Hilbert-style calculi for $\vdash_{\mathbf{BPL}}$ and $\vdash_{\mathbf{FPL}}$

Félix Bou*

Abstract

A. Visser, in [Vis81], introduces the consequence relations $\vdash_{\mathbf{BPL}}$ and $\vdash_{\mathbf{FPL}}$. The latter is obtained by interpreting implication as a formal provability. He presents natural deduction calculi for both. Since then, some Gentzen-style calculi have also been given, but no Hilbert-style calculi. We present Hilbert-style calculi for both consequence relations.

Let us take the intuitionistic language $\mathcal{L}_{int} = \langle \wedge, \vee, \rightarrow, \bot \rangle$ and the modal language $\mathcal{ML} = \langle \wedge, \vee, \bot, \supset, \Box \rangle$[a]. In the language $\mathcal{L}_{int}$ we define the following three connectives: i) $\top = \bot \rightarrow \bot$, ii) $\neg\varphi = \varphi \rightarrow \bot$, iii) $\Box\varphi = \top \rightarrow \varphi$[b]. We define the map $\tau : Fm\mathcal{L}_{int} \longrightarrow Fm\mathcal{ML}$ as

$$\text{i) } \tau(p_i) = \Box p_i, \qquad \text{ii) } \tau(\bot) = \Box\bot, \qquad \text{iii) } \tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi),$$
$$\text{iv) } \tau(\varphi \vee \psi) = \tau(\varphi) \vee \tau(\psi), \qquad \text{v) } \tau(\varphi \rightarrow \psi) = \Box(\tau(\varphi) \supset \tau(\psi)).$$

It is well known that τ is an embedding of

- the intuitionistic propositional logic $\vdash_{\mathbf{IPL}}$ into $\vdash_{\mathbf{S4}}$[c], i.e., $\Gamma \vdash_{\mathbf{IPL}} \varphi$ iff $\tau[\Gamma] \vdash_{\mathbf{S4}} \tau(\varphi)$[d],
- the intuitionistic propositional logic $\vdash_{\mathbf{IPL}}$ into $\vdash_{\mathbf{Grz}}$, i.e., $\Gamma \vdash_{\mathbf{IPL}} \varphi$ iff $\tau[\Gamma] \vdash_{\mathbf{Grz}} \tau(\varphi)$,
- the classical propositional logic $\vdash_{\mathbf{CPL}}$ into $\vdash_{\mathbf{S5}}$, i.e., $\Gamma \vdash_{\mathbf{CPL}} \varphi$ iff $\tau[\Gamma] \vdash_{\mathbf{S5}} \tau(\varphi)$.

The consequence relation $\vdash_{\mathbf{FPL}}$ (*formal propositional logic*) in $\mathcal{L}_{int}$ was first considered by A. Visser in [Vis81]. He introduced it as the consequence relation embeddable into the provability logic $\vdash_{\mathbf{GL}}$ through the map τ. That is, $\Gamma \vdash_{\mathbf{FPL}} \varphi$ iff $\tau[\Gamma] \vdash_{\mathbf{GL}} \tau(\varphi)$. Therefore, the consequence relation $\vdash_{\mathbf{FPL}}$ can be interpreted inside the Peano Arithmetic.

In [Vis81] A. Visser also introduced the consequence relation $\vdash_{\mathbf{BPL}}$ (*basic propositional logic*) in $\mathcal{L}_{int}$. It is defined as the consequence relation embeddable into $\vdash_{\mathbf{K4}}$ (sometimes

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[a]In this paper the symbol $\rightarrow$ will refer to the intuitionistic implication (strict implication) and $\supset$ to the material implication.

[b]The reason to adopt this abbreviation is given by the Kripke semantic for intuitionistic propositional logic; see [CZ97]. Under this semantic the validity of $\top \rightarrow \varphi$ in a world of a Kripke model is equivalent to the validity of φ in all the worlds that are accessible from this one.

[c]If $\mathbf{L}$ is a normal modal logic, by $\vdash_{\mathbf{L}}$ we understand the consequence relation that has the elements in $\mathbf{L}$ as axioms, and modus ponens as its only proper rule (that is, we do not consider the necessitation rule). Therefore, the consequence relation $\vdash_{\mathbf{L}}$ has the deduction-detachment theorem, i.e., $\Gamma, \varphi \vdash_{\mathbf{L}} \psi$ iff $\Gamma \vdash_{\mathbf{L}} \varphi \supset \psi$. All the modal consequence relations that are discussed in the present paper can be found elsewhere, for instance [CZ97]. For $\vdash_{\mathbf{GL}}$ two excellent references are [Boo93, Smo85].

[d]By the notation $\tau[\Gamma]$ we understand $\bigcup\{\tau(\gamma) : \gamma \in \Gamma\}$, i.e., the image of the set Γ. We will use the same notation for the image under arbitrary maps (not only for τ).

called *basic modal logic*, as in [Smo85]), i.e., $\Gamma \vdash_{\mathbf{BPL}} \varphi$ iff $\tau[\Gamma] \vdash_{\mathbf{K4}} \tau(\varphi)$. It is clear that $\vdash_{\mathbf{FPL}}$ and $\vdash_{\mathbf{IPL}}$ are extensions of $\vdash_{\mathbf{BPL}}$. Although $\vdash_{\mathbf{BPL}}$ was devised for technical reasons, over the last decade this consequence relation has acquired an interest of its own. It appears if we replace the Brouwer-Heyting-Kolmogorov interpretation of implication with the weaker interpretation

- a proof of $\varphi \to \psi$ is a construction that uses the assumption φ to produce a proof of ψ.

Further explanations can be found in [Rui91, Rui93].

[Vis81] gives natural deduction calculi for $\vdash_{\mathbf{FPL}}$ and $\vdash_{\mathbf{BPL}}$; in [AR98] M. Ardeshir and W. Ruitenburg present Gentzen-style calculi for both; and a different Gentzen-style calculus for $\vdash_{\mathbf{BPL}}$ is presented by K. Sasaki in [Sas99][e]. However, to our knowledge, no Hilbert-style calculus has been produced for either consequence (see [SWZ98, page 324]). The difficulty lies in the fact that modus ponens does not hold in $\vdash_{\mathbf{FPL}}$ (and therefore does not hold in $\vdash_{\mathbf{BPL}}$). In fact, $\top, \top \to p_0 \nvdash_{\mathbf{FPL}} p_0$. In this paper we solve this problem. We define $\mathcal{H}_{BPC}$ as the following Hilbert-style calculus over $\mathcal{L}_{int}$;

$$(Ru1)\ \frac{p_0 \wedge p_1}{p_0}, \quad (Ru2)\ \frac{p_0 \wedge p_1}{p_1 \wedge p_0}, \quad (Ru3)\ \frac{p_0 \quad p_1}{p_0 \wedge p_1}, \quad (Ru4)\ \frac{p_0}{p_0 \vee p_1},$$

