On uncertainty and Kripke modalities in t-norm fuzzy logics

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Dedicated to Professor Petr Hájek
on the occasion of his 70th anniversary

1 Introduction and some remembrances

Five years ago, in the scientific event "The Beauty of Logic" organized by the Czech Academy of Sciences (CS AR) in Prague in honour of Professor Hájek in occasion of his 65th anniversary, we already stressed that Petr Hájek is recognized as a prominent scientist not only for his outstanding theoretical contributions in many different fields but also as a researcher that has never forgotten a more applied perspective. Indeed, part of his scientific work has been devoted to develop theoretical basis of the GUHA system [HH78], a methodology to systematically generate all hypotheses (relations among properties of objects) interesting with respect to given data, to provide mathematical grounds to process uncertain information [HHJ92], and to justify from a theoretical point of view some successful applications like those based on fuzzy logic [Háj98b].

In fact, when for the first time Lluís Godo (and Carles Sierra) met Petr Hájek in Prague in June 1987 (thanks to a contact with the late prof. Jiři Bečváš), the main motivation was to talk about uncertainty inference models for expert systems. In that time, expert systems were very popular Artificial Intelligence tools with many applications in different fields, and Petr Hájek, together with some colleagues of his (among them his wife Maria Hájková, the late Tomáš Havránek, and Milan Daniel) was developing the expert system shell EQUANT-PC. It was also in that period that Petr was supervising the doctoral work of J.J. Valdés analyzing the inference models of the two pioneer expert systems like MYCIN and PROSPECTOR and relating them, for the first time as far as we know, to ordered Abelian groups. This subject was extensively developed in the doctoral dissertation of J.J. Valdés and in a chapter of the book "Uncertain Information Processing in Expert Systems" by Hájek, Havránek and Jiroušek [HHJ92].

On the other hand, in the same period, in the Department of Artificial Intelligence of the Center for Advanced Studies of Blanes (CEAB), belonging to the Spanish National Research Council (CSIC), some researchers (among them Lluis and Carles), with the supervision of R. López de Mantaras, were developing the shell for expert systems MILORD (with a fuzzy logic based inference model) together with some medical applications (see [GdMSV89]).

After that first contact, three years later the Czechoslovak Academy of Sciences (CSAS) and the Spanish CSIC signed a collaboration agreement, and the authors could make a short visit to Petr in Prague in 1991, when Petr still was at the Mathematical Institute of the Czechoslovak Academy. By that time, Petr's office was under reconstruction and we could only have some short but intensive meetings at the Mazanka residence together with a seminar at the Mathematical Institute where we talked about some initial research on formalization of fuzzy logic we were involved in. The same year, a few months later, Petr made a first visit to our institute CEAB. That visit actually was for us the beginning of a long, illuminating, fruitful and still lasting collaboration between Petr and us, which has been extended along the years to colleagues of both teams.

And, as far as we are aware, it was in that time that Petr got really interested in fuzzy logic, even more when after that visit he attended the first European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty ECSQARU'91 in Marseille (October 1991).

Soon after, we started our scientific collaboration and in 1994 we published our first joint paper [HHE+94], with Dagmar Harmancová and Pere Garcia, about comparative possibilistic logic. The second joint paper was in 1995 [HGE95] about probability and fuzzy logic and the third was [HGE96] where Product logic was introduced. After Łukasiewicz and Gödel-Dummett logics studied in the late fifties, Product logic was the third basic residuated fuzzy logic whose semantics are given by the three distinguished continuous t-norms and their residua. Three years after Hájek published his celebrated book [Háj98b] about Metamathematics of Fuzzy logic, where he defined the Basic Fuzzy logic BL, which is proved to be the logic of continuous t-norms [Háj98a, CEGT00]. Two years later, the authors introduced the Monoidal t-norm logic MTL [EG01], proved to be the logic of the left-continuous t-norms in [JM02]. Since then, the new discipline of *Mathematical Fuzzy logic* has become a nice and deep research area and has attracted the interest of many scholars, most of them integrating the EUSFLAT working group on Mathematical Fuzzy logic MATHFUZZLOG[1].

As a contribution to this volume as tribute to Petr Hájek we have chosen to overview a topic that we have jointly worked on all over the past years: modalities in t-norm based fuzzy logics, mainly in extensions of BL and possibly expanded with truth constants[2]. On one hand we will review the work

[1] See http://www.mathfuzzlog.org/.

[2] The expansion of Łukasiewicz logic with truth constants was revisited by Hájek in his

done about logics to reason about uncertainty by means of fuzzy modalities. This topic was initiated with a joint paper with Petr Hájek (see e.g. [HGE95]) where probability and other measures of uncertainty on classical propositions were modelled as modalities over Rational Pavelka logic. This approach has been extended later to modalities to model different uncertainty measures over both classical and many-valued propositions. On the other hand we will also review recent papers studying modal many-valued logics following the definition of modal operators over finite Heyting algebras in the papers by Fitting [Fit92a, Fit92b]. We will restrict ourselves to review the results on minimum modal logics over a finite BL-chain. Actually, they are in fact two different kinds of modalities, arising from different motivations and semantics and with few common features. While the former uncertainty modalities are introduced in a restricted language where e.g. nested modalities are not allowed, the latter are proper generalizations of classical modalities with Kripke semantics to a fuzzy or many-valued framework. It is worth noticing that our very first joint paper [HHE$^+$94] was about some many-valued modalities with Kripke semantics but to model qualitative possibilistic uncertainty.

