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Fuzzy Description Logics and *t*-norm based fuzzy logics

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ABSTRACT

Description Logics (DLs) are knowledge representation languages built on the basis of classical logic. DLs allow the creation of knowledge bases and provide ways to reason on the contents of these bases. Fuzzy Description Logics (FDLs) are natural extensions of DLs for dealing with vague concepts, commonly present in real applications. Hájek proposed to deal with FDLs taking as basis *t*-norm based fuzzy logics with the aim of enriching the expressive possibilities in FDLs and to capitalize on recent developments in the field of Mathematical Fuzzy Logic. From this perspective we define a family of description languages, denoted by $\mathcal{ALC}^*(\mathbf{S})$, which includes truth constants for representing truth degrees. Having truth constants in the language allows us to define the axioms of the knowledge bases as sentences of a predicate language in much the same way as in classical DLs. On the other hand, taking advantage of the expressive power provided by these truth constants, we define a graded notion of satisfiability, validity and subsumption of DL concepts as the satisfiability, validity and subsumption of *evaluated formulas*. In the last section we summarize some results concerning fuzzy logics associated with these new description languages, we analyze aspects relative to general and canonical semantics, and we prove some results relative to canonical standard completeness for some FDLs considered in the paper.

1. Introduction: from Description Logic to Fuzzy Description Logic

Description Logics (DLs) are knowledge representation languages particularly suited to specifying ontologies, creating knowledge bases and reasoning with them. DLs have been studied extensively over the last two decades. A full reference manual of the field is [1]. The vocabulary of DLs consists of *concepts*, which denote sets of individuals, and *roles*, which denote binary relations among individuals. From atomic concepts and roles and by means of *constructors*, DL systems allow us to build complex descriptions of both concepts and roles. These complex descriptions are used to describe a domain through a knowledge base (KB) containing the definitions of relevant domain concepts or some hierarchical relationships among them (*Terminological Box* or *TBox*), and a specification of properties of the domain instances (*Assertional Box* or *ABox*).[1] One of the main issues of DLs is the fact that the semantics is given in a Tarski-style presentation and the statements in both the *TBox* and the *ABox* can be identified with formulas in first-order logic or an extension of it; therefore we can use reasoning to obtain implicit knowledge from the explicit knowledge in the KB.

Nevertheless, the knowledge used in real applications is usually imperfect and has to address situations of uncertainty, imprecision and vagueness. From a real world viewpoint, vague concepts like "patient with a high fever" and "person living near Paris" have to be considered. A natural generalization to cope with vague concepts and relations consists in interpreting

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[1] Sometimes, for very expressive DLs the knowledge base also has an *RBox*, containing specific knowledge about roles.

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concepts and roles as fuzzy sets and fuzzy relations respectively. Fuzzy sets and fuzzy logics were born to deal with the problem of approximate reasoning [2,3]. Their first developments were characterized by the applications that gave rise to various semantic approaches to this problem. In recent times, formal logic systems have been developed for such semantics, and the logics based on triangular norms (t-norms) have become a central paradigm in fuzzy logic. The development of that field is intimately linked to the book *Metamathematics of Fuzzy Logics* [4], published in 1998, where Hájek shows the connection of fuzzy logic systems with many-valued residuated lattices based on continuous t-norms. He proposes a Hilbert-style calculus called Basic fuzzy Logic (*BL*) and conjectures that this logic is sound and complete with respect to the structures defined in the unit real interval $[0,1]$ by continuous t-norms and their residua. The conjecture was proved in [5]. In [6] Esteva and Godo introduced the logic *MTL*, a weakening of *BL*, which is proved in [7] to be the logic of left continuous t-norms and their residua. Since then, the field of t-norm based fuzzy logics has grown very quickly and it is the subject matter of intensive research (see http://www.mathfuzzlog.org for an exhaustive list of works and researchers in this area).

As regards fuzzy interpretations for description logic languages, with the exception of a paper by Yen [8] published in 1991, it was at the end of the last decade (from 1998) when several proposals of Fuzzy Description Logics (FDLs) were introduced (e.g., the ones by Tresp and Molitor [9] and by Straccia [10]). However, these early works on FDLs use a limited fuzzy logic apparatus called either "*minimalistic*" by Hájek in [11] or *Zadeh Logic* in [12,13]. In this logic the connectives are interpreted as follows: the intersection and the union are interpreted as *min* and *max* respectively; the complementation as $1-x$; and the interpretation of the universal quantified expression uses the Kleene–Dienes implication $\max(1-x,y)$. This interpretation is directly inspired by both the classical interpretation of the language $\mathcal{ALC}$ and Zadeh's initial proposal giving $\max, \min$ and the operation $1-x$ as the respective interpretations of union, intersection and complementation of fuzzy sets. It is worth observing that the Kleene–Dienes implication is not residuated. An important feature of the residuated implications is their good relationship with the order in the sense that $x \to y = 1$ is an equivalent way of saying $x \leqslant y$. However, the Kleene–Dienes implication behaves poorly with respect to the order, and it is difficult to interpret it as an implication in the logical setting. For example, in Zadeh Logic the implication $\varphi \to \psi$ is valid if and only if, for every interpretation, either the interpretation of φ has the value 0 or the interpretation of ψ is 1 (for a discussion of the counter-intuitive effects of the Kleene–Dienes implication in FDLs see for instance [11,14]). We should also mention other lines of investigation in FDLs composed of works where the term *fuzzy* is considered in the *broad sense*, according to the terminology of Zadeh. The languages considered in these papers (see for instance [15,16]) can handle, for example, fuzzy modifiers and fuzzy quantifiers, which are not considered in our approach.

In the paper *Making Fuzzy Description Logic more general* [11], published in 2005, Hájek proposes to deal with FDLs taking as basis t-norm based fuzzy logics with the aim of enriching the expressive possibilities in FDLs (see also [17]). This change of view gives a wide number of choices on which a DL can be based: for every particular problem we can consider the fuzzy logic that seems best suited and thus benefit from the recent advances in the setting of *Mathematical Fuzzy Logic* where the residuated (fuzzy) logics are widely studied. As an example, in [11] Hájek studies the FDLs associated with the description language $\mathcal{ALC}$. Since then, several researchers on FDLs have developed approaches based on the spirit of Hájek's paper (see for instance [12,18–20], the survey [21], or [22] dealing with DL programs). Thus, in particular, these studies interpret the constructor of intersection as a continuous t-norm, and the interpretation of the universal quantified expression $\forall R.C$ uses the residuated implication function associated with the t-norm which parameterize the constructor of intersection. So, the expressions of the description languages considered in these papers can be seen as instances of formulas built in the language of t-norm based fuzzy predicate logics. Nevertheless, these approaches mainly deal with the expressiveness of the languages and reasoning algorithms rather than with logical foundations.

The current paper explores the logical foundations of FDLs. Roughly speaking, each Fuzzy Description Logic can be seen as a logical system related to a fragment of a particular first order t-norm based logic presented with a well defined Hilbert-style calculus. This fact provides FDLs with powerful tools from a *metamathematical* point of view and allows them to exploit methods and results from Mathematical Fuzzy Logic. So the present paper is a first step in the direction proposed by Hájek in [11] concerning the analysis of the relationships between FDLs and t-norm based fuzzy logics. We deal with the (fuzzy) description logics associated to the language $\mathcal{ALC}$. A first requirement is to build a t-norm based fuzzy logic with the adequate logical tools to define an $\mathcal{ALC}$-like description language. To this end, we take the logic of a continuous t-norm and we add, if necessary, an involutive negation in order to capture the complementation needed in an $\mathcal{ALC}$-like description language; then we also add truth constants in order to capture the graded formulas used in the knowledge bases.

More explicitly, for each continuous t-norm $*$ there is a finitely axiomatizable propositional logic, presented by means of a Hilbert-style calculus and denoted by L^*, which is complete in the sense that theorems are equal to tautologies of the standard algebra defined by the t-norm and its residuum. Taking as basis the logic L^* we add:

(a) an involutive negation (if needed) by means of a finite set of axioms,
(b) a countable set of truth constants by means of an also countable set of axioms (the so-called *book keeping* axioms),

in order to define the logics denoted by $L^*_\sim(\mathbf{S})$. In this framework we define the first order logics $L^*_\sim(\mathbf{S})\forall$ corresponding to each one of these propositional logics by adding the two "classical" quantifiers (universal and existential) and both a finite number of axioms dealing with the quantifiers and the generalization inference rule corresponding to the universal quantifier.

On the other hand, given a continuous t-norm and a countable subalgebra $\mathbf{S}$ of the standard algebra $[0,1]_*$ expanded with the standard involutive negation function $1-x$, an $\mathcal{ALC}$-like description logic denoted by $\mathcal{ALC}^*(\mathbf{S})$ is defined in association with the logic $L^*_\sim(\mathbf{S})\forall$, in the following way: Firstly, we distinguish a subset of formulas of the language of $L^*_\sim(\mathbf{S})\forall$ as *instances* of $\mathcal{ALC}^*(\mathbf{S})$-concepts. Then, the $\mathcal{ALC}^*(\mathbf{S})$-logic is the logical calculus semantically defined in the usual way over the *canonical standard algebra* restricted to the *instances* of $\mathcal{ALC}^*(\mathbf{S})$-concepts. Note that this algebra is obtained by adding to the standard algebra the elements of S as distinguished elements, in such a way that the truth constants are interpreted by their defining values.

Notice that the semantically defined logics associated to our $\mathcal{ALC}$-like description languages are related with certain fragments of the logics $L^*_\sim(\mathbf{S})\forall$ in much the same way as the semantical calculus associated with the classical $\mathcal{ALC}$ is related with a fragment of the Hilbert-style axiomatization in which the classical First Order Logic (FOL) is presented. Nevertheless, the situation in the classical and fuzzy framework is not always the same. In the classical case the semantical calculus (the $\mathcal{ALC}$-logic) coincides with the corresponding fragment because of the completeness theorem; however, it is well known that this is not the case for all the logics in the family $L^*_\sim(\mathbf{S})\forall$. On the one hand we have that, when $L^*_\sim(\mathbf{S})\forall$ is canonical (standard) complete, the semantically defined description logic $\mathcal{ALC}^*(\mathbf{S})$ coincides with the corresponding fragment of the Hilbert-style calculus defining $L^*_\sim(\mathbf{S})\forall$ (Fig. 1). Thus, in Section 6.3 we prove that the logic $G_\sim(\mathbf{S})\forall$ (i.e., the first order Gödel logic with truth constants and an involutive negation) is strong canonical complete for finite theories, and the same can be proved for the logics $L^*_\sim(\mathbf{S})\forall$ when the t-norm is finite. It is also well known that both first order Łukasiewicz and Product logics are not canonical complete (tautologies on the real unit interval in both logics are a set that strictly contains the set of theorems of the corresponding First Order Logic). In such cases the $\mathcal{ALC}^*(\mathbf{S})$-logic does not necessarily coincide with the fragment of the first order fuzzy logic $L^*_\sim(\mathbf{S})\forall$ and it has to be studied in a different way: for instance as Hájek did in [11] for the $\mathcal{ALC}$-logic relative to Łukasiewicz Logic (see Section 6.2).

With respect to the inclusion of truth constants in the description languages, (as Hájek proposed) our motivation is the following: since the axioms of knowledge bases in FDLs include truth degrees, a natural choice seems to be the inclusion of symbols (truth constants) for these degrees in both the description language and the t-norm based logic where that language is interpreted. The topic of t-norm based fuzzy logics with truth constants in the language was firstly studied by Pavelka in [23]. In recent years it has received a renewed impulse and it has been the focus of exhaustive analysis (see [4,24–26]). In [27] Hájek assesses the computational complexity of propositional fuzzy logics with rational truth constants and analyzes some consequences for fuzzy description logics concerning satisfiability and validity. In particular, he defines the fuzzy description languages denoted by R-$\mathcal{ALC}*$ ($*$ being a continuous t-norm) with a truth constant for each rational number, and proves that the witnessed satisfiability of R-$\mathcal{ALC}*$-concepts is decidable.

The present paper is organized as follows. In Section 2 we recall some notions and results concerning t-norm based fuzzy logics necessary to define the family of description languages presented in Section 3. In Section 4 we describe the notions of *ABox* and *TBox* for the considered family of languages and we give their semantics, illustrating the differences between the crisp and the fuzzy cases with an example. Having truth constants in the language, we can handle graded general inclusion axioms in addition to the graded assertional axioms as is usually done in FDLs. Section 5 deals with reasoning in both classic $\mathcal{ALC}$ and its extension to finitely graded or fuzzy cases. Again, taking advantage of having truth constants, we can define graded notions of validity, satisfiability and subsumption from a syntactic perspective. We complete the section with some illustrative examples. In Section 6 we give some logical results about the logics $L^*_\sim(\mathbf{S})\forall$ and their consequences for the description logics $\mathcal{ALC}^*(\mathbf{S})$. Section 7 is devoted to the conclusions and suggestions for future work.

2. Fuzzy logics: general and standard (canonical) semantics

In this section we introduce the logics that are the formal counterpart of the description languages which will be introduced in Section 3. The propositional fragments of these logics are axiomatic extensions or conservative expansions of the Basic fuzzy Logic BL [4]. Thus, first we introduce the logic BL, proved in [5] to be the logic of all continuous t-norms and their residua, and we recall the *general semantics* for the logics which are axiomatic extensions or expansions of BL that interest us. Then, after recalling the notions of t-norm and its residuum, we introduce the notion of *divisible finite t-norm*, which extends the notion of continuous t-norm to the framework of finite chains. We then introduce the logics L^* which are the logics of a continuous t-norm $*$ (or a divisible finite t-norm $*$) and its residuum. Then, we define the logics $L^*_\sim$ (by introducing an involutive negation), the logics $L^*_\sim(\mathbf{S})$ (by introducing a set of truth constants S), and their first order extensions $L^*_\sim(\mathbf{S})\forall$.

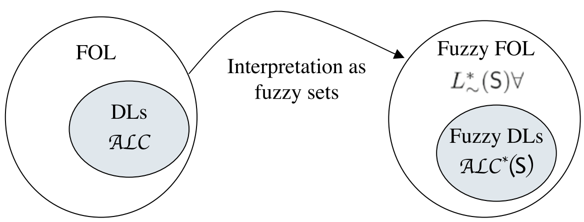

Fig. 1. When the first order logic $L^*_\sim(S)\forall$ (syntactically defined by means of a Hilbert-style calculus) is standard canonical complete, the $\mathcal{ALC}^*(S)$-logic (semantically defined) coincides with the corresponding fragment of $L^*_\sim(S)\forall$ in the same way as the logic associated with the classical $\mathcal{ALC}$ coincides with a fragment of the Hilbert-style calculus defining FOL.

We also discuss the general and the canonical semantics for these logics and we stress their interest for Fuzzy Description Logics.

2.1. Propositional fuzzy logics: general semantics

The Basic fuzzy Logic (*BL*) (defined in [4]) has the following basic connectives: *multiplicative conjunction* ($\&$), *implication* ($\rightarrow$), and *falsity* ($\bar{0}$). *BL* is defined by the following schemata (taking $\rightarrow$ as the least binding connective):

(BL1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(BL2) $\varphi \& \psi \rightarrow \varphi$
(BL3) $\varphi \& \psi \rightarrow \psi \& \varphi$
(BL4) $\varphi \& (\varphi \rightarrow \psi) \rightarrow \psi \& (\psi \rightarrow \varphi)$
(BL5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$
(BL5b) $(\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(BL6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(BL7) $\bar{0} \rightarrow \varphi$

The only deduction rule of *BL* is, as in Classical Propositional Logic, *Modus Ponens*. The notions of *proof*, *provability*, *theorem*, *consequence relation*, *theory*, etc., are defined in the usual way. Further connectives are defined as follows:

$$\varphi \wedge \psi := \varphi \& (\varphi \rightarrow \psi), \quad \varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi),$$

$$\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi),$$

$$\neg \varphi := \varphi \rightarrow \bar{0}, \quad \bar{1} := \neg \bar{0}$$

In [6] Esteva and Godo defined the logic *MTL* as a generalization of the logic *BL*. In fact *BL* is actually the extension of *MTL* obtained by adding the divisibility axiom: $\varphi \wedge \psi \rightarrow \varphi \& (\varphi \rightarrow \psi)$. *MTL* is proved to be the most general *t*-norm based fuzzy logic (see [7]). Łukasiewicz, Product and Gödel Logics can be obtained as axiomatic extensions of *BL* with the following schemata:

- $\neg\neg\varphi \rightarrow \varphi$ for Łukasiewicz Logic,
- $\varphi \wedge \neg\varphi \rightarrow \bar{0}$, and $\neg\neg\chi \rightarrow ((\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi))$ for Product Logic, and
- $\varphi \rightarrow \varphi \& \varphi$ for Gödel Logic.

