'_i)$, then $rg(\delta_{\mathbf{B}_1}(a_1, \ldots, a_n)) = rg(\delta_{\mathbf{B}_1}(a'_1, \ldots, a'_n))$. Let us assume that for every $0 < i \leq n$, $rg(a_i) = rg(a'_i)$. Since $dom(T) = B_1$, for every $0 < i \leq n$, we choose $b_i \in B_2$ such that $(a_i, b_i) \in T$. Thus, by assumption, since $rg(a_i) = rg(a'_i)$, we have also that $(a'_i, b_i) \in T$. By condition 1.(a) of the definition of relative relation,

$$(\delta_{\mathbf{B}_1}(a_1, \ldots, a_n), \delta_{\mathbf{B}_2}(b_1, \ldots, b_n)) \in T$$

and $(\delta_{\mathbf{B}_1}(a'_1, \ldots, a'_n), \delta_{\mathbf{B}_2}(b_1, \ldots, b_n)) \in T$, finally, by condition 1.(c) of the definition of relative relation, we have that

$$rg(\delta_{\mathbf{B}_1}(a_1, \ldots, a_n)) = rg(\delta_{\mathbf{B}_1}(a'_1, \ldots, a'_n)).$$

In order to show that θ_2 is a congruence, we can follow an analogous proof, using condition 1.(b) of the definition of relative relation, instead of 1.(c). Now we define a mapping $g : \mathbf{B}_1/\theta_1 \to \mathbf{B}_2/\theta_2$. First we fix enumerations (possibly with repetitions) $(a_i : i \in I)$ and $(b_i : i \in I)$ of $\mathbf{B}_1$ and $\mathbf{B}_2$ respectively, with the property that, for every $i \in I$, $(a_i, b_i) \in T$. And then let, for every $i \in I$, $g([a_i]_{\theta_1}) = [b_i]_{\theta_2}$. By using the definition of relative relation it is easy to check that g is well-defined, and it is an L-isomorphism.

Now, since $dom(R) = M_1$ and $rg(R) = M_2$, we can fix enumerations (possibly with repetitions) $(d_j : j \in J)$ and $(e_j : j \in J)$ of M_1 and M_2 respectively, with the property that, for every $j \in J$, $(d_j, e_j) \in R$. And then let, for every $j \in J$,

$$f([d_j]_{\Omega(\mathbf{M_1}, \mathbf{B_1}, \theta_1)}) = [e_j]_{\Omega(\mathbf{M_2}, \mathbf{B_2}, \theta_2)}$$

Let us see that f is well defined. Let $(d_j, d'_j) \in \Omega(\mathbf{M_1}, \mathbf{B_1}, \theta_1)$, we show that $(e_j, e'_j) \in \Omega(\mathbf{M_2}, \mathbf{B_2}, \theta_2)$. Let $\phi(y, x_1, \ldots, x_n)$ be an atomic formula, and a sequence of elements $k_1, \ldots, k_n \in M_2$. Since $rg(R) = M_2$, we can choose $l_1, \ldots, l_n \in M_1$ such that for every $0 < i \leq n$, $(l_i, k_i) \in R$. Remark that, since $(d_j, e_j) \in R$ and for every $0 < i \leq n$, $(l_i, k_i) \in R$, by conditions 2.(a) and 2.(b) of the definition of relative relation, we have for every Γ-term t, $(t_{M_1}(d_j, l_1, \ldots, l_n), t_{M_2}(e_j, k_1, \ldots, k_n)) \in R$. Assume that the formula $\phi(y, x_1, \ldots, x_n)$ is of the form $P(t_1, \ldots, t_s)(y, x_1, \ldots, x_n)$, where P is a s-ary predicate symbol and $t_1, \ldots, t_s$ are Γ-terms. Since $(d_j, d'_j) \in \Omega(\mathbf{M_1}, \mathbf{B_1}, \theta_1)$, $(\|P(t_1, \ldots, t_s)(d_j, l_1, \ldots, l_n)\|^{\mathbf{B}_1}_{\mathbf{M}_1}$, $\|P(t_1, \ldots, t_s)(d'_j, l_1, \ldots, l_n)\|^{\mathbf{B}_1}_{\mathbf{M}_1}) \in \theta_1$, thus, by 2.(c) of the definition of relative relation,
$\|P(t_1, \ldots, t_s)(e_j, k_1, \ldots, k_n)\|^{\mathbf{B}_2}_{\mathbf{M}_2}$ and
$\|P(t_1, \ldots, t_s)(e'_j, k_1, \ldots, k_n)\|^{\mathbf{B}_2}_{\mathbf{M}_2}$
$\in rg(\|P(t_1, \ldots, t_s)(d_j, l_1, \ldots, l_n)\|^{\mathbf{B}_1}_{\mathbf{M}_1})$

and by 1.(b), $dom(\|P(t_1, \ldots, t_s)(e_j, k_1, \ldots, k_n)\|^{\mathbf{B}_2}_{\mathbf{M}_2}) = dom(\|P(t_1, \ldots, t_s)(e'_j, k_1, \ldots, k_n)\|^{\mathbf{B}_2}_{\mathbf{M}_2})$ and we obtain the desired result:

$$(\|\phi(e_j, k_1, \ldots, k_n)\|^{\mathbf{B}_2}_{\mathbf{M}_2}, \|\phi(e'_j, k_1, \ldots, k_n)\|^{\mathbf{B}_2}_{\mathbf{M}_2}) \in \theta_2.$$

In an analogous way it is easy to check that f is one-to-one and that (f, g) is an isomorphism.

