On conjectures in t-norm based fuzzy logics

Francesc Esteva[1], **Itziar García-Honrado**[2], **Lluís Godo**[1]

[1] *Artificial Intelligence Research Institute (IIIA) - CSIC*
Campus UAB s/n 08193 Bellaterra, Spain, {esteva,godo}@iiia.csic.es
[2] *Department of Statistics, Operational Research and Mathematical Education. University of Oviedo,* Spain
Affiliated researcher at European Centre for Soft Computing, garciaitziar@uniovi.es

Abstract

In this paper we study elements of models of ordinary reasoning like conjectures, hypothesis and speculations in the framework of t-norm based fuzzy logics that arise from two natural families of consequence operators definable in these logics, with special emphasis in three particular logics: Gödel, Product and Łukasiewicz logics.

Keywords: CHC models, consequence operators, t-norm based fuzzy logics, conjectures

1 Introduction

The paper deals with models of ordinary reasoning defined by Trillas et al. [2, 7, 4] through the definition of the set of conjectures, and its partition into consequences, hypotheses and speculations. Usually, these models are defined from a prefixed algebraic structure, for instance there are many publications regarding orthomodular lattices [2], but more recent papers deal with very general mathematical structure called Basic Flexible Algebras (BFA), that have as particular cases Boolean algebras, ortolattices, orthomodular lattices, De Morgan algebras and standard algebras of fuzzy sets.

Originally, in earlier papers the different models of ordinary reasoning, also called CHC models (from "Consequences, Hypotheses and Conjectures" [6]) were directly defined from a Tarski consequence operator, independently from the order of the underlying lattice. However, in [8] the CHC model is extended to residuated lattices where the notion of consequence is already based on the implication operation, which is a graded preorder, and in [7] the models are generalized to the setting of to pre-ordered sets.

In this paper we study the sets of conjectures in relation to two natural different consequence operators in the context of t-norm-based fuzzy logics [5, 3], namely the ones associated to the truth-preserving and degree-preserving notions of logical consequence [1]. We show they verify similar the properties of those described in [4] and it can be divided into consequences, hypotheses and speculations as well.

The paper is organized as follows. After some preliminaries on CHC models and consequence operators in the setting of t-norm-based fuzzy logics, we study consistency for sets of formulas relative to every consequence operator. Finally we study general the sets of conjectures, consequences, hypothesis and speculations for the main logics of a continuous t-norms, i.e., for Łukasiewicz, Gödel and Product logics.

2 Preliminaries

2.1 Preliminaries on CHC models

CHC models tries to collect the main properties of some of the basic types of ordinary reasoning: deduction, induction, abduction and speculative reasoning. These types of reasoning are represented, respectively, by the sets of consequences, conjectures, hypotheses and speculations. And all of them can be defined from a consequence operator in the sense of Tarski.

Any reasoning process starts from a body of information or, in logical terms, from a set of premises. If this set is finite, to obtain their consequences is usually reduced to look for the consequences of the conjunction (or meet in algebraic terms) of all the premises. This is done for instance in [7], in the setting of preordered sets, with the consequence operator $C_\wedge$ for which the consequences are the elements greater or equal (with respect to a given preorder) from the intersection of premises.

Notwithstanding, there are different ways of defining a consequence operator, even in that general setting. For instance in [7], the authors also consider the operator

$C_\cup$ for which the consequences of a set of premises is taken as the union of the consequences of each premise.

Once the set of consequences is defined, one can try to characterize the information that is not inconsistent with the set of premises P, where consistency refers to the impossibility of deducing the negation from an already deduced element (i.e. if $q \in C(P)$, then *not-q* $\notin C(P)$), assuming a negation *not* is available in the framework. Then the set of conjectures for a set of premises P are those elements consistent with P.

On the other hand, the hypotheses for a set of premises P will be those elements which allow to deduce every premise in P. Finally, the set of speculations for P is defined as the set of those conjectures that are neither consequences nor hypotheses. In this way, one obtains a partition of the set of conjectures in terms of consequences, hypotheses and speculations.