Note that Classical Propositional Logic can be obtained as the axiomatic extension of *BL* adding the *excluded middle* axiom $\neg\varphi \vee \varphi$. *BL* belongs to the class of the *core fuzzy logics* introduced and studied by Hájek and Cintula in [28,29]. Briefly stated, propositional core fuzzy logics are expansions (extensions with, possibly, some extra connectives) of *MTL* whose additional connectives satisfy a congruence condition with respect to the double implication, and satisfying – as *MTL* does – the Local Deduction Theorem, that is, for every set of formulas Γ and formulas φ, ψ,

$$\Gamma, \varphi \vdash \psi \text{ iff there exists a natural number } n \text{ such that } \Gamma \vdash \varphi^n \rightarrow \psi.$$

General results contained in this section are formulated for core fuzzy logics as a general framework. We now recall the notion of *BL*-algebra and the notion of *L*-algebra for the case of a core fuzzy logic *L* expanding *BL*.

Definition 2.1. A *BL*-algebra is a divisible and prelinear commutative integral bounded residuated lattice, that is, an algebra $\mathbf{A} = \langle A, \vee^{\mathbf{A}}, \wedge^{\mathbf{A}}, *^{\mathbf{A}}, \rightarrow^{\mathbf{A}}, \bar{0}^{\mathbf{A}}, \bar{1}^{\mathbf{A}} \rangle$ with four binary operations and two distinguished elements, satisfying:

1. $\langle A, \vee^{\mathbf{A}}, \wedge^{\mathbf{A}}, \bar{0}^{\mathbf{A}}, \bar{1}^{\mathbf{A}} \rangle$ is a bounded lattice with minimum element $\bar{0}^{\mathbf{A}}$, and maximum element $\bar{1}^{\mathbf{A}}$.
2. $\langle A, \&^{\mathbf{A}}, \bar{1}^{\mathbf{A}} \rangle$ is a commutative monoid with unit $\bar{1}^{\mathbf{A}}$.
3. The operation $\&^{\mathbf{A}}$ is residuated and the operation $\rightarrow^{\mathbf{A}}$ is its residuum, i.e.,

$$\text{for every } a, b, c \in A, \; a \&^{\mathbf{A}} b \leqslant c \text{ iff } b \leqslant a \rightarrow^{\mathbf{A}} c,$$

where $\leqslant$ is the order associated to the lattice reduct.

4. For every $a, b \in A$, $(a \rightarrow^{\mathbf{A}} b) \vee^{\mathbf{A}} (b \rightarrow^{\mathbf{A}} a) = \bar{1}^{\mathbf{A}}$. (Prelinearity)
5. For every $a, b \in A$, $a \&^{\mathbf{A}} (a \rightarrow^{\mathbf{A}} b) = a \wedge^{\mathbf{A}} b$. (Divisibility)

A unary operation, the *negation* operator, is defined in this algebra $\mathbf{A}$ in the following way: $\neg^{\mathbf{A}} x := x \rightarrow^{\mathbf{A}} \bar{0}^{\mathbf{A}}$. [2]

[2] We will omit superscripts in the operations of the algebras when clear from the context.

Table 1
The three continuous t-norms, their residua and their associated negation.

$*$	Minimum (Gödel)	Product (of real numbers)	Łukasiewicz
$x*y$	$\min(x,y)$	$x\cdot y$	$\max(0,x+y-1)$
$x\rightarrow_* y$	$\begin{cases}1, & \text{if } x\leqslant y\\ y, & \text{otherwise}\end{cases}$.	$\begin{cases}1, & \text{if } x\leqslant y\\ y/x, & \text{otherwise}\end{cases}$.	$\min(1,1-x+y)$
n_*	$\begin{cases}1, & \text{if } x=0\\ 0, & \text{otherwise}\end{cases}$.	$\begin{cases}1, & \text{if } x=0\\ 0, & \text{otherwise}\end{cases}$.	$1-x$

Let L be a core fuzzy logic obtained by expansion of BL, and let $\mathcal{I}$ be the set of connectives of L. Let $\mathbf{A}=\langle A,\vee^{\mathbf{A}},\wedge^{\mathbf{A}},\&^{\mathbf{A}},\rightarrow^{\mathbf{A}},\langle \iota^{\mathbf{A}}\rangle_{\iota\in\mathcal{I}},\bar{0}^{\mathbf{A}},\bar{1}^{\mathbf{A}}\rangle$ be a structure such that $\langle A,\vee^{\mathbf{A}},\wedge^{\mathbf{A}},\&,\rightarrow^{\mathbf{A}},\bar{0}^{\mathbf{A}},\bar{1}^{\mathbf{A}}\rangle$ is a BL-algebra and such that, for every connective $\iota\in\mathcal{I}$ of arity k, $\iota^{\mathbf{A}}$ is a k-ary operation defined on A. An *evaluation* of the propositional variables in $\mathbf{A}$ is a mapping e assigning to each propositional variable p a *truth value* $e(p)\in A$. The evaluation e is inductively extended to a mapping $e_{\mathbf{A}}$ from the set of formulas of the language of L into the algebra $\mathbf{A}$ in the following way:

$$e_{\mathbf{A}}(\varphi\&\psi)=e_{\mathbf{A}}(\varphi)\&^{\mathbf{A}}e_{\mathbf{A}}(\psi);\ e_{\mathbf{A}}(\varphi\rightarrow\psi)=e_{\mathbf{A}}(\varphi)\rightarrow^{\mathbf{A}}e_{\mathbf{A}}(\psi);\ e_{\mathbf{A}}(\bar{0})=\bar{0}^{\mathbf{A}};$$

and, for every $\iota\in\mathcal{I}$, $e_{\mathbf{A}}(\iota(\varphi_1\ldots\varphi_k))=\iota^{\mathbf{A}}(e_{\mathbf{A}}(\varphi_1),\ldots,e_{\mathbf{A}}(\varphi_k))$.

A formula φ in the language of L is an $\mathbf{A}$-*tautology* if for every evaluation $e, e_{\mathbf{A}}(\varphi)=\bar{1}^{\mathbf{A}}$.

Definition 2.2 (*L-algebra*). Let L be a core fuzzy logic obtained by expanding BL, and let $\mathcal{I}$ be the set of basic additional connectives of L. An *L-algebra* is a structure $\mathbf{A}=\langle A,\vee^{\mathbf{A}},\wedge^{\mathbf{A}},\&^{\mathbf{A}},\rightarrow^{\mathbf{A}},\langle \iota^{\mathbf{A}}\rangle_{\iota\in\mathcal{I}},\bar{0}^{\mathbf{A}},\bar{1}^{\mathbf{A}}\rangle$ such that:

1. $\langle A,\vee^{\mathbf{A}},\wedge^{\mathbf{A}},\&,\rightarrow^{\mathbf{A}},\bar{0}^{\mathbf{A}},\bar{1}^{\mathbf{A}}\rangle$ is a BL-algebra.
2. Each additional axiom of L is an $\mathbf{A}$-tautology.

If the lattice reduct of $\mathbf{A}$ is linearly ordered we say that $\mathbf{A}$ is an *L-chain*.

Let Γ be a set of formulas in the language of L and $\mathbf{A}$ be an L-algebra. We say that an evaluation e is an $\mathbf{A}$-*model* of Γ if $e_{\mathbf{A}}(\gamma)=\bar{1}^{\mathbf{A}}$ for every $\gamma\in\Gamma$. We say that e is an $\mathbf{A}$-*model* of a formula φ if it is an $\mathbf{A}$-*model* of the set $\{\varphi\}$. All the logics that we will consider in the following sections are core fuzzy logics. Each logic L of this family enjoys strong completeness with respect to the class of L-chains (see [30,31]).

Theorem 2.3 (*Strong completeness theorem*). *Let L be a core fuzzy logic. Let Γ be a set of formulas (i.e., a theory) and φ a formula. The following conditions are equivalent:*

1) $\Gamma\vdash_L\varphi$
2) $e_{\mathbf{A}}(\varphi)=\bar{1}^{\mathbf{A}}$ for each L-algebra $\mathbf{A}$ and each $\mathbf{A}$-model e of Γ.
3) $e_{\mathbf{A}}(\varphi)=\bar{1}^{\mathbf{A}}$ for each L-chain $\mathbf{A}$ and each $\mathbf{A}$-model e of Γ.

This last theorem states that every core fuzzy logic L has strong completeness with respect the *general semantics*, i.e., with respect to the full class of L-algebras. Moreover, due to the prelinearity condition, we have also completeness with respect to L-chains since the class of L-chains generates all L-algebras. It is easy to see that if the language is countable (i.e., finite or numerable), condition (3) in the theorem can be restricted to countable L-chains.

2.2. Triangular norm based fuzzy logics

A *triangular norm* (or t-norm) is a binary operation defined on the real interval $[0,1]$ satisfying the following properties: associative, commutative, non decreasing in both arguments, and having 1 as unit element. A left continuous t-norm $*$ is characterized by the existence of a unique operation $\rightarrow_*$ satisfying, for all $a,b,c\in[0,1]$, the condition $a*c\leqslant b \iff c\leqslant a\rightarrow_* b$. This operation is called the *residuum* of the t-norm. Thus, in particular, all continuous t-norms have a residuum. A *negation* on $[0,1]$ is a unary operation $n:[0,1]\rightarrow[0,1]$ satisfying the following properties: $n(0)=1, n(1)=0$ and, for all $a,b\in[0,1], a\leqslant b\Rightarrow n(b)\leqslant n(a)$ (antimonotonicity). We say that n is *involutive* if, for all $a\in[0,1], n(n(a))=a$. We can also associate to each continuous t-norm $*$ a negation defined as follows: $n_*(x)=x\rightarrow_* 0$. Table 1 shows the main continuous t-norms (Minimum, Product and Łukasiewicz) with their residua and the corresponding associated negations. These three t-norms are the basic ones since any continuous t-norm can be expressed as an *ordinal sum* of copies of these three t-norms [32,33].

Definition 2.4. Let I be a countable (i.e., finite or enumerable) set of indexes, and let $\{[a_i,b_i]: i\in I\}$ be a family of closed subintervals of $[0,1]$ such that their interiors are pairwise disjoint. For every $i\in I$, let $*_i$ be a t-norm defined on $[a_i,b_i]$.[3] The *ordinal sum* of this family of t-norms is the operation on $[0,1]$ defined as follows:

[3] Given two real numbers a,b, with $a<b$, the name *t-norm* is also applied to operations defined in $[a,b]$ satisfying the same conditions of a t-norm, but in this case b is the unit element of the operation and a the zero element.

Table 2
The dual t-conorms corresponding to the three main continuous t-norms.

$\oplus$	Maximum (Gödel)	Sum	Łukasiewicz
$x \oplus y$	$\max(x, y)$	$(x + y) - (x \cdot y)$	$\min(1, x + y)$

$$x * y := \begin{cases} x *_m y, & \text{if } x, y \in [a_m, b_m], \text{ for some } m \in I, \\ \min\{x, y\}, & \text{otherwise.} \end{cases}$$

It is straightforward to prove that the ordinal sum of a family of continuous t-norms is a continuous t-norm.

A *triangular conorm* (or t-conorm) is a binary operation defined on $[0, 1]$ associative, commutative, non decreasing in both arguments, and having 0 as unit element. We say that a t-norm $*$ and a t-conorm $\oplus$ are *dual* with respect to a negation n if, for every $a, b \in [0, 1]$, the following conditions hold (De Morgan laws):

1. $n(a * b) = n(a) \oplus n(b)$,
2. $n(a \oplus b) = n(a) * n(b)$.

Given a t-norm $*$ and an involutive negation n, the operation defined by $x \oplus y := n(n(x) * n(y))$ is a t-conorm and $*$ and $\oplus$ are dual with respect to n. Table 2 shows the dual t-conorms of the three basic continuous t-norms with respect to the so-called *standard* involutive negation $N(x) = 1 - x$.

The concept of t-norm can be extended to finite chains in $[0, 1]$ with 0 and 1 as first and last element respectively (see [34,35]). We will call this kind of operation *finite t-norm*. These finite t-norms have to fulfill the same properties as t-norms, but to generalize continuity it is necessary to determine what "continuity" means in the case of finite chains. In [34] the authors propose the use of the property called *smoothness*, which can be easily proved to be equivalent to what is called *divisibility* in the residuated setting. Given a finite t-norm $*$ on a finite chain, the *divisibility* is defined as follows:

For every $a, b \in C$ such that $a > b$, exists $c \in C$ satisfying $b = c * a$.

Obviously, every finite t-norm $*$ on a finite chain has a residuum. If we denote the residuum by $\rightarrow_*$, we can see that the divisibility condition is equivalent to the satisfaction of the identity $x * (x \rightarrow_* y) = \min(x, y)$.[4] In [34] the authors prove that the divisible finite t-norms over a chain of n elements are the finite t-norms called Łukasiewicz and Minimum, and their finite ordinal sums. In fact, the structures defined by a divisible finite t-norm and its residuum are isomorphic to finite subalgebras of the algebraic structures defined by continuous t-norms on $[0, 1]$. For instance, the structure defined by the Łukasiewicz (resp. Minimum) finite t-norm over a chain of n elements is isomorphic to the subchain of the unit real interval when restricting the Łukasiewicz (resp. Minimum) t-norm to the set $C_n = \{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$.

Given a continuous t-norm $*$, it is easy to see that the structure

$$[0, 1]_* = \{[0, 1], \max, \min, *, \rightarrow_*, 0, 1\}$$

is a BL-chain. And given a divisible finite t-norm $*$ over a chain of n elements we also have that the structure

$$C_n^* = \{C_n, \max, \min, *, \rightarrow_*, 0, 1\}$$

is a BL-chain. Each one of these algebras is called the *standard canonical chain* relative to $*$. From now on and when no confusion is possible we will refer to it simply as the *canonical chain* of $*$.

Definition 2.5 (*Logic of a family of t-norms*). A core fuzzy logic L is said to be the *logic of a family of continuous t-norms T and their residua* (*the logic of T*, for short) if it is complete with respect to the class of canonical chains defined by the t-norms in T, that is, the set of theorems of L coincides with the set of tautologies of all canonical chains defined by a t-norm in T.

Definition 2.6 (*t-norm based logic*). A core fuzzy logic L expanding BL is said to be a *t-norm based logic* when there exists a family T of continuous t-norms such that L is the logic of T.

In an analogous way we can define *the logic of a family T of finite t-norms*. From now on, we will say that *L is the logic of a family T* without indicating whether T is a family of either t-norms or finite t-norms.

2.3. The logic L^* of a continuous t-norm

It is well known that BL is the logic of all continuous t-norms and their residua (see [5]). It is also known (see for instance [4]) that Łukasiewicz (resp. Gödel and Product) Logic is the logic of Łukasiewicz (resp. Minimum and Product) t-norm and its residuum. The main logics of a divisible finite t-norm over a chain of n elements are the logics $Ł_n$ and G_n corresponding to the finite t-norms of Łukasiewicz and Minimum.

[4] Notice that this equality is the same as the one that expresses the condition of continuity when $*$ is a t-norm and $\rightarrow_*$ is its residuum (see [4]).