$$(Ru5)\ \frac{p_0 \vee p_1}{p_1 \vee p_0}, \quad (Ru6)\ \frac{p_0 \vee (p_1 \vee p_2)}{(p_0 \vee p_1) \vee p_2}, \quad (Ru7)\ \frac{p_0 \vee (p_1 \wedge p_2)}{(p_0 \vee p_1) \wedge (p_0 \vee p_2)},$$

$$(Ru8)\ \frac{(p_0 \vee p_1) \wedge (p_0 \vee p_2)}{p_0 \vee (p_1 \wedge p_2)}, \quad (Ru9)\ \frac{p_0 \vee p_0}{p_0},$$

$$(7^*)\ \frac{((p_0 \to p_1) \wedge (p_1 \to p_2)) \vee p_3}{(p_0 \to p_2) \vee p_3}, \quad (8^*)\ \frac{((p_0 \to p_1) \wedge (p_0 \to p_2)) \vee p_3}{(p_0 \to (p_1 \wedge p_2)) \vee p_3},$$

$$(9^*)\ \frac{((p_0 \to p_2) \wedge (p_1 \to p_2)) \vee p_3}{((p_0 \vee p_1) \to p_2) \vee p_3}, \quad (1^*)\ \frac{\bot \vee p_1}{p_0 \vee p_1}, \quad (N^*)\ \frac{p_0 \vee p_1}{\Box p_0 \vee p_1},$$

(Ax 1) $p_0 \to p_0$,
(Ax 2) $p_0 \to \top$,
(Ax $\widetilde{Ru1}$) $(p_0 \wedge p_1) \to p_0$,
(Ax $\widetilde{Ru2}$) $(p_0 \wedge p_1) \to (p_1 \wedge p_0)$,
(Ax $\widetilde{Ru4}$) $p_0 \to (p_0 \vee p_1)$,
(Ax $\widetilde{Ru5}$) $(p_0 \vee p_1) \to (p_1 \vee p_0)$,
(Ax $\widetilde{Ru8}$) $((p_0 \vee p_1) \wedge (p_0 \vee p_2)) \to (p_0 \vee (p_1 \wedge p_2))$,
(Ax $\widetilde{7}$) $((p_0 \to p_1) \wedge (p_1 \to p_2)) \to (p_0 \to p_2)$,
(Ax $\widetilde{8}$) $((p_0 \to p_1) \wedge (p_0 \to p_2)) \to (p_0 \to (p_1 \wedge p_2))$,
(Ax $\widetilde{9}$) $((p_0 \to p_2) \wedge (p_1 \to p_2)) \to ((p_0 \vee p_1) \to p_2)$,
(Ax $\widetilde{1}$) $\bot \to p_0$,
(Ax $\widetilde{N}$) $p_0 \to \Box p_0$.

The main result, Theorem 14, says i) $\mathcal{H}_{BPC}$ is a Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$, ii) $\mathcal{H}_{BPC} \cup \{(\text{Ax } \widetilde{L})\}$ is a Hilbert-style calculus for $\vdash_{\mathbf{FPL}}$ where
(Ax $\widetilde{L}$) $(\Box p_0 \to p_0) \to \Box p_0$.

The first section introduces the notions and results that are already known and are necessary for the proof; and in the second section the proof is given.

[e]There the consequence relation $\vdash_{\mathbf{BPL}}$ is called $\vdash_{\mathbf{VPL}}$ (*Visser's propositional logic*) in honor of Visser.

In the literature, the set of theorems of $\vdash_{\mathbf{BPL}}$ and $\vdash_{\mathbf{FPL}}$ are known as **BPL** and **FPL**. Although modus ponens is not a valid rule in either $\vdash_{\mathbf{BPL}}$ or $\vdash_{\mathbf{FPL}}$, it is an admissible rule in both cases. That is, if $\varphi \in \mathbf{BPL}$ and $\varphi \to \psi \in \mathbf{BPL}$ then $\psi \in \mathbf{BPL}$ (and the same for **FPL**). Thanks to this fact a Hilbert-style calculus with modus ponens and the same theorems as $\vdash_{\mathbf{BPL}}$ has been presented; see [SO93, Sas01]. It is easy to see that if we add the axiom (Ax $\widetilde{L}$) we obtain a Hilbert-style calculus with modus ponens and the same theorems as $\vdash_{\mathbf{FPL}}$.

Kripke semantics have been given for both $\vdash_{\mathbf{BPL}}$ and $\vdash_{\mathbf{FPL}}$. As we do not need them to prove Theorem 14 we will not introduce them here; the reader can find details in [Vis81, AR98, SWZ98]. From these semantics it is straightforward that the $\langle \wedge, \vee \rangle$-restriction of $\vdash_{\mathbf{BPL}}$ coincides with the $\langle \wedge, \vee \rangle$-restriction of $\vdash_{\mathbf{CPL}}$, and also with the $\langle \wedge, \vee \rangle$-restriction of $\vdash_{\mathbf{FPL}}$.

1 Preliminaries

For the rest of the paper let us fix an infinite set $Var = \{p_n : n \in \omega\}$, whose elements are called *variables*, such that if $n \neq m$ then $p_n \neq p_m$. Given a propositional language $\mathcal{L}$ the set of *$\mathcal{L}$-formulas* is defined as usual and is denoted by $Fm\mathcal{L}$. Its elements will be denoted by $\varphi, \psi, \delta, \alpha \ldots$; and the subsets of $\mathcal{L}$-formulas by $\Gamma, \Delta, \Sigma, \ldots$ For the rest of the paper we assume that an enumeration $\langle \rho_n : n \in \omega \rangle$ of the set $Fm\mathcal{L}$ is fixed. If $\wedge$ is a connective of our propositional language we will write $\varphi_1 \wedge \ldots \wedge \varphi_n$ as an abbreviation of the formula $(\ldots((\varphi_1 \wedge \varphi_2) \wedge \varphi_3) \wedge \ldots) \wedge \varphi_n$; and for every non-empty finite set Γ of $\mathcal{L}$-formulas $\bigwedge \Gamma$ will be an abbreviation of the formula $\rho_{n_0} \wedge \ldots \wedge \rho_{n_m}$, where $\Gamma = \{\rho_{n_0}, \ldots, \rho_{n_m}\}$, $n_0 < \ldots < n_m$.