2.2 Preliminaries on t-norm based logics

(Continuous) T-norm-based fuzzy logics[1] are a family of logics whose language $\mathcal{L}$ is built from a countable set of propositional variables using different connectives $\&, \wedge, \vee, \rightarrow, \neg$ and truth constants $\bar{0}, \bar{1}$ for truth and falsity. Semantically, they correspond to logical calculi with the real interval $[0,1]$ as set of truth-values and taking the conjunction $\&$, the implication $\rightarrow$ and the truth-constant $\bar{0}$ as primitive. Further connectives are definable as: $\varphi \wedge \psi = \varphi\&(\varphi \rightarrow \psi)$, $\varphi \vee \psi = ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, $\neg\varphi = \varphi \rightarrow \bar{0}$ and $\bar{1} = \neg\bar{0}$.

In this framework, each continuous t-norm $*$ uniquely determines a logic $L_\star$ as a propositional calculus over formulas interpreting the conjunction $\&$ by the t-norm $\star$, the implication $\rightarrow$ by its residuum $\Rightarrow$ and the truth-constant $\bar{0}$ by the value 0. More precisely, evaluations of propositional variables are mappings e assigning to each propositional variable p a truth-value $e(p) \in [0,1]$, which extend univocally to compound formulas as follows:

$$\begin{aligned} e(\bar{0}) &= 0 \\ e(\varphi\&\psi) &= e(\varphi) \star e(\psi) \\ e(\varphi \rightarrow \psi) &= e(\varphi) \Rightarrow e(\psi) \end{aligned}$$

From these defintions, it follows that $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$, $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$, $e(\neg\varphi) = e(\varphi) \Rightarrow 0$ and $e(\varphi \equiv \psi) = e(\varphi \rightarrow \psi) * e(\psi \rightarrow \varphi)$.

For each logic $L_\star$, we consider two kinds of finitary consequence relations, $\models$ and $\models^{\leq}$, defined as follows, where $\Gamma \cup \{\varphi\}$ is a finite set of formulas from $\mathcal{L}$:

[1] In fact the framework also cover logics defined by left-continuous t-norms but in this work we only deal with the case of continuous t-norms.

- $\Gamma \models \varphi$ if $e(\varphi) = 1$ for every evaluation $e : \mathcal{L} \rightarrow [0,1]_*$ such that $e(\psi) = 1$ for every $\psi \in \Gamma$.
- $\Gamma \models^{\leq} \varphi$ if $\min\{e(\psi) \mid \psi \in \Gamma\} \leq e(\varphi)$ for every evaluation (morphism) $e : \mathcal{L} \rightarrow [0,1]_*$.

The consequence relation $\models$ is usually called "1-preserving" while $\models^{\leq}$ is called "degree-preserving", for obvious reasons. Observe that $\{\psi_1, \ldots, \psi_n\} \models^{\leq} \varphi$ iff $\models (\psi_1 \wedge \ldots \wedge \psi_n) \rightarrow \varphi$, so that deductions from premises with $\models^{\leq}$ can be translated to deductions of theorems with $\models$.

Well-known axiomatic systems, like Łukasiewicz logic (Ł), Gödel logic (G) or Product logic (Π), syntactically capture the "1-preserving" consequence relation for $L_\star$ when $\star$ is Łukasiewicz, min or product t-norm respectively [5]. It is worth noticing that Gödel logic is the only t-norm based fuzzy logic such that $\models$ coincides with $\models^{\wedge}$.

3 Consequence operators on $L_\star$ logics

Given a logic $L_\star$, we consider the consequence operators associated to logical consequences $\models$ and $\models^{\leq}$:

- $C(\Gamma) = \{\varphi \mid \Gamma \models \varphi\}$, and
- $C^{\leq}(\Gamma) = \{\varphi \mid \Gamma \models^{\leq} \varphi\}$.

For each consequence operator C and $C^{\leq}$ we also consider the consequence operators $C_\cup$ and $C_\cup^{\leq}$ (used in Trillas et al.'s paper [4]) defined by

- $C_\cup(\Gamma) = \{\varphi \mid \gamma \models \varphi$ for some $\gamma \in \Gamma\}$, and
- $C_\cup^{\leq}(\Gamma) = \{\varphi \mid \gamma \models^{\leq} \varphi$ for some $\gamma \in \Gamma\}$.

Since $\models$ is a stronger notion of consequence than $\models^{\leq}$, we have the following chains of inclusions among these operators:

$$C_\cup^{\leq} \subseteq C^{\leq} \subseteq C \subseteq \mathbb{CL} \quad (1)$$

$$C_\cup^{\leq} \subseteq C_\cup \subseteq C \subseteq \mathbb{CL} \quad (2)$$

where $\mathbb{CL}$ denotes the consequence operator of classical propositional logic (CL) in the language $\mathcal{L}$ where we identify the connectives $\&$ and $\wedge$. In the particular case of Gödel logic ($\star = \min$), since $\models$ coincides with $\models^{\leq}$, it turns out that $C = C^{\leq}$ and $C_\cup = C_\cup^{\leq}$.

