# Twofold integral and Multi-step Choquet integral

**Yasuo Narukawa**[1], **Vicenç Torra**[2]
[1] Toho Gakuen
3-1-10 Naka, Kunitachi, Tokyo, 186-0004 Japan
e-mail: narukawa@d4.dion.ne.jp
and [2]Institut d'Investigació en Intel·ligència Artificial - CSIC
Campus UAB s/n, 08193 Bellaterra, Catalonia, Spain
e-mail: vtorra@iiia.csic.es

## Summary

In this work we study the twofold integral. First, we prove that the twofold integral as well as the Sugeno integral can be represented as a 2-step Choquet integral. Then, we study which is the twofold integral of a crisp set and establish some relationships between this integral and the Choquet and Sugeno ones.

## 1 Introduction

Choquet and Sugeno integrals are one of the most well known integrals to operate with fuzzy measures. In both cases, the functional calculates the integral of a function with respect to a fuzzy measure.

In 1991, Murofushi and Sugeno [6] proposed the fuzzy t-conorm integral to unify both integrals in a single framework. The generalization is based on the definition of a t-conorm system for integration that generalizes the following pairs of operations: the product and sum (used in the Choquet integral) and the minimum and maximum (used in the Sugeno integral). t-norm-like and t-conorm operators are used for this generalization.

The twofold integral, proposed in [10], is an alternative generalization. Roughly speaking, the generalization process is as follows. Instead of building the new integral in terms of operators generalizing both ($\cdot$ and $min$) and ($+$ and $max$), it defines the integral considering all these terms and, additionally, two fuzzy measures (the one used in the Sugeno integral and the one in the Choquet integral).

The rationale of the approach is that the semantics of both measures are different. In particular, the one in the Choquet integral is seen as a "probabilistic-flavor" measure and the one in the Sugeno integral is seen as a "fuzzy-flavor" measure. Due to their semantic difference, the generalization - the twofold integral - considers both. Then, it was proven [10] that with a particular selection of these measures, the twofold integral either reduces to the Choquet integral or to the Sugeno integral.

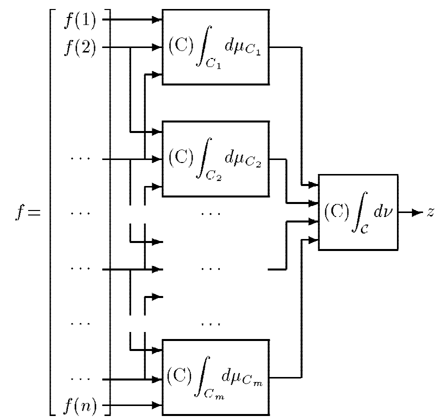


Figure 1: 2-step Choquet integral

In this work we further study this integral. In particular, we show that this integral as well as the Sugeno integral can be represented as a 2-step Choquet integral with constant and then we study some of its properties. In particular, we study the integral of a crisp set and some of the relationships with Sugeno and Choquet integrals.

The structure of the paper is as follows. In Section 2 we define the twofold integral and prove its relation with 2-step Choquet integrals. Then, in Section 3 we present some properties about the integral. Finally, in Section 4 we finish with some conclusions.

## 2 Multi-step Choquet integral and Twofold integral

We consider below the definition of the integral.

First, we present the basic definitions of a fuzzy measure and Choquet and Sugeno integral with respect to a fuzzy measure.

**Definition 1.** [9] A fuzzy measure $\mu$ on $(X, 2^X)$ is a real valued set function,

$$\mu : 2^X \longrightarrow [0, 1]$$

with the following properties.

1. $\mu(\emptyset) = 0$, $\mu(X) = 1$
2. $\mu(A) \leq \mu(B)$ whenever $A \subset B$, $A, B \in 2^X$.

**Definition 2.** [3, 5] Let $\mu$ be a fuzzy measure on $(X, 2^X)$. The Choquet integral of $f \in R_+^n$ with respect to $\mu$ is defined by

$$C_\mu(f) := \int_0^\infty \mu_f(r)dr,$$

where $\mu_f(r) = \mu(\{i|f(i) \geq r\})$.

**Definition 3.** *Define the delta measure $\delta_x$ of $x$ for $x \in X$ by*

$$\delta_x(E) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if o.w.} \end{cases}$$

*for $E \in 2^X$.*

Then we have the following property: $C_{\delta_x}(f) = f(x)$.