We aim to give a Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$ and $\vdash_{\mathbf{FPL}}$. First of all we need to define what we understand by a Hilbert-style calculus. A *Hilbert-style rule over* $\mathcal{L}$ is a pair $\langle \Gamma, \varphi \rangle$ where $\Gamma \cup \{\varphi\}$ is a finite subset of $Fm\mathcal{L}$. Hilbert-style rules that have the form $\langle \emptyset, \varphi \rangle$ are called *Hilbert-style axioms*, and the ones that do not are called *Hilbert-style proper rules*. A *Hilbert-style calculus* $\mathcal{H}$ *over* $\mathcal{L}$ is a set of Hilbert-style rules over $\mathcal{L}$ that is recursive (i.e., there exists a procedure to determine the elements in $\mathcal{H}$). It can be infinite. A *substitution instance of a Hilbert-style rule* $\langle \Gamma, \varphi \rangle$ over $\mathcal{L}$ is a pair of the form $\langle s[\Gamma], s(\varphi) \rangle$ where s is a $\mathcal{L}$-substitution. Given a Hilbert-style calculus $\mathcal{H}$ over $\mathcal{L}$ we can define the notion of a *$\mathcal{H}$-proof of φ from the assumptions* Γ as a finite sequence $\langle \varphi_0, \ldots, \varphi_n \rangle$, $n \in \omega$ such that $\varphi_n = \varphi$ and for each $i \in \{0, \ldots, n\}$ at least one of the following two conditions holds:

1. $\varphi_i \in \Gamma$,
2. there exists $\Delta \subseteq \{\varphi_j : j < i\}$ such that $\langle \Delta, \varphi_i \rangle$ is a substitution instance of a Hilbert-style rule in $\mathcal{H}$.

Given a Hilbert-style calculus $\mathcal{H}$ over $\mathcal{L}$ it is possible to define a consequence relation between the $\mathcal{L}$-formulas. Given $\Gamma \subseteq Fm\mathcal{L}$ and $\varphi \in Fm\mathcal{L}$, we define

$$\Gamma \vdash_{\mathcal{H}} \varphi \quad \text{iff} \quad \text{there exists a } \mathcal{H}\text{-proof of } \varphi \text{ from } \Gamma.$$

Therefore, when we say that we add a rule $\langle \Gamma, \varphi \rangle$ to a Hilbert-style calculus implicitly we are saying that we add all its substitution instances, that is, we treat the rules as schemata. When we say that a Hilbert-style calculus $\mathcal{H}$ over $\mathcal{L}$ is *a calculus for a consequence relation* $\vdash$ we mean that $\vdash = \vdash_{\mathcal{H}}$.

For many consequence relations finite Hilbert-style calculi are known. For instance, finite Hilbert-style calculi for $\vdash_{\mathbf{CPL}}$ and $\vdash_{\mathbf{IPL}}$ can be found in [CZ97]. These calculi use modus ponens as their unique Hilbert-style proper rule. In most cases in which a Hilbert-style calculus is known for a consequence relation, modus ponens holds. The following example is one of the few exceptions.

Example 1 *Let* $\vdash_{\mathbf{CPL}^\bullet}$ *be the* $\langle\wedge,\vee\rangle$*-restriction of the classical propositional logic. And let the Hilbert-style calculus* $\mathcal{H}_{CPC^\bullet}$ *over* $\mathcal{L}_{\wedge\vee} = \langle\wedge,\vee\rangle$ *be the one given by the first nine proper rules of* $\mathcal{H}_{BPC}$ *on page 21, i.e.,* $\mathcal{H}_{CPC^\bullet} = \{(Ru1),(Ru2),\ldots,(Ru9)\}$ *(there is no axiom in* $\mathcal{H}_{CPC^\bullet}$*). It is known that* $\mathcal{H}_{CPC^\bullet}$ *is a Hilbert-style calculus for* $\vdash_{\mathbf{CPL}^\bullet}$ *(see [DP80]; the remaining rules of [DP80] are proved to be derivable in [FGV91]).*

Lemma 2 (invariance under substitutions) *Let* $\mathcal{H}$ *be a Hilbert-style calculus over a language* $\mathcal{L}$ *such that* $\Gamma \vdash_{\mathcal{H}} \varphi$*. If* s *is a* $\mathcal{L}$*-substitution then* $s[\Gamma] \vdash_{\mathcal{H}} s(\varphi)$*.*

Proof: Straightforward. ∎

To prove our main theorem we will use the Gentzen-style calculus given by M. Ardeshir and W. Ruitenburg in [AR98] for $\vdash_{\mathbf{BPL}}$ and $\vdash_{\mathbf{FPL}}$. Thus, we need to introduce the notions of sequent and Gentzen-style calculus. We will give the definition only for the case of the intuitionistic language, the only case that we need.

A $\mathcal{L}_{int}$*-sequent* is a pair $\langle\varphi,\psi\rangle$ where $\varphi,\psi \in Fm\mathcal{L}_{int}$[f]. We will write them as $\varphi \Rightarrow \psi$, and the set of all $\mathcal{L}_{int}$-sequents will be denoted by $Seq\mathcal{L}_{int}$. The subsets of $Seq\mathcal{L}_{int}$ will be denoted by $\Pi, \Upsilon \ldots$

A *Gentzen-style rule over* $\mathcal{L}_{int}$ is a pair $\langle\Pi, \varphi \Rightarrow \psi\rangle$ where $\Pi \cup \{\varphi \Rightarrow \psi\}$ is a finite subset of $Seq\mathcal{L}_{int}$. Gentzen-style rules that have the form $\langle\emptyset\ ,\ \varphi \Rightarrow \psi\rangle$ are called *Gentzen-style axioms*, and the ones that do not are called *Gentzen-style proper rules.* A *Gentzen-style calculus* $\mathcal{G}$ *over* $\mathcal{L}_{int}$ is a recursive set of Gentzen-style rules over $\mathcal{L}_{int}$. A *substitution instance of a Gentzen-style rule* $\langle\Pi\ ,\ \varphi \Rightarrow \psi\rangle$ over $\mathcal{L}_{int}$ is a pair of the form $\langle s[\Pi]\ ,\ s(\varphi) \Rightarrow s(\psi)\rangle$ where s is a $\mathcal{L}_{int}$-substitution. Given a Gentzen-style calculus $\mathcal{G}$ over $\mathcal{L}_{int}$ we can define the notion of a $\mathcal{G}$*-proof of* $\varphi \Rightarrow \psi$ as a finite sequence $\langle\varphi_0 \Rightarrow \psi_0\ ,\ \ldots\ ,\ \varphi_n \Rightarrow \psi_n\rangle$, $n \in \omega$ such that $\varphi_n = \varphi$ and $\psi_n = \psi$ and for each $i \in \{0,\ldots,n\}$ there exists $\Upsilon \subseteq \{\varphi_j \Rightarrow \psi_j : j < i\}$ such that $\langle\Upsilon\ ,\ \varphi_i \Rightarrow \psi_i\rangle$ is a substitution instance of a Gentzen-style rule in $\mathcal{G}$. Given a Gentzen-style calculus $\mathcal{G}$ over $\mathcal{L}_{int}$ it is possible to define a consequence relation between the $\mathcal{L}_{int}$-formulas. Given $\Gamma \subseteq Fm\mathcal{L}_{int}$ and $\varphi \in Fm\mathcal{L}_{int}$, we define

$$\Gamma \vdash_{\mathcal{G}} \varphi \quad \text{iff} \quad \begin{cases} \text{exists } \gamma_0,\ldots,\gamma_n \in \Gamma \cup \{\top\},\ n \in \omega \text{ such that} \\ \text{there is a } \mathcal{G}\text{-proof of } \gamma_0 \wedge \ldots \wedge \gamma_n \Rightarrow \varphi. \end{cases}$$

Therefore, when we say that we add a rule $\langle\Pi, \varphi \Rightarrow \psi\rangle$ to a Gentzen-style calculus implicitly we are saying that we add all its substitution instances, that is, we treat the rules as schemata. When we say that a Gentzen-style calculus $\mathcal{G}$ over $\mathcal{L}_{int}$ is *a calculus for a consequence relation* $\vdash$ we mean that $\vdash = \vdash_{\mathcal{G}}$.