Observe that, unlike C, the operators $C_\cup^{\leq}$ and $C_\cup$ are not closed by modus ponens, which makes the associated notion of inference quite weak. Actually $C^{\leq}$ is neither closed by modus ponens but its closed by a restricted version of modus ponens: if $\varphi \rightarrow \psi$ is a theorem, from φ derive ψ.

All these operators are consequence operators in the sense of Tarski, that is, any $C^* \in \{C, C^{\leq}, C_{\cup}, C_{\cup}^{\leq}\}$ verifies:

- $\forall\Gamma,\ \Gamma \subseteq C^*(\Gamma)$.
- $\forall\Gamma_1, \Gamma_2$, if $\Gamma_1 \subseteq \Gamma_2$ then $C^*(\Gamma_1) \subseteq C^*(\Gamma_2)$.
- $\forall\Gamma,\ C^*(C^*(\Gamma)) = C^*(\Gamma)$.

The following lemmas highlight several properties of the above defined consequence operators. In what follows, given a finite set of formulas Γ, we will write $\Gamma^\wedge$ for $\bigwedge\{\psi \mid \psi \in \Gamma\}$. Moreover, to simplify notation we will also write $C^*(\varphi)$ for $C^*(\{\varphi\})$ for any $C^* \in \{C, C^{\leq}, C_{\cup}, C_{\cup}^{\leq}\}$.

Lemma 1 *For $C^* \in \{C, C^{\leq}\}$, it holds that $C^*(\Gamma) = C^*(\Gamma^\wedge)$.*

Lemma 2 *C and $C^{\leq}$ are closed by the weak conjunction $\wedge$: if $\varphi \in C^{\leq}(\Gamma)$ and $\psi \in C^{\leq}(\Gamma)$ then $\varphi \wedge \psi \in C^{\leq}(\Gamma)$.*

Lemma 3 *C is closed by the strong conjunction &: if $\varphi \in C(\Gamma)$ and $\psi \in C(\Gamma)$ then $\varphi \& \psi \in C(\Gamma)$.*

These results do not hold in general for the operators $C_{\cup}$ and $C_{\cup}^{\leq}$, while Lemma 3 does not even hold for $C^{\leq}$.

On the other hand, by definition, it is clear that the $C_{\cup}$ and $C_{\cup}^{\leq}$ operators satisfy the following property.

Lemma 4 *Let $C^* \in \{C_{\cup}, C_{\cup}^{\leq}\}$. Then $C^*(\Gamma) = \bigvee_{\varphi\in\Gamma} C^*(\varphi)$.*

It is known that in the family of logics $L_\star$, C and $C^{\leq}$ (and also $C_{\cup}$ and $C_{\cup}^{\leq}$) coincide if and only if $\star$ is the min t-norm. For Product and Łukasiewicz logics it is easy to compute that $\varphi\&\varphi$ belongs to $C(\varphi)$ and does not belong to $C^{\leq}(\varphi)$, see Lemma 3.

4 Notions of consistency relative to the consequence operators C, $C^{\leq}$, $C_{\cup}$ and $C_{\cup}^{\leq}$

In this section we study the notion of consistency with respect to our four consequence operators C, $C^{\leq}$, $C_{\cup}$ and $C_{\cup}^{\leq}$ within the framework of $L_\star$ logics.

In the setting of the consequence relation $\models$, in "mathematical fuzzy logic" [3] it is customary to define a set of premises Γ to be consistent whenever $\Gamma \not\models \bar{0}$. However this notion of consistency does not make much sense in the setting of the consequence relation $\models^{\leq}$, since $\models^{\leq}$ is paraconsistent, i.e. it is not always the case that $\{\varphi, \neg\varphi\} \models^{\leq} \bar{0}$. Because of that we will adopt the following general definition.

Definition 5 *Let $C^* \in \{C, C^{\leq}, C_{\cup}, C_{\cup}^{\leq}\}$. We say that a set of premises Γ is C^*-consistent whenever the following condition holds: if $\varphi \in C^*(\Gamma)$ then $\neg\varphi \notin C^*(\Gamma)$.*

Notice that if Γ is C^*-consistent, then so is $C^*(\Gamma)$.

