# Towards an algebraic framework for many-valued conditional probability


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## Abstract


We investigate probability functions defined over many valued conditional events. During the last decade or so considerable research effort has been directed towards the understanding of what (subjective) conditional probability might look like in the context of many-valued logics. Given its centrality in the area, a particularly well-studied case is that of Łukasiewicz's infinite-valued logic $Ł_\infty$ (cf. [1, 5]). Among the initial attempts to define conditionals on such logics let us recall Gerla's [4] and Kroupa's [6, 7], whilst more recent characterizations are due to Mundici [12, 13] and Montagna [10].

A particularly hard problem in defining MV-conditional probability originates from the fact that MV-logics have no unique notion of conjunction. So, if we are to define conditional probability via the so-called rule of compound probabilities $P(\theta \mid \phi) = \frac{P(\theta \wedge \phi)}{P(\phi)}$ we face the problem of *justifying* one particular choice of conjunction. This is by no means trivial unless we are happy to settle with a purely conventional (i.e. arbitrary) choice. Yet conditionals do have an intuitive semantics and our work aims at providing a formalization of probabilistic conditionals which goes some way towards capturing those intuitions.

The main idea of our work consists in treating conditional probability as a simple (i.e. unconditional) probability on a conditional algebra. In particular we are interested in defining a notion of probability satisfying two sorts of intuitive constraints. The former concerns the semantics of the underlying logic which we assume is many-valued. This allows us to model the concept of *conditional many-valued event*. The latter sort of constraints we shall be looking at, concerns the properties of the probability function, which we take to represent an agent's subjective degree of belief in the spirit of de Finetti's work [2, 3]. This opens the problem of defining a suitable notion conditional bet. We propose the *rejection constraint* according to

which a bet is defined provided that the conditioning event gets a strictly positive truth-value.

We capture these constraints formally in what we call a *conditional MV-algebra* $\mathcal{F}(k) \mid \mathcal{F}(k)^+$. Our construction of $\mathcal{F}(k) \mid \mathcal{F}(k)^+$ is similar to Mundici's MV-algebraic tensorial product, and it works as follows: let $\mathcal{F}(k)$ be the $k$-free generated MV-algebra and let $\mathcal{F}(k)^+$ be $\mathcal{F}(k) \setminus \{\bot\}$ ($\bot$ being the bottom element of $\mathcal{F}(k)$). Then $\mathcal{F}(k) \mid \mathcal{F}(k)^+$ is defined as the quotient algebra $\mathcal{F}(\mathcal{F}(k) \mid \mathcal{F}(k)^+)/I$, where $\mathcal{F}(\mathcal{F}(k) \mid \mathcal{F}(k)^+)$ is the MV-algebra freely generated by the pairs $(a, b)$ in the Cartesian product $\mathcal{F}(k) \times \mathcal{F}(k)^+$, and $I$ is the MV-ideal of $\mathcal{F}(\mathcal{F}(k) \mid \mathcal{F}(k)^+)$, generated by those MV-terms ensuring that our constraints are satisfied. In fact, rather then an (internal) algebraic operation to be add to a free MV-algebra $\mathcal{F}(k)$, we retain that a conditional can be better described as an *external* operation between $\mathcal{F}(k)$, and $\mathcal{F}(k)^+$.

As a first significant result obtained with this framework we show that conditional probability on MV-algebras in the sense of Gerla [4] can be reduced to a special notion of (unconditional) probability (state) on our conditional MV-algebras.

We conclude by noting that the conditional algebra framework has important applications also in the classical (boolean) case. Taking conditional probability as unconditional probability over conditional algebras in fact, allows us to move the conditional operation outside the scope of the probability function. By doing this we can secure our framework against the unpleasant consequences of taking the conditioning operation as an object-level connective, chiefly Lewis's Triviality result [9].

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