**Definition 4.** *[5] The Choquet integral with constant $b$ of $x \in R_+^n$ with respect to $\mu$ is defined by*

$$C_{\mu,b}(x) := \int_0^\infty \mu_x(r)dr + b,$$

*where $\mu_x(r) = \mu(\{i|x_i \geq r\})$ and $b$ is a real number.*

**Definition 5.** *[4, 7, 5]* Let $X$ be a finite set with $|X| = n$ and $\mathcal{I}$ be a functional $\mathcal{I} : R_+^n \to R_+$. A 1-step Choquet integral with constant is defined by

$$\mathcal{I}(x) = C_{\mu,b}(x)$$

for $x \in R_+^n$, and a given $\mu$. The functional $\mathcal{I}$ is said to be a k-step Choquet integral with constant if there exist a natural number $m_k$, $k_j (k_j < k)$ step Choquet integrals with constant $\mathcal{I}_j : R_+^n \to R_+$ for $j = 1, \ldots, m_k$ and a fuzzy measure $\mu_k$ on $2^{\{1,\ldots,m_k\}}$ such that $k = \max\{k_j | j = 1, \ldots, m_k\} + 1$ and

$$\mathcal{I}(x) = C_{\mu_{\mu_k, b_k}}(\mathcal{I}_j(x)).$$

Figure 1 gives a graphical representation of 2-step Choquet integrals.

**Example 1.** *Let $X$ be $X = \{1, 2, 3, 4\}$. Consider the function $f : X \to R$. $f(1) \leq f(2) \leq f(3) \leq f(4)$. Suppose that the Choquet integral with constant $b_{1i}; i = 1, 2, 3$ with respect to the fuzzy measures $\mu_{11}$,$\mu_{12}$ and $\mu_{13}$ satisfy*

$$C_{\mu_{\mu_{11},b_{11}}}(f) \leq C_{\mu_{\mu_{12},b_{12}}}(f) \leq C_{\mu_{\mu_{13},b_{13}}}(f).$$

*Let $M_2 := \{1, 2, 3\}$ and $A_{21} := \{1, 2, 3\}$,$A_{22} := \{2, 3\}$,$A_{23} := \{3\}$ and $\mu_{2,j}; j = 1, 2$ be fuzzy measure on $2^{M_2}$. The 2-step Choquet integral $TwC_{\mu_{\mu_{2i},b_{2i}}}(f); i = 1, 2$ with constant $b_{2i}$ is defined by*

$$TwC_{\mu_{\mu_{2i},b_{2i}}}(f) := \sum_{j=1}^{3} C_{\mu_{\mu_{1j},b_{1j}}}(f)(\mu_{2i}(A_{2j}) - \mu_{2i}(A_{2\ j+1})) + b_{2i}$$

*where $i = 1, 2$ and $A_{24} := \emptyset$. Suppose that $TwC_{\mu_{\mu_{21},b_{21}}}(f) \leq TwC_{\mu_{\mu_{22},b_{22}}}(f)$. Let $M_3 := \{1, 2\}$ and $A_{31} := \{1, 2\}$,$A_{32} := \{2\}$,$A_{33} := \emptyset$ and $\mu_3$ be a fuzzy measure on $2^{M_3}$. The 3-step Choquet integral $ThrC_{\mu_{\mu_3,b_3}}(f)$ with constant $b_{2i}$ is defined by*

$$ThrC_{\mu_{\mu_3,b_3}}(f) := \sum_{j=1}^{2} C_{\mu_{\mu_{2j},b_{2j}}}(f)(\mu_3(A_{3j}) - \mu_3(A_{3\ j+1})).$$

**Definition 6.** *[9] The Sugeno integral of a function $f : X \to [0, 1]$ with respect to $\mu$ is defined by*

$$S_\mu(f) := \sup_{r \in [0,1]} [r \wedge \mu_f(r)]$$

*where $\mu_f(r) := \mu(\{x|f(x) > r\})$.*

**Proposition 1.** *Sugeno integral is represented as a 2-step Choquet integral with constant.*

*Proof.* Let $\mu$ be a fuzzy measure on X with $|X| = N$. Denote the Sugeno integral of a function $f : X \to [0, 1]$ with respect to the fuzzy measure $\mu$ by:

$$S_\mu(f) = \bigvee_{j=1}^{N} f(x_{s(j)}) \wedge \mu(A_{s(j)}) \tag{1}$$

where $f(x_{s(i)})$ indicates that the indices have been permuted so that $0 \leq f(x_{s(1)}) \leq \cdots \leq f(x_{s(N)}) \leq 1$, $A_{s(i)} = \{x_{s(i)}, \cdots, x_{s(N)}\}$, $A_{s(N+1)} = \emptyset$. Let $X' := X \cup \{2^X \setminus \emptyset\}$. Define $0-1$ fuzzy measure on $(X', 2^{X'})$ by