Next we provide equivalent conditions for different particular cases of consequence operators and particular choices of the t-norm $\star$ in the logic $L_\star$.

4.1 The case of C and $C^{\leq}$ operators

Lemma 6 *For any logic $L_\star$, the following conditions are equivalent:*

- *Γ is C-consistent*
- *$\Gamma \not\models \bar{0}$ (i.e. there exists an evaluation e such that $e(\psi) = 1$ for all $\psi \in \Gamma$)*
- *$\Gamma^\wedge \not\models \bar{0}$ (i.e. there exists an evaluation e such that $e(\Gamma^\wedge) = 1$)*

The proof is an immediate consequence of Lemma 3 since if Γ is C-inconsistent then $\Gamma \models \varphi\&\neg\varphi$, and $\varphi\&\neg\varphi$ is equivalent to $\bar{0}$.

Lemma 7 *For any logic $L_\star$, the following conditions are equivalent:*

- *Γ is $C^{\leq}$-consistent*
- *For all formula φ, if $\models \Gamma^\wedge \to \varphi$ then $\not\models \Gamma^\wedge \to \neg\varphi$.*

This follows directly from the definition of $\models^{\leq}$ in terms of $\models$.

Next we show some specific conditions for different choices of the t-norm $\star$ in the logic $L_\star$, namely $\star = \min$ (Gödel logic), $\star =$ product t-norm (Product logic) and $\star =$ Łukasiewicz t-norm (Łukasiewicz logic).

Lemma 8 *For $L_\star$ being Gödel or Product logic, C-consistency and $C^{\leq}$-consistency coincide.*

Proof: The case of Gödel logic has already been mentioned. For Product logic it is true by using the following equivalences: Γ is $C^{\leq}$-consistent iff $\Gamma \not\models^{\leq} 0$ iff there is an evaluation e such that $e(\Gamma^\wedge) > 0$ iff there is an evaluation e' ($e'(p) = e(\neg\neg p)$ for each propositional variable p) [2] such that $e'(\Gamma^\wedge) = 1$ iff $\Gamma \not\models 0$ iff Γ is C-consistent ⊣

[2]Recall that in Gödel or Product algebras $\neg\neg x = 1$ if $x > 0$ and $\neg\neg 0 = 0$.

Lemma 9 *For $L_\star$ being Gödel or Product logic, let $C^* \in \{C, C^{\leq}\}$. Then the following conditions are equivalent:*

(i) Γ *is* C^**-consistent*

(ii) there exists an $\{0,1\}$*-evaluation* e *s.t.* $e(\Gamma^\wedge) = 1$

Proof: The proof is easy taking into account that the evaluation e' defined in the proof of previous lemma is crisp ($e'(\varphi) \in \{0,1\}$ for al formula φ). ⊣

This previous lemma amounts to say that, in the case of Gödel or Product logic, Γ is C^*-consistent if and only if Γ is *classically* consistent (identifying the weak and strong conjunctions).

However, in Łukasiewicz logic C-consistency is not equivalent to $C^{\leq}$-consistency and condition (ii) is not satisfied neither for C nor for $C^{\leq}$.

Lemma 10 *For $L_\star$ being Łukasiewicz logic, the following conditions are equivalent:*

- Γ *is* $C^{\leq}$*-consistent*
- *there exists an* $L_\star$*-evaluation* e *s.t.* $e(\Gamma^\wedge) > 1/2$

Proof: Assume there exists an evaluation e such that $e(\Gamma^\wedge) > 1/2$, then if $\Gamma \models^{\leq} \varphi$, necessarily $e(\varphi) \geq e(\Gamma^{\leq}) > 1/2$, and thus $e(\neg\varphi) = 1 - e(\varphi) < 1/2$ and $\Gamma \not\models^{\leq} \neg\varphi$.

Conversely, assume $e(\Gamma^\wedge) \leq 1/2$ for any evaluation e. Then, we would have both $\Gamma \models^{\leq} \Gamma^\wedge$ (by definition of $\models^{\leq}$), but also $\Gamma \models^{\leq} \neg\Gamma^\wedge$ (by hypothesis), and hence Γ would be $C^{\leq}$-inconsistent. ⊣

To see that C-consistency is not equivalent to $C^{\leq}$-consistency take for example $\Gamma = \{p, \neg q, p \to q\}$, then Γ is obviously C-inconsistent (there is not e such that $e(p \wedge \neg q \wedge (p \to q) = 1)$ while it is $C^{\leq}$-consistent (take the evaluation e such that $e(p) = 0.6, e(q) = 0.4$. Then $e(p \wedge \neg q \wedge (p \to q) = 0.6 > 0.5)$.