Example 3 *Consider the Gentzen-style calculus over* $\mathcal{L}_{int}$ *given by*

[f] Sometimes $\mathcal{L}_{int}$-sequents are defined as pairs $\langle\Gamma,\psi\rangle$ where $\Gamma \cup \{\psi\}$ is a finite subset of $\mathcal{L}_{int}$-formulas (see for instance [Sas01]). We have adopted the other definition, since this is how [AR98] presents the Gentzen-style calculus that we are going to use. Nevertheless, since in our language we have $\wedge$ and $\top$ we may think that both definitions are more or less the same (because the finite sets of $\mathcal{L}_{int}$-formulas can be seen as a single $\mathcal{L}_{int}$-formula). Another reason to adopt this definition of $\mathcal{L}_{int}$-sequent is the fact that the Gentzen-style calculi given for $\vdash_{\mathbf{BPL}}$ and $\vdash_{\mathbf{FPL}}$ are finite. This does not hold if we adopt the other definition.

$$\begin{array}{ll}
(0)\ p_0 \Rightarrow p_0, & (5)\ p_1 \Rightarrow p_0 \vee p_1, \\
(1)\ \bot \Rightarrow p_0, & (6)\ p_0 \wedge (p_1 \vee p_2) \Rightarrow (p_0 \wedge p_1) \vee (p_0 \wedge p_2), \\
(2)\ p_0 \wedge p_1 \Rightarrow p_0, & (7)\ (p_0 \to p_1) \wedge (p_1 \to p_2) \Rightarrow p_0 \to p_2, \\
(3)\ p_0 \wedge p_1 \Rightarrow p_1, & (8)\ (p_0 \to p_1) \wedge (p_0 \to p_2) \Rightarrow p_0 \to (p_1 \wedge p_2), \\
(4)\ p_0 \Rightarrow p_0 \vee p_1, & (9)\ (p_0 \to p_2) \wedge (p_1 \to p_2) \Rightarrow (p_0 \vee p_1) \to p_2,
\end{array}$$

$$(Cut)\ \frac{p_0 \Rightarrow p_1 \qquad p_1 \Rightarrow p_2}{p_0 \Rightarrow p_2}, \qquad (\Rightarrow \wedge)\ \frac{p_0 \Rightarrow p_1 \qquad p_0 \Rightarrow p_2}{p_0 \Rightarrow p_1 \wedge p_2},$$

$$(\vee \Rightarrow)\ \frac{p_0 \Rightarrow p_2 \qquad p_1 \Rightarrow p_2}{p_0 \vee p_1 \Rightarrow p_2}, \qquad (DT)\ \frac{p_0 \wedge p_1 \Rightarrow p_2}{p_0 \Rightarrow p_1 \to p_2}.$$

It is known that this Gentzen-style calculus is a calculus for $\vdash_{\mathbf{BPL}}$ *([AR98, Theorem 3.5]). It is also known that if we add the Gentzen-style axiom*

$$(L) \quad \top \Rightarrow (\Box p_0 \to p_0) \to \Box p_0$$

to the above calculus we obtain a calculus for $\vdash_{\mathbf{FPL}}$ *([AR98, Lemma 2.10]).*

Example 4 *Consider the Gentzen-style calculus over* $\mathcal{L}_{int}$ *given by*

$$\top \Rightarrow p_0 \to p_0, \quad \top \Rightarrow p_0 \to (p_1 \to p_0), \quad \top \Rightarrow p_0 \to (p_1 \to (p_0 \wedge p_1)),$$

$$\top \Rightarrow \bot \to p_0, \quad \top \Rightarrow (p_0 \wedge p_1) \to p_0, \quad \top \Rightarrow (p_0 \wedge p_1) \to p_1,$$

$$\top \Rightarrow p_0 \to (p_0 \vee p_1), \quad \top \Rightarrow p_1 \to (p_0 \vee p_1),$$

$$\top \Rightarrow ((p_1 \to p_2) \wedge (p_0 \to p_1)) \to (p_0 \to p_2),$$

$$\top \Rightarrow ((p_0 \to p_1) \wedge (p_0 \to p_2)) \to (p_0 \to (p_1 \wedge p_2)),$$

$$\top \Rightarrow ((p_0 \to p_2) \wedge (p_1 \to p_2)) \to ((p_0 \vee p_1) \to p_2),$$

$$\top \Rightarrow (p_0 \wedge (p_1 \vee p_2)) \to ((p_0 \wedge p_1) \vee (p_0 \wedge p_2)),$$

$$(Cut)\ \frac{p_0 \Rightarrow p_1 \qquad p_1 \Rightarrow p_2}{p_0 \Rightarrow p_2}, \qquad (\Rightarrow \wedge)\ \frac{p_0 \Rightarrow p_1 \qquad p_0 \Rightarrow p_2}{p_0 \Rightarrow p_1 \wedge p_2},$$

$$(mp)\ \frac{\top \Rightarrow p_0 \to p_1}{p_0 \Rightarrow p_1}.$$

It is known that this Gentzen-style calculus is a calculus for $\vdash_{\mathbf{BPL}}$ *([Sas99, Theorem 2.2]). Although* K. Sasaki *says in [Sas01] that this is a Hilbert-style formalization for* $\vdash_{\mathbf{BPL}}$ *it is not a Hilbert-style calculus (the rule* (mp) *is a Gentzen-style rule that cannot easily be written as a Hilbert-style rule).*

2 Proof

For our proof we will use the Gentzen-style calculi given in [AR98] with some simplifications. Using this fact, necessary and sufficient conditions for being a Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$ and $\vdash_{\mathbf{FPL}}$ will be straightforwardly obtained in Proposition 6.

The Gentzen-style calculus $\mathcal{G}_{BPC}$ is defined as the calculus over $\mathcal{L}_{int}$ in Example 3 replacing the Gentzen-style rule (DT) with the following three Gentzen-style rules

$$(10)\ \frac{\emptyset}{p_0 \Rightarrow \top}, \qquad (N)\ \frac{\emptyset}{p_0 \Rightarrow \Box p_0}, \qquad (DT_0)\ \frac{p_0 \Rightarrow p_1}{\top \Rightarrow p_0 \to p_1}.$$

It is straightforward to see that this replacement does not change the provable sequents.