In [36] the logic of each continuous t-norm $*$ and its residuum, denoted by L^*, is proved to be finitely axiomatizable as an axiomatic extension of BL, and an algorithm to find a finite set of axioms characterizing each logic L^* is given. Similar results are also true when $*$ is a finite t-norm: the corresponding axiomatizations can be easily obtained from our current knowledge of BL-algebras, mainly using the results in [37]. Moreover, using results in [26,36], it can be easily proved that these logics L^* (when $*$ is either a t-norm or a finite t-norm) satisfy the *finite strong canonical completeness*, that is, for every finite set of formulas Γ and every formula $\varphi, \Gamma \vdash_{L^*} \varphi$ if and only if every evaluation over the canonical chain of $*$ that is a model of Γ, is also a model of φ. In fact, in the case of a finite t-norm $*$, the logic L^* enjoys the strong canonical completeness, i.e., the completeness holds for any set Γ of premises, not only for finite ones.

2.4. Adding an involutive negation: the logics $L^*_\sim$

In order to define a description language having a complementation (as is done, for instance, in classical $\mathcal{ALC}$), it would be reasonable to have an involutive negation in the underlying logic. Therefore, when the negation $\neg\varphi := \varphi \to \bar{0}$ defined in L^* is not involutive, a new logic $L^*_\sim$ obtained by expanding L^* with an involutive negation should be considered. This negation could be introduced, as is done in the context of intuitionistic logic (see [38]) or in the context of Gödel logic (cf. [39]), by adding to L^* a new unary connective $\sim$ satisfying the following axioms:

$$\begin{array}{ll} (\sim 1) & \sim\sim \varphi \to \varphi \\ (\sim 2) & \sim (\varphi \vee \psi) \leftrightarrow (\sim \varphi \wedge \sim \psi) \\ (\sim 3) & \neg\varphi \to \sim \varphi \end{array} \tag{1}$$

From now on we denote by $L^*_\sim$ both the logics L^* such that the defined negation $\neg\varphi := \varphi \to \bar{0}$ is involutive and, when this is not the case, the logics obtained from L^* by expanding the language with the connective $\sim$ and by adding to its axiomatization the axioms (1).

When $*$ is a continuous t-norm, the truth function of the involutive negation over the canonical chain is not unique (any strictly decreasing bijection $n : [0,1] \to [0,1]$ satisfy the axioms of (1)). We define the *canonical* $L^*_\sim$*-chain* as the chain obtained by adding the negation function $N(x) = 1 - x$ to the canonical L^*-chain. Observe that, when $*$ is a finite t-norm, there is only one obvious way to obtain the canonical $L^*_\sim$-chain because there is only one possible involutive negation function definable over the canonical L^*-chain.

Having an involutive negation in the logic enriches the representational power of the logical language in a non-trivial way because:

(a) A multiplicative (or strong) disjunction $\varphi \veebar \psi$ is definable as $\sim (\sim \varphi \& \sim \psi)$, its truth function being the t-conorm defined by $x \oplus y := n(n(x) * n(y))$, where n is the involutive negation function defined in $[0,1]$ as truth function of $\sim$. Thus in this logic we have two disjunctions: the multiplicative one $\veebar$ defined above, and the additive one $\vee$ related to the order.
(b) Using these disjunctions, two new connectives can be defined by $\sim \varphi \veebar \psi$ and $\sim \varphi \vee \psi$, its truth functions being $n(x) \oplus y$ and $\max(n(x), y)$ respectively. This second function is the so-called Kleene–Dienes implication, the one used in the Zadeh Logic and in the first papers on FDLs.[5]

If the logic $L^*_\sim$ is canonical complete the definability of the Kleene–Dienes implication in the logic $L^*_\sim$ implies that the Zadeh Logic can be seen as a sublogic of $L^*_\sim$ for any t-norm $*$. This is the case when $*$ is the Minimum or Łukasiewicz t-norm. The logics $L^*_\sim$ such that the t-norm $*$ satisfies $\min(x \to_* 0, x) = 0$ (which is the case, for example, for Minimum and Product t-norms) were studied in [39].

2.5. Adding truth constants: the logics $L^*_\sim(\mathbf{S})$

Each one of the logics $L^*_\sim$ is a many-valued logic because the truth values are in $[0,1]$ (or in C_n). Intermediate values represent the different degrees of truth, i.e., the *partial truth* of a formula. These logics can also be seen as logics of *comparative truth* in the following sense: a formula $\varphi \to \psi$ is a logical consequence of a theory Γ if the truth degree of φ is at most as high as the truth degree of ψ in any interpretation that is a $*$-model of the theory. [6] This is because the residuum $\to_*$ of a continuous t-norm $*$ satisfies the condition $x \to_* y = 1$ if and only if $x \leqslant y$ for all $x, y \in [0,1]$. Thus, for any finite set of formulas Γ, we have $\Gamma \vdash_{L^*} \varphi \to \psi$ if and only if

$$\text{for each } e, \text{ if } e_*(\gamma) = 1 \text{ for every } \gamma \in \Gamma, \text{ then } e_*(\varphi \to \psi) = 1,$$

[5] Implication functions of this kind are called *S-implications* (Strong implications) in the literature on fuzzy sets and fuzzy logics (see for instance [40]) and they are generalizations of the truth function for the classical implication $\neg\varphi \vee \psi$. In these frameworks, the implication functions given by the residuum of a continuous (or left continuous) t-norm are examples of *R-implications* (Residuated implications) defined, given a t-norm $*$, as $x \to_* y := \sup\{x \in [0,1] : x * z \leqslant y\}$.

[6] For the sake of simplicity, we will use the notations e_*, instead of either $e_{[0,1]_*}$ or $e_{C^*_n}$; and $*$-model instead of either $[0,1]_*$-model or C^*_n-model.

but $e_*(\varphi \to \psi) = 1$ is equivalent to $e_*(\varphi) \to_* e_*(\psi) = 1$ which, by the property of the residuum mentioned above, is equivalent to $e_*(\varphi) \leqslant e_*(\psi)$.

The semantic deduction of formulas in many-valued logics (and in particular in FDLs) only takes into account the truth (i.e., the degree 1) but not partial truth degrees. Current approaches use a truth-preserving consequence relation in the same way as in the classical logic, i.e., true formulas are deduced from sets of true formulas. An elegant way to take advantage of the many-valued approach is to introduce truth constants into the language, as is done by Pavelka in [23] and more recently in [4,24–27]. The approach considered in the current paper is based on these ideas.

Given a logic L^*, let $\mathbf{S} = \langle S, *, \to_*, \max, \min, 0, 1\rangle$ be a *countable* (i.e., finite or enumerable) *subalgebra* (i.e., a subset closed under the operations) of the corresponding canonical chain. From L^* and $\mathbf{S}$, we define the logic $L^*(\mathbf{S})$ as follows:

(*i*) the language of $L^*(\mathbf{S})$ is the one of L^* plus a truth constant $\bar{r}$ for each $r \in S \setminus \{0, 1\}$,
(*ii*) the axioms and rules of $L^*(\mathbf{S})$ are those of L^* plus the so-called *book-keeping* axioms: for each $r, s \in S \setminus \{0, 1\}$,
- $\bar{r} \& \bar{s} \leftrightarrow \overline{r * s}$
- $(\bar{r} \to \bar{s}) \leftrightarrow \overline{r \to_* s}$.

When the negation associated to $*$ is not involutive, we can also define the logic $L^*_\sim(\mathbf{S})$ by combining the introduction of an involutive negation with the addition of truth constants. In this case $\mathbf{S}$ has to be a countable subalgebra of the canonical $L^*_\sim$-chain and we need to add, for every $r \in S \setminus \{0, 1\}$, a book-keeping axiom for the involutive negation: $\sim \bar{r} \leftrightarrow \overline{1 - r}$.

Definition 2.7. We define the *canonical* $L^*_\sim(\mathbf{S})$*-chain* as the chain obtained by adding a constant $\bar{r}$ for every element $r \in S$ to the canonical $L^*_\sim$-chain, and interpreting every $\bar{r}$ by r.

Remark 2.8 (*About completeness and canonical completeness*). The logics $L^*, L^*_\sim, L^*(\mathbf{S})$ and $L^*_\sim(\mathbf{S})$ all are core fuzzy logics. Thus they satisfy the strong completeness theorem relative to the general semantics (Theorem 2.3) in the sense that deductions in a logic L coincide with deductions with respect to evaluations in countable L-chains. But the semantics used in FDLs is the so-called *canonical semantics*. This semantics is obtained when we restrict ourselves to evaluations over the canonical $L^*_\sim(\mathbf{S})$-chain. Therefore, we are really interested in *canonical completeness*: that is, when both the general and the canonical semantics coincide at the level of either tautologies or deductions from a finite set of formulas. Canonical completeness results for the propositional logics $L^*(\mathbf{S})$, when $*$ is a continuous t-norm, have been fully studied in [26].

2.6. The predicate fuzzy logics $L^*_\sim(\mathbf{S})\forall$

In this section we introduce predicate versions of the propositional fuzzy logics described in previous sections. The basic notions and results are taken from [4,29,31]. Given a core fuzzy logic L in a propositional language $\mathcal{L}$, a *predicate language* (without functional symbols) consists of a countable set of *predicate symbols* $\mathcal{P} = \{P, Q, \ldots\}$, each one with arity $k \geqslant 0$, and a countable set of *object constants* $\mathcal{C} = \{c, d, \ldots\}$. The logical symbols are: a countable set of *object variables* $\{x, y, \ldots\}$, the *connectives* of the propositional language $\mathcal{L}$, and the *quantifiers* $\forall$ and $\exists$. *Terms* are object constants and object variables. An *atomic formula* is an expression of the form $P(t_1, \ldots, t_k)$, where P is a predicate symbol of arity k, and $t_1, \ldots, t_k$ are terms. The set of predicate formulas is defined as in the propositional case adding the rule stating that if φ is a formula, and x is a variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas. The notions of *free variable*, *open formula* (i.e., with free variables) and *closed formula* or *sentence* (i.e., without free variables) are defined in the usual way.

Given a logic $L^*_\sim(\mathbf{S})$, the corresponding Predicate Fuzzy Logic, denoted by $L^*_\sim(\mathbf{S})\forall$, is the expansion of $L^*_\sim(\mathbf{S})$ with the two quantifiers $\forall$ and $\exists$. The axioms of $L^*_\sim(\mathbf{S})\forall$ are the ones of $L^*_\sim(\mathbf{S})$ plus the following axioms on quantifiers (see [4]):

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$),
($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$),
($\forall$2) $(\forall x)(\varphi \to \psi) \to (\varphi \to (\forall x)\psi)$ (x not free in φ),
($\exists$2) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to \psi)$ (x not free in ψ),
($\forall$3) $(\forall x)(\varphi \vee \psi) \to (\forall x)\varphi \vee \psi$ (x not free in ψ).

Deduction rules of $L^*_\sim(\mathbf{S})\forall$ are (as in Classical Logic) *Modus Ponens* and *Generalization*. The notions of *proof*, *provability*, *theory*, etc., are defined in the usual way. Notice that the Classical Predicate Logic can be obtained as the axiomatic extension of any logic $L^*_\sim(\mathbf{S})\forall$ with the *excluded middle* axiom.

2.7. General and canonical semantics for $L^*_\sim(\mathbf{S})\forall$

Given a logic $L^*_\sim(\mathbf{S})$ and an $L^*_\sim(\mathbf{S})$-algebra $\mathbf{A}$, an $\mathbf{A}$-*interpretation* for the corresponding predicate language is a tuple

$$\mathbf{M} = \langle M, \{a^{\mathbf{M}} : a \in \mathcal{C}\}, \{P^{\mathbf{M}} : P \in \mathcal{P}\}\rangle$$

where

- M is a non-empty set,
- for each object constant $a \in \mathcal{C}, a^{\mathbf{M}}$ is an element of M,
- for each k-ary predicate symbol $P \in \mathcal{P}, P^{\mathbf{M}}$ is an $\mathbf{A}$-fuzzy k-ary relation defined on M, that is, a function $P^{\mathbf{M}} : M^k \to A$.

Given an $\mathbf{A}$-interpretation $\mathbf{M}$, a map v assigning an element $v(x) \in M$ to each variable x is called an *evaluation of the variables in* $\mathbf{M}$ (an $\mathbf{M}$-*evaluation*). Given $\mathbf{M}$ and v, the *value of a term t* in $\mathbf{M}$, denoted by $\|t\|_{\mathbf{M},v}$, is defined as $v(x)$ when t is a variable x, and as $a^{\mathbf{M}}$ when t is a constant a.

In order to emphasize that a formula α has its free variables in $\{x_1, \ldots, x_n\}$, we will denote it by $\alpha(x_1, \ldots, x_n)$. Let v be an $\mathbf{M}$-evaluation such that $v(x_1) = b_1, \ldots, v(x_n) = b_n$. The *truth value in* $\mathbf{M}$ *over* $\mathbf{A}$ *of the formula* $\varphi(x_1, \ldots, x_n)$ *for the evaluation* v, denoted by $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v}$ or by $\|\varphi(b_1, \ldots, b_n)\|^{\mathbf{A}}_{\mathbf{M}}$, is a value in A defined inductively as follows:

$P^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_k\|_{\mathbf{M},v})$	if $\varphi = P(t_1, \ldots, t_k)$,
$\|\alpha\|^{\mathbf{A}}_{\mathbf{M},v} \&^{\mathbf{A}} \|\beta\|^{\mathbf{A}}_{\mathbf{M},v}$	if $\varphi = \alpha \& \beta$,
$\|\alpha\|^{\mathbf{A}}_{\mathbf{M},v} \to^{\mathbf{A}} \|\beta\|^{\mathbf{A}}_{\mathbf{M},v}$	if $\varphi = \alpha \to \beta$,
$\sim^{\mathbf{A}} \|\alpha\|^{\mathbf{A}}_{\mathbf{M},v}$	if $\varphi = \sim \alpha$,
$\bar{r}^{\mathbf{A}}$	if $\varphi = \bar{r}$,
$\bar{0}^{\mathbf{A}}$	if $\varphi = \bar{0}$,
$\bar{1}^{\mathbf{A}}$	if $\varphi = \bar{1}$,
$\inf\{\|\alpha(a, b_1, \ldots, b_n)\|^{\mathbf{A}}_{\mathbf{M}} : a \in M\}$	if $\varphi = (\forall x)\alpha(x, y_1, \ldots, y_n)$,
$\sup\{\|\alpha(a, b_1, \ldots, b_n)\|^{\mathbf{A}}_{\mathbf{M}} : a \in M\}$	if $\varphi = (\exists x)\alpha(x, y_1, \ldots, y_n)$.

If the infimum or supremum does not exist, we take its value as undefined. We say that $\mathbf{M}$ is *safe* if $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v}$ is defined for each formula φ and each $\mathbf{M}$-evaluation v. A safe $\mathbf{A}$-interpretation $\mathbf{M}$ is an $\mathbf{A}$-*model* of a set of formulas Γ if for each $\varphi \in \Gamma$, and each $\mathbf{M}$-evaluation v, $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v} = 1$. If $\Gamma = \{\varphi\}$, we say that $\mathbf{M}$ is an $\mathbf{A}$-model of φ. A formula φ is an $\mathbf{A}$-*tautology* if every safe $\mathbf{A}$-interpretation is an $\mathbf{A}$-model of φ. If $\mathbf{A}$ is the canonical chain we say that φ is a *canonical tautology*. Obviously, every safe $\mathbf{A}$-interpretation is an $\mathbf{A}$-model of the empty set of formulas. Thus, from now on, we will sometimes use the name *model* instead of *safe interpretation*.

A first general completeness result for these logics is their strong completeness with respect to chain-valuated models.

Theorem 2.9 (Strong completeness theorem). *Let Γ be a set of formulas (i.e., a theory) and φ a formula of the language of $L^*_\sim(\mathbf{S})\forall$. The following conditions are equivalent:*

(1) $\Gamma \vdash_{L^*_\sim(\mathbf{S})\forall} \varphi$.
(2) For each $L^*_\sim(\mathbf{S})$-chain $\mathbf{A}$, every $\mathbf{A}$-model of Γ is an $\mathbf{A}$-model of φ.
(3) For each countable $L^*_\sim(\mathbf{S})$-chain $\mathbf{A}$, every $\mathbf{A}$-model of Γ is also an $\mathbf{A}$-model of φ.