4.2 The case of $C_\cup$ and $C_\cup^{\leq}$ operators

Lemma 11 *For any logic $L_\star$, the following are equivalent:*

- Γ *is* $C_\cup$*-inconsistent*
- *there exists* $\psi, \chi \in \Gamma$*, such that* $\psi, \chi \models \overline{0}$
- *there exists* $\psi, \chi \in \Gamma$ *and* $n \in \mathbb{N}$ *such that* $\neg(\chi^n) \in C^\cup(\psi)$*, where* $\chi^n = \chi \& \overset{n}{\ldots} \& \chi$*; or equivalently,* $\psi \models \neg(\chi^n)$*.*
- *there exists* $\psi, \chi \in \Gamma$ *and* $n \in \mathbb{N}$ *such that for every evaluation* e*, if* $e(\psi) = 1$ *then* $e(\chi) \star \overset{n}{\ldots} \star e(\chi) = 0$*.*

Proof: If Γ is $C_\cup$-inconsistent, there exist $\chi, \psi \in \Gamma$ such that $\chi \models \varphi$ and $\psi \models \neg\varphi$, then $\chi, \psi \models \overline{0}$. Moreover, this is is equivalent in turn to the fact of the existence of $n \in \mathbb{N}$ such that $\psi \models \neg\chi^n$.[3]

Reciprocally, if $\psi, \chi \in \Gamma$ are such that there exists $n \in \mathbb{N}$ verifying $\psi \models \neg\chi^n$, then $\chi \models \chi^n$ and $\psi \models \neg\chi^n$. Therefore, Γ is $C_\cup$-inconsistent. ⊣

In the case of Lukasiewicz logic, it can be shown the possibility of having $\chi, \psi \in \Gamma$, such that $\chi, \psi \models \overline{0}$, with $\psi \not\models \neg\chi$ and $\chi \not\models \neg\psi$. For instance, it is enough to take $\chi = p\&(q \equiv p \otimes p)$ and $\psi = r\&(\neg q \equiv r \otimes r)$. Therefore, for Lukasiewicz logic the previous lemma does not hold with $n = 1$ in the third and fourth items. However, this holds true for Gödel and Product logics.

Lemma 12 *For $L_\star$ being Gödel or Product logic, let $C^* \in \{C_\cup, C_\cup^{\leq}\}$. Then the following conditions are equivalent:*

- Γ *is* C^**-consistent*
- *for any* $\psi, \chi \in \Gamma$*,* $\chi \not\models \neg\psi$*.*
- *for any* $\psi, \chi \in \Gamma$*, there exists an evaluation* e *such that* $e(\psi \wedge \chi) > 0$*.*
- *for any* $\psi, \chi \in \Gamma$*, there exists an evaluation* e *such that* $e(\psi \wedge \chi) = 1$

Proof: We prove the case $C^* = C_\cup$, the case of $C^* = C_\cup^{\leq}$ being similar. If Γ is $C_\cup$-inconsistent, there exist $\psi, \chi \in \Gamma$ such that $\psi \models \varphi$ and $\chi \models \neg\varphi$ for some φ. Let us prove that $\chi \models \neg\psi$. The hypothesis implies that for all evaluation e, if $e(\psi) = 1$ then $e(\varphi) = 1$, and if $e(\chi) = 1$, then $e(\varphi) = 0$. Therefore, $e(\chi) = 1$ implies $e(\psi) < 1$. Then we have two cases: either $e(\psi) = 0$ and we are done, or $e(\psi) > 0$. In the latter case, define a new evaluation e' by putting $e'(p) = e(\neg\neg p)$ for all propositional variable p. It can be checked (see e.g. [5]) that $e'(\varphi) = e(\neg\neg\varphi)$ for any formula φ, and hence[4] $e'(\chi) = e'(\varphi) = e'(\psi) = 1$, contradicting the hypothesis that $\chi \models \neg\varphi$.

Reciprocally, assume that for all evaluation e such that $e(\chi) = 1$, we have $e(\neg\psi) = 1$. Therefore, Γ is $C_\cup$-inconsistent, since $\psi \models \psi$ and $\chi \models \neg\psi$.

