# Twofold integral: A Choquet integral and Sugeno integral generalization

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**Abstract**

Sugeno and Choquet integrals are well-known fuzzy integrals that are commonly in use for aggregation purposes. They integrate a function with respect to a fuzzy measure. Such fuzzy measure is used to represent some background knowledge about the information sources being aggregated.

Murofushi and Sugeno defined in 1991 the fuzzy t-conorm integral, an integral that generalizes both Sugeno and Choquet integrals. Such generalization is based on the use of t-conorm and product-like operators that generalize addition/maximum and product/minimum present in the former integrals. In this way, fuzzy t-conorm integral corresponds to an integral of a function with respect to a single fuzzy measure. According to its construction, integral particularizations are achieved appropriately selecting t-conorm and t-norm like operators.

In this work we introduce the *twofold integral*. This is an alternative operator that generalizes both Sugeno and Choquet integral. The

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approach is based on the use of two fuzzy measures instead of one. Particular selection of the fuzzy measures permits the reduction of the integral to either Sugeno or Choquet integrals. The motivation of our approach is to keep in a single operator the semantics of both measures.

**Keywords:** Information fusion, aggregation operators, Sugeno integral, Choquet integral, Fuzzy t-conorm integral, twofold integral.

# 1 Introduction

At present, there exist a large number of aggregation operators for numerical data. Some of the most well-known ones are the arithmetic mean, weighted mean, OWA, median. Most of these operators are encompased in two large families of operators: the Choquet integral family and the Sugeno integral family.

These families take their name on the Choquet integral and the Sugeno integral. These integrals combine data from a set of sources $X$ taking into account some background knowledge. More specifically, these integrals integrate a function (that associates each source with its value) with respect to a fuzzy measure (that represents the background knowledge – following Artificial Intelligence terminology). For particular fuzzy measures, the operators reduce to the operators enumerated above.

In 1991, Murofushi and Sugeno defined the fuzzy t-conorm integral to put in a common framework all these operators. Their integral generalizes both Choquet and Sugeno integral and, therefore, it also generalize all the particular aggregation operators given above.

The generalization was based on the use of t-conorms (a well-known operator in fuzzy sets) and product-like operators. This construction achieved a compact form that using particular t-conorms and product-like operators reduce to Choquet or Sugeno integrals.

The generalization relies on an implicit assumption that the fuzzy measure present in both Choquet and Sugeno integral has the same semantics and, therefore, the generalized integral only has one fuzzy measure.

In this work, we also build a generalization of both Sugeno and Choquet integral. However, our basic assumption is radically different. We start considering that the fuzzy measure present in these integrals have a different semantics and, therefore, the new generalization requires two fuzzy measures. One fuzzy measure would correspond, from the semantics point of view, to the one used in a Sugeno integral and the other would correspond to the one used in a Choquet integral. Naturally, besides of that, we are interested on an integral that can be reduced to either the Choquet or Sugeno one when particular fuzzy measures are selected.

The structure of this work is as follows. In Section 2, we describe fuzzy measures and some of their interpretations and, also, Sugeno, Choquet and fuzzy t-conorm (by Murofushi and Sugeno) integrals. Then, in Section 3, we introduce the twofold integral and prove that this is a proper generalization of both Sugeno and Choquet and that satisfies the usual properties considered for aggregation operators. The paper finishes in Section 4 with the conclusions.

## 2 Preliminaries

In this section we review some definitions and results about fuzzy measures and integrals. We limit our description to the case of having a finite set of sources or variables $X$. This is the case considered in the rest of the paper and the usual case in most applications related with aggregation or information fusion.

The section is divided in two subsections. The first one devoted to fuzzy measures and the second one devoted to fuzzy integrals.

## 2.1 Fuzzy measures

We give below the definition of fuzzy measures together with an example. Then, some interpretations (semantics) are considered. For details and a state-of-the-art description of the subject see [2].

**Definition 1** *A fuzzy measure (or capacity) $\mu$ on a finite set $X$ is a set function $\mu : \wp(X) \rightarrow [0,1]$ satisfying the following axioms:*

*(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$ (boundary conditions)*

*(ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ (monotonicity)*

As an example of fuzzy measure, we define below the one corresponding to complete ignorance.

**Definition 2** *The fuzzy measure representing complete ignorance, denoted by $\mu^*$, is defined as follows:*

$$\mu^*(A) = \begin{cases} 1 & \textit{for all } A \neq \emptyset \\ 0 & \textit{if } A = \emptyset \end{cases}$$

Fuzzy measures generalize probability measures and also necessity and possibility measures. At present there exist several interpretations (semantics) for fuzzy measures. We underline the following ones:

**a)** A fuzzy measure $\mu(A)$ corresponds, according to Sugeno [11], to "the situation that one guesses whether an ill-located element $x$ in a universe $X$ belongs to a subset $A$ of $X$, and he regarded a fuzzy measure of $A$ as the grade of $x \in A$" (see [8]). Usually, Sugeno integrals are used for fuzzy measure induced from possibility distributions like possibility and necessity measures.

**b)** A fuzzy measure is a kind of generalization of probability distributions. Belief and plausibility measures, two kind of dual fuzzy measures, lead this relationship. We underline three ways of relating fuzzy measures to probability distributions (see [4], [12] for details).