Proposition 5

1. *$\mathcal{G}_{BPC}$ is a Gentzen-style calculus for $\vdash_{\mathbf{BPL}}$.*

2. *$\mathcal{G}_{BPC} \cup \{(L)\}$ is a Gentzen-style calculus for $\vdash_{\mathbf{FPL}}$.*

Proof: By Example 3 we only need to prove that these three rules are interderivable with the rule (DT). It is really simple to observe that they can be obtained using (DT). For the other direction here is given a sketch.

$$\dfrac{\dfrac{\dfrac{\dfrac{\emptyset}{p_0 \Rightarrow p_1 \to \top} \quad \dfrac{\emptyset}{p_0 \Rightarrow \top \to p_0}}{p_0 \Rightarrow p_1 \to p_0} \quad \dfrac{\dfrac{\emptyset}{\top \Rightarrow p_1 \to p_1}}{p_0 \Rightarrow p_1 \to p_1}}{p_0 \Rightarrow p_1 \to (p_0 \wedge p_1)} \quad \dfrac{\dfrac{p_0 \wedge p_1 \Rightarrow p_2}{\top \Rightarrow (p_0 \wedge p_1) \to p_2}}{p_0 \Rightarrow (p_0 \wedge p_1) \to p_2}}{p_0 \Rightarrow p_1 \to p_2}$$

∎

Let $\langle \{\varphi_1 \Rightarrow \psi_1, \ldots, \varphi_n \Rightarrow \psi_n\}, \varphi \Rightarrow \psi \rangle$ be a Gentzen-style rule over $\mathcal{L}_{int}$. We will say that a Hilbert-style calculus $\mathcal{H}$ over $\mathcal{L}_{int}$ is *a model of this Gentzen-style rule* iff for every $\mathcal{L}_{int}$-substitution s holds

$$\text{if for every } i \in \{1, \ldots, n\}\ s(\varphi_i) \vdash_{\mathcal{H}} s(\psi_i), \text{ then } s(\varphi) \vdash_{\mathcal{H}} s(\psi).$$

For instance, the fact that $\mathcal{H}$ is a model of the Gentzen-style rule (DT_0) means that for every $\varphi, \psi \in Fm\mathcal{L}_{int}$ it is the case that

$$\text{if } \varphi \vdash_{\mathcal{H}} \psi \text{ then } \top \vdash_{\mathcal{H}} \varphi \to \psi.$$

And the fact that $\mathcal{H}$ is a model of the Gentzen-style rule $(\vee \Rightarrow)$ means that for every $\varphi, \psi, \delta \in Fm\mathcal{L}_{int}$ holds

$$\text{if } \varphi \vdash_{\mathcal{H}} \delta \text{ and } \psi \vdash_{\mathcal{H}} \delta \text{ then } \varphi \vee \psi \vdash_{\mathcal{H}} \delta.$$

Given a Hilbert-style calculus $\mathcal{H}$, some properties are true and others false. We are interested in the following five properties.

(P1) If $\langle \Gamma, \varphi \rangle \in \mathcal{H}$ then $\Gamma \vdash_{\mathbf{BPL}} \varphi$.

(P2)[g] $\bot \vdash_{\mathcal{H}} p_0$, $\quad p_0 \wedge p_1 \dashv\vdash_{\mathcal{H}} \{p_0, p_1\}$, $\quad p_0 \vdash_{\mathcal{H}} p_0 \vee p_1$, $\quad p_1 \vdash_{\mathcal{H}} p_0 \vee p_1$, $\quad p_0 \wedge (p_1 \vee p_2) \vdash_{\mathcal{H}} (p_0 \wedge p_1) \vee (p_0 \wedge p_2)$, $\quad (p_0 \to p_1) \wedge (p_1 \to p_2) \vdash_{\mathcal{H}} p_0 \to p_2$, $\quad (p_0 \to p_1) \wedge (p_0 \to p_2) \vdash_{\mathcal{H}} p_0 \to (p_1 \wedge p_2)$, $\quad (p_0 \to p_2) \wedge (p_1 \to p_2) \vdash_{\mathcal{H}} (p_0 \vee p_1) \to p_2$, $\quad p_0 \vdash_{\mathcal{H}} \top$, $\quad p_0 \vdash_{\mathcal{H}} \Box p_0$.

(P3) $\mathcal{H}$ is a model of the Gentzen-style rules $(\vee \Rightarrow)$ and (DT_0).

(P4) $\emptyset \vdash_{\mathcal{H}} (\Box p_0 \to p_0) \to \Box p_0$.

(P5) If $\langle \Gamma, \varphi \rangle \in \mathcal{H}$ then $\Gamma \vdash_{\mathbf{FPL}} \varphi$.

A simple consequence of Proposition 5 (see [FJ96, Proposition 4.4(3)]) is the following.

Proposition 6 *Let $\mathcal{H}$ be a Hilbert-style calculus over $\mathcal{L}_{int}$. Then,*

1. *(P1), (P2) and (P3) holds in $\mathcal{H}$ iff $\vdash_{\mathcal{H}} = \vdash_{\mathbf{BPL}}$.*

[g] This property (P2) means that $\mathcal{H}$ is a model of all the rules in $\mathcal{G}_{BPC} \setminus \{(\vee \Rightarrow), (DT_0)\}$.

2. (P2), (P3), (P4) and (P5) holds in $\mathcal{H}$ iff $\vdash_{\mathcal{H}} = \vdash_{\mathbf{FPL}}$.

For each of these properties except (P3) it is clear how to obtain a Hilbert-style calculus where the property holds. In the following two lemmas we will consider conditions about a Hilbert-style calculus $\mathcal{H}$ that allow us to conclude (P3). It is clear that the hypotheses of both lemmas are true if we replace $\vdash_{\mathcal{H}}$ with $\vdash_{\mathbf{BPL}}$.

Lemma 7 *Let $\mathcal{H}$ be a Hilbert-style calculus over $\mathcal{L}_{int}$ such that*
(a) $\emptyset \vdash_{\mathcal{H}} p_0 \to p_0$, *(b)* $(p_0 \to p_1) \wedge (p_0 \to p_2) \vdash_{\mathcal{H}} p_0 \to (p_1 \wedge p_2)$,
(c) $\emptyset \vdash_{\mathcal{H}} p_0 \to \top$, *(d)* $(p_0 \to p_1) \wedge (p_1 \to p_2) \vdash_{\mathcal{H}} p_0 \to p_2$,
(e) $p_0, p_1 \dashv\vdash_{\mathcal{H}} p_0 \wedge p_1$, *(f)* $p_0 \vdash_{\mathcal{H}} \Box p_0$.
The following conditions are equivalent.