The last item is an easy consequence of previous one. ⊣

[3] Here we are using the local deduction theorem that is valid for all t-norm based logics, namely $\Gamma \cup \{\varphi\} \models \psi$ iff there exists $n \in \mathbb{N}$ such that $\Gamma \models \varphi \& \overset{n}{\ldots} \& \varphi \to \psi$ (see e.g. [5, 3]).

[4] Recall that in Gödel or Product algebras $\neg\neg x = 1$ if $x > 0$ and $\neg\neg 0 = 0$.

In fact this lemma simply says that in Gödel and Product logics, a set of formulas is consistent with respect to the consequence operators $C_\cup$ and $C^{\leq}_{\cup}$ iff they are pairwise consistent in the usual sense of the operators C and $C^{\leq}$ respectively, which is very natural according to the definition of the consequence operators $C_\cup$ and $C^{\leq}_{\cup}$.

Lemma 13 *For $L_\star$ being Lukasiewicz logic, Γ is $C^{\leq}_{\cup}$-consistent*

- *iff for all $\psi, \chi \in \Gamma$, $\neg\chi \notin C^{\leq}_{\cup}(\psi)$*
- *for all $\psi, \chi \in \Gamma$ there exists an evaluation e such that $e(\psi) > e(\neg\chi)$, i.e. $e(\psi \& \chi) > 0$.*

Proof: If Γ is $C^{\leq}_{\cup}$-consistent, there exist $\psi, \chi \in \Gamma$ and φ such that, for all evaluation e, $e(\psi) \leq e(\varphi)$, and $e(\chi) \leq e(\neg\chi)$. Therefore, $e(\psi) \leq e(\varphi) \leq e(\neg\chi)$.
Reciprocally, if there exist $\psi, \chi \in \Gamma$ such that $e(\psi) \leq e(\neg\chi)$, then Γ is inconsistent since obviously $e(\chi) \leq e(\chi)$. ⊣

5 Hypothesis and conjectures

In this section we study how the notions of conjecture, hypothesis and speculation can be characterized under the different notions of consequence operators and logics we have considered. We start by recalling the usual definitions adapted to our framework.

Definition 14 *Let Γ be a set of premises. We respectively define the set of hypotheses, conjectures and speculations of Γ wrt $C^* \in \{C, C^{\leq}, C_\cup, C^{\leq}_{\cup}\}$ as follows:*

- $Conj_{C^*}(\Gamma) = \{\varphi \mid \Gamma \cup \{\varphi\}$ *is C^*-consistent*$\}$
- $Hyp^{+}_{C^*}(\Gamma) = \{\varphi \mid \varphi$ *is C^*-consistent and* $\Gamma \subseteq C^*(\varphi)\}$

  $Hyp_{C^*}(\Gamma) = Hyp^{+}_{C^*}(\Gamma) \setminus C^*(\Gamma)$
- $Spec_{C^*}(\Gamma) = Conj_{C^*}(\Gamma) \setminus (C^*(\Gamma) \cup Hyp_{C^*}(\Gamma))$.

From this definition, it readily follows the next inclusions for any set of formulas Γ:

$$C^*(\Gamma) \cup Hyp_{C^*}(\Gamma) \subseteq Conj_{C^*}(\Gamma).$$

The following general properties also trivially hold:

- $Conj_{C^*}(\Gamma)$ may be not C^*- consistent
- $Conj_{C^*s}(\Gamma) = \bigcup\{T \mid \Gamma \subseteq T, T$ is maximally C^*-consistent$\}$
- If $\Gamma_1 \subseteq \Gamma_2$, then $Conj_{C^*}(\Gamma_2) \subseteq Conj_{C^*}(\Gamma_1)$.
- $(Conj_{C^*}, \vee, 1)$ is a $\vee$-semilattice.
- If $\Gamma_1 \subset \Gamma_2$, then $Hyp_{C^*}(\Gamma_2) \subset Hyp_{C^*}(\Gamma_1)$
- If $C_1 \subseteq C_2$ then $Conj_{C_2}(\Gamma) \subseteq Conj_{C_1}(\Gamma)$
- If $C_1 \subseteq C_2$ then $Hyp_{C_1}(\Gamma) \subseteq Hyp_{C_2}(\Gamma)$
- If $C_1 \subseteq C_2$ then $Spec_{C_2}(\Gamma) \subseteq Spec_{C_1}(\Gamma)$

From the chains of inclusions (1) and (2), and the last three items we have corresponding chains of inclusion for Conjectures, Hypothesis and Speculations with respect to the of the consequence operators $C, C^{\leq}, C_\cup$ and $C^{\leq}_{\cup}$.