2. $\Rightarrow$ 1. Let (f, g) be an isomorphism. Define (R, T) in the following way: for every $b_1 \in B_1, b_2 \in B_2$,$(a, b) \in T$ iff $g([a]_{\theta_1}) = [b]_{\theta_2}$ and for every $d \in M_1, e \in M_2$, $(d, e) \in R$ iff $f([d]_{E_1}) = [e]_{E_2}$. Using the fact that (f, g) is an isomorphism, it is easy to check that $(R, T) : (\mathbf{M}_1, \mathbf{B}_1) \sim (\mathbf{M}_2, \mathbf{B}_2)$ is a relative relation. □

Remark that our approach differs from [6] because the relative relation is not a measure of the degree of similarity between structures. We left for future work the study of the relationship between these two notions.

Definition 15 (R, T) *is an* elementary *relative relation if for every formula* $\phi(x_1, \ldots, x_n) \in \Gamma$ *and for every* $0 < i \leq n$*, if* $(a_i, b_i) \in R$*, then*

$$(\|\phi(a_1, \ldots, a_n)\|^{\mathbf{B}_1}_{\mathbf{M}_1}, \|\phi(b_1, \ldots, b_n))\|^{\mathbf{B}_2}_{\mathbf{M}_2}) \in T$$

By induction on the complexity of the formulas it is straightforward to show that, in case (E_1, θ_1) and (E_2, θ_2) are elementary congruences in Theorem 14, then (R, T) is also an elementary relative relation and $(\mathbf{M}_1, \mathbf{B}_1) \equiv (\mathbf{M}_2, \mathbf{B}_2)$. The following corollary show us that, when we study structures over the same algebra, we can improve Theorem 14.

Corollary 16 *Let* $(\mathbf{M}_1, \mathbf{B})$ *and* $(\mathbf{M}_2, \mathbf{B})$ *be two* Γ*-structures. The following are equivalent:*

1. *There is a relative relation* $(R, Id_{\mathbf{B}}) : (\mathbf{M}_1, \mathbf{B}) \sim (\mathbf{M}_2, \mathbf{B})$
2. $(\mathbf{M}_1, \mathbf{B})^r \cong (\mathbf{M}_2, \mathbf{B})^r$

Proof: By the proof of Theorem 14. □

Let us see now an example of two fuzzy equivalence relations that are relatives but there is no homomorphism from one onto the other. Let $(\mathbf{M}_1, \mathbf{B})$ and $(\mathbf{M}_2, \mathbf{B})$ be defined as follows: the domains of the structures are $M_1 = \{d_1, d_2, e_1, e_2\}$ and $M_2 = \{d'_1, d'_2, d'_3, e'_1\}$ respectively. The fuzzy equivalence relation E_1 is defined by: for every $i, j \in \{1, 2\}$, $E_1(d_i, d_j) = 1 = E_1(e_i, e_j)$ and $E_1(d_i, e_j) = r$, where $r \neq 1$ is a fixed element of $\mathbf{B}$. And the fuzzy equivalence relation E_2 is defined by: for every $i, j \in \{1, 2, 3\}$, $E_2(d'_i, d'_j) = 1 = E_2(e'_1, e'_1)$ and $E_2(d'_i, e'_1) = r$. It is easy

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to check that $(R, Id_{\mathbf{B}}) : (\mathbf{M}_1, \mathbf{B}) \sim (\mathbf{M}_2, \mathbf{B})$, where R is the relation $R = \{(d_i, d'_j), (e_i, e'_j) : i, j \in \{1, 2, 3\}\}$.

The relation of being either a homomorphic image or a homomorphic counter-image is not in general transitive. Its transitivization is precisely the relative relation, as the following propositions show.

Lemma 17 *Given an L-algebra $\mathbf{B}$, $\sim$ is an equivalence relation in the class of $\mathbf{B}$-structures.*

Proof: By Corollary 16, because $\cong$ is an equivalence relation on the class of reduced structures. □

Notation: Given Γ-structures $(\mathbf{N}, \mathbf{B})$ and $(\mathbf{O}, \mathbf{B})$, we denote by $(\mathbf{N}, \mathbf{B}) \in \mathbf{H}(\mathbf{O}, \mathbf{B})$ the fact that there exists a mapping f from O onto N such that $(f, Id_{\mathbf{B}})$ is a homomorphism.