2 Uncertainty modalities: belief degrees versus truth degrees

Since the first contacts, a topic of common interest was the interpretation of certainty values associated to rules and facts in expert systems (or more in general in knowledge based systems).

Indeed, in that time, it was common in expert systems to allow rules and facts be qualified by certainty degrees, providing a measure of the strength of the statements. The way those certainty values were interpreted and the compositional way they were handled by the inference procedures varied from one system to another. However, in most of the systems, there was a mismatch between the intended semantics of the certainty degrees and the way they were used. Indeed, in some systems the certainty values were interpreted probabilistically (in some form or another), like in Prospector or Mycin, but the propagation rules were either not sound or they were making too strong conditional independence assumptions. On the other hand, other systems interpreted certainty degrees as truth degrees in a truth-functional many-valued of fuzzy logic setting. Here the problem was the misuse of partial degrees of truth as belief degrees. This kind of confusion was quite common, but Petr Hájek had already clear this distinction in mind since our first meetings in the beginning of the nineties, put forward later during the European Coper-

book (he called it RPL, for Rational Pavelka logic), and the general case of any logic of a continuous t-norm and its residuum has been addressed in [EGGN07]. Later, Hájek has returned to this topic in the paper [Háj06b] where he studied the complexity issues of these logics with truth constants.

nicus 10053 project MUM (*Management of Uncertainty in Medicine*) where we jointly participated, and stressed by himself in many different occasions to the fuzzy logic community.

In his excellent monograph [Háj98b] Petr Hájek deals with this problem. In the introduction he writes *Fuziness is imprecision (vagueness); a fuzzy proposition may be true in a some degree* and in the same paragraph he adds *the truth of a fuzzy proposition is a matter of degree.* He also argues a very important issue to differentiate uncertainty measures and truth values in a logical setting. He states that *most many-valued logics are truth functional*[3] but this is not true for uncertainty measures that, as it is well known, are not truth functional.

However, as Petr Hájek and Dagmar Harmancová noticed in [HH95], one can safely interpret a probability degree on a Boolean proposition φ as a truth degree, but not of φ itself but of another proposition $P\varphi$, read as "φ is *probable*". The point is that "being probable" is actually a fuzzy predicate, which can be more or less true, depending on how much probable is φ. Hence, it is meaningful to take the truth-degree of $P\varphi$ as the probability-degree of φ. The second important observation is the fact that the standard Łukasiewicz logic connectives provide a proper modelling of the Kolmogorov axioms of finitely additive probabilities. Indeed, these were the key issues that are behind the probability logic first defined in our joint UAI'95 paper [HGE95] as a theory over Rational Pavelka logic, and later described with an improved presentation in Hájek's monograph [Háj98b, Chapter 8] where P is introduced as a modality.

In the following subsections we recall the definition of the basic probability logic and then succinctly describe some other fuzzy probability logics we have jointly studied extending the original approach.

2.1 The basic probability logic FP(CPL, Ł)

The basic probability logic, as defined in [Háj98b], takes formulas of classical propositional logic φ and defines atomic probability formulas to be of the form $P\varphi$, and compound probability formulas are then defined from the atomic ones by means of Łukasiewicz logic connectives. So, the language is not a full modal language, actually it contains two kind of formulas, Boolean formulas of classical propositional logic (CPL) and probability formulas. Notice that nesting of the P operator is not allowed as well as formulas combining Boolean and probability subformulas. The axioms of FP(CPL, RPL), standing for fuzzy probability logic to reason about the probability of CPL formulas using Łukasiewicz logic, are the following: Axioms and rules of CPL for Boolean

[3]This is also true for the combination of membership degrees in fuzzy set theory since from Zadeh most of fuzzy set systems assume functional combination of membership degrees

formulas, axioms and rules of Łukasiewicz logic for probability formulas, plus the following three probabilistic axioms:

(FP1) $(P\varphi \,\&\, P(\varphi \to \psi)) \to_{Ł} P\psi$

(FP2) $P(\neg\varphi) \equiv_{Ł} \neg_{Ł} P\varphi$

(FP3) $P(\varphi \vee \psi) \equiv_{Ł} (P\varphi \to_{Ł} P(\varphi \wedge \psi)) \to_{Ł} P\psi.$

and the necessitation rule for P: if φ is a theorem of CPL, then derive $P\varphi$.

The crucial point is that axiom (FP3) exactly captures finite additivity. In fact, an easy checking shows that a Łukasiewicz truth evaluation e satisfying (FP3) is such that $e(P(\varphi\vee\psi)) = e(P\varphi)+e(P\psi)-e(P(\varphi\wedge\psi))$. Moreover, any e satisfying axioms (FP1), (FP2) and (FP3) and respecting the necessitation rule (i.e. making $e(P\varphi) = 1$ for each CPL theorem φ) induces a probability μ_e on Boolean formulas, namely, if we define $\mu_e(\varphi) = e(P\varphi)$, then $\mu_e : \mathcal{L} \to [0, 1]$ is a finitely additive probability preserving classical logical equivalence on the set of Boolean formulas $\mathcal{L}$. And conversely, given such a probability on formulas μ, it induces a Łukasiewicz truth-evaluation of atomic probability formulas e_μ, by defining $e_\mu(P\varphi) = \mu(\varphi)$, which is a model of the above axioms.