The above theorem is a result relative to *general semantics*, with respect to valuations over the general class of L-chains and it was proved in [29]. The equivalence between (2) and (3) is not stated in the cited paper but it is obvious since the language of our logics is countable. However, the semantic interesting for FDLs is the *canonical semantics* with respect to models over the canonical chain. For the sake of simplicity, when the algebra of truth values $\mathbf{A}$ is the canonical $L^*_\sim(\mathbf{S})$-chain we will use the notations $*$-interpretation and $*$-model instead of $\mathbf{A}$-interpretation $\mathbf{A}$-model respectively. Given a $*$-interpretation $\mathbf{M}$, the truth value of a formula φ in $\mathbf{M}$ for a valuation v will be denoted by $\|\varphi\|^*_{\mathbf{M},v}$.

Definition 2.10 (*Canonical completeness*). We say that $L^*_\sim(\mathbf{S})\forall$ enjoys the *(finite) strong canonical completeness*, if for every (finite) theory Γ and each formula φ, the following conditions are equivalent:

(1) $\Gamma \vdash_{L^*_\sim(\mathbf{S})\forall} \varphi$.
(2) Every $*$-model of Γ is also a $*$-model of φ.

We say that $L^*_\sim(\mathbf{S})\forall$ enjoys *canonical completeness* if the above equivalence holds to the empty theory.

As we have already pointed out, general and canonical semantics do not always coincide. The problem of canonical completeness of the logics $L^*(\mathbf{S})\forall$ is addressed in [41,42] and that of the logics $L^*_\sim(\mathbf{S})\forall$ is considered in Section 6.

The logics $L^*_\sim(\mathbf{S})\forall$ will be the basis of our proposal for the description languages presented in the next section. These logics are truth preserving in the sense that true formulas are deduced from sets of true formulas. The novelty here is the introduction of truth constants in the language allowing us to write sentences like $\bar{r} \to \varphi$ or $\varphi \to \bar{r}$ which, when they are true, means that the truth value of φ is *greater or equal* or *less or equal* than r respectively. This is the main idea behind the so-called *evaluated formulas*, i.e., formulas of type $\bar{r} \to \varphi$ or $\varphi \to \bar{r}$ where the formula φ has no truth constants different from 0 or 1 (see for instance [26] and references therein). Evaluated formulas use truth constants as a way to compare the truth value of a formula without constants with a value r. In turn this comparison allows us both to use and to reason with *partial* *truth*. In Section 4 evaluated formulas are used to define knowledge bases corresponding to the FDL languages defined in Section 3.

3. The Description Logics $\mathcal{ALC}^*(\mathbf{S})$

In the tradition of Description Logic literature, the language $\mathcal{ALC}$ (see [43]) is presented using: (*a*) the symbols in $\{\sqcup, \sqcap, \neg, \bot, \top\}$ which, from the first order logic point of view, can be understood as the propositional connectives in $\{\vee, \wedge, \neg, \bar{0}, \bar{1}\}$; and (*b*) the symbols $\exists$ and $\forall$ used in the denotation of the constructors of concepts $\exists R.C$ and $\forall R.C$ (*existential* and *universal quantification* respectively) which can also be read as a particular kind of quantified first order formulas (cf. [44]).

In what follows we will introduce, for each logic $L^*_\sim(\mathbf{S})\forall$, the $\mathcal{ALC}$-like corresponding description logic, denoted by $\mathcal{ALC}^*(\mathbf{S})$. In the literature on *t*-norm based fuzzy logics, it is common to use the symbols $\&$ and $\veebar$ for the *multiplicative conjunction* and *disjunction* respectively, and to reserve the symbols $\wedge$ and $\vee$ for the *additive conjunction* and *disjunction* respectively (cf. [4]). Accordingly, we will use as primitive connectives the *conjunction* (denoted by $\&$), the *disjunction* (denoted by $\veebar$), the *implication* (denoted by $\rightarrow$), the *involutive negation* (denoted by $\sim$), the *falsity* ($\bar{0}$), the *truth* ($\bar{1}$) and a truth constant $\bar{r}$ for each $r \in S \setminus \{0, 1\}$.

3.1. The languages $\mathcal{ALC}^*(\mathbf{S})$

In abstract notation, the symbol A is used for atomic concepts, the symbol R is used for atomic roles, and the symbols C and D are used for descriptions of concepts. After fixing a continuous *t*-norm or a divisible finite *t*-norm $*$ and a subalgebra $\mathbf{S}$ of the canonical $L^*_\sim$-chain, the complex descriptions of concepts in $\mathcal{ALC}^*(\mathbf{S})$ are built using the constructors in $\{\veebar, \&, \rightarrow, \sim, \bar{0}, \bar{1}\} \cup \{\bar{r} : r \in S \setminus \{0, 1\}\}$, the quantifiers $\forall, \exists$, and the point . as an auxiliary symbol, in accordance with the following syntactic rules:

$C, D \rightsquigarrow$	$A\vert$	(atomic concept)
	$\bar{0}\vert$	(empty concept)
	$\bar{1}\vert$	(universal concept)
	$\bar{r}\vert$	(constant concept)
	$\sim C\vert$	(complementary concept)
	$C\veebar D\vert$	(concept union)
	$C\& D\vert$	(concept intersection)
	$C \rightarrow D\vert$	(concept implication)
	$\forall R.C\vert$	(universal quantification)
	$\exists R.C\vert$	(existential quantification)

Observe that by cutting this set of syntax rules the ones corresponding to the constants $\bar{r}$ we obtain, up to notation, the classical language $\mathcal{ALC}$. Now, considering a predicate language (as introduced in Section 2.6) defined using the set of connectives of $L^*_\sim(\mathbf{S})$ and only predicate symbols of arity $k \leqslant 2$, we read

- each atomic concept A as a unary predicate symbol,
- each atomic role R as a binary predicate symbol,
- the constructors in $\{\&, \rightarrow, \sim, \bar{0}, \bar{1}\} \cup \{\bar{r} : r \in S \setminus \{0, 1\}\}$ as the connectives of the language of $L^*_\sim(\mathbf{S})$, and the constructor $\veebar$ as the strong disjunction defined by $\varphi\veebar\psi := \sim(\sim\varphi \,\&\, \sim\psi)$ (see Section 2.4).

Next we define the notion of *instance* of both a concept and an atomic role, which allows us to read the formulas of $\mathcal{ALC}^*(\mathbf{S})$ as predicate formulas.

Definition 3.1 (*Instance of a concept*). Given a term t and an $\mathcal{ALC}^*(\mathbf{S})$-concept D, the *instance* $D(t)$ of D is defined as follows:

$A(t)$	if D is an atomic concept A,
$\sim C(t)$	if $D = \sim C$,
$C_1(t)\veebar C_2(t)$	if $D = C_1\veebar C_2$,
$C_1(t)\& C_2(t)$	if $D = C_1\& C_2$,
$C_1(t) \rightarrow C_2(t)$	if $D = C_1 \rightarrow C_2$,
$(\forall y)(R(t, y) \rightarrow C(y))$	if $D = \forall R.C$,
$(\exists y)(R(t, y)\& C(y))$	if $D = \exists R.C$,
$\bar{0}$	if $D = \bar{0}$,
$\bar{1}$	if $D = \bar{1}$,
$\bar{r}$	if $D = \bar{r}$,

where y is a variable not occurring in $C(t)$.

Definition 3.2 (*Instance of a role*). Given two terms t_1 and t_2 and an atomic role R every atomic formula $R(t_1, t_2)$ is called an *instance* of R.

3.2. The logics $\mathcal{ALC}^*(\mathbf{S})$

At this point one option could be to define the logic $\mathcal{ALC}^*(\mathbf{S})$-logic as the fragment of the Hilbert style calculus defining $L^*_\sim(\mathbf{S})\forall$ corresponding to the instances of $\mathcal{ALC}^*(\mathbf{S})$-concepts. As we explained above, this logic coincides with the logical calculus semantically defined by interpretations in $L^*_\sim(\mathbf{S})$-chains, that is, in the *general* semantics for the formulas of the language of $L^*_\sim(\mathbf{S})\forall$. Nevertheless, as we have pointed above, this approach and the one given by the canonical semantics do not always coincide, and so, since our intended semantics is the canonical one, we will define the semantics for the instances of $\mathcal{ALC}^*(\mathbf{S})$-concepts using the interpretations over the canonical chain.

Let $\mathcal{C}$ be the set of object constants and let $\mathcal{A}$ and $\mathcal{R}$ be the sets of predicate symbols of arity 1 (concepts) and 2 (roles) respectively. Let $\mathbf{M}$ be an interpretation for our predicate language, that is,

$$\mathbf{M} = \langle M, \{a^\mathbf{M} : a \in \mathcal{C}\}, \{A^\mathbf{M} : A \in \mathcal{A}\}, \{R^\mathbf{M} : R \in \mathcal{R}\}\rangle$$

where

- M is a non-empty set,
- for each constant $a \in \mathcal{C}, a^\mathbf{M}$ is an element of M,
- for each atomic concept $A \in \mathcal{A}, A^\mathbf{M}$ is a function $M \to [0,1]$, i.e., a fuzzy set on M,
- for each atomic role $R \in \mathcal{R}, R^\mathbf{M}$ is a function $M \times M \to [0,1]$, i.e., a fuzzy binary relation on M.

Notice that if $*$ is a finite t-norm we need to replace the codomain of these functions by the carrier of the canonical chain, i.e., by C_n. In this case we obtain an *n-graded* (resp. *crisp* if $n = 2$) interpretation for the atomic concepts and roles.

Let $D(t)$ be an instance of an $\mathcal{ALC}^*(\mathbf{S})$-concept D. Given either a continuous t-norm or a divisible finite t-norm $*$, according with the definitions given in Section 2.6, given an $*$-interpretation (i.e., an interpretation over the canonical $L^*_\sim(\mathbf{S})$-chain) $\mathbf{M}$, the *truth value* $\|D(t)\|^*_{\mathbf{M},v}$ for an $\mathbf{M}$-evaluation v is given by

$A^\mathbf{M}(\|t\|_{\mathbf{M},v})$	if D is an atomic concept A,
$N(\|C(t)\|^*_{\mathbf{M},v})$	if $D = \sim C$,
$\|C_1(t)\|^*_{\mathbf{M},v} * \|C_2(t)\|^*_{\mathbf{M},v}$	if $D = C_1 \& C_2$,
$\|C_1(t)\|^*_{\mathbf{M},v} \to_* \|C_2(t)\|^*_{\mathbf{M},v}$	if $D = C_1 \to C_2$,
$\|C_1(t)\|^*_{\mathbf{M},v} \oplus \|C_2(t)\|^*_{\mathbf{M},v}$	if $D = C_1 \veebar C_2$,
$\inf\{\|R(t,b) \to C(b)\|^*_\mathbf{M} : b \in M\}$	if $D = \forall R.C$,
$\sup\{\|R(t,b) \& C(b)\|^*_\mathbf{M} : b \in M\}$	if $D = \exists R.C$.
0	if $D = \bar{0}$,
1	if $D = \bar{1}$,
r	if $D = \bar{r}$,

where N is the standard involutive negation $N(x) = 1 - x$ and $\oplus$ is the t-conorm defined by $x \oplus y := N(N(x) * N(y))$.

Definition 3.3 (*The description logics* $\mathcal{ALC}^*(\mathbf{S})$). Given a logic $L^*_\sim(\mathbf{S})$ let $\Gamma \cup \{\varphi\}$ be a finite set of instances of $\mathcal{ALC}^*(\mathbf{S})$-concepts. We define the $\mathcal{ALC}^*(\mathbf{S})$-logic in the following way:

$$\Gamma \models_{\mathcal{ALC}^*(\mathbf{S})} \varphi \text{ if and only if every } *\text{-model of } \Gamma \text{ is also a } *\text{-model of } \varphi.$$

When $S = \{0,1\}$, the canonical $L^*_\sim(\mathbf{S})$-chain is equal to the canonical $L^*_\sim$-chain. In this case the logic is denoted by $\mathcal{ALC}^*$ and the consequence relation by $\models_{\mathcal{ALC}^*}$.

For every $*$-interpretation $\mathbf{M}$, each $\mathcal{ALC}^*(\mathbf{S})$-concept C determines a fuzzy set $C^\mathbf{M}$ defined as follows:

$$\text{for every } a \in M, C^\mathbf{M}(a) := \|C(a)\|^*_\mathbf{M}$$

Thus, every interpretation and every continuous t-norm (or divisible finite t-norm) associate to complex descriptions the following fuzzy (or n-graded) sets:

$$(\sim C)^\mathbf{M}(a) = 1 - C^\mathbf{M}(a),$$
$$(C \& D)^\mathbf{M}(a) = C^\mathbf{M}(a) * D^\mathbf{M}(a)$$
$$(C \to D)^\mathbf{M}(a) = C^\mathbf{M}(a) \to_* D^\mathbf{M}(a)$$
$$(C \veebar D)^\mathbf{M}(a) = C^\mathbf{M}(a) \oplus D^\mathbf{M}(a)$$
$$(\forall R.C)^\mathbf{M}(a) = \inf\{R^\mathbf{M}(a,b) \to_* C^\mathbf{M}(b) : b \in M\}$$
$$(\exists R.C)^\mathbf{M}(a) = \sup\{R^\mathbf{M}(a,b) * C^\mathbf{M}(b) : b \in M\}$$
$$\bar{0}^\mathbf{M}(a) = 0$$
$$\bar{1}^\mathbf{M}(a) = 1$$
$$\bar{r}^\mathbf{M}(a) = r$$

Notice that these interpretations are generalizations of the classical case in the sense that taking $*$ as a finite t-norm and $n = 2$ we obtain the classical interpretation for the language $\mathcal{ALC}$.

Remark 3.4. Observe that, by the definition of residuum, in the classical case, i.e., taking $*$ as the classical truth function for the conjunction, denoted here by $\wedge$, we have:

$$(C \to D)^{\mathbf{M}}(a) = \sup\{b \in \{0,1\} : C^{\mathbf{M}}(a) \wedge b \leqslant D^{\mathbf{M}}(a)\} \tag{2}$$

Let us recall that in the classical case the truth function of the residuated implication coincides with the truth function of the material implication that is given by $\max(1 - x, y)$. Therefore, in the classical case the identity (2) is equivalent to

$$(C \to D)^{\mathbf{M}}(a) = \max(1 - C^{\mathbf{M}}(a), D^{\mathbf{M}}(a)) \tag{3}$$

This interpretation of the connective of implication is the usual one in early work on FDLs (it corresponds to the Kleene–Dienes implication function) and it is directly inspired in the classical interpretation of the $\mathcal{ALC}$ language, as is clearly seen when comparing this interpretation with the classical one given by the expression (2). However it is worth noting that, in general, the expressions (2) and (3) above are not equivalent for n-graded and fuzzy cases.

4. Representing knowledge bases in $\mathcal{ALC}$ and in $\mathcal{ALC}^*(\mathbf{S})$

In this section we will explain how to represent knowledge bases in our languages $\mathcal{ALC}^*(\mathbf{S})$. We describe the notions of *ABox* and *TBox* for the considered family of languages and we give their semantics, illustrating the differences between the crisp and the fuzzy cases with an example. In particular, since we are interested in reasoning on partial truth of formulas, we will restrict ourselves to using evaluated formulas for representing the knowledge contained in knowledge bases. With truth constants in the language, we can handle graded general inclusion axioms in addition to graded assertional axioms, as is usually done in FDLs.