1. *$\mathcal{H}$ is a model of the Gentzen-style rule (DT_0).*
2. *For every Hilbert-style proper rule $\langle \Gamma, \varphi \rangle \in \mathcal{H}$ holds $\emptyset \vdash_{\mathcal{H}} (\bigwedge \Gamma) \to \varphi$.*

Proof: Straightforward. ∎

Lemma 8 *Let $\mathcal{H}$ be a Hilbert-style calculus over $\mathcal{L}_{int}$ such that*
(a) $p_0 \vdash_{\mathcal{H}} p_0 \vee p_1$, *(b)* $p_0 \vee p_1 \vdash_{\mathcal{H}} p_1 \vee p_0$,
(c) $p_0 \vee p_0 \vdash_{\mathcal{H}} p_0$, *(d)* $p_0 \vee p_2, p_1 \vee p_2 \vdash_{\mathcal{H}} (p_0 \wedge p_1) \vee p_2$.
The following conditions are equivalent.

1. *$\mathcal{H}$ is a model of the Gentzen-style rule $(\vee \Rightarrow)$.*
2. *For every Hilbert-style proper rule $\langle \{\gamma_1, \dots, \gamma_n\}, \varphi \rangle \in \mathcal{H}$, $n \geq 1$ holds $\{\gamma_1 \vee p_k, \dots, \gamma_n \vee p_k\} \vdash_{\mathcal{H}} \varphi \vee p_k$ where p_k is the first variable that doesn't appear in $\{\gamma_1, \dots, \gamma_n, \varphi\}$.*

Proof: Straightforward. ∎

From Lemmas 7 and 8 and Proposition 6 this corollary follows:

Corollary 9 *If $\mathcal{H}$ is Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$ then $\mathcal{H} \cup \{(Ax\ \widetilde{L})\}$ is a Hilbert-style calculus for $\vdash_{\mathbf{FPL}}$.*

Therefore, it only remains to obtain a Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$, i.e., a Hilbert-style calculus satisfying (P1), (P2) and (P3). If we allow the use of an infinite Hilbert-style calculus it is possible to obtain a calculus for $\vdash_{\mathbf{BPL}}$ straightforwardly. Let us explain the method.

First of all we define two maps der_0, der_1 from the set of Hilbert-style proper rules over $\mathcal{L}_{int}$ to the set of Hilbert-style rules over $\mathcal{L}_{int}$. Consider a Hilbert-style proper rule $\langle \Gamma, \varphi \rangle$ over $\mathcal{L}_{int}$. We define $der_0(\langle \Gamma, \varphi \rangle)$ as the Hilbert-style axiom $\langle \emptyset, (\bigwedge \Gamma) \to \varphi \rangle$; and $der_1(\langle \Gamma, \varphi \rangle)$ is defined as $\langle \{\gamma_1 \vee p_k, \dots, \gamma_n \vee p_k\}, \varphi \vee p_k \rangle$ where $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ and p_k is the first variable that does not appear in $\{\gamma_1, \dots, \gamma_n, \varphi\}$. If a Hilbert-style proper rule is called (R) then in this paper we use the names (Ax $\widetilde{R}$) and (R^*) to refer to $der_0(\langle \Gamma, \varphi \rangle)$ and $der_1(\langle \Gamma, \varphi \rangle)$ respectively[h].

[h]This method has been used for some of the names for the calculus $\mathcal{H}_{BPC}$ on page 21; $(7^*), (8^*), (9^*), (N^*), (1^*)$,(Ax $\widetilde{7}$),(Ax $\widetilde{8}$),(Ax $\widetilde{9}$),(Ax $\widetilde{1}$),(Ax $\widetilde{N}$). The rules $(7), (8), (9), (1), (N)$ have been defined as Gentzen-style axioms, but it is clear that we can think of them as Hilbert-style proper rules (see these rules on page 29).

Suppose $\mathcal{H}$ is an arbitrary Hilbert-style calculus over $\mathcal{L}_{int}$. The sequence $\langle \mathcal{H}^n : n \in \omega \rangle$ is defined recursively as i) $\mathcal{H}^0 = \mathcal{H}$, ii) $\mathcal{H}^{n+1} = \mathcal{H}^n \cup \{der_i(r) : i \in \{0,1\}, r$ is a Hilbert-style proper rule in $\mathcal{H}^n\}$. And the Hilbert-style calculus $\mathcal{H}^{der}$ over $\mathcal{L}_{int}$ is defined as $\bigcup\{\mathcal{H}^n : n \in \omega\}$. It is clear that $\mathcal{H}^{der}$ is the closure of $\mathcal{H}$ under the maps der_0 and der_1. That is, $\mathcal{H}^{der}$ is the smaller extension of $\mathcal{H}$ such that if r is a Hilbert proper rule in $\mathcal{H}^{der}$ then $der_0(r) \in \mathcal{H}^{der}$ and $der_1(r) \in \mathcal{H}^{der}$.

Take $\mathcal{H}_1$ as the Hilbert-style calculus over $\mathcal{L}_{int}$ given by the Hilbert-style rules in (P2) and in the hypotheses of lemmas 7 and 8. A moment of reflection allows us to say that $(\mathcal{H}_1)^{der}$ is a Hilbert-style calculus where (P1), (P2) and (P3) hold (for the last property use lemmas 7 and 8). Therefore, $(\mathcal{H}_1)^{der}$ is a Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$. The problem is that this is an infinite Hilbert-style calculus. How can we replace this calculus with a finite one? To answer this question let us see how our Hilbert-style calculus $\mathcal{H}_{BPC}$ has been obtained.

It is clear that if we consider a set of Hilbert-style rules $\mathcal{H}_2$ over $\mathcal{L}_{int}$ satisfying (P1) then $(\mathcal{H}_1 \cup \mathcal{H}_2)^{der}$ is also a Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$. Now, our idea is to look for a set of rules $\mathcal{H}_2$ satisfying (P1) such that for a certain $n \in \omega$ the Hilbert-style calculus $(\mathcal{H}_1 \cup \mathcal{H}_2)^n$ gives the same consequence relation as the Hilbert-style calculus $(\mathcal{H}_1 \cup \mathcal{H}_2)^{der}$. This holds if we consider the Hilbert-style calculus $\mathcal{H}_{CPC^\bullet}$ described in Example 1[i]. That is, it is easy to verify that $(\mathcal{H}_1 \cup \mathcal{H}_{CPC^\bullet})^1$ gives the same consequence relation as $(\mathcal{H}_1 \cup \mathcal{H}_{CPC^\bullet})^{der}$; the trick is based on the fact that we know $(p_0 \vee p_1) \vee p_2 \dashv\vdash_{\mathcal{H}_{CPC^\bullet}} p_0 \vee (p_1 \vee p_2)$. Therefore, $(\mathcal{H}_1 \cup \mathcal{H}_{CPC^\bullet})^1$ is a Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$ and is finite. This Hilbert-style calculus can be simplified. Some of its rules can be obtained from the others, and so can be removed. If we do so we obtain our calculus $\mathcal{H}_{BPC}$.

Let us see that the ideas explained before are adequate to solve the problem. That is, let us see that the Hilbert-style calculus $\mathcal{H}_{BPC}$ verifies (P1), (P2) and (P3).