These considerations, combined together with Łukasiewicz logic finite standard completeness gives the following probabilistic completeness of FP(CPL, Ł): any finite theory T of probability formulas proves in FP(CPL, Ł) a probability formula Φ iff for any probability on formulas μ such that e_μ is a model of all formulas of T, e_μ is also a model of Φ.

If one wants to reason with explicit probability degrees one can switch from Łukasiewicz logic to Rational Pavelka logic by introducing rational truth-constants into the language of probability formulas, and adding the corresponding book-keeping axioms to the FP(CPL, Ł) logic. In this way we obtain the logic FP(CPL, RPL), which inherits the Pavelka-style completness of RPL, stating that the truth degree of a probability formula Φ over an arbitrary theory T of probability formulas equals its provability degree over FP(CPL, RPL).

Let us mention that the same approach can be used to reason about other uncertainty measures, like possibility and necessity measures [Háj98b] or upper and lower probability measures [Mar08]. The case of belief functions is recalled in the next subsection.

On the other hand, if one is interested in reasoning about conditional probabilities, then one definitely needs to replace Ł, or RPL, by the more expressive logic $Ł\Pi\frac{1}{2}$ [EGM01], putting together Łukasiewicz and Product logic connectives. If $\to_\Pi$ denotes Product logic implication, then one can mimic the usual definition of conditional probability in terms of the 1-place probability by defining $P(\varphi \mid \psi)$ as a shorthand for

$$P\psi \rightarrow_{\Pi} P(\varphi \wedge \psi).$$

The corresponding logic FP(CPL, $Ł\Pi\frac{1}{2}$) was defined and studied in a joint paper with Petr Hájek [GEH00]. Another alternative is to start with conditional probability as a primitive notion, so to introduce a binary probability operator $P(\cdot \mid \cdot)$, redefine accordingly the set of probability formulas and to replace the above axioms (FP1)-(FP3) by a suitable set of axioms for conditional probability. This approach was taken in [GM06] and the corresponding logic was denoted FCP(CPL, $Ł\Pi\frac{1}{2}$). An alternative approach to reason about conditional probability, based on non-standard probabilities, has been developed by Flaminio and Montagna in [FM05], where they define the logic FP(SŁΠ), and further studied in [Fla07]. In this framework of fuzzy modal logics of conditional measures, let us mention as well the paper by Marchioni [Mar06] where he defines a logic to reason about conditional possibilities.

2.2 FP(S5, RPL): a logic for belief functions

In [GHE03] we generalized the approach to deal with belief functions, i.e. we took the following identity

$$\text{belief degree of } \varphi = \text{truth degree of } B\varphi,$$

where now belief degrees are related to Dempster-Shafer belief functions, and $B\varphi$ stands for the fuzzy proposition "φ is believed". For getting a complete axiomatization we used one of possible definitions of Dempster-Shafer belief functions, namely as probability of knowledge, in the epistemic logic sense (see for instance [Rus87, HKR94, Háj96]. This enabled us to combine the above approach to probabilistic reasoning with the modal logic S5, the most common logic for knowledge, i.e. to define our belief formulas $B\varphi$ to be defined as $P\Box\varphi$, where $\Box$ is a S5 modality and φ is a propositional modality-free formula.

We recall that a belief function on Boolean formulas is a mapping into $[0, 1]$ preserving classical logical equivalence and satisfying the following properties:

(B1) $bel(\top) = 1$,
(B2) $bel(\bot) = 0$,
(B3) $bel(\varphi_1 \vee \ldots \vee \varphi_n) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} bel(\bigwedge_{i \in I} \varphi_i)$, for each n.

As it was proved in [HKR94, FHM90] for the case of finitely-many propositional variables and generalized in [GHE03] for the countable case, all belief functions on formulas are given by probabilities on S5-formulas. More precisely, for each belief function on formulas *bel* there is a $\Box$-probabilistic Kripke model $K = (W, e, R, \mu)$, where (W, e, R) is a S5 Kripke model and μ is a probability measure on a subalgebra of 2^W such that $bel(\varphi) = \mu(\{w \in W \mid e(w, \Box\varphi) = 1\})$.

In the logic FP(S5, RPL) we distinguish again two kinds of *formulas*:

(1) *Boolean formulas* (or S5-formulas) are built from propositional variables $p_0, p_1, \ldots$ using connectives $\to, \vee, \wedge, \neg$ (say) and the modality $\Box$ as described above. *Modality-free* formulas do not contain $\Box$; a formula is *closed* if each propositional variable is in the scope of a modality. Closed formulas without nested modalities are called *normal;* they are just built from formulas of the form $\Box\varphi$, where φ if modality-free, using the connectives above.
(2) *P-formulas* (or many-valued formulas). Atomic P-formulas have the form $P\varphi$, where φ is a normal S5-formula. P-formulas are built from atomic P-formulas using the connectives of Łukasiewicz logic, i.e. $\to_{\text{Ł}}, \&$, and truth constants $\overline{r}$ for each rational $r \in [0, 1]$. As mentioned above, $B\varphi$ is an abbreviation for $P(\Box\varphi)$, where φ is a modality-free formula; Formulas built from these using Ł-connectives and truth-constants are called *belief formulas* or B-formulas.

The deduction rule of FP(S5, RPL) is modus ponens for Łukasiewicz implication $\to_{\text{Ł}}$ and its axioms are:

- (FB1) φ, for each S5-formula φ provable in S5
- (FB2) $P\varphi$, for each normal S5-formula φ provable in S5
- (FB3) the schemes (FP1)–(FP3) above for P-formulas
- (FB4) axioms of RPL for P-formulas.