Proposition 18 *Let $(\mathbf{M}, \mathbf{B})$ and $(\mathbf{N}, \mathbf{B})$ be two Γ-structures. The following are equivalent:*

1. $(\mathbf{M}, \mathbf{B}) \sim (\mathbf{N}, \mathbf{B})$.
2. *There is a natural number n and Γ-structures $(\mathbf{O}_1, \mathbf{B}), \ldots, (\mathbf{O}_n, \mathbf{B})$ such that $(\mathbf{M}, \mathbf{B}) = (\mathbf{O}_1, \mathbf{B})$, $(\mathbf{N}, \mathbf{B}) = (\mathbf{O}_n, \mathbf{B})$ and for every $0 < i < n$, either $(\mathbf{O}_{n+1}, \mathbf{B}) \in \mathrm{H}(\mathbf{O}_n, \mathbf{B})$ or $(\mathbf{O}_n, \mathbf{B}) \in \mathrm{H}(\mathbf{O}_{n+1}, \mathbf{B})$.*
3. *There is a Γ-structure $(\mathbf{O}, \mathbf{B})$ such that $(\mathbf{M}, \mathbf{B}), (\mathbf{N}, \mathbf{B}) \in \mathrm{H}(\mathbf{O}, \mathbf{B})$.*
4. *There is a Γ-structure $(\mathbf{O}, \mathbf{B})$ such that $(\mathbf{O}, \mathbf{B}) \in \mathrm{H}(\mathbf{M}, \mathbf{B})$ and $(\mathbf{O}, \mathbf{B}) \in \mathrm{H}(\mathbf{N}, \mathbf{B})$.*

Proof: 4. $\Rightarrow$ 2. and 3. $\Rightarrow$ 2. are clear. 1. $\Rightarrow$ 4. By Corollary 16. 2. $\Rightarrow$ 1. By the definition of relative relation, given two Γ-structures $(\mathbf{O}, \mathbf{B})$ and $(\mathbf{D}, \mathbf{B})$, if $(\mathbf{O}, \mathbf{B}) \in \mathbf{H}(\mathbf{D}, \mathbf{B})$, then $(\mathbf{O}, \mathbf{B}) \sim (\mathbf{D}, \mathbf{B})$ therefore we can apply the transitive property of the relative relation (Lemma 17).

1. $\Rightarrow$ 3. Assume that there is a relative relation $(R, Id_{\mathbf{B}}) : (\mathbf{M}, \mathbf{B}) \sim (\mathbf{N}, \mathbf{B})$. Since $dom(R) = M$ and $rg(R) = N$, we can fix enumerations (possibly with repetitions) $(d_j : j \in J)$ and $(e_j : j \in J)$ of M and N respectively, with the property that, for every $j \in J$, $(d_j, e_j) \in R$. Now we define a structure $(\mathbf{O}, \mathbf{B})$ and homomorphisms $(f^M, Id_{\mathbf{B}})$ and $(f^N, Id_{\mathbf{B}})$ from $(\mathbf{O}, \mathbf{B})$ onto $(\mathbf{M}, \mathbf{B})$ and $(\mathbf{N}, \mathbf{B})$ respectively.

The algebraic reduct of $(\mathbf{O}, \mathbf{B})$ is the algebra Ter_J of Γ-terms generated by the set of variables $V_J = \{v_j : j \in J\}$. We define the function $f_0^M : V_J \to M$ as follows: for every $j \in J$, $f_0^M(v_j) = d_j$. Then we extend f_0^M in the usual way, to a homomorphism f^M from Ter_J onto M. Finally we define the interpretation of the predicate symbols in $(\mathbf{O}, \mathbf{B})$: for each n-ary predicate $P \in \Gamma$ and elements $t_1, \ldots, t_n \in O$, $P_O(t_1, \ldots, t_n) = P_M(f^M(t_1), \ldots, f^M(t_n))$. So defined $(f^M, Id_{\mathbf{B}})$ is clearly a homomorphism onto $(\mathbf{M}, \mathbf{B})$. Now we define f_0^N by: for every $j \in J$, $f_0^N(v_j) = e_j$ and we extend f_0^N as before, to a homomorphism f^N of the terms algebra onto N. Since for every $j \in J$, $(d_j, e_j) \in R$, for every Γ-term t,

$$(t_M(d_{j_1}, \ldots, d_{j_n}), t_N(e_{j_1}, \ldots, e_{j_n})) \in R$$

and using this fact it is easy to check that $(f^N, Id_{\mathbf{B}})$ is also a homomorphism onto $(\mathbf{N}, \mathbf{B})$. □

5 Conclusions

Work in progress includes the development of usual tools of model theory such as the method of diagrams or ultraproducts in order to work in fuzzy predicate logic. The use of a reduced semantics will allow us to show when one structure is either embeddable or elementarily embeddable in another in terms of extensions of the usual diagrams with special sentences. By using relative relations we could define new operations among structures, such as *ultrafilter-products*, more suitable for working with equality-free languages. Future work will be devoted also to provide different characterizations of the relation of elementary equivalence and some strenghtenings of this notion.

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