On the complexity of de Finetti coherence of Łukasiewicz events

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Abstract

De Finetti foundation of probability theory relies on the *coherence* of betting odds as follows [2, 3, 4]: Let $\phi_1, \dots, \phi_k$ be classical events and let $\mathbf{a} : \{\phi_1, \dots, \phi_k\} \to [0, 1]$ be an assessment of $\phi_1, \dots, \phi_k$. Then $\mathbf{a}$ is said to be coherent if and only if there is no system of reversible bets on the events leading to a win independently on the truth of $\phi_1, \dots, \phi_k$. Precisely, the assessment $\mathbf{a}$ is coherent if and only if, for every $\mathbf{b} : \{\phi_1, \dots, \phi_k\} \to \mathbb{R}$, there exists a Boolean valuation $\mathbf{v}\colon \{\phi_1, \dots, \phi_k\} \to \{0, 1\}$ such that

$$\sum_{i=1}^{k} \mathbf{b}(\phi_i)(\mathbf{a}(\phi_i) - \mathbf{v}(\phi_i)) \geq 0. \tag{1}$$

The celebrated de Finetti theorem states that an assessment $\mathbf{a}$ is coherent if and only if $\mathbf{a}$ coincides with the restriction to $\{\phi_1, \dots, \phi_k\}$ of a finitely additive and normalized function P from the free Boolean algebra generated by the ϕ_i's to $[0, 1]$. In this case, we say that P is a probability measure *extending* $\mathbf{a}$, or that $\mathbf{a}$ *extends* to a probability measure P. The problem of checking whether or not a *rational* assessment $\mathbf{a}\colon \{\phi_1, \dots, \phi_k\} \to \mathbb{Q} \cap [0, 1]$ is coherent is NP-complete [11].

A natural generalization of de Finetti coherence criterion is obtained allowing an infinite-valued interpretation of events $\phi_1, \dots, \phi_k$, instead of their classical two-valued interpretation. A first attempt in this direction has been made by Paris, who firstly extended de Finetti theorem to deal with a generalization of the classical Boolean semantics of the events, namely the semantics of $(n + 1)$-valued Łukasiewicz logic [1]: An assessment $\mathbf{a}\colon \{\phi_1, \dots, \phi_k\} \to [0, 1]$ is coherent if and only if $\mathbf{a}$ extends to a *state* on the finite $(n + 1)$-valued MV-algebra over $\{0, 1/n, \dots, 1\}$ freely generated by the ϕ_i's, if and only if for every $\mathbf{b}\colon \{\phi_1, \dots, \phi_k\} \to \mathbb{R}$, there exists a valuation $\mathbf{v}\colon \{\phi_1, \dots, \phi_k\} \to \{0, 1/n, \dots, 1\}$ satisfying (1). As a straightforward consequence of [8, Theorem 1] and [5, Theorem 4.4.1], deciding the coherence of $\mathbf{a}$ above is an NP-complete problem. In light of Paris work,

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in [10] Mundici approaches the infinite-valued semantics for the events, showing that the coherence of an assessment $\mathbf{a}\colon \{\phi_1, \ldots, \phi_k\} \to [0,1]$ with respect to $[0,1]$-valued Łukasiewicz valuations is characterized by the existence of a state on the free MV-algebra generated by the ϕ's, extending $\mathbf{a}$. In recent work [9], Mundici and Kühr further extend this result to every $[0,1]$-valued algebraizable logic with continuous connectives.

In [10], Mundici shows that the coherence of rational Łukasiewicz assessments is decidable, and, as regards to the computational complexity of the problem, Hájek shows that the problem is in PSPACE [8]. We settle the computational complexity issue, showing that the problem is NP-complete.

Theorem 0.1. *The set* $\mathsf{COH\text{-}LUK\text{-}ASS} = \{\langle \mathbf{a} \rangle \mid \mathbf{a}$ *is a coherent rational Łukasiewicz assessment*$\}$ *is NP-complete.*

In light of this, we obtain NP-completeness results for the satisfiability problem of several classes of formulas of probabilistic logics introduced in [6, 7], settling a problem raised by [8, 7].

References

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