Lemma 10 *$\mathcal{H}_{BPC}$ verifies (P1).*

Proof: Straightforward. ∎

Lemma 11

1. *If $\Gamma \vdash_{\mathbf{CPL}} \varphi$ with $\Gamma \cup \{\varphi\} \subseteq Fm\mathcal{L}_{\wedge\vee}$ then $\Gamma \vdash_{\mathcal{H}_{BPC}} \varphi$*
2. *$\mathcal{H}_{BPC}$ verifies (P2).*

Proof: 1) It is obvious using Example 1, $\mathcal{H}_{CPC^\bullet} \subseteq \mathcal{H}_{BPC}$ and $Fm\mathcal{L}_{\wedge\vee} \subseteq Fm\mathcal{L}_{int}$.

2) Since the first point holds, it only remains to prove that $\bot \vdash_{\mathcal{H}_{BPC}} p_0$, $(p_0 \to p_1) \wedge (p_1 \to p_2) \vdash_{\mathcal{H}_{BPC}} p_0 \to p_2$, $(p_0 \to p_1) \wedge (p_0 \to p_2) \vdash_{\mathcal{H}_{BPC}} p_0 \to (p_1 \wedge p_2)$, $(p_0 \to p_2) \wedge (p_1 \to p_2) \vdash_{\mathcal{H}_{BPC}} (p_0 \vee p_1) \to p_2$, $p_0 \vdash_{\mathcal{H}_{BPC}} \top$, $p_0 \vdash_{\mathcal{H}_{BPC}} \Box p_0$. Using (Ax 1) it is clear that $p_0 \vdash_{\mathcal{H}_{BPC}} \top$.

All the other cases are proved in the same way. Suppose we are in a case $\varphi \vdash_{\mathcal{H}_{BPC}} \psi$. Then, it holds that $\langle \varphi \vee p_k, \psi \vee p_k \rangle \in \mathcal{H}_{BPC}$ where p_k is the first variable that doesn't appear in $\{\varphi, \psi\}$. Thus, $\varphi \vee \varphi \vdash_{\mathcal{H}_{BPC}} \psi \vee \varphi$ and $\varphi \vee \psi \vdash_{\mathcal{H}_{BPC}} \psi \vee \psi$ by the invariance under substitutions. Therefore, using also $(Ru4)$, $(Ru5)$ and $(Ru9)$,

$$\varphi \vdash_{\mathcal{H}_{BPC}} \varphi \vee \varphi \vdash_{\mathcal{H}_{BPC}} \psi \vee \varphi \vdash_{\mathcal{H}_{BPC}} \varphi \vee \psi \vdash_{\mathcal{H}_{BPC}} \psi \vee \psi \vdash_{\mathcal{H}_{BPC}} \psi$$ ∎

[i] A reason for considering this set of Hilbert-style rules is the fact that the $\langle\wedge,\vee\rangle$-restriction of $\vdash_{\mathbf{BPL}}$ coincides with the $\langle\wedge,\vee\rangle$-restriction of $\vdash_{\mathbf{CPL}}$, for which $\mathcal{H}_{CPC^\bullet}$ is a Hilbert-style calculus.

Lemma 12 *$\mathcal{H}_{BPC}$ is a model of (DT_0).*

Proof: We know that the Hilbert-style calculus $\mathcal{H}_{BPC}$ satisfies the hypotheses of Lemma 7 (remember Lemma 11). Therefore, we only need to see that for every Hilbert-style proper rule $\langle\Gamma,\varphi\rangle \in \mathcal{H}_{BPC}$ it holds $\emptyset \vdash_{\mathcal{H}_{BPC}} (\bigwedge \Gamma) \to \varphi$. Let us look at all the Hilbert-style proper rules in $\mathcal{H}_{BPC}$.

Suppose we consider a Hilbert-style rule (X) in $\{(Ru1),(Ru2),(Ru4),\ (Ru5),(Ru8)\}$. Then, it has the form $\langle\{\gamma\},\varphi\rangle$. By the Hilbert-style axiom (Ax $\widetilde{X}$) we obtain $\langle\emptyset,\gamma \to \varphi\rangle \in \mathcal{H}_{BPC}$. Thus, $\emptyset \vdash_{\mathcal{H}_{BPC}} \gamma \to \varphi$.

We consider the rule $(Ru3)$. We know $\emptyset \vdash_{\mathcal{H}_{BPC}} (p_0 \wedge p_1) \to (p_o \wedge p_1)$ and $\emptyset \vdash_{\mathcal{H}_{BPC}} (p_1 \wedge p_0) \to (p_o \wedge p_1)$. From this we obtain what we want.

For the rule $(Ru6)$ it is necessary to see that $\emptyset \vdash_{\mathcal{H}_{BPC}} (p_0 \vee (p_1 \vee p_2)) \to ((p_0 \vee p_1) \vee p_2)$. An easy remark[j] shows that $\emptyset \vdash_{\mathcal{H}_{BPC}} p_0 \to ((p_0 \vee p_1) \vee p_2)$, $\emptyset \vdash_{\mathcal{H}_{BPC}} p_1 \to ((p_0 \vee p_1) \vee p_2)$ and $\emptyset \vdash_{\mathcal{H}_{BPC}} p_2 \to ((p_0 \vee p_1) \vee p_2)$. And by Lemma 11(2) we know that $(p_0 \to p_2) \wedge (p_1 \to p_2) \vdash_{\mathcal{H}_{BPC}} (p_0 \vee p_1) \to p_2$; i.e., $(\varphi \to \delta) \wedge (\psi \to \delta) \vdash_{\mathcal{H}_{BPC}} (\varphi \vee \psi) \to \delta$ for every $\varphi,\psi,\delta \in Fm\mathcal{L}_{int}$. Using the last two sentences we obtain what we want.

The cases in $\{(Ru7),(Ru9),(7^*),(8^*),(9^*),(1^*),(N^*)\}$ are treated in the same way than $(Ru6)$; i.e., we observe that the premise is a disjunction and we use the fact that $(\varphi \to \delta) \wedge (\psi \to \delta) \vdash_{\mathcal{H}_{BPC}} (\varphi \vee \psi) \to \delta$ for every $\varphi,\psi,\delta \in Fm\mathcal{L}_{int}$. ∎

Lemma 13 *$\mathcal{H}_{BPC}$ is a model of $(\vee \Rightarrow)$.*

Proof: We know that the Hilbert-style calculus $\mathcal{H}_{BPC}$ satisfies the hypotheses of Lemma 8 (remember Lemma 11(1)). Therefore, we only need to see that for every Hilbert-style proper rule $\langle\{\gamma_1,\ldots,\gamma_n\},\varphi\rangle \in \mathcal{H}_{BPC}$ $(n \geq 1)$ it holds that $\{\gamma_1 \vee p_k,\ldots,\gamma_n \vee p_k\} \vdash_{\mathcal{H}_{BPC}} \varphi \vee p_k$ where p_k is the first variable that doesn't appear in $\{\gamma_1,\ldots,\gamma_n,\varphi\}$. Let us examine all the Hilbert-style proper rules in $\mathcal{H}_{BPC}$.