**(i)** Belief and plausibility as inner and outer measures. In a probability space $(X, Y, Pr)$ where $Y \subseteq \wp(X)$ is a $\sigma$-algebra of subsets of $X$, the probability measure $Pr$ is not necessarily defined on $\wp(X)$, but only on $Y$. In this case, $Pr$ can be extended to $\wp(X)$ as follows:

$$\mu_{+}(A) = sup\{Pr(y) | y \subseteq A \text{ and } y \in Y\}$$

$$\mu^{+}(A) = inf\{Pr(y) | y \supseteq A \text{ and } y \in Y\}$$

$\mu_{+}$ and $\mu^{+}$ are fuzzy measures. In fact, $\mu_{+}$ is a belief function and $\mu^{+}$ is a plausibility function.

**(ii)** Probability interval induced by a belief measure. Each belief measure $Bel$, with its dual plausibility measure $Pl$ induces a probability interval $P_{Bel}$:

$$P_{Bel} = \{p | p \text{ is a probability and } Bel(A) \leq p(A) \leq Pl(A) \text{ for all } A \in \wp(X)\}$$

**(iii)** Fuzzy measures as a probability distribution suffering from an information loss process. Here, information loss is defined in terms of a sequence of basic information loss steps defined, in turn, over basic probability assignments (bpa). See [12] for details on this semantics and [10] for bpa definition. The basic information loss step is defined as follows: Given $m_1$ the bpa of a measure $\mu_1$ and $m_2$ the bpa of a measure $\mu_2$, it is said that the set $A_0$ has lost information in favour of $A_1$ in bpa $m_2$ iff:

$m_2(A) = m_1(A)$ for all $A \subseteq X$ such that $A \neq A_0$ and $A \neq A_1$

$m_2(A_0) = m_1(A_0) - k$ with $m_1(A_0) \geq k$

$m_2(A_1) = m_1(A_1) + k$

Note that the fuzzy measure $\mu^*$ given in Definition 2 for representing complete ignorance fits to all these interpretations.

## 2.2 Integrals

Now, we review the Sugeno and Choquet integrals. Both integrate a function with respect to a fuzzy measure. While any fuzzy measure can be used in both operators, Sugeno integral seems more appropriate for measures with the fuzzy-related interpretation and Choquet integral with the probability-related interpretation.

The section also includes some propositions describing properties of these integrals.

**Definition 3** *[11] Let $\mu$ be a fuzzy measure on $X$, then the* Sugeno integral *(SI) of a function $f : X \to [0,1]$ with respect to $\mu$ is defined by:*

$$(S)\int f d\mu = \bigvee_{i=1,N} \big(f(x_{s(i)}) \wedge \mu(A_{s(i)})\big) \tag{1}$$

*where $f(x_{s(i)})$ indicates that the indices have been permuted so that $0 \leq f(x_{s(1)}) \leq ... \leq f(x_{s(N)}) \leq 1$, $A_{s(i)} = \{x_{s(i)}, ..., x_{s(N)}\}$ and $f(x_{s(0)}) = 0$.*

In this definition $\vee$ corresponds to the maximum and $\wedge$ corresponds to the minimum. For the sake of simplicity, we will also use $SI_\mu(a_1, \ldots, a_N)$ to denote the Sugeno integral of $f(x_i) = a_i$.

**Definition 4** *[1] Let $\mu$ be a fuzzy measure on $X$, then the* Choquet integral *(CI) of a function $f : X \to [0, 1]$ with respect to the fuzzy measure $\mu$ is defined by:*

$$(C) \int f d\mu = \sum_{i=1}^{N} [f(x_{s(i)}) - f(x_{s(i-1)})] \mu(A_{s(i)}) \tag{2}$$

*where $f(x_{s(i)})$ indicates that the indices have been permuted so that $0 \leq f(x_{s(1)}) \leq \cdots \leq f(x_{s(N)}) \leq 1$, $A_{s(i)} = \{x_{s(i)}, \cdots, x_{s(N)}\}$ and $f(x_{\sigma(0)}) = 0$.*

Similar to the case of the Sugeno integral, we will also use $CI_\mu(a_1, \ldots, a_N)$ to denote Choquet integral for $f(x_i) = a_i$. An important property of both Sugeno and Choquet integrals is that they are monotonic with respect to the function $f$.

**Proposition 1** *Let $f_1$ and $f_2$ be two functions over $X$ such that $f_1(x) \geq f_2(x)$ for all $x \in X$. Then, the following equations hold:*

$$(S) \int f_1 d\mu \geq (S) \int f_2 d\mu$$

$$(C) \int f_1 d\mu \geq (C) \int f_2 d\mu$$

Another property of Sugeno and Choquet integrals that is relevant for aggregation purposes is that the output of the integrals is a value in the interval defined by the minimum and the maximum of the values they aggregate.

**Proposition 2** *Let $f_1$ a function over $X$. Then, the following hold:*

$$\min f \leq (S) \int f d\mu \leq \max f$$

$$\min f \leq (C) \int f d\mu \leq \max f$$

Propositions 1 and 2 are reported e.g. in [3] or [13].

Now, we turn into Murofushi and Sugeno fuzzy t-conorm integral. This integral, defined in [9], is to generalize both Sugeno and Choquet integrals. Before giving its definition, we need some basic elements to build latter on the integral.

**Definition 5** *[9] $\mathcal{F} = (\Delta, \perp, \underline{\perp}, \otimes)$ is a* t-conorm system for integration *if and only if:*

1. *$\Delta, \perp, \underline{\perp}$, are continuous t-conorms, which are the maximum or Archimedean.*
2. *$\otimes : ([0,1], \Delta) \times ([0,1], \perp) \to ([0,1], \underline{\perp})$ is a product-like operation fulfilling:*
   - *(a) $\otimes$ is continuous on $(0,1]^2$*
   - *(b) $a \otimes x = 0$ if and only if $a = 0$ or $x = 0$*