$$\nu(E) := \begin{cases} 1 & \text{if} A \cup \{A\} \subset E \text{ for some } A \in 2^X \setminus \{\emptyset\} \\ 0 & \text{o.w.} \end{cases}$$

for $E \in 2^{X'}$ Then we have

$$C_\nu(F) = \sup_{A \in 2^X \setminus \{\emptyset\}} \inf_{x \in A \cup \{A\}} F(x),$$

where $F : X' \to [0,1]$. Note that in fact, it follows from the definition of $\nu$ that

$$C_\mu(F) = \sup_{A \in 2^X \setminus \{\emptyset\}} C_{\mu_A}(F)$$

where $\mu_A$ is a 0-1 a possibility measure defined by

$$\nu_A(E) := \begin{cases} 1 & \text{if} A \cup \{A\} \subset E \\ 0 & \text{o.w.} \end{cases}$$

for $E \in 2^{X'}$. Then, since $C_{\mu_A}(F) = \inf_{x \in A} F(x)$, we have

$$C_\nu(F) = \sup_{A \in 2^X \setminus \{\emptyset\}} \inf_{x \in A \cup \{A\}} F(x).$$

Next, for each $\omega$ in $2^{X'}$ we define a Choquet integral denoted $F_f(\omega)$. This is a Choquet integral with constant of the function $f$ with respect to the measure $\nu_\omega$. The Choquet integrals are defined as follows:

$$F_f(\omega) := \begin{cases} C_{\nu_\omega}(f) & \text{if} \omega = x \in X \\ C_{\nu_\omega}(f) + \mu(A) & \text{if} \omega = A \in 2^X \setminus \{\emptyset\} \end{cases}$$

and where the fuzzy measure $\nu_\omega$ on $(X, 2^X)$ are defined by

$$\nu_\omega := \begin{cases} \delta_x & \text{if} \omega = x \in X \\ 0 & \text{if} \omega = A \in 2^X \setminus \{\emptyset\} \end{cases}$$

where $\delta_x$ is as in Definition 3.

Note that $F_f(\omega) = \mu(A)$ for all $\omega = A \in 2^X \setminus \{\emptyset\}$. Also, if $f(x) \le \mu(A)$ for $x \in A$, we have

$$\inf_{x \in A} f(x) = \inf_{x \in A} F_{\nu_x}(f).$$

If $f(x) \ge \mu(A)$ for $x \in A$, we have

$$\mu(A) = \inf_{x \in A} F_{\nu_x}(f).$$

Therefore we have

$$S_\mu(f) = C_\nu(F_f(\cdot)).$$

□

**Definition 7.** A functional $\mathcal{I} : R_+^n \to R_+$ is said to be piecewise linear (for short PL) if there exists a finite family $\mathcal{D}_j : j \in J$ of closed domains such that

$$\bigcup_{j \in J} \mathcal{D}_j = R^n$$

and the restriction of $\mathcal{I}$ on every $\mathcal{D}_j$ is linear. A unique linear functional $\mathcal{J}_j$ on $R_+^n$ which coincided with $\mathcal{I}$ on a given $\mathcal{D}_j, j \in J$ is said to be a component of $\mathcal{I}$.

It is obvious that the Choquet integral is a PL functional. Ovchinikov [8] shows that the Sugeno integral [9] is a PL functional.

**Definition 8.** *[10] Let $\mu_C$ and $\mu_S$ be two fuzzy measures on X, then the* twofold integral *of a function $f : X \to [0,1]$ with respect to the fuzzy measures $\mu_S$ and $\mu_C$ is defined by:*

$$TI_{\mu_S,\mu_C}(f) =$$

$$\sum_{i=1}^{N} \Big( \big( \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \big) \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big) \Big) \qquad (2)$$

*where $f(x_{s(i)})$ indicates that the indices have been permuted so that $0 \le f(x_{s(1)}) \le \cdots \le f(x_{s(N)}) \le 1$, $A_{s(i)} = \{x_{s(i)}, \cdots, x_{s(N)}\}$, $A_{s(N+1)} = \emptyset$.*

Applying Proposition 1, it is obvious that the twofold integral is represented as 3-step Choquet integral with constant. Moreover since 3-step Choquet integral with constant is piecewise linear, the twofold integral is represented as 2-step integral with constant.