The main completeness result for FP(S5, RPL) in [GHE03] reads as follows. Let T be a finite set of belief formulas and Φ a belief formula. Then, T proves Φ in FP(S5, RPL) iff $\|\Phi\|_{bel} = 1$ for each belief function *bel* which is a model of T.

2.3 Logics combining belief and fuzziness

An extension of the notion of probability to the framework of fuzzy sets was early defined by Zadeh in order to represent and reasoning with sentences like *the probability that the traffic in Rome will be chaotic tomorrow is 0.7.* Clearly, the modeling of this kind of knowledge cannot be done using the classical approach to probability since, given the un-sharp nature of events like *chaotic traffic*, the structure of such fuzzy events cannot be considered to be a Boolean algebra any longer. The study of finitely-additive measures in the context of MV-algebras was started by Mundici in [Mun95] and has been further developed in the last years, see e.g. [RM02, Kro06]. Similar studies of probabilities on Gödel algebras have been carried out by Aguzzoli, Gerla and Marra [AGM08].

Therefore, a fuzzy logical approach to reason about the probability of fuzzy events is a natural generalization of the previous works. In logical terms, this can be approached by assuming that the logic of events is a (suitable) many-valued logic and by defining and axiomatizing appropriate probability-like measures on top of the many-valued propositions.

In [Háj98b] Hájek himself already proposed a logic built up over the Łukasiewicz predicate calculus Ł∀ allowing a treatment of (simple) probability of fuzzy events. To model this kind of probability, Hájek introduced in Ł∀ a generalized fuzzy quantifier standing for *most* together with a set of characteristic axioms and denoted his logic by $Ł\forall\int$. In his monograph Hájek also proposed two (Kripke-style) probabilistic semantics for this logic, called *weak* and *strong*, that can be roughly described as follows:

- A weak probabilistic model for $Ł\forall\int$ evaluates a modal formula $P(\varphi)$ by means of a finitely additive measure (or *state*) defined over the MV-algebra of provably equivalent Łukasiewicz formulas.

- A strong probabilistic model for $Ł\forall\int$ consists on a probability distribution σ over the set of all the evaluations of the events (remember that an event is now a formula of the Łukasiewicz calculus). Then the truth value of a modal formula $P(\varphi)$ is defined as the *integral* of the fuzzy-set of all the evaluations of φ under the measure σ.

Hájek showed that his logic is Pavelka-style complete w.r.t. weak probabilistic models, but the issue of completeness w.r.t. to the strong semantics remains as an open problem. More recently, Flaminio and one of the authors have defined in [FG07] a logic to reason about the probability of many-valued events of finitely-valued Łukasiewicz logic, following the approach described in the previous subsections and called it FP($Ł_n$,Ł). Here the basic difference with the previous logic FP(CPL, Ł) is of course the inner logic of events. Indeed, the axioms FP($Ł_n$,Ł) are those of $Ł_n$ for non-modal formulas, axioms of Ł for modal formulas, the following version of the above probabilistic axioms

$$\begin{array}{ll}
\text{(FP1)} & P(\varphi \to_Ł \psi) \to_Ł (P\varphi \to_Ł P\psi) \\
\text{(FP2)} & P(\neg_Ł \varphi) \equiv_Ł \neg_Ł P\varphi \\
\text{(FP3)} & P(\varphi \oplus \psi) \equiv_Ł [(P\varphi \to_Ł P(\varphi \& \psi)) \to_Ł P\psi]
\end{array}$$

together with the rule of modus ponens and the rule of necessitation for P: if φ is a theorem of $Ł_n$ then derive $P\varphi$. This logic was shown to be complete for finite theories of modal formulas with respect to both weak and strong probabilistic models as defined above, i.e. defined by finitely additive measures on $Ł_n$-formulas (and hence respecting $Ł_n$ logical equivalence) and by probability distributions on sets of worlds (which may be identified with Ł$_n$-evaluations). Completeness of the analogous logic over events in the infinitely-valued Łukasiewiz logic Ł remains as an open problem.

It is worth pointing out that Petr Hájek has studied in [Háj07] the complexity of a variety of fuzzy probability logics, extending first results of Hájek and Tulipani for FP(CPL, Ł) and FP(CPL, ŁΠ) [HT01] showing that the corresponding sets of 1-satisfiable formulas are NP-complete and in PSPACE respectively. Indeed, he considers arbitrary fuzzy probability logics FP(L_1, L_2), where L_1, the logic of events, can be either Boolean logic (CPL) or a fuzzy logic, and L_2 is a fuzzy logic. The fuzzy probability logic FP(L_1, L_2) is then a fuzzy modal logic whose non-modal formulas are those of L_1, atomic modal formulas have the form $P\varphi$ and other modal formulas are formed from atomic ones by means of connectives of the logic L_2. Probabilistic Kripke models have the form $M = (W, e, \sigma)$ where W is a non-empty (at most countable) set of possible worlds, e evaluates in each possible world all atomic formulas by truth values from the standard set of truth values of L_1 and $\sigma : W \to [0, 1]$ with $\Sigma_{w\in W}\ \sigma(w) = 1$. For each L_1-formula φ, the evaluation of $P\varphi$ in the model is the probability of φ defined as $\|P\varphi\|_M = \Sigma_{w\in W}\ \sigma(w)\cdot e_{L_1}(\varphi, w)$. Hájek then shows, among others, complexity results for different sets of FP(L_1, L_2) satisfiable formulas. In particular, he shows that the sets of all satisfiable formulas of FP(Ł$_n$, Ł) and FP(G_n, Ł), where G_n denotes the n-valued Gödel logic, are NP-complete. Also, the set of satisfiable formulas of FP(G, L_2), for L_2 arbitrary suitable[4], and the set of 1-tautologies of FP(L_1, Ł) for L_1 arbitrary suitable, are in PSPACE.

