A characterization of admissible Łukasiewicz assessments through t-SMV-algebras.

M. Fedel[1] T. Flaminio[2]

[1]Department of Mathematics and Computer Science
Univeristy of Siena, Pian dei Mantellini 44, Siena 53100. Italy
email: fedel@unisi.it

[2]Artificial Intelligence Research Institute (IIIA - CSIC),
Campus UAB, Bellaterra 08193, Spain.
email: tommaso@iiia.csic.es

Abstract

In [2], it is studied a betting situation (called *non-reversible betting game*) where the bookmaker fixes his betting odds on some fuzzy events and bettor can bet on them, but it is not allowed to interchange the roles of the players. In that paper, events are identified with MV-terms, or equivalence classes of formulas of Łukasiewicz logic, and the following result is proved:

Theorem 1. *Let $\phi_1, \ldots, \phi_n$ be events, and let $\mathbf{a} : \{\phi_1, \ldots, \phi_n\} \to [0, 1]$ be a rational-valued assessment in a non-reversible betting game. Then the following are equivalent:*

(i) The assessment $\mathbf{a}$ does not admit a bad bet*, that is, a bet for which there is an alternative system of bets which guarantees a strictly better payoff, independently of the truth values of the events involved.*

(ii) There is a $t \leq n$, and a set of states $\{s_1, \ldots, s_t\}$ over the Lindenbaum algebra of Łukasiewicz logic generated by the propositional variables occurring in $\phi_1, \ldots, \phi_k$, such that, for every $i = 1, \ldots, n$,

$$\mathbf{a}(\phi_i) = \max\{s_j([\phi_i]) \mid j = 1, \ldots, t\}.$$

Let us call *admissible* a rational-valued Łukasiewicz assessment $\mathbf{a} : \{\phi_1, \ldots, \phi_n\} \to [0, 1]$ in a non-reversible betting game that avoids bad bets and let us denote by $\mathsf{Luk\text{-}Adm}$ the set of all admissible assessments.

In order to investigate this betting games in an algebraic framework, we introduce the variety of t-SMV-algebras. These structures are a generalization of SMV-algebras (cf. [3]) and allow us to treat and characterize admissible assessments in terms of satisfiability of suited defined equations in their language. In [1], the authors prove that the problem of deciding satisfiable equations in SMV-algebra is NP-complete. Here we use a similar technique to show that the problem to check satisfiability of equations in t-SMV-algebras is also NP-complete. As a consequence of this result, we obtain that Luk-Adm is NP-complete.

Definition 1. For every $t \geq 1$ a t-SMV-algebra is an algebra $\mathcal{A} = (A, \sigma_1, \ldots, \sigma_t)$ where A is an MV-algebra and, for every $i, k, h = 1, \ldots, t$, the following equations are satisfied:

(i) $\sigma_i(\perp) = \perp$

(ii) $\sigma_i(\neg x) = \neg\sigma_i(x)$

(iii) $\sigma_i(\sigma_k(x) \oplus \sigma_h(y)) = \sigma_k(x) \oplus \sigma_h(y)$

(iv) $\sigma_i(x \oplus y) = \sigma_i(x) \oplus \sigma_i(y \ominus (x \odot y))$

Theorem 2. *Let $\phi_1, \ldots, \phi_t$, be* t-SMV*-terms in k variables and let $\mathbf{a} : \phi_i \mapsto k_i/z_i$ (for $i = 1, \ldots, t$) be a rational-valued Łukasiewicz assessment. Then, the following are equivalent:*

(i) $\mathbf{a}$ *is admissible.*

(ii) There exists a t-SMV*-algebra $(A, \sigma_1, \ldots, \sigma_t)$ satisfying, for all $i = 1, \ldots, t$, the equations ε_i: $(z_i - 1)x_i = \neg x_i$ and δ_i: $k_i x_i = \left(\bigvee_{j=1}^{t} \sigma_j(\phi_i)\right)$, where the variables x_i's are fresh.*

In other words the admissibility of $\mathbf{a}$ is witnessed by the satisfiability, in the class of t-SMV-algebras,
of the set of equations

$$\Phi = \{\varepsilon_i, \delta_i : i = 1, \ldots, t\}. \tag{1}$$

Then we study the computational complexity for the problem of checking satisfiability of equations in t-SMV-algebras and we prove the following.

Theorem 3. *The problem of checking the satisfiability of an equation in a* t-SMV*-algebra is NP-complete.*

As a consequence, we have an algorithm for checking admissibility of an assessment. Given a rational-valued Łukasiewicz assessment $\mathbf{a} : \phi_i \mapsto k_i/z_i$ (with $i = 1, \ldots, t$), we do the following:

(i) For all $i = 1, \ldots, t$, define the terms ε_i and δ_i as in Theorem 2.

(ii) Define the set of equations Φ as in (1).

(iii) Check the satisfiability of Φ in the class of t-SMV-algebras.

Being the binary encoding of Φ polynomial in t, the above algorithm joint with Theorem 3 ensures that:

Lemma 1. Luk-Adm *is in NP.*

NP-hardness of Luk-Adm immediately follows from the NP-hardness of satisfiability of MV-equations. Therefore:

Theorem 4. *Deciding* Luk-Adm *is NP-complete.*

References

[1] S. Bova, T. Flaminio: The coherence of Łukasiewicz assessment is NP-complete. International Journal of Approximate Reasoning 51, 294–304, 2010.

[2] M. Fedel, K. Keimel, F. Montagna, W. Roth.: Imprecise probabilities, bets and functional analytic methods in Łukasiewicz logic. Submitted.

[3] T. Flaminio, F. Montagna: MV-algebras with Internal States and Probabilistic Fuzzy Logics. International Journal of Approximate Reasoning 50, 138–152, 2009.