5.1 The case of C and $C^{\leq}$ operators

From Lemma 1 it follows that, for $C^* \in \{C, C^{\leq}\}$, $Conj_{C^*}(\Gamma) = Conj_{C^*}(\Gamma^\wedge)$ and $Hyp_{C^*}(\Gamma) = Hyp_{C^*}(\Gamma^\wedge)$.

Moreover for $L_\star$ being Gödel or Product logics the notion of consistency for C and $C^{\leq}$ coincide and thus the following results hold.

Lemma 15 *Let $C^* \in \{C, C^{\leq}\}$ and let $L_\star$ be Gödel or Product logics. Then*

- *$\varphi \in Conj_{C^*}(\Gamma)$ iff there exists an evaluation e such that $e(\Gamma^\wedge \wedge \varphi) = 1$.*
- *$Conj_{C^*}(\Gamma) = \alpha^{[-1]}(Conj_{\mathbb{CL}}(\alpha(\Gamma)))$*

where α is the mapping formulas of the logic $L_\star$ to formulas of classical logic obtained by identifying the weak and strong conjunctions.

Obviously α is the identity in Gödel Logic. Therefore, for this logic, the set $Conc_{C^*}(\Gamma)$ is the same than in classical logic.

Lemma 16 *Let $C^* \in \{C, C^{\leq}\}$ and let $L_\star$ be Gödel or Product logics. Then $Conj_{C^*}(\Gamma) = \mathcal{L} \setminus Hyp^{+}_{C^*}(\neg\Gamma^\wedge)$.*

Proof: A formula φ does not belong to $Conj_{C^*}(\Gamma)$ iff $\{\Gamma^\wedge, \varphi\} \models \bar{0}$ iff there is a natural n such that $\varphi \models (\Gamma^\wedge)^n \to \bar{0}$, i.e, $\varphi \models \neg((\Gamma^\wedge)^n)$ iff $\varphi \models \neg(\Gamma^\wedge)$ iff $\varphi \in Hyp^{+}_{C^*}(\neg(\Gamma^\wedge))$. ⊣

For $L_\star$ being Łukasiewicz Logic we need to distinguish the cases of consequence operator C and $C^{\leq}$ since their notion of consistency are different. In that logic we have.

Lemma 17 *Let $L_\star$ be Łukasiewicz logic. The following conditions hold:*

- $\chi \in Conj_C(\Gamma)$ *iff there exists an evaluation* e *such that* $e(\Gamma^\wedge \wedge \chi) = 1$
- $\chi \in Conj_{C^\leq}(\Gamma)$ *iff there exists an evaluation* e *such that* $e(\Gamma^\wedge \wedge \chi) > \frac{1}{2}$.

5.2 The case of $C_\cup$ and $C_\cup^\leq$ operators

From the notion of $C_\cup$-consistency, it follows that the set of conjectures of a set formulas Γ is $Conj_{C_\cup}(\Gamma) = \{\varphi \mid \forall\psi \in \Gamma\ \varphi, \psi \not\models \overline{0}\} = \{\varphi \mid \forall\psi \in \Gamma,\ \forall n,\ \varphi \not\models \neg\psi^n\}$.

In the case of Gödel or Product logic it is enough to take $n = 1$, and then the above expression could be simplified to

$$Conj_{C_\cup}(\Gamma) = \{\varphi \mid \forall\psi \in \Gamma, \varphi \not\models \neg\psi\}. \qquad (3)$$

It turns out that this condition is also true for $C_\cup^\leq$ since, for Gödel and Product logics, $C_\cup^\leq$-consistency coincides with $C_\cup$-consistency. However, (3) is not valid for Łukasiewicz logic as we have shown in Section 4.2.

The results of Lemmas 15 and 16 translate into the next characterizations, where we denote by $C_{CPC}^\cup$ the following consequence operator related to classical logic: $\mathbb{CL}_\cup(\Gamma) = \{\psi \mid \psi \in \mathbb{CL}(\varphi)$ for some $\varphi \in \Gamma\}$.

Lemma 18 *Let* $C^* \in \{C_\cup, C_\cup^\leq\}$ *and let* $L_\star$ *be Gödel or Product logics. Then,*

- $Conj_{C^*}(\Gamma) = \alpha^{[-1]}(\mathbb{CL}_\cup(\alpha(\Gamma)))$
- $Conj_{C^*}(\Gamma) = \mathcal{L} \setminus \bigcup_{\psi\in\Gamma} Hyp^+_{C^*}(\neg\psi)$.