To close this section, let us mention that for the case of *possibilistic* uncertainty, some logics have been also defined to reason about the necessity of fuzzy events. Indeed, in our very first joint paper with Petr Hájek and Dagmar Harmancová [HHE$^+$94], we dealt with logics of necessity of events of Ł$_n$, a very similar topic of the more recent paper [FGM], while in [DGM09] several necessity logics are defined over events of Gödel logic.

3 Kripke modalities and fuzzy logic

In this section we review some results about modal many-valued logics understood as logics defined by Kripke frames (including the possibility of many-valued accessibility relations) where every world follows the rules of a many-valued logic, this many-valued logic being the same for every world.

One can find in the literature some attempts and approaches to generalize modal logic formalisms to the many-valued setting. Roughly speaking, we can classify the approaches in three groups depending how the corresponding

[4]Hájek defines a fuzzy logic to be suitable when its standard set of truth values is the real unit interval [0, 1] and the truth functions of its (finitely many) connectives are definable by open formulas in the (ordered) field of reals.

Kripke frames look like, in the sense of how many-valuedness affects the worlds and the accesibility relations. Next we describe these three groups and comment about our work in each of them.

A first group (see e.g. [Gob70, CF92, EGGR97, Suz97]) is formed by those logical systems whose class of Kripke frames are such that their *worlds are classical* (i.e. they follow the rules of classical logic) but their *accessibility relations are many-valued*, with values in some suitable lattice A. In such a case, the usual approach to capture the many-valuedness of an accessibility relation $R : W \times W \rightarrow A$ is by considering the induced set of classical accessibility relations $\{R_a \mid a \in A\}$ defined by the different level-cuts of R, i.e. $\langle w, w' \rangle \in R_a$ iff $R(w, w') \geq a$. At the syntactical level, one then introduces as many (classical) necessity operators $\Box_a$ (or possibility operators $\Diamond_a$) as elements a of the lattice A, interpreted by (classical) relations R_a. Therefore, in this kind of approach, one is led to a multi-modal language but where (both modal and non modal) formulas are Boolean in each world.

In this setting the work of our team is related to models of similarity-based reasoning developed in [EGCG94, DEG+95, DPE+97]. The starting point is the paper by Ruspini [Rus91] about a possible semantics for fuzzy set theory. He develops the idea that we could represent a fuzzy concept by its set of prototypical elements (which will have fully membership to the corresponding fuzzy set) together with a similarity relation giving the degree of similarity of each element of the universe to the closest prototype. This degree is taken then as the membership to the fuzzy set. From this basic idea, we have developed and characterized some forms of graded entailments that can be represented in multi-modal systems with frames where the (graded) accessibility is given by a similarity relation on pairs of worlds, and for which we have proved completeness in several cases [EGGR98].

A second group of approaches are the ones whose corresponding Kripke frames have *many-valued worlds*, evaluating propositional variables in a suitable lattice of truth-values A, but with *classical accessibility relations* (see e.g. [HH96, Háj98b, HT06, Háj]). In this case, we have languages with only one necessity and/or possibility operator ($\Box, \Diamond$), but whose truth-evaluation rules in the worlds is many-valued, so modal (and non-modal formulas) are many-valued.

Main Hájek's contributions to many-valued modal logic in the sense reviewed in this section fall into this group, see e.g. [HH96, Háj98b, Háj]. We have also worked in this setting but only as a particular case of the general modal many-valued setting that we will review after introducing the third group.

Finally, a third group of approaches are *fully many-valued*, in the sense that in their Kripke frames, both worlds and accessibility relations are many-valued, again over a suitable lattice A. In that case, some approaches (like

[Fit92a, Fit92b, KNP02, CR]) have a language with a single necessity/ possibility operator ($\Box, \Diamond$), and some (like [Mir05]) consider a multi-modal language with a family of indexed operators $\Box_a$ and $\Diamond_a$ for each $a \in A$, interpreted in the Kripke models via the level-cuts R_a of a many-valued accessibility relations R. Actually, these two kinds of approaches are not always equivalent, in the sense that the operator $\Box$ and the set of operators $\{\Box_a \mid a \in A\}$ are not always interdefinable (or analogously with the possibility operators).

In what follows we review the main results contained in the recent papers [BEG08, BEGR09] falling in this third group. To do so we first introduce the logic of a finite BL-chain and then we focus on the minimum modal and multi-modal systems over these logics.