4.1. Knowledge bases for $\mathcal{ALC}$

Description Logics can be used to build knowledge representation systems since they allow the creation of *knowledge bases* (KBs) and provide ways to reason on the contents of these bases. A *KB* is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, where the first component is a *TBox* and the second one is an *ABox*. From a general point of view, the *TBox* introduces complex concepts and models the hierarchy of domain concepts by introducing the vocabulary of an application domain. The *ABox* models a concrete description of the domain.

For the classical interpretation of $\mathcal{ALC}$, a *TBox* is a finite set of concept inclusion axioms. A *concept inclusion axiom* is a sentence of the form $(\forall x)(C(x) \to D(x))$ (abbreviatedly $C \sqsubseteq D$). We use $C \equiv D$ as an abbreviation for the two axioms $C \sqsubseteq D$ and $D \sqsubseteq C$. An *ABox* is a finite set of assertion axioms. An *assertion axiom* is a sentence either $C(a)$ or $R(a,b)$. In the tradition of DLs, it is usual to denote these two kinds of sentence by $a : C$ and $(a,b) : R$ respectively.

According to the semantics for first order classical logic, it is said that an interpretation $\mathbf{M}$ satisfies an axiom α of a knowledge base $\mathcal{K}$, written $\mathbf{M} \models \alpha$, if and only if it satisfies the corresponding sentence. That is,

$\mathbf{M} \models C \sqsubseteq D$	iff $\inf\{\max(1 - C^{\mathbf{M}}(x), D^{\mathbf{M}}(x)) : x \in M\} = 1$
$\mathbf{M} \models C(a)$	iff $C^{\mathbf{M}}(a^{\mathbf{M}}) = 1$
$\mathbf{M} \models R(a,b)$	iff $R^{\mathbf{M}}(a^{\mathbf{M}}, b^{\mathbf{M}}) = 1$

4.2. An example: the Robots data set

We will illustrate the notions introduced above using an application domain that is a free adaptation of the Monks data set from the Machine Learning Repository of Irvine's University (http://archive.ics.uci.edu/ml/). Our data set is composed of nine robots, each one with either the same or a different shape of head and body (i.e., they are homogeneous or not homogeneous respectively), they can or cannot wear a tie, they can or cannot smile (i.e., they are happy or not happy respectively), and they hold an object. Objects such as swords or axes are considered unfriendly and the other ones (i.e., flags, balloons and flowers) are considered friendly. Taking all these characteristics into account, robots can be classified as *friendly* or *unfriendly*. The domain of interpretation is the set

$$M_{\mathcal{R}} = \{r_i : 1 \leqslant i \leqslant 9\} \cup \{o_i : 1 \leqslant i \leqslant 9\},$$

where the r_i are the robots in Fig. 2 and each o_i is the object that the robot r_i holds (e.g., the robot r_4 holds the object o_4 that is a flower). Atomic concepts of the language are the following: Robot, Happy, Object, FriendlyObject, Homogeneous, Balloon, Flag, Flower, Sword, Ax and WearsTie. There are two atomic roles: hasObject and hasnoObject. Notice that this latter role is introduced to avoid the use of negation of atomic roles, which is not allowed in $\mathcal{ALC}$. A *TBox* for the example of the robots is the following:

TBox for the Little Robots

$\text{Friendly} \equiv \text{Robot} \ \& \ (\exists \text{hasObject.FriendlyObject}) \ \& \ (\text{Happy} \underline{\vee} \text{Homogeneous})$
$\text{Unfriendly} \equiv \text{Robot} \ \& \sim \text{Friendly}$
$\text{FriendlyObject} \sqsubseteq \text{Object}$
$\text{UnfriendlyObject} \equiv \text{Object} \ \& \sim \text{FriendlyObject}$
$\text{Robot} \ \& \ \text{Object} \sqsubseteq \bar{0}$
$\bar{1} \sqsubseteq \text{Robot} \underline{\vee} \text{Object}$
$\text{Homogeneous} \sqsubseteq \text{Robot}$
$\text{Happy} \sqsubseteq \text{Robot}$
$\text{WearsTie} \sqsubseteq \text{Robot}$
$\text{Flower} \sqsubseteq \text{FriendlyObject}$
$\text{Balloon} \sqsubseteq \text{FriendlyObject}$
$\text{Flag} \sqsubseteq \text{FriendlyObject}$
$\text{Sword} \sqsubseteq \text{UnfriendlyObject}$
$\text{Ax} \sqsubseteq \text{UnfriendlyObject}$

Notice that this *TBox* includes definitions for classifying a robot as either *friendly* or *unfriendly*. A robot is considered *friendly* when it holds a friendly object and when it is either happy or homogeneous. An *unfriendly* robot is defined using the negation of the definition of *friendly* robot. Following this definition, robots r_1, r_2, r_6 and r_9 are friendly whereas the remaining ones are unfriendly (since at least one of the conditions of the conjunctive expression defining friendly robots is not satisfied).

The *ABox* contains the assertions describing the robots in Fig. 2:

ABox for the Little Robots

For each $i, 1 \leqslant i \leqslant 9, \text{Robot}(r_i), \text{hasObject}(r_i, o_i)$
For each $j \neq i, 1 \leqslant i, j \leqslant 9, \ \text{hasnoObject}(r_i, o_j)$
$\text{Homogeneous}(r_1), \text{Balloon}(o_1), \text{Happy}(r_1), \text{WearsTie}(r_1)$
$\text{Homogeneous}(r_2), \text{Flag}(o_2), \text{Happy}(r_2), \text{WearsTie}(r_2)$
$\sim \text{Homogeneous}(r_3), \text{Sword}(o_3), \text{Happy}(r_3), \text{WearsTie}(r_3)$
$\sim \text{Homogeneous}(r_4), \text{Flower}(o_4), \sim \text{Happy}(r_4), \sim \text{WearsTie}(r_4)$
$\sim \text{Homogeneous}(r_5), \text{Sword}(o_5), \sim \text{Happy}(r_5), \sim \text{WearsTie}(r_5)$
$\sim \text{Homogeneous}(r_6), \text{Flag}(o_6), \text{Happy}(r_6), \sim \text{WearsTie}(r_6)$
$\text{Homogeneous}(r_7), \text{Ax}(o_7), \text{Happy}(r_7), \text{WearsTie}(r_7)$
$\sim \text{Homogeneous}(r_8), \text{Ax}(o_8), \text{Happy}(r_8), \text{WearsTie}(r_8)$
$\text{Homogeneous}(r_9), \text{Balloon}(o_9), \text{Happy}(r_9), \sim \text{WearsTie}(r_9)$

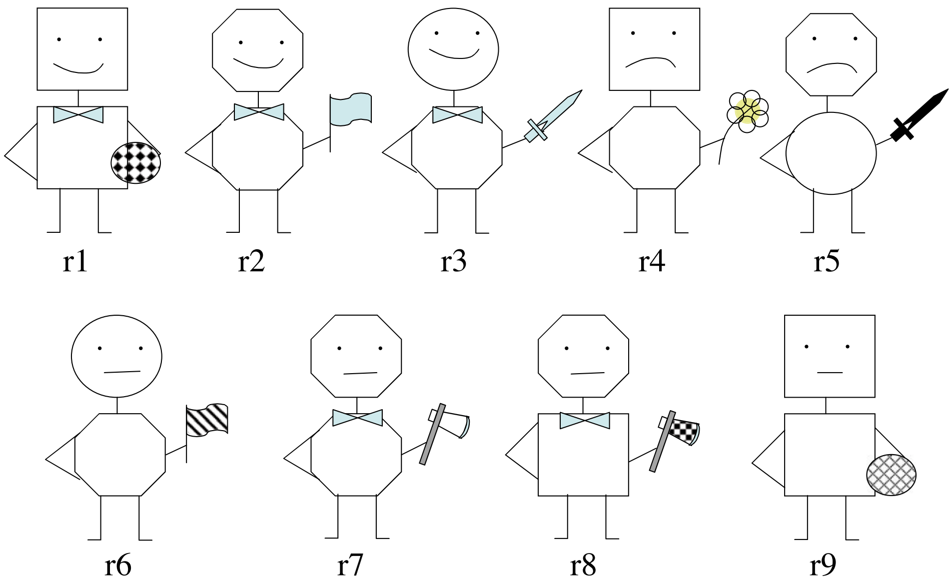

Fig. 2. The nine little robots of our data set.

The two first definitions mean that there are nine robots and each one of them holds one and only one object. Other definitions describe each particular robot. For instance robot r_1 is homogeneous, it holds a balloon, it is happy and it wears a tie. Observe in Fig. 2 that robots r_1, r_4 and r_8 have different shapes of mouth. Here we considered that both r_1 and r_8 are happy whereas r_4 is not happy (since it is clearly not smiling). We will see later that by interpreting the concepts as either n-graded or fuzzy sets these three shapes of mouth can be clearly distinguished.

The knowledge contained in both the *TBox* and the *ABox* can be used, for instance, to assess whether or not robots are friendly. According to the definition of Friendly contained in the *TBox*, and taking into account the semantic interpretation to $\&, \veebar$ and to the existential quantification, we have that for every object x, $\mathsf{Friendly}(x)$ is the minimum of the following three items:

(a) $\mathsf{Robot}(x)$,
(b) $\sup\{\min(\mathsf{hasObject}(x,y), \mathsf{FriendlyObject}(y)) : y \in M_{\mathcal{R}}\}$ and,
(c) $\max(\mathsf{Happy}(x), \mathsf{Homogeneous}(x))$.

Notice that, concerning item (b), when x is a robot, say r_i, the only value of y for which $\mathsf{hasObject}\ (x,y)$ takes as value 1 is the object o_i held by r_i.

Now, let us consider the friendliness of some robots in our example. For instance,

- Is r_1 friendly? For $x = r_1$, items (a), (b), and (c) above take value 1. In particular, item (b) takes value 1 because the object o_1 held by r_1 is a balloon, i.e., a FriendlyObject according to the *TBox*. Consequently, r_1 is friendly.
- Is r_4 friendly? For $x = r_4$, item (c) is 0, therefore $\mathsf{Friendly}(r_4)$=0, i.e. r_4 is not friendly. Consequently, according to the definition of Unfriendly we have that, for this robot, $\mathsf{Unfriendly}\ (r_4)$=1.
- Is r_8 friendly? For $x = r_8$, item (b) takes value 0 because the object o_8 held by r_8 is an ax, i.e., it is not a FriendlyObject according to the *TBox*, therefore $\mathsf{Friendly}(r_8)$=0. Consequently, $\mathsf{Unfriendly}\ (r_8)$=1.

4.3. Knowledge bases for $\mathcal{ALC}^*(\mathbf{S})$

In this section we define the notions concerning knowledge bases for $\mathcal{ALC}^*(\mathbf{S})$. As before, a KB has two components: *TBox* and *ABox*. In the *TBox* we use the graded notion of inclusion between fuzzy sets defined as follows: $degree(C \subseteq D) = \inf_x(C(x) \to_* D(x))$. Of course this degree is 1 if and only if $C(x) \leqslant D(x)$ for all x; and when the supports [7] of the two fuzzy sets are disjoint, then it is 0. Notice that having the truth constants in the language allows us to associate expressions like, for instance, "$degree\ (C \subseteq D) \leqslant r$" with first order formulas such as $(\forall x)(C(x) \to D(x)) \to \bar{r}$.

Given a logic $L^*_\sim(\mathbf{S})\forall$, let $r \in S$. An *evaluated formula* of this logic is a formula of one of the types

$$\bar{r} \to \varphi,\ \ \varphi \to \bar{r},\ \ \bar{r} \leftrightarrow \varphi,$$

where φ does not contain any occurrence of truth constants different than $\bar{0}$ or $\bar{1}$. Notice that the last axiom above is, in fact, the conjunction of the other two. When φ is a sentence, we use the name *evaluated sentence* for the above formulas.

Let C, D be $\mathcal{ALC}^*(\mathbf{S})$-concepts without occurrences of any truth constant different than $\bar{0}$ or $\bar{1}$; R be an atomic role; and a, b be constant objects. Let $r \in S$.

A *fuzzy concept inclusion axiom* is an evaluated sentence of one of the forms:

- $\bar{r} \to (\forall x)(C(x) \to D(x))$
- $(\forall x)(C(x) \to D(x)) \to \bar{r}$
- $\bar{r} \leftrightarrow (\forall x)(C(x) \to D(x))$

A *fuzzy concept assertion axiom* is an evaluated sentence of one of the forms:

- $\bar{r} \to C(a)$
- $C(a) \to \bar{r}$
- $\bar{r} \leftrightarrow C(a)$

A *fuzzy role assertion axiom* is an evaluated sentence of the form:

- $\bar{r} \to R(a,b)$

A *TBox* for $\mathcal{ALC}^*(\mathbf{S})$ is a finite set of fuzzy concept inclusion axioms. An *ABox* for $\mathcal{ALC}^*(\mathbf{S})$ is a finite set of fuzzy concept assertion axioms and fuzzy role assertion axioms.

Remark 4.1 (*About KB axioms*). Notice that fuzzy concept inclusion axioms are in general not instances of $\mathcal{ALC}^*(\mathbf{S})$-concepts; fuzzy concept assertion axioms always are, and fuzzy role assertion axioms never are. Observe also that in the *ABox* we do not allow sentences of the form $R(a,b) \to \bar{r}$. This choice is made in order to define the fuzzy KB associated to the language $\mathcal{ALC}^*(\mathbf{S})$ as a generalization of the KB associated to the classic $\mathcal{ALC}$. Allowing sentences such as $R(a,b) \to \bar{r}$ implies the possibility of allowing negation of atomic roles in the *ABox*, which is not allowed in classic $\mathcal{ALC}$. Thus, for instance, if $L^*_\sim$ is

[7] The *support* of a fuzzy set is the set of elements whose membership degree is greater than 0.

Lukasiewicz Logic, $R(a,b) \to \bar{0}$ is equivalent to $\sim R(a,b)$. But the negation is allowed for concepts (formulas of type $C(a) \to \bar{0}$) as in the classical case.

All the axioms in the *KB* are *evaluated sentences* in the language of the logic $L^*_{\sim}(\mathbf{S})\forall$. Thus from this syntactic notion of *KB* both the *TBox* and the *ABox* can be seen as theories of the logic $L^*_{\sim}(\mathbf{S})\forall$.

The left part of Table 3 shows sentences for the axioms of the KBs for $\mathcal{ALC}$. The right part of the same table shows the notations we propose for the corresponding evaluated sentences of $\mathcal{ALC}^*(\mathbf{S})$. Notice that the graded notation of these evaluated formulas is similar to the notation used in some papers on FDLs (see for instance [19]); however, in our framework these expressions correspond to sentences of our first order fuzzy logics. Therefore, the fact that an interpretation M satisfies an axiom of a knowledge base is equivalent to saying that $\mathbf{M}$ satisfies the corresponding first order sentence. Thus, for instance, $\mathbf{M} \models \langle C \sqsubseteq D, \succcurlyeq \bar{r}\rangle$ is equivalent to $\mathbf{M} \models \bar{r} \to (\forall x)(C(x) \to D(x))$ which, in turn, is equivalent to $\inf\{C^{\mathbf{M}}(x) \to_* D^{\mathbf{M}}(x) : x \in \mathrm{M}\} \geqslant r$.