Proof: The first item basically follows by the same reasoning used in the first property in Lemma 15. As for the second one, we have the following equivalences: $\varphi \notin Conj_{C^*}(\Gamma)$ iff there exists $\psi \in \Gamma$ such that $\varphi \models \neg\psi$ iff $\varphi \in \bigcup_{\psi\in\Gamma} Hyp^+_{C^*}(\neg\psi)$. ⊣

Finally we present a summary of results organized by logics: For Gödel logic it holds that, for any Γ:

(i) $C^\leq(\Gamma) = C(\Gamma) \subsetneq \mathbb{CL}(\Gamma)$
(ii) $Conj_{C^\leq} = Conj_C = Conj_{\mathbb{CL}}$
(iii) $Hyp_{C^\leq} = Hyp_C \subsetneq Hyp_{\mathbb{CL}}$
(iv) The same inequalities hold for the $C_\cup$ and $C_\cup^\leq$ operators and the corresponding classical ones,

while for Product logic it holds that:

(i') $C^\leq(\Gamma) \subsetneq C(\Gamma) \subsetneq \alpha^{-1}(\mathbb{CL}(\alpha[\Gamma]))$
(ii') $Conj_{C^\leq} = Conj_C = \alpha^{-1}(Conj_{\mathbb{CL}}(\alpha()))$
(iii') $Hyp_{C^\leq} \subsetneq Hyp_C \subsetneq Hyp_{\mathbb{CL}}$
(iv') The same inequalities hold for the $C_\cup$ and $C_\cup^\leq$ operators and the corresponding classical ones.

6 Concluding remarks

Regarding the notion of consistency, in this paper we have obtained some results very similar to those of [7] in two aspects. First, some spaces for sets of premises are defined in order to preserve the consistency of a Tarski consequence operator:

- The space $\{\triangle;\ \triangle^\wedge \not\models \overline{0}\}$, where a set of premises Γ belongs to in case there exists an evaluation e such that $e(\Gamma^\wedge) = 1$ or that $e(\Gamma^\wedge) > 0$. (Product and Gödel logics.)
- The space $\{\triangle;\ \triangle^\wedge \not\models^\leq \neg\triangle^\wedge\}$, where a set of premises Γ belongs to just in case there exists an evaluation e such that $e(\Gamma^\wedge) > 1/2$. (Łukasiewicz logic.)

Second, the sets of conjectures are defined based on the notion of consistency, since conjectures are those elements that are "not inconsistent" with the body of information.

As for future research, we plan to get a deeper insight into CHC models for t-norm based logics, with more general characterizations and new applications.

Acknowledgments This work has been partially supported by the Spanish projects TIN2012-39348-C02-01 (Esteva and Godo) and TIN2011-29827-C02-01 (García-Honrado).

References

[1] F. Bou, F. Esteva,J.M. Font, A. Gil, L. Godo, A. Torrens, and V. Verdú. Logics preserving degrees of truth from varieties of residuated lattices. *Journal of Logic and Computation*, 19(6):1031–1069, 2009.

[2] E. Castiñeira, E. Trillas, S. Cubillo. On conjectures in orthocomplemented lattices. *Artificial Intelligence*, 117:255–275, 2000.

[3] P. Cintula, P. Hájek, and C. Noguera, editors. *Handbook of Mathematical Fuzzy Logic (in 2 volumes)*, vols. 37 and 38 of *Studies in Logic, Mathematical Logic and Foundations*. College Publications, London, 2011.

[4] I. García-Honrado and E. Trillas. On an attempt to formalize guessing. In *Soft Computing in Humanities and Social Sciences* (Eds. R. Seising and V. Sanz) Springer-Verlang, 273:237–255, 2012.

[5] P. Hájek. *Metamathematics of Fuzzy Logic*, Trends in Logic, vol. 4, Kluwer, Dordercht, 1998.

[6] D. Qiu. A note on Trillas' CHC models. *Artificial Intelligence* 171 (2007) 239-254.

[7] E. Trillas, I. García-Honrado and A. Pradera. Consequences and conjectures in preordered sets. *Information Sciences*, 180(19): 3573–3588, 2010.

[8] M. Ying, H. Wang. Lattice-theoretic models of conjectures, hypotheses and consequences. *Artificial Intelligence* 139 (2002) 253-267.