3.1 The logic of a finite BL-chain

Let $\mathbf{A} = \langle A, \wedge, \vee, \odot, \rightarrow, 1, 0\rangle$ be a finite BL-chain and define the corresponding propositional logic, denoted $\Lambda(\mathbf{A})$, as the logic such that for all (finite) set of formulas Γ and a formula φ,

$$\Gamma \models \varphi \text{ if and only if } e(\varphi) = 1 \text{ for any } A\text{-evaluation } e \text{ that is a model of } \Gamma$$

It is known that t$\Lambda(\mathbf{A})$ has a finite axiomatization, see e.g. [AM03]. Now consider the logic $\Lambda(\mathbf{A}^c)$ obtained from $\Lambda(\mathbf{A})$ by adding a set of truth constants in the language (one $\overline{a}$ for each $a \in \mathbf{A}$) and the following axioms and rules of inference:

- *Book-keeping axioms*

$$(\overline{a} * \overline{b}) \leftrightarrow \overline{a * b} \quad \text{where } a, b \in A \text{ and } * \in \{\odot, \rightarrow\}.$$

- *Witnessing axiom*

$$\bigvee_{a \in A} (\varphi \leftrightarrow \overline{a})$$

- And the rule of inference

$$\bar{r} \vee \varphi \vdash \varphi$$

  where $r \in \mathbf{A}$ is the coatom of the finite chain $\mathbf{A}$.

In this setting one can prove that the logic $\Lambda(\mathbf{A}^c)$ is finite strong canonical complete, i.e. complete with respect to evaluations over the chain $\mathbf{A}^c$ and interpreting each constant by its corresponding element of the chain.

3.2 The minimum modal many-valued logic

The main goal of [BEGR09] was to define and study $\Box$-modal systems over the logic $\Lambda(\mathbf{A}^c)$, i.e., starting from an axiomatization of the logic $\Lambda(\mathbf{A}^c)$, to obtain an axiomatization of the modal logics over it. The basic definition of the minimum modal logic over the logic of a finite BL-chain is the following:

Modal language.- The modal language is the expansion of the language of either $\Lambda(\mathbf{A}^c)$ by a new unary connective: the *necessity operator* $\Box$.

Kripke frames.- A Kripke frame is a pair $F = \langle W, R\rangle$ where W is a non empty set (the *worlds*) and R is a binary relation valued in A (i.e., $R : W \times W \longrightarrow A$) called *accessibility relation*. The Kripke frame is said to be *crisp* if the range of R is included in $\{0, 1\}$ and *idempotent* if included in the set of idempotent elements. The classes of Kripke frames, idempotent frames and crisp Kripke frames are denoted, respectively, by $Fr(\mathbf{A})$, $IFr(\mathbf{A})$ and $CFr(\mathbf{A})$ (or simply Fr, IFr and CFr if there is no ambiguity).

Kripke models.- A Kripke model is a 3-tuple $M = \langle W, R, V\rangle$ where $\langle W, R\rangle$ is an $\mathbf{A}$-valued Kripke frame and V is a map, called *valuation*, assigning to each variable in Var and each world in W an element of A (i.e., $V : Var \times W \longrightarrow A$). The map V can be uniquely extended to arbitrary formulas by interpreting connectives and truth-constants as usual by the corresponding operations in the chain $\mathbf{A}$ and stipulating

- $V(\Box\varphi, w) = \bigwedge\{R(w, w') \to V(\varphi, w') : w' \in W\}$.

Let us denote by $\Lambda(Fr, B)$, $\Lambda(IFr, B)$ and $\Lambda(CFr, B)$ the set of valid formulas in the class of frames Fr, IFr and CFr respectively over B, where B is either $\mathbf{A}$ or $\mathbf{A^c}$.

Some available results that can be found in the literature about many-valued systems are the following:

- Fitting in [Fit92a, Fit92b] studied $\Lambda(Fr, A^c)$ when $\mathbf{A}$ is a finite Heyting Algebra.
- Caicedo and Rodríguez in [CR] study $\Lambda(Fr, [0, 1]_G)$ and proved its equivalence with $\Lambda(CFr, [0, 1]_G)$
- Metcalfe and Olivetti in [MO09] give a proof theory for $\Lambda(Fr, [0, 1]_G)$

But none of them deals with the general case, studying the minimum modal many-valued logic in a systematic way. Among the difficulties to find such an axiomatization is that while the meet-distributivity axiom

(MD) $(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi)$

is valid (as in the classical modal case), in general this is not the case in the many-valued modal setting for the normality axiom

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

In fact the normality axiom K is only valid if either we restrict to idempotent or crisp Kripke frames (IFr or CFr respectively) or when **A** is a Heyting algebra (when the interpretation of the strong conjunction coincides with the meet).

In [BEGR09], using the canonical model method, we have proved completeness of the axiomatization of $\Lambda(Fr, A^c)$, the minimum modal logic over a finite BL-chain **A**, given below.

Axiomatization of the set $\Lambda(Fr, A^c)$ when **A** is a finite BL-chain

- the set of axioms is the smallest set closed under substitutions containing
  - an axiomatic basis for $\Lambda(\mathbf{A})$,
  - the witnessing axiom $\bigvee_{a\in A}(\varphi \leftrightarrow \overline{a})$
  - the book-keeping axioms $(\overline{a_1} * \overline{a_2}) \leftrightarrow \overline{a_1 * a_2}$ (for every $a_1, a_2 \in A$ and every $* \in \{\wedge, \vee, \odot, \rightarrow\}$),
  - $\Box\overline{1}$, $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ and $\Box(\overline{a} \rightarrow \varphi) \leftrightarrow (\overline{a} \rightarrow \Box\varphi)$ (for every $a \in A$),
- Modus Ponens plus the rule $\overline{k} \vee \varphi \vdash \varphi$ (where k is the coatom of **A**) and the Monotonicity rule $\varphi \rightarrow \psi \vdash \Box\varphi \rightarrow \Box\psi$.