Suppose $\langle\{\gamma_1,\ldots,\gamma_n\},\varphi\rangle \in \mathcal{H}_{CPC^\bullet}$ and p_k is the first variable that does not appear in $\{\gamma_1,\ldots,\gamma_n,\varphi\}$. Then, $\{\gamma_1 \vee p_k,\ldots,\gamma_n \vee p_k\} \vdash_{\mathbf{CPL^\bullet}} \varphi \vee p_k$. By Lemma 11(1) we conclude that $\{\gamma_1 \vee p_k,\ldots,\gamma_n \vee p_k\} \vdash_{\mathcal{H}_{BPC}} \varphi \vee p_k$.

Suppose we consider a rule in $\{(7^*),(8^*),(9^*),(1^*),(N^*)\}$. Then, it has the form $\langle\{\gamma \vee p_j\},\varphi \vee p_j\rangle$ for some $\gamma,\varphi \in Fm\mathcal{L}_{int}$ and $p_j \in Var$ which does not appear in $\{\gamma,\varphi\}$. Let p_k be the first variable that does not appear in $\{\gamma \vee p_j,\varphi \vee p_j\}$. We must prove that $(\gamma \vee p_j) \vee p_k \vdash_{\mathcal{H}_{BPC}} (\varphi \vee p_j) \vee p_k$. By Lemma 11(1) and the invariance under substitutions it is known that $(\gamma \vee p_j) \vee p_k \dashv\vdash_{\mathcal{H}_{BPC}} \gamma \vee (p_j \vee p_k)$ and $(\varphi \vee p_j) \vee p_k \dashv\vdash_{\mathcal{H}_{BPC}} \varphi \vee (p_j \vee p_k)$. Therefore, it only remains to prove $\gamma \vee (p_j \vee p_k) \vdash_{\mathcal{H}_{BPC}} \varphi \vee (p_j \vee p_k)$. This is an immediate consequence of the Hilbert-style rule $\langle\{\gamma \vee p_j\},\varphi \vee p_j\rangle$ that we are considering (use the invariance under substitutions). ∎

Theorem 14

1. *$\mathcal{H}_{BPC}$ is a finite Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$.*
2. *$\mathcal{H}_{BPC} \cup \{(Ax\ \widetilde{L})\}$ is a finite Hilbert-style calculus for $\vdash_{\mathbf{FPL}}$.*

[j] Lemma 11(2) says that $(p_0 \to p_1) \wedge (p_1 \to p_2) \vdash_{\mathcal{H}_{BPC}} p_0 \to p_2$. Therefore, $(\varphi \to \psi) \wedge (\psi \to \delta) \vdash_{\mathcal{H}_{BPC}} (\varphi \to \delta)$ for every $\varphi,\psi,\delta \in Fm\mathcal{L}_{int}$.

Proof: The first point is deduced from Proposition 6 and Lemmas 10, 11, 12, 13. The second point is a consequence of Corollary 9. ∎

A similar trick is used by J. Rebagliato and V. Verdú in [RV94] to obtain a Hilbert-style calculus over $\langle \wedge, \vee, \neg \rangle$ for the $\langle \wedge, \vee, \neg \rangle$-restriction of $\vdash_{\mathbf{IPL}}$.

In the Hilbert-style calculus $\mathcal{H}_{BPC}$ we have the rules $(7^*), (8^*), (9^*), (1^*), (N^*)$ in place of the simpler Hilbert-style rules

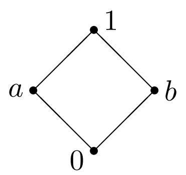
$$(7)\ \frac{(p_0 \to p_1) \wedge (p_1 \to p_2)}{(p_0 \to p_2)}, \qquad (8)\ \frac{(p_0 \to p_1) \wedge (p_0 \to p_2)}{p_0 \to (p_1 \wedge p_2)},$$

$$(9)\ \frac{(p_0 \to p_2) \wedge (p_1 \to p_2)}{(p_0 \vee p_1) \to p_2}, \qquad (1)\ \frac{\bot}{p_0}, \qquad (N)\ \frac{p_0}{\Box p_0}.$$

We cannot simultaneously replace our five rules in $\mathcal{H}_{BPC}$ with these simpler ones. In fact, if we take (R) as one of the rules in $\{(7), (1), (N)\}$ then the Hilbert-style calculus over $\mathcal{L}_{int}$ obtained from $\mathcal{H}_{BPC}$ replacing the Hilbert-style rule (R^*) with (R) is not a Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$. If this holds for the rules in $\{(8), (9)\}$ is still an open question. Let us see the proof for the case (N). The same method, changing the algebra, can be used for the cases (7) and (1).

Proposition 15 *Let $\mathcal{H}$ be the Hilbert-style calculus over $\mathcal{L}_{int}$ obtained from $\mathcal{H}_{BPC}$ replacing the Hilbert-style rule (N^*) with (N). Then, $\mathcal{H}$ is not a Hilbert-style calculus for $\vdash_{\mathbf{BPL}}$.*

Proof: We define the $\mathcal{L}_{int}$-algebra $\mathbf{A} = \langle \{0, a, b, 1\}, \wedge, \vee, \to, 0 \rangle$ where $\wedge, \vee$ are the operations of the lattice in the picture, and $\to$ is the operation defined below.



$\to$	0	a	b	1
0	1	1	1	1
a	1	1	1	1
b	0	0	1	1
1	0	0	1	1

An easy induction proves that if $\Gamma \vdash_{\mathcal{H}} \varphi$ then

for every homomorphism h of $\mathbf{Fm}\boldsymbol{\mathcal{L}_{int}}$ into $\mathbf{A}$ such that $h[\Gamma] \subseteq \{1\}$ holds $h(\varphi) = 1$.[k]

Using that $a \vee b = 1$ and $\Box a \vee b = 0 \vee b = b \neq 1$ it is clear that $p_0 \vee p_1 \nvdash_{\mathcal{H}} \Box p_0 \vee p_1$. As $p_0 \vee p_1 \vdash_{\mathbf{BPL}} \Box p_0 \vee p_1$ we conclude that $\mathcal{H}$ is not a calculus for $\vdash_{\mathbf{BPL}}$. ∎

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[k] In the terminology of Abstract Algebraic Logic this means that the matrix $\langle \mathbf{A}, \{1\} \rangle$ is a model for $\vdash_{\mathbf{BPL}}$; two good references in this subject are [Cze01, FJ96].

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Félix Bou. *Departament de Lògica, Història i Filosofia de la Ciència.*
Facultat de Matemàtiques. Universitat de Barcelona.
Gran Via de les Corts Catalanes 585. 08007 Barcelona, Spain.
E-mail: bou@mat.ub.es