**Theorem 1.** *The twofold integral is represented as the 2-step Choquet integral with constant.*

*Proof.* Let $f \in R^n$ and $S_{\mu_{S_i}}(f) := \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})$. Suppose that $f \le g$ for $f, g \in R^n$, we have $S_{\mu_{S_i}}(f) \le S_{\mu_{S_i}}(g)$. Since Choquet integral is monotone, we have $C_{\mu_C}(S_{\mu_{S_i}}(f)) \le C_{\mu_C}(S_{\mu_{S_i}}(g))$. That is, Twofold integral is monotone. Since Sugeno integral is piecewise linear, there exists a set $\mathcal{D}$ of functions such that $S_{\mu_{S_i}}(f+g) = S_{\mu_{S_i}}(f) + S_{\mu_{S_i}}(g)$ for $f, g \in D$. Suppose that $\mathcal{D}' \subset R^n$ is a set of functions satisfying $S_{\mu_{S_i}}(f) < S_{\mu_{S_j}}(f) \Rightarrow S_{\mu_{S_i}}(g) \le S_{\mu_{S_j}}(g)$. Then the twofold integral $TI$ is piecewise linear on $\mathcal{D} \cap \mathcal{D}'$. Therefore applying Ovchinikov's theorem $TI$ has a max-min representation. Then the max-min representation is represented by 2-step Choquet integral with constant. □

## 3 Properties

In this section we study some new properties of this integral. We start considering the integration of the characteristic function of a set $A$ and proving that in this case, the integral is the product of the two measures for this set.

**Proposition 2.** *Let $A$ be a subset of $X$ and let $f$ be the characteristic function of $A$ (this is, $f$ is defined for $x$ as one if and only if $x \in A$), then the twofold integral of $f$ with respect to the two measures $\mu_S$ and $\mu_C$ is equal to:*

$$TI_{\mu_S,\mu_C}(f) = \mu_S(f) \cdot \mu_C(f)$$

*Proof.* First, let us note that $f(x_{s(j)})$ is ordered with respect to $s$ so that $f(x_{s(i)}) = 0$ for $i < |X| - |A| + 1$ and that $f(x_{s(i)}) = 1$ for all $i \ge |X| - |A| + 1$.

Therefore, the terms

$$\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})$$

are equal to 0 for $i < |X| - |A| + 1$ and equal to $\mu_S(A)$ for $i \geq |X| - |A| + 1$.

Now, replacing these values in the twofold integral we get:

$$\sum_{i=1}^{|X|-|A|} \Big(0\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big)\Big) +$$
$$\sum_{i=|X|-|A|+1}^{N} \Big(\mu_S(A)\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big)\Big)$$

In this expression, the first term is zero and the second can be rewritten as:

$$\mu_S(A)\big(\mu_C(A_{s(|X|-|A|+1)}) - \mu_C(A_{s(N+1)})\big)$$

That, being $\mu_C(A_{s(N+1)}) = 0$ because $A_{s(N+1)} = \emptyset$, and being $\mu_C(A_{s(|X|-|A|+1)}) = \mu_C(A)$ because $A_{s(|X|-|A|+1)} = A$, is equivalent to:

$$\mu_S(A)\mu_C(A_{s(|X|-|A|+1)}) = \mu_S(A)\mu_C(A)$$

Therefore, the proposition is proven. □

**Proposition 3.** *For all $f$, the following inequality holds:*

$$TI_{\mu_S,\mu_C}(f) \leq SI_{\mu_S}(f)$$

*where $SI$ stands for the Sugeno integral.*

*Proof.* Let us define $\alpha$ as follows: $\alpha := SI_{\mu_S}(f) = \bigvee_{j=1}^{N} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})$. From this definition, it is clear that the following inequality holds:

$$\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq \alpha$$

for all $i$.
¿From this inequality, we can proof the following:

$$\sum_{i=1}^{N} \Big(\big(\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})\big)\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big)\Big) \leq$$
$$\sum_{i=1}^{N} \Big(\alpha\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big)\Big)$$

As the right hand side of this inequality is equal to $\alpha$, the proposition is proven. □

**Proposition 4.** *For all $f$, the following inequality holds:*

$$TI_{\mu_S,\mu_C}(f) \leq CI_{\mu_C}(f)$$

*where $CI$ stands for the Choquet integral.*

*Proof.* First, let us proof that $\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq f(x_{s(i)})$. This is so, because $f(x_{s(1)}) \leq f(x_{s(2)}) \leq \cdots \leq f(x_{s(i)})$ and, additionally, $f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq f(x_{s(j)})$.

Therefore, the following holds:

$$\sum_{i=1}^{N} \Big(\big(\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})\big)\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big)\Big) \leq$$
$$\sum_{i=1}^{N} \Big(f(x_{s(i)})\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big)\Big)$$

As the right hand side of this expression is the Choquet integral of f with respect to the measure $\mu_C$ the proposition is proven. □

Since monotone convergence theorem is valid for both Choquet and Sugeno integral with finite universal set $X$, it is also valid for the twofold integral.