Notice that this axiomatization is over $\Lambda(\mathbf{A}^c)$, i.e., having truth constants in the language. The problem of axiomatizing the minimum modal logic over a finite residuated chain without truth constants in the language is still an open problem except for the case of the minimum modal logic over finite Łukasiewicz logic that was also axiomatized in [BEGR09] but in a rather complicated way.

The case of idempotent frames was easily solved because they are characterized as the general frames where the axiom K is valid. From that the proof that the logic $\Lambda(IFr, \mathbf{A}^c)$ is axiomatized by the axioms and rules of $\Lambda(Fr, \mathbf{A}^c)$ adding the axiom K is easy and proved also in [BEGR09].

If we restrict ourselves to *Crisp* Kripke models then the axiom K is valid even though it does not characterize them. The found axiomatization of $\Lambda(CFr, A^c)$ is given below. Notice that, due to axiom K, in this case one can replace the monotonicity rule for the more usual and simple necessitation rule.

Axiomatization of the set $\Lambda(CFr, \mathbf{A}^c)$ when $\mathbf{A}$ is a finite BL-chain

- the set of axioms is the smallest set closed under substitutions containing
  - an axiomatic basis for $\Lambda(\mathbf{A})$,
  - the witnessing axiom $\bigvee_{a \in A}(\varphi \leftrightarrow \overline{a})$
  - the book-keeping axioms $(\overline{a_1} * \overline{a_2}) \leftrightarrow \overline{a_1 * a_2}$ (for every $a_1, a_2 \in A$ and every $* \in \{\wedge, \vee, \odot, \to\}$),
  - $\Box\overline{1}$, $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ and $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  - $\Box(\overline{a} \to \varphi) \leftrightarrow (\overline{a} \to \Box\varphi)$ (for every $a \in A$) and $\Box(\overline{k} \vee \varphi) \to (\overline{k} \vee \Box\varphi)$ (where k is the coatom of $\mathbf{A}$),
- Modus Ponens plus the rule $\overline{k} \vee \varphi \vdash \varphi$ (where k is the coatom of $\mathbf{A}$) and the Necessity rule $\varphi \vdash \Box\varphi$.

3.3 Multi-modal many-valued versus Modal many-valued

As in the first group of approaches mentioned above, one way to capture the many-valuedness of the accessibility relation R is by considering the induced set of classical accessibility relations defined by the different level-cuts of R. Namely it is well-known the correspondence between a fuzzy binary relation R over A, i.e. a function $R : W \times W \longrightarrow A$, and an $A \setminus \{0\}$-indexed and decreasing family of crisp binary relations, i.e. a family $\{R_a : a \in A, a \neq 0\}$ such that if $a \leq b$ then $R_b \subseteq R_a \subseteq W \times W$. This correspondence is given by the following identities:

(1) $R_a = \{(w_1, w_2) \in W \times W : R(w_1, w_2) \geq a\}$

(2) $R(w_1, w_2) = \max\{a \in A : (w_1, w_2) \in R_a\}$

In the same way, an $\mathbf{A}$-valued Kripke frame $\mathcal{F} = (W, R)$ can be transformed into a family of crisp Kripke frames

$$F_a = (W, R_a))_{a \in A \setminus \{0\}}.$$

At the syntactical level, one then introduces as many necessity operators $\Box_a$ as elements a of the lattice A interpreted by the (classical) relations R_a. The advantage is that each modal operator $\Box_a$ satisfies the axiom K. Therefore, in this kind of approach, one is led to a multi-modal language which is built from the set of propositional variables and connectives adding the truth constants $\{\bar{a} \mid a \in A\}$ and the family of modal operators $\{\Box_a \mid a \in A\}$. The Kripke frames $\mathcal{M} = (W, R, V)$ are defined as in the precedent case and the valuation V is extended to modal formulas by the following condition:

$$V(\Box_a \varphi, w) = \bigwedge_{w' \in W} R_a(w, w') \to V(\varphi, w') = \bigwedge_{w' \in W} \{V(\varphi, w') : R(w, w') \geq a\}.$$

The resulting multi-modal system, denoted as $M\Lambda(Fr, \mathbf{A}^c)$, has been axiomatized in [BEGR09] by the following axioms and rules of inference:

Axiomatization of the set $M\Lambda(Fr, \mathbf{A}^c)$ when $\mathbf{A}$ is a finite BL-chain

- the set of axioms is the smallest set closed under substitutions containing
  - Axiomatic basis for $\Lambda(A^c)$,
  - $\Box_b(\varphi \to \psi) \to (\Box_b \varphi \to \Box_b \psi)$, for every $b \in A \setminus \{0\}$).
  - $\Box_b(\overline{a} \to \varphi) \leftrightarrow (\overline{a} \to \Box_b \varphi)$, for every $a \in A$ and $b \in A \setminus \{0\}$),
  - $\Box_b(\overline{k} \vee \varphi) \to (\overline{k} \vee \Box_b \varphi)$, for every $b \in A \setminus \{0\}$, where k is the coatom of $\mathbf{A}$),
  - $\Box_{b_1} \varphi \to \Box_{b_2} \varphi$ for every $b_1, b_2 \in A \setminus \{0\}$ such that $b_1 \leq b_2$),
- Modus Ponens plus the rule $\overline{k} \vee \varphi \vdash \varphi$ (where k is the coatom of $\mathbf{A}$) and the Necessity rules: $\varphi \vdash \Box_b \varphi$ for every $b \in A \setminus \{0\}$.