4.4. A fuzzy KB for the Robots domain

The use of the graded notation allows us to define the knowledge base of a domain in a more refined way than in the crisp case. For instance, the following *TBox* is a graded refinement of those given in Section 4.2:

Graded TBox for the Little Robots

$\mathsf{Friendly} \equiv \mathsf{Robot}\ \&\ (\exists \mathsf{hasObject.FriendlyObject})\ \&\ (\mathsf{Happy} \veebar \mathsf{Homogeneous})$
$\langle \mathsf{Robot}\ \&\ \mathsf{Object} \sqsubseteq \bar{0}, \approx \bar{1}\rangle$
$\langle \bar{1} \sqsubseteq \mathsf{Robot} \veebar \mathsf{Object}, \approx \bar{1}\rangle$
$\langle \mathsf{Homogeneous} \sqsubseteq \mathsf{Robot}, \approx \bar{1}\rangle$
$\langle \mathsf{Happy} \sqsubseteq \mathsf{Robot}, \approx \bar{1}\rangle$
$\langle \mathsf{WearsTie} \sqsubseteq \mathsf{Robot}, \approx \bar{1}\rangle$
$\langle \mathsf{FriendlyObject} \sqsubseteq \mathsf{Object}, \approx \bar{1}\rangle$
$\langle \mathsf{Flower} \sqsubseteq \mathsf{FriendlyObject}, \approx \bar{1}\rangle$
$\langle \mathsf{Balloon} \sqsubseteq \mathsf{FriendlyObject}, \approx \overline{0.75}\rangle$
$\langle \mathsf{Flag} \sqsubseteq \mathsf{FriendlyObject}, \approx \overline{0.50}\rangle$
$\langle \mathsf{Sword} \sqsubseteq \mathsf{FriendlyObject}, \approx \overline{0.25}\rangle$
$\langle \mathsf{Ax} \sqsubseteq \mathsf{FriendlyObject}, \approx \bar{0}\rangle$

Notice that we use the expression $C \equiv D$ as an abbreviation for the conjunction of the axioms $\langle C \sqsubseteq D, \approx \bar{1}\rangle$ and $\langle D \sqsubseteq C, \approx \bar{1}\rangle$. The definition of the concept Friendly is the same as the one given in the example of the classical case. Notice that we do no longer need to define the concept Unfriendly because it can be seen as equivalent to saying that a robot belongs to the concept Friendly with degree 0. With regard to the definition of *friendly and unfriendly objects* the situation is the same. Now we only need to associate a *friendliness* degree to each object. For instance, we consider that an ax is less friendly (i.e., its associated truth degree is 0) than a sword (with associated truth degree 0.25). Similarly, we consider that the flower is the object with the highest degree of friendliness.

With regard to the *ABox*, as in the classical case, it contains the definitions of the robots shown in Fig. 2. The difference is that now each concept instance is associated to a truth degree. The following *ABox* is a graded refinement of the one given in Section 4.2:

Table 3
The left part shows the usual KB axioms in $\mathcal{ALC}$ and the right part shows the corresponding axioms using evaluated formulas and the graded notation we propose for them.

$\mathcal{ALC}$ axioms	$\mathcal{ALC}^*(\mathbf{S})$ axioms	notation
$(\forall x)(C(x) \to D(x))$	$\bar{r} \to (\forall x)(C(x) \to D(x))$	$\langle C \sqsubseteq D, \succcurlyeq \bar{r}\rangle$
(*denoted by* $C \sqsubseteq D$)	$(\forall x)(C(x) \to D(x)) \to \bar{r}$	$\langle C \sqsubseteq D, \preccurlyeq \bar{r}\rangle$
	$\bar{r} \leftrightarrow (\forall x)(C(x) \to D(x))$	$\langle C \sqsubseteq D, \approx \bar{r}\rangle$
$C(a)$	$\bar{r} \to C(a)$	$\langle C(a), \succcurlyeq \bar{r}\rangle$
	$C(a) \to \bar{r}$	$\langle C(a), \preccurlyeq \bar{r}\rangle$
	$\bar{r} \leftrightarrow C(a)$	$\langle C(a), \approx \bar{r}\rangle$
$R(a,b)$	$\bar{r} \to R(a,b)$	$\langle R(a,b), \succcurlyeq \bar{r}\rangle$

Graded ABox for the Little Robots

For each $i, 1 \leqslant i \leqslant 9, \langle \mathsf{Robot}(r_i), \approx \bar{1}\rangle, \langle \mathsf{hasObject}(r_i, o_i), \succcurlyeq \bar{1}\rangle$
For each $i \neq j, 1 \leqslant j \leqslant 9, \langle \mathsf{Robot}(r_i), \approx \bar{1}\rangle, \langle \mathsf{hasnoObject}(r_i, o_j), \succcurlyeq \bar{1}\rangle$
$\langle \mathsf{Homogeneous}(r_1), \approx \bar{1}\rangle, \langle \mathsf{Balloon}(o_1), \approx \bar{1}\rangle, \langle \mathsf{Happy}(r_1), \approx \bar{1}\rangle, \langle \mathsf{WearsTie}(r_2), \approx \bar{1}\rangle$
$\langle \mathsf{Homogeneous}(r_2), \approx \bar{1}\rangle, \langle \mathsf{Flag}(o_2), \approx \bar{1}\rangle, \langle \mathsf{Happy}(r_2), \approx \bar{1}\rangle, \langle \mathsf{WearsTie}(r_2), \approx \bar{1}\rangle$
$\langle \mathsf{Homogeneous}(r_3), \approx \overline{0.75}\rangle, \langle \mathsf{Sword}(o_3), \approx \bar{1}\rangle, \langle \mathsf{Happy}(r_3), \approx \bar{1}\rangle, \langle \mathsf{WearsTie}(r_3), \approx \bar{1}\rangle$
$\langle \mathsf{Homogeneous}(r_4), \approx \overline{0.50}\rangle, \langle \mathsf{Flower}(o_4), \approx \bar{1}\rangle, \langle \mathsf{Happy}(r_4), \approx \bar{0}\rangle, \langle \mathsf{WearsTie}(r_4), \approx \bar{0}\rangle$
$\langle \mathsf{Homogeneous}(r_5), \approx \overline{0.50}\rangle, \langle \mathsf{Sword}(o_5), \approx \bar{1}\rangle, \langle \mathsf{Happy}(r_5), \approx \bar{0}\rangle, \langle \mathsf{WearsTie}(r_5), \approx \bar{0}\rangle$
$\langle \mathsf{Homogeneous}(r_6), \approx \overline{0.75}\rangle, \langle \mathsf{Flag}(o_6), \approx \bar{1}\rangle, \langle \mathsf{Happy}(r_6), \approx \overline{0.50}\rangle, \langle \mathsf{WearsTie}(r_6), \approx \bar{0}\rangle$
$\langle \mathsf{Homogeneous}(r_7), \approx \bar{1}\rangle, \langle \mathsf{Ax}(o_7), \approx \bar{1}\rangle, \langle \mathsf{Happy}(r_7), \approx \overline{0.50}\rangle, \langle \mathsf{WearsTie}(r_7), \approx \bar{1}\rangle$
$\langle \mathsf{Homogeneous}(r_8), \approx \overline{0.75}\rangle, \langle \mathsf{Ax}(o_8), \approx \bar{1}\rangle, \langle \mathsf{Happy}(r_8), \approx \overline{0.50}\rangle, \langle \mathsf{WearsTie}(r_8), \approx \bar{1}\rangle$
$\langle \mathsf{Homogeneous}(r_9), \approx \bar{1}\rangle, \langle \mathsf{Balloon}(o_9), \approx \bar{1}\rangle, \langle \mathsf{Happy}(r_9), \approx \overline{0.50}\rangle, \langle \mathsf{WearsTie}(r_9), \approx \bar{1}\rangle$

The definitions of the robots are the same as those given in the classical case. However, now we can provide more information about the different aspects of a robot. For instance, in the classical case, robots were only homogeneous or not homogeneous, whereas now, using truth constants, we can assess different degrees of homogeneity according to the shape of both the head and the body. Notice that robots r_1, r_2, r_7 and r_9, considered homogeneous in the classical case, now have truth degree 1. We subjectively assess the truth degree of the other robots considering that a combination of round shapes of head and body (for instance, a circle and an octagon) give a more homogeneous aspect to the robot than combining round and square shapes. Thus, robots r_6 and r_8 are considered more homogeneous than robot r_4. Similarly, the shapes of the robots' mouths give them different degree of happiness (i.e., robot r_1 is assessed as more happy than robots r_8 and r_4).

Let us illustrate how to use the knowledge contained in both the *T Box* and the *A Box* to assess the degree of friendliness of the robots. The definition of $\mathsf{Friendly}$ contained in the *T Box* is the following:

$$\mathsf{Friendly} \equiv \mathsf{Robot} \& (\exists \mathsf{hasObject.FriendlyObject}) \& (\mathsf{Happy} \veebar \mathsf{Homogeneous})$$

Given a continuous *t*-norm $*$ and its dual continuous *t*-conorm $\oplus$, according to the semantic interpretation of $\&, \veebar$ and the existential quantification, we have that, for every $x \in M_{\mathcal{R}}$,

$$\mathsf{Friendly}(x) = \mathsf{Robot}(x) * sup\{\mathsf{hasObject}(x,y) * \mathsf{FriendlyObject}(y) : y \in M_{\mathcal{R}}\} * (\mathsf{Happy}(x) \oplus \mathsf{Homogeneous}(x))$$

If x is a robot, according to the *ABox*, $\mathsf{Robot}(x) = 1$ and $\mathsf{hasObject}(x,y) = 1$, where y is the object held by x. Therefore we have

$$\mathsf{Friendly}(x) = \mathsf{FriendlyObject}(y) * (\mathsf{Happy}(x) \oplus \mathsf{Homogeneous}(x))$$

The friendliness degree of a particular robot x will depend on the particular *t*-norm we choose. Table 4 shows the friendliness degrees of the robots in Fig. 2 with the usual *t*-norms (i.e., Minimum, Product and Łukasiewicz) and their dual *t*-conorms with respect to the standard involutive negation.

For instance, the reader can compare the friendliness degree of robots r_4, r_8 and r_9 with those obtained in the classical case. In particular r_4 and r_8 were unfriendly. Notice that now, in the fuzzy case, r_8 is also unfriendly because it holds an ax (i.e., $\mathsf{Friendly}(r_8)$ takes value 0). However, robot r_4 has friendliness degree 0.50: although it does not smile, it holds a flower (considered as the most friendly object) and it has some homogeneity degree. Similarly, robot r_9, considered friendly in the classical case ($\mathsf{Friendly}(r_9) = 1$) now has a lower friendliness degree (i.e., 0.75) because it does not smile ($\mathsf{Happy}(r_9) = 0.50$).

Table 4
Friendliness degree of the robots introduced in Section 4.4 using the Minimum, Product and Łukasiewicz *t*-norms and their dual *t*-conorms.

Robot	Expression	Min/Max	Prod/Sum	$*_Ł/\oplus_Ł$
r_1	$0.75 * (1 \oplus 1)$	0.75	0.75	0.75
r_2	$0.50 * (1 \oplus 1)$	0.50	0.50	0.50
r_3	$0.25 * (1 \oplus 0.75)$	0.25	0.25	0.25
r_4	$1 * (0 \oplus 0.50)$	0.50	0.50	0.50
r_5	$0.25 * (0 \oplus 0.50)$	0.25	0.125	0
r_6	$0.50 * (0.50 \oplus 0.75)$	0.50	0.4375	0.50
r_7	$0 * (0.50 \oplus 1)$	0	0	0
r_8	$0 * (0.50 \oplus 0.75)$	0	0	0
r_9	$0.75 * (0.50 \oplus 1)$	0.75	0.75	0.75

Particularly interesting are the results of robots r_5 and r_6. In these cases the friendliness degree depends on the *t*-norm and *t*-conorm chosen; the friendliness degree of r_5 goes from 0 using the Łukasiewicz *t*-norm to 0.25 using the Minimum *t*-norm. Depending on the domain, the user can analyze which of the possible *t*-norms is the most appropriate.

5. Reasoning in DL and in FDL

Common tasks when performing reasoning on concepts in classical description logics are the following:

- to check the *satisfiability* of a concept C. This task is equivalent to the problem of *consistency* of an assertion $C(a)$, where a is an object constant.
- to check the *validity* of a concept C.
- to determine whether or not a concept description is more general than another one, i.e., the *subsumption* of concepts.

Reasoning in Fuzzy Description Logics involves the same kind of tasks but their results depend on the continuous *t*-norm $*$ chosen. Let us introduce the notions corresponding to these tasks formally.

- C is $*$-*satisfiable* iff there exists a $*$-model $\mathbf{M}$ of the formula $C(x)$, i.e., a $*$-interpretation $\mathbf{M}$ and an evaluation v such that $v(x) = a$ and $\|C(a)\|^*_{\mathbf{M}} = 1$.[8]
- C is $*$-*valid* iff $\|(\forall x)C(x)\|^*_{\mathbf{M}} = 1$ for every interpretation $\mathbf{M}$, i.e., all the interpretations are $*$-models of the formula $(\forall x)C(x)$.
- C is $*$-*subsumed* by D iff the concept $C \rightarrow D$ is $*$-valid.

Notice that these definitions are also valid for the crisp and graded cases: in the crisp case $*$ is min, in the graded case $*$ is a divisible finite *t*-norm, and in the fuzzy case $*$ is a continuous *t*-norm. The notions of $*$-satisfiability, $*$-subsumption and $*$-validity defined above are associated with the truth value 1. For instance, the concept $A\& \sim A$ is not satisfiable because there is no interpretation $\boldsymbol{M}$ such that $\|(A\& \sim A)(a)\|^*_{\mathbf{M}} = 1$.

One of the advantages of introducing truth constants in the language is the possibility of defining the graded versions of the notions of $*$-satisfiability, $*$-subsumption and $*$-validity without modifying the semantics. In addition, these notions also allow the reasoning on knowledge bases (i.e., on a set of evaluated formulas). Let us now introduce these graded notions.

Given a concept C, a continuous *t*-norm $*$, a subalgebra $\mathbf{S}$ of the corresponding canonical algebra, and a truth value $r \in S$,

- C is $*$-*satisfiable to a degree greater or equal than r* iff the concept $\bar{r} \rightarrow C$ is $*$-satisfiable.
- C is $*$-*valid to a degree greater or equal than r* iff $\bar{r} \rightarrow C$ is $*$-valid.
- C is $*$-*subsumed* by D *to a degree greater or equal than r* iff $\bar{r} \rightarrow (C \rightarrow D)$ is $*$-valid.

On the other hand, we can analogously define the notions for low thresholds. For instance, a concept C is $*$-*satisfiable to a degree lower or equal than r* iff $C \rightarrow \bar{r}$ is $*$-satisfiable. Moreover, it is also possible to define the notions of satisfiability, validity, and subsumption to a degree belonging to an interval of truth values $r, s \in S$, being $r \leqslant s$. For instance, a concept C is $*$-*satisfiable to a degree in the closed interval* $[r, s]$ if and only if $(\bar{r} \rightarrow C)\&(C \rightarrow \bar{s})$ is $*$-satisfiable. In particular, when $r = s$ we will say that C is *satisfiable to a degree equal to r*.

Let us consider the concept $A\& \sim A$. This concept is not $*$-satisfiable since $\|(A\& \sim A)(a)\|^*_{\mathbf{M}} \neq 1$ for any $*$ and any interpretation $\mathbf{M}$. However $A\& \sim A$ can be $*$-satisfiable in some degree. For instance, in the robots example, we can assess that $\mathsf{Friendly}(r_3) * (1 - \mathsf{Friendly}(r_3)) = 0.25$ taking $*$ as the Minimum *t*-norm. This means that for the robots model the concept $(A\& \sim A) \leftrightarrow 0.25$ is $*$-satisfiable when $*$ is the Minimum *t*-norm.

We can also define the notions of $*$-satisfiability, $*$-validity and $*$-subsumption (and their corresponding versions with degrees) with respect to a knowledge base $\mathcal{K}$ in the following way. A concept C *is $*$-satisfiable with respect to* $\mathcal{K}$ if there exists some $*$-model of the axioms of $\mathcal{K}$. With regard to *entailment*, we say that a fuzzy assertion α is $*$-*entailed* by a knowledge base $\mathcal{K}$ if every $*$-model of $\mathcal{K}$ is also a $*$-model of α. Let us illustrate all these notions with some examples from our robots dataset.

Example 5.1 (Satisfiability of concepts). Let us analyze the satisfiability of the concept

$$C = \mathsf{Homogeneous}\&\exists \mathsf{hasObject.FriendlyObject}\&\mathsf{WearsTie}$$

with respect to the KBs defined in the examples of Sections 4.2 and 4.4.