**Theorem 2.** *Let $\mu_C$ and $\mu_S$ be two fuzzy measures on $(X, 2^X)$. If the monotone increasing sequence $\{f_n\}$ of functions $f_n : X \to [0,1]$ converge to a function $f$, that is, $f_n \uparrow f$, then*

$$TI_{\mu_S,\mu_C}(f_n) \uparrow TI_{\mu_S,\mu_C}(f)$$

*as $n \to \infty$.*

*Proof.* Since $X$ is finite, the fuzzy measure $\mu_S$ is continuous. So we have

$$\bigvee_{j=1}^{i} f_n(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \uparrow \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})$$

as $n \to \infty$. Since $\mu_C$ is also continuous, we have

$$C_{\mu_C}(\bigvee_{j=1}^{i} f_n(x_{s(j)}) \wedge \mu_S(A_{s(j)})) \uparrow C_{\mu_C}(\bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}))$$

as $n \to \infty$.

□

**Remark.** Benvenutti and Mesiar [1] proposed the hypothesis below:
*Hypothesis* (Benvenuti and Mesiar [1])
A lower semi continuous functional $\mathcal{I} : R_+^n \to R_+$ which is monotone, homogeneous and additively homogeneous (for short, Add-Hom functional) can be represented as an $n$-step Choquet integral, where lower semi continuous means that $x_n \uparrow x$ implies $\mathcal{I}(x_n) \uparrow \mathcal{I}(x)$. We say that the functional $I$ on $R_+^n$ is additively homogeneous if $I(f + a) = I(f) + a$ for $f \in R_+^n$ and $a \geq 0$ is constant. Narukawa and Murofushi [7] show that the hypothesis above is not true, using the relations of the piecewise linear functional and n-step Choquet integral. Theorem 2 shows that the twofold integral is lower semi-continuous in the Benvenutti sense. Since $\mu(X) = 1$ for a fuzzy measure in this paper, we have

$$\begin{aligned} C_{\mu,b}(f+a) &= C_\mu(f+a) + b \\ &= C_\mu(f) + a + b = C_{\mu,b}(f) + a, \end{aligned}$$

that is, Choquet integral with constant is Add-Hom functional. But since Sugeno integral is not Add-Hom, 2-step Choquet integral is not always Add-Hom. Twofold integral is one of the examples of 2-step Choquet integrals that are lower semincontinuous but not Add-Hom.

## 4 Conclusions

In this work we have studied the twofold integral. We have proven a theorem about the representation of the integral in terms of 2-step Choquet integral. Then, we have compared the twofold integral and Sugeno and Choquet ones.

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## References

[1] P. Benvenuti & R. Mesiar, A note on Sugeno and Choquet integrals, *Proc. 8th Int. Conf. Information Processing and Management of Uncertainty in Knowledge-Based Systems* (2000) 582–585.

[2] R. Mesiar & D. Vivona, Two-step integral with respect to fuzzy measure, *Tatra Mt. Math. Publ.* **16** (1999) 359–368.

[3] T. Murofushi & M. Sugeno, An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, *Fuzzy Sets and Systems*, **29** (1989) 201–227.

[4] T. Murofushi & Y. Narukawa, A characterization of multi-step discrete Choquet integral, *6th Int. Conf. Fuzzy Sets Theory and Its Applications, Abstracts*, (2002) p. 94.

[5] T. Murofushi & Y. Narukawa, A characterization of multi-level discrete Choquet integral over a finite set (in Japanese). *Proc. of 7th Workshop on Evaluation of Heart and Mind* (2002) pp.33-36.

[6] T. Murofushi & M. Sugeno, Fuzzy t-conorm integral with respect to fuzzy measures: generalization of Sugeno integral and Choquet integral, *Fuzzy Sets and Systems*, 42:1 (1991) 57-71.

[7] Y. Narukawa & T. Murofushi, The $n$-step Choquet integral on finite spaces, *Proc. 9th Int. Conf. Information Processing and Management of Uncertainty in Knowledge-Based Systems* (2002) 539–543.

[8] S. Ovchinnikov, Max-min representation of piecewise linear functions, *Contributions to Algebra and Geometry*, **43** (2002) 297–302.

[9] M. Sugeno, *Theory of fuzzy integrals and its application*, Thesis, Tokyo Institute of Technology, 1974.

[10] V. Torra, Twofold integral: A Choquet integral and Sugeno integral generalization, submitted.