An interesting problem is to study and compare the expressive power of many-valued modal and multi-modal systems. To this purpose it is useful to introduce the following concepts. Given two pointed[5] Kripke models $\langle M, w \rangle$ and $\langle M', w' \rangle$, we say that they are *modally equivalent* in the case that $V(\varphi, w) = V'(\varphi, w')$ for every modal formula φ. Analogously we will talk about *multimodally equivalent* and *full modally equivalent* in the case we focus, respectively, on multi-modal formulas or full modal formulas.

A first remark comparing the expressive power is that in every finite chain (and even in every finite residuated lattice) $\mathbf{A}$ it holds that

$$(3) \quad \Box \varphi \equiv \bigwedge \{\overline{a} \to \Box_a \varphi : a \in A \setminus \{0\}\}$$

Therefore, the modality $\Box$ is explicitly definable using the modalities $\Box_a$'s (and these last ones have the advantage that satisfy the normality axiom).

[5] By a *pointed Kripke model* we mean a Kripke model together with a distinguished point or world.

In other words, the expressive power of the modal language is smaller than the one of the multi-modal one. Thus, $\Lambda(Fr, A^c)$ can be seen as a fragment of $M\Lambda(Fr, A^c)$. It is obvious that if two pointed Kripke models are multimodally equivalent then they are also modally equivalent.

But the inverse sense is not valid in general. As a matter of fact it is necessary to distinguish between having an empty set Var of propositional variables or not:

- If $Var = \emptyset$ and two pointed Kripke models are modally equivalent, then they are also multimodally equivalent.
- If $Var \neq \emptyset$, let $\mathbf{A}$ be the ordinal sum $\mathbf{A}_1 \oplus \mathbf{A}_2$ of two finite BL chains such that $\mathbf{A}_1$ and $\mathbf{A}_2$ are non trivial (i.e., $\min\{|A_1|, |A_2|\} \geq 2$). Then, there are two pointed Kripke models that are modally equivalent but not multimodally equivalent (in [BEGR09] a counterexample proving this fact is shown).

However, the expressive power of both logics become the same in case $\mathbf{A}$ is a finite MV-chain since in such a case we have the following inter-definability result: let $\mathbf{A}$ be the finite Łukasiewicz chain of cardinal n; then, for every $a \in A \setminus \{0\}$ it holds that

$$\Box_a \varphi \;\equiv\; \bigwedge \left\{ \left(\overline{a} \to \neg\Box\neg((\varphi \leftrightarrow \overline{b})^{n-1})\right)^{n-1} \to \overline{b} : b \in A \right\}$$

The proof is based on the fact that φ^{n-1} only takes crisp values (i.e., in $\{0,1\}$). Indeed, φ^{n-1} takes value 1 when φ takes value 1, and φ^{n-1} takes value 0 elsewhere. Hence, φ^{n-1} takes the same value than $\Delta\varphi$, where Δ is the well-known Baaz projection (see [Háj98b]).

This leads to the following general result, for $\mathbf{A}$ being a finite BL-chain, stating that the following statements are equivalent:

1. $\mathbf{A}$ is a finite MV-chain (i.e., the only idempotent elements of A are 0 and 1),
2. The modalities $\Box_a$'s are explicitly definable in the modal language.
3. Two pointed Kripke models are modally equivalent iff they are multimodally equivalent

The problem of axiomatizing the modal and multi-modal systems obtained by adding either a possibility operator $\Diamond$ or a family of possibility operators $\Diamond_a$ for $a \in \mathbf{A}$ are open problems. Of course, it is solved in the case of $\mathbf{A}$ being a finite MV chain since possibility modal operators are definable as in the classical case as $\Diamond \equiv \neg\Box\neg$ and the same for $\Diamond_a$ and $\Box_a$. Moreover in [CR] the axiomatization of the modal system obtained adding either $\Box$ or $\Diamond$ over Gödel Logic are given. But the problem of the axiomatization of a system obtained by adding both modal operators over Gödel Logic is still open.

4 And the collaboration is going on

The relation initiated by Petr Hájek (with Dagmar Darmancová and Milan Daniel) with the authors almost twenty years ago is not only very alive but also has improved with the incorporation of young colleagues of both groups, that have a very good and fruitful collaboration both at the scientific and personal level. In this contribution we have overviewed only a common topic in our research, leaving aside many other topics that are of our common interest. Nevertheless one of the first common topics of interest, the difference between degrees of truth and uncertainty measures has been deeply studied and the first part of this paper is devoted to this subject. The second part (that deals with many-valued modal logics) is very related to Fuzzy Description logics, a topic that has been studied by our teams in the last period. Following the first ideas of Hájek in [Háj05, Háj06a] we have developed the logical basis for Fuzzy Description languages in [GCAE]. It is well known that description logics have a nice translation to modal logic in the classical case and the same remains valid for the fuzzy case. In this sense the study of many-valued modal logics we think will be very useful for the theoretical study of fuzzy description logis.

We believe that the collaboration between the two teams has been really fruitful, we have learnt a lot from Petr Hájek, and during all this time we have enjoyed his personality, ideas and friendship. We dedicate this paper to him in the occasion of his seventieth anniversary. Thank you Petr for being a reference, for your stimulating scientific ideas and for your friendship, in summary, for being as you are.

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