- C is satisfied with respect to the KB of the robots in the crisp case (Section 4.2). In accordance with the crisp *TBox*, the goal is to search for a robot that is holding a friendly object, is homogeneous and is wearing a tie. The reader can easily see in the crisp *ABox* that robots r_1 and r_2 are instances of this concept. Therefore C is satisfiable with respect to this KB.
- It is easy to see that the concept C is not $*$-satisfied in degree 1 with respect to the KB of the robots in the fuzzy case (Section 4.4). However, taking as $*$ any of the three main continuous *t*-norms, is C $*$-satisfied in some degree (different than 0)? The answer will be affirmative if we can find (at least) a robot x such that the expression

[8] The common definition used in FDL for satisfiability is $C(a) > 0$. We do not use this definition because it does not allow us to define the graded notion of satisfiability as we do later.

$$\mathsf{Homogeneous}(x) * \sup_{y \in M_{\mathcal{R}}} \{\mathsf{hasObject}(x,y) * \mathsf{FriendlyObject}(y)\} * \mathsf{WearsTie}(x)$$

takes a value $r \neq 0$. For the robot r_1, C is $*$-satisfied in degree 0.75 because it is homogeneous in degree 1, it holds a balloon that – as the *TBox* states – is a friendly object with degree 0.75, and it wears a tie (i.e.,$\mathsf{WearsTie}(r_1) = 1$). Thus, because 1 is the unity element of any *t*-norm, r_1 has friendliness degree 0.75 independently of the *t*-norm considered.

Notice that the introduction of the notion of $*$-satisfiability to a degree allows us to handle the notion of partial satisfiability of a concept. The example illustrates the situation where a concept is not $*$-satisfiable with respect to the KB, meaning that it is not satisfied (in degree 1) with respect to the KB. However, using degrees, when we assess that a concept is $*$-satisfiable to a *degree* r, this can be interpreted as saying that it is *partially* satisfied with respect to the KB.

Example 5.2 ($*$-*Subsumption of concepts*). Let us analyze the degree r of the subsumption

$$\mathsf{Homogeneous} \sqsubseteq \mathsf{Happy}$$

with respect to the KB of Section 4.4.

By definition, the subsumption above holds to a degree equal to r if and only if the following sentence is $*$-valid w.r.t. the KB.

$$\bar{r} \leftrightarrow (\forall x)(\mathsf{Homogeneous}(x) \to \mathsf{Happy}(x))$$

According with the semantics, this means that in every model **M** satisfying the KB the truth value of the sentence above is 1. That is,

$$r = \inf_{x \in M} \{\mathsf{Homogeneous}(x) \to_* \mathsf{Happy}(x)\}$$

Table 5 shows the values of Homogeneous, Happy, and the value of the residuum for each one of the robots of the *ABox*. Notice that for robots r_1, r_2, r_3, r_7 and r_9 the value of the residuum does not depend on the *t*-norm considered. However, for the other robots the value of the residuum depends on the *t*-norm. For instance for robots r_4 and r_5, the residuum $0.50 \to_* 0$ is 0 when using the Minimum and Product *t*-norm, and it is 0.50 using the Łukasiewicz *t*-norm. Similarly, for robots r_6 and r_8 the residuum $0.75 \to_* 0.50$ is 0.50, 2/3 and 0.75 using respectively the Minimum, Product and Łukasiewicz *t*-norms. Consequently,

- Minimum *t*-norm : $r = \inf_x\{1, 0, 0.50\} = 0$
- Product *t*-norm: $\inf_x\{1, 0, 2/3\} = 0$
- Łukasiewicz *t*-norm : $\inf_x\{1, 0.50, 0.75\} = 0.50$

Thus using the Minimum and the Product *t*-norms the concept Homogeneous is subsumed w.r.t. the KB by the concept Happy in degree 0. Using the Łukasiewicz *t*-norm Homogeneous is subsumed w.r.t. the KB by Happy to a degree 0.50.

Example 5.3 ($*$-*Entailment*). The *ABox* from Section 4.4 $*$-entails the assertion $\langle \mathsf{Friendly}(r_6), \succcurlyeq 0.50\rangle$ with respect to the *TBox* using either Łukasiewicz or Minimum *t*-norms. However, using the Product *t*-norm the above $*$-entailment does not hold. Indeed, according to the definition of Friendly, we have to calculate

$$\mathsf{Robot}(x) * \sup_{y \in M_{\mathcal{R}}} \{\mathsf{hasObject}(x,y) * \mathsf{FriendlyObject}(y)\} * (\mathsf{Happy} \oplus \mathsf{Homogeneous})(x)$$

As Table 4 shows, the friendliness of r_6 is 0.50 using both the Łukasiewicz and Minimum *t*-norms, whereas it is 0.4375 using the Product *t*-norm. This means that the proposed assertion is $*$-entailed when using either the Łukasiewicz or the Minimum *t*-norm.

Example 5.4 ($*$-*Entailment*). Let us analyze the values of r for which the assertion

$$\alpha = \langle \mathsf{Homogeneous} \sqsubseteq \mathsf{Happy}, \succcurlyeq \bar{r}\rangle$$

Table 5
Calculation of the truth value of the sentence $(\forall x)$ $(\mathsf{Homogeneous}(x) \to \mathsf{Happy}(x))$ where the range of x are the robots of Fig. 2.

Robot	Homogeneous	Happy	$\to_*$
r1	1	1	1
r2	1	1	1
r3	0.75	1	1
r4	0.50	0	$0.50 \to_* 0$
r5	0.50	0.50	$0.50 \to_* 0$
r6	0.75	0.50	$0.75 \to_* 0.50$
r7	1	0.50	$1 \to_* 0.50$
r8	0.75	0.50	$0.75 \to_* 0.50$
r9	1	0.50	$1 \to_* 0.50$

can be $*$-entailed from the *ABox* of Section 4.4 taking as $*$ one of the three main continuous t-norms. By definition, the assertion α is the sentence

$$\bar{r} \to (\forall x)(\mathsf{Homogeneous}(x) \to \mathsf{Happy}(x))$$

According to the semantics, we have that the truth value of this sentence is

$$r \to_* \inf_{x \in M_{\mathcal{R}}} \{\mathsf{Homogeneous}(x) \to_* \mathsf{Happy}(x)\}$$

The reader can easily calculate that, for instance, taking the Minimum t-norm, the expression $\mathsf{Homogeneous}(x) \to_* \mathsf{Happy}(x)$ for the different robots takes as values 0, 0.50 or 1. Thus, $\inf_{x \in M_{\mathcal{R}}} \{\mathsf{Homogeneous}(x) \to_* \mathsf{Happy}(x)\} = 0$. Similarly, the reader can see that using the Product t-norm the infimum is 0, whereas taking the Łukasiewicz t-norm the infimum is 0.50. Therefore, we have that the assertion α takes the following values:

- Minimum and Product: $r \to_* 0$
- Łukasiewicz: $r \to_* 0.50$

Thus: When using either Minimum or Product t-norms, the assertion α cannot be $*$-entailed from the *ABox* when $r > 0$, since $r \to_* 0 = 0$. When using the Łukasiewicz t-norm, we have that $r \to_* 0.50 = 1$ if and only if $r \leqslant 0.50$; therefore in this case the assertion α is $*$-entailed from the *ABox*.

Example 5.5 ($*$-*Entailment*). Given, two values s_1 and s_2, let us suppose that we want to analyze whether or not the assertion $\alpha = \langle \mathsf{Happy}(a), \succcurlyeq \bar{s_1} \rangle$ is $*$-entailed from the *ABox* plus the assertion $\beta = \langle \mathsf{Homogeneous}(a), \succcurlyeq \bar{s_2} \rangle$. In fact, what we want to determine if all robots that are homogeneous to a degree equal or higher than s_1 are also happy to a degree equal or higher than s_2.

- Let us suppose that $s_1 = 0.75$ and $s_2 = 0.50$. The robots that are homogeneous to a degree equal or higher than 0.75 are $r_1, r_2, r_3, r_6, r_7, r_8, r_9$ and all of them are also happy to a degree equal or higher than 0.50. Thus, the assertion α is $*$-entailed from $ABox \cup \{\beta\}$.
- Let us suppose that $s_1 = 0.75$ and $s_2 = 0.75$. The robots that are homogeneous to a degree equal or higher than 0.75 are $r_1, r_2, r_3, r_6, r_7, r_8, r_9$ and only r_1, r_2, r_3 are happy to a degree equal or higher than 0.75. Thus, the assertion α is not $*$-entailed from $ABox \cup \{\beta\}$.

The examples above show that the three main continuous t-norms have different behavior. In particular, Example 5.4 shows that depending on the chosen t-norm an assertion can or cannot be $*$-entailed from the KB. The main conclusion from this is that when facing a problem it is important to analyze the behavior of the t-norms on it, and then to choose the most appropriate one.

6. Logical results related to the languages $\mathcal{ALC}^*(\boldsymbol{S})$

In this section, we summarize the main logical results concerning canonical completeness for the logics $L^*_\sim(\mathbf{S})\forall$. We consider the cases when $*$ is a divisible finite t-norm (Section 6.1); when $*$ is the Łukasiewicz t-norm (Section 6.2); and when $*$ is the Minimum t- norm (Section 6.3). Finally, we end with some remarks about the Zadeh Logic and the corresponding FDL. In what follows we use the notion of *witnessed interpretation*. Let us recall its definition (cf. [11]).

Definition 6.1 (*Witnessed interpretation*). Let $*$ be either a divisible finite t-norm or a continuous t-norm, and let $\mathbf{M}$ be an $*$-interpretation.

- A closed formula (a sentence) $(\forall x)\varphi(x)$ is *witnessed* in $\mathbf{M}$ if $\|(\forall x)\varphi(x)\|^*_{\mathbf{M}} = \|\varphi(x)\|^*_{\mathbf{M}}$ for some $a \in M$, i.e., when the infimum of values of M-instances of $\varphi(x)$ is in fact a minimum.
- More generally, an open formula $(\forall x)\varphi(x, y_1, \ldots, y_n)$ is *witnessed* in $\mathbf{M}$ if for any choice $b_1, \ldots, b_n \in M$ of values of $y_1, \ldots, y_n$, and for some $a \in M$, $\|(\forall x)\varphi(x, b_1, \ldots, b_n)\|^*_{\mathbf{M}} = \|\varphi(a, b_1, \ldots, b_n)\|^*_{\mathbf{M}}$.
- The notion of witnessed existential quantified formulas is analogously defined with the obvious changes, i.e., using supremum (resp. maximum) instead of infimum (resp. minimum).
- A $*$-interpretation $\mathbf{M}$ is *witnessed* if all quantified formulas are witnessed in $\mathbf{M}$.

We say that a first order fuzzy logic satisfies the *Witnessed Model Property* with respect to a continuous t-norm (or divisible finite t-norm) $*$ if it is complete with respect to all the witnessed $*$-interpretations.

6.1. The case of the logic of a divisible finite t-norm over a finite chain

It is known that any divisible finite t-norm $*$ over a chain of n elements is either the Łukasiewicz finite t-norm (denoted as $Ł_n$), or the Minimum (denoted as G_n) or any finite ordinal sum of copies of them. The propositional logics $L^*_\sim(\mathbf{S})$ corresponding to these chains are defined as the logics whose theorems coincide with the tautologies over the canonical $L^*_\sim(\mathbf{S})$-chain. Their corresponding first order logics $L^*_\sim(\mathbf{S})\forall$ are finitely axiomatizable and enjoy the strong canonical completeness, i.e., for any set of formulas Γ, with $*$ being the divisible finite t-norm defining the logic,

$$\Gamma\vdash_{L^*_\sim(\mathbf{S})\forall}\varphi \text{ if and only if every } *\text{-model of } \Gamma \text{ is also a } *\text{-model of } \varphi.$$

The proof of this completeness is easy since the logic $L^*_\sim(\mathbf{S})\forall$ is complete with respect to $L^*_\sim(\mathbf{S})$-chains and, mainly using results from [37], it is not difficult to see that any $L^*_\sim(\mathbf{S})$-chain is embeddable into the canonical chain. This completeness result could also be obtained as a particular case of a more general result in [45]. There it is proved that, for any given finite algebra defined by truth value functions, a finite and complete system of natural deduction for the corresponding first order many-valued logic is given. Moreover, an automated method to build a finite and complete Gentzen System for these logics, called MUtlog, is given in [46].

Thus, for each divisible finite t-norm $*$, each description logic $\mathcal{ALC}^*(\mathbf{S})$, which is semantically defined by interpretations in models valuated over the canonical $L^*_\sim(\mathbf{S})$-chain, coincides with the corresponding fragment of the Hilbert-style calculus for $L^*_\sim(\mathbf{S})\forall$. An interesting problem that remains open is to find Hilbert-style calculi and/or Gentzen systems for these fragments. Moreover, since the chain C_n is finite, all first order models are witnessed because the infimum (respectively the supremum) becomes the minimum (respectively the maximum) and, therefore, the logic is obviously complete with respect to witnessed models. Thus, by an easy adaptation of the results in [11, Section 4] we have that the satisfiability (resp. validity) problem for *A*Boxes (when dealing with empty *T*Boxes) for the logic $\mathcal{ALC}^*(\mathbf{S})$ is decidable and enjoys the Finite Model Property. Notice also that the algorithms for satisfiability given in [19,20] for FDL languages over infinite-valued Łukasiewicz and Gödel logics (the later with an added involutive negation), are adaptable to the finite-valued case.

Finally, it is worth saying that the restriction to finite chains is not a hard one since in most applications of FDLs it is usual to have only a finite number of truth values.

6.2. The case of Łukasiewicz infinite-valued logic

It is well known that Łukasiewicz predicate logic is not axiomatizable in the usual sense: there is no recursive system of axioms and deduction rules for which the set of theorems coincides with the set of $[0,1]$-tautologies (see [4]). Thus, we have that Ł$\forall$ is complete with respect to the general semantics although it is not standard canonical complete. [9] Hájek in [11] analyzes the$\mathcal{ALC}$-like description logic over Łukasiewicz directly; he proves that the satisfiability and validity problems are decidable for *A*Boxes [10] and proves also that they are equivalent, respectively, to satisfiability and validity on finite models (i.e., the $\mathcal{ALC}^{Ł}$-logic considered by Hájek has the Finite Model Property). The generalization of these results to acyclic *T*Boxes or the expansion with truth constants (as Hájek did for rational truth constants in [27]) is not difficult, and therefore we can obtain the same results. A satisfiability algorithm for the fuzzy description logic of Łukasiewicz is given in [47].

Logical results about witnessed models and about decidability for the satisfiability problem in the description logic $\mathcal{ALC}^{Ł}(\mathbf{S})$ are closely related to the fact that all truth functions associated to Łukasiewicz connectives over $[0,1]$ are continuous. Thus these results do not seem be generalizable to the logics $\mathcal{ALC}^*(\mathbf{S})$ when $*$ is different from the Łukasiewicz t-norm. On the other hand, it is important to stress that, even staying in the Łukasiewicz setting, the Finite Model Property fails when allowing general concept inclusions, as is shown in [48].

6.3. The case of infinite-valued Gödel logic

In [41] canonical completeness of the logic $G(\mathbf{S})\forall$ is proved. Nevertheless in the same paper authors prove that this logic does not have the finite strong canonical completeness, although this kind of completeness holds when we restrict ourselves to evaluated formulas of the form $\bar{r}\to\varphi$. [11]

However, having an involutive negation, we can prove the finite strong canonical completeness for $G_\sim(\mathbf{S})\forall$ without the restriction to evaluated formulas.

Theorem 6.2. *The logic $G_\sim(\mathbf{S})\forall$ has the Finite Strong Canonical Completeness, i.e., for every finite set of formulas Γ and every formula φ, the following conditions are equivalent:*

(1) $\Gamma\vdash_{G_\sim(\mathbf{S})\forall}\varphi$
(2) Every min-*model of Γ is also a* min-*model of φ,*

where min-model *means an interpretation $\mathbf{M}$ valuated over the canonical $G_\sim(\mathbf{S})$-chain, that is, the algebra*

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[10] Hájek gives an algorithm that transforms a satisfiability problem in the considered $\mathcal{ALC}$-like description logic to a satisfiability problem in the infinite-valued Łukasiewicz propositional logic.

[11] Technically speaking, the problem is due to the fact that in a $G(\mathbf{S})$-chain L, each element $x\in L$ defines two filters: the intervals $(x,1]$ and $[x,1]$; and it is known (see [26]) that the truth constants can be interpreted according to them, i.e., for any filter F the mapping

$$f(\bar{r}) := \begin{cases} 1, & \text{if } r\in S\cap F \\ r, & \text{otherwise.} \end{cases}$$

is an interpretation of the truth constants over the real unit interval satisfying the book-keeping axioms. However, having an involutive negation, the situation changes. Any $G_\sim(\mathbf{S})$-chain is simple, i.e., it has no congruence filters different from $\{1\}$ and the full chain.

$$\langle [0,1], \max, \min, \rightarrow_{\min}, N, \langle r \rangle_{r \in S \setminus \{0,1\}}, 0, 1\} \rangle,$$

N being the standard involutive negation function $N(x) := 1 - x$.

Proof. Soundness is obvious. For the converse direction we will argue by contraposition, i.e., we have to prove that if $\Gamma \nvdash \varphi$, then there is a safe min-interpretation **M** and an evaluation v such that, for all $\gamma \in \Gamma, \|\gamma\|^{\min}_{\mathbf{M},v} = 1$ and $\|\varphi\|^{\min}_{\mathbf{M},v} < 1$. By completeness with respect to the general semantics (Theorem 2.9), $\Gamma \nvdash \varphi$ implies that there exist a countable $G_{\sim}(\mathbf{S})$-chain **C**, a safe **C**-interpretation

$$\mathbf{M}' = \langle M', \{a^{\mathbf{M}'} : a \in \mathcal{C}\}, \{P^{\mathbf{M}'} : P \in \mathcal{P}\} \rangle,$$

and an evaluation v', such that, for all $\gamma \in \Gamma, \|\gamma\|^{\mathbf{C}}_{\mathbf{M}',v'} = 1$ and $\|\varphi\|^{\mathbf{C}}_{\mathbf{M}',v'} < 1$. Let X be the finite set of elements $r \in S$ appearing in $\Gamma \cup \{\varphi\}$ and let $\mathbf{S}'$ be the finite subalgebra of **S** generated by X. Let $r_i, i = 1, 2, \ldots, m$, be the elements of S' ordered so that $r_i < r_{i+1}$ for all i. Then we define a mapping $f : C \rightarrow [0,1]$ in the following way:

$$f(x) := \begin{cases} f^+(x), & \text{if } x \text{ is positive}, \\ N(f^+(\sim^{\mathbf{C}} x)), & \text{otherwise}, \end{cases}$$

where f^+ is a mapping from positive elements of C (i.e, those satisfying that $\sim^{\mathbf{C}} x \leqslant x$) into the positive elements of $[0,1]$, in such a way that f^+ is an order-embedding preserving all the existing infima and suprema and satisfying $f^+(\bar{r}_i^{\mathbf{C}}) = r_i$, for all positive $r_i \in S'$. It is easy to see that such a mapping always exists since the elements $\bar{r}_i^{\mathbf{C}}$ divide the set of positive elements of C in a finite number of intervals. Then the mapping f^+ restricted to each of these intervals has to be an order-embedding, preserving the existing infima and suprema, into the corresponding interval of positive elements of the canonical chain. Finally, as the intervals of C are countable and the ones of $[0,1]$ continuous, such a mapping always exists.

Finally taking $\mathbf{M} = \langle M, \{a^{\mathbf{M}} : a \in \mathcal{C}\}, \{P^{\mathbf{M}} : P \in \mathcal{P}\} \rangle$ such that $M = M', a^{\mathbf{M}} = f \circ a^{\mathbf{M}'}$ and $P^{\mathbf{M}} = f \circ P^{\mathbf{M}'}$, it is obvious that **M** is a safe structure over the canonical $G_{\sim}(S)$-chain. Moreover, taking $v = v'$, for all $\varphi, \|\varphi\|^{\min}_{\mathbf{M},v} = f(\|\varphi\|^{\mathbf{C}}_{\mathbf{M}',v'})$. Then, for all $\gamma \in \Gamma, \|\gamma\|^{\min}_{\mathbf{M},v} = 1$ and $\|\varphi\|^{\min}_{\mathbf{M},v} < 1$. Thus the proof is finished. □

Therefore, the $\mathcal{ALC}^{\min}(\mathbf{S})$-logic coincides with the corresponding fragment of the Hilbert-style calculus for $G_{\sim}(\mathbf{S})\forall$. Until now, no Hilbert-style axiomatization or Gentzen system is known for that fragment. The decidability of the $\mathcal{ALC}^{\min}(\mathbf{S})$-logic is also an open problem.

Thus, since the logic $G_{\sim}(\mathbf{S})\forall$ is canonical complete, a deduction such as

$$\Gamma \vdash_{\mathcal{ALC}^{\min}(\mathbf{S})} \langle \varphi, \preceq \bar{r} \rangle$$

can be equivalently expressed as the first order Gödel deduction

$$\Gamma' \vdash_{G_{\sim}(\mathbf{S})\forall} \varphi \rightarrow \bar{r},$$

where Γ is a finite set of graded description formulas and Γ' is the set of the corresponding evaluated formulas.

In the next proposition we prove that in $G_{\sim}(\mathbf{S})\forall$ there are formulas whose validity is equivalent to the fact that the truth value of a formula φ is strictly greater or strictly less than a certain truth value $r \in S$. These formulas correspond to the ones in some papers of FDLs which are denoted by formal expressions as $\langle \varphi < r \rangle$or $\langle \varphi > r \rangle$. We will call them *strictly graded formulas* and, in accordance with the graded notation used in this paper, we will denote them by $\langle \varphi, \prec \bar{r} \rangle$ and $\langle \varphi, \succ \bar{r} \rangle$ respectively.

Proposition 6.3. *For every safe interpretation* **M** *over the canonical* $G_{\sim}(\mathbf{S})$*-chain, the following conditions are equivalent:*

(a) $\|\varphi\|^{\min}_{\mathbf{M},v} < r$,
(b) $\|\neg\neg \sim (\bar{r} \rightarrow \varphi)\|^{\min}_{\mathbf{M},v} = 1$.

Proof. The proof is based on the following equivalences: $\|\varphi\|^{\min}_{\mathbf{M},v} < r$ if and only if

$$\|\varphi\|^{\min}_{\mathbf{M},v} \ngeq r$$

if and only if

$$\|\bar{r} \rightarrow \varphi\|^{\min}_{\mathbf{M},v} \neq 1$$

if and only if

$$\| \sim (\bar{r} \rightarrow \varphi)\|^{\min}_{\mathbf{M},v} \neq 0$$

if and only if

$$\|\neg\neg \sim (\bar{r} \rightarrow \varphi)\|^{\min}_{\mathbf{M},v} = 1.$$

The basic idea is the fact that in G (and thus in G ∀ and in any expansion) a formula has a value different from 0 (i.e $e(\varphi) \neq 0$) if and only if its double negation is 1 (i.e., $e(\neg\neg\varphi) = 1$). Notice that this property holds in every logic of a continuous t-norm whose associated negation n is the Gödel negation function which satisfies that $x \neq 0$ implies $n(n(x)) = 1$. □

Analogously we have the following.

Proposition 6.4. *For every safe interpretation* **M** *over the canonical* $G_{\sim}(\mathbf{S})$*-chain, the following conditions are equivalent:*

(a) $\|\varphi\|^{\min}_{\mathbf{M},v} > r$,

(b) $\|\neg\neg \sim (\varphi \to \bar{r})\|^{\min}_{\mathbf{M},v} = 1$.

Thus, by the finite strong canonical completeness (Theorem 6.2) and the previous propositions, we have that

(1) The validity of an evaluated formula such as $\langle \varphi, \succ \bar{r} \rangle$, with the intended meaning that the truth value of φ is strictly greater than r, is equivalent to the derivability of the formula $\neg\neg \sim (\varphi \to \bar{r})$ in $G_{\sim}(\mathbf{S})\forall$.

(2) A deduction such as $\Gamma \models_{\mathcal{ALC}^{\min}(\mathbf{S})} \langle \varphi, \succ \bar{r} \rangle$ is equivalent to the derivation $\Gamma' \vdash_{G_{\sim}(\mathbf{S})\forall} \neg\neg \sim (\varphi \to \bar{r})$, where Γ is a finite set of graded description formulas and Γ' is the set of the corresponding formulas obtained by the transformation given in Propositions 6.3 and 6.4.

In Section 5 we have introduced the graded notions of $*$-satisfiability, $*$-validity and $*$-subsumption (greater or equal to some degree) as logical properties of formulas in the underlying logic. In the last proposition we have seen that the validity of the graded expressions using $\prec$ and $\succ$ can also be expressed in terms of derivability of logical formulas. Now we turn our attention to the relation between satisfiability and subsumption.

Proposition 6.5. *Let C and D concepts of the language* $\mathcal{ALC}^{\min}(\mathbf{S})$. *The concept C is* min*-subsumed by D to a degree strictly less than r, i.e.,* $\langle C \sqsubseteq D, \prec r \rangle$ *is* min*-valid, if and only if the formula* $\neg\neg \sim (\bar{r} \to (C(x) \to D(x)))$ *is* min*-satisfiable to a degree 1.*

Proof. To say that $\langle C \sqsubseteq D, \prec r \rangle$ is min-valid is equivalent to saying that C is not min-subsumed by D to a degree greater or equal than r, that is,

$$\|\bar{r} \to (\forall x)(C(x) \to D(x))\|^{\min}_{\mathbf{M}} < 1.$$

and this is equivalent to the existence of both a min-interpretation **M** and an element $a \in M$ such that

$$\|(C(a) \to D(a))\|^{\min}_{\mathbf{M}} < r.$$

Then, the following chain of equivalences holds: $\|(C(a) \to D(a))\|^{\min}_{\mathbf{M}} < r$ if and only if

$$\|\bar{r} \to (C(a) \to D(a))\|^{\min}_{\mathbf{M}} < 1$$

if and only if

$$\| \sim (\bar{r} \to (C(a) \to D(a)))\|^{\min}_{\mathbf{M}} > 0$$

if and only if

$$\|\neg\neg \sim (\bar{r} \to (C(a) \to D(a)))\|^{\min}_{\mathbf{M}} = 1.$$

Thus the proposition is proved. □

Notice that the formulas used in the last propositions are close to being evaluated formulas, but in fact are not. Thus the language of evaluated formulas is not enough when we want to cope with strictly graded notions in the description $\mathcal{ALC}^{\min}(\mathbf{S})$-logic.

6.4. Remark about zadeh logic

The first FDL systems were associated to Zadeh's initial proposal for fuzzy sets operations. In fact, the logic underlying this proposal, Zadeh Logic, is the logic associated to the calculus over $[0,1]$ defined by the functions $\min, \max, N(x) = 1 - x$, and $x \to y = \max(1 - x, y)$ (the Kleene–Dienes implication function) in the usual way. A consideration of this logic is beyond the scope of our paper because it is not a residuated many-valued logic. However, the propositional fragment of Zadeh Logic without truth constants was studied in [49] as the implication-free fragment of $G_{\sim}$ (the logic of $\min, \max$ and $1 - x$, where the Kleene–Dienes implication is definable). The results obtained in that paper could be easily adapted when adding truth constants. On the other hand, the first order logic associated to Zadeh Logic can be seen as a sublogic of $L^*_{\sim}(\mathbf{S})\forall$ for any t-norm $*$ (see Section 2.4). We could take the simplest case, that is when $*$ is the Minimum t-norm (which in addition has the advantage of being finite strong canonical complete), and study the $\mathcal{ALC}$ description language over Zadeh Logic as the corresponding fragment of $G_{\sim}(\mathbf{S})\forall$. Therefore the FDLs studied in earlier papers are a fragment of the previously defined $\mathcal{ALC}^{\min}(\mathbf{S})$ -logic obtained by restricting the connectives to the ones with truth functions $\min, \max$ and $1 - x$.

7. Conclusions and future work

This paper is a first step in the direction proposed by Hájek. Our approach is based on the idea that the field of t-norm based fuzzy logics is an appropriate logical framework for FDLs which allows us to exploit of the recent developments of mathematical fuzzy logics. We have explained the necessity of introducing both truth constants and an involutive negation (when necessary) in the$\mathcal{ALC}$-like description language. Consequently, we present a new family of languages, denoted by $\mathcal{ALC}^*(\mathbf{S})$, and we give a general setting to relate each one of these languages with a fragment of a first order t-norm based (fuzzy) logic, in much the same way as the language $\mathcal{ALC}$ is studied as a fragment of classical first order logic. Having truth constants in the language, we can deal with graded assertional axioms, as is commonly done in FDLs, and we can also handle graded terminological axioms. Another advantage provided by the truth constants is that they allow us to introduce graded notions of satisfiability, validity and subsumption in terms of non-graded notions of these reasoning tasks by using evaluated formulas.

In general, classical DL languages have a consolidated logical background as fragments of classical first order logic. In this framework, logical studies refer to the relationships between the description logic under consideration (depending on the expressiveness of the DL language) and the corresponding fragment of first order logic and its computational complexity. However, a first and basic difference between the fuzzy and the classical case is the fact that, in general, first order residuated many-valued logics are not canonical complete. Therefore the $\mathcal{ALC}$-like Fuzzy Description Logic may not coincide with the related fragment of the corresponding first order logic. We have seen that canonical completeness of $L^*_{\sim}(\mathbf{S})\forall$ is satisfied when $*$ is either a divisible finite t-norm or the Minimum t-norm. Thus, in these cases it is possible to define *a priori* the $\mathcal{ALC}$-like description logic as the corresponding fragment of the Hilbert-style calculus for $L^*_{\sim}(\mathbf{S})\forall$. Notice that the problem of finding axiomatizations for these fragments remains open (we know an axiomatization of the first order logic but not an axiomatization of the fragment that interests us). On the other hand, when the logic is not canonical complete, we can try to study the$\mathcal{ALC}$-like description logic directly as, for example, Hájek did for the Łukasiewicz case (see Section 6.2).

Several research lines could be followed in future work. For instance, we are interested in analyzing the relevant fragments of first order fuzzy logics in connection with less expressive languages than $\mathcal{ALC}$ such as the families of languages $\mathcal{FL}$ and $\mathcal{AL}$. We also intend to analyze the decidability for the considered fuzzy (or graded) description logics $\mathcal{ALC}^*(\mathbf{S})$. Another interesting topic is to study the relationship between more expressive languages than $\mathcal{ALC}^*(\mathbf{S})$ and different fragments of t-norm based predicate fuzzy logics.

An other interesting line of future research is the translation of the description logic $\mathcal{ALC}^*(\mathbf{S})$ into modal logics in much the same way as in classical DLs. As in the classical framework, modal operators in FDLs can be defined from roles in the setting of t-norm based fuzzy logics. Nevertheless, in contrast to the classical case, there are not many results on modal logics over t-norm-based logics (see [50] for recent advances in the topic). Interesting studies in this direction are [51], which gives an axiomatization of modal operators $\Box$ and $\Diamond$ separately over Gödel Logic; [52], which gives axiomatizations of $\Box$ over the logic of any finite residuated lattice; and [53], which extends the work done in [51], by providing proof systems for the Gödel Modal Logic with $\Box$, the broader aim being to start a general investigation into the proof theory of modal fuzzy logics. We will try to relate the topics covered in these studies with the approach proposed in this paper.

A very interesting topic that we have not covered here concerns to the satisfiability and subsumption algorithms and their computational complexity. There is a large amount of literature on this topic. Nevertheless their relation with the metamathematical framework of Fuzzy Logics has not been analyzed in depth. We think that the research on FDLs can obtain fruitful results from the collaboration of researchers working on reasoning algorithms with those working on logical foundations. The current paper is written in the belief that logic can help the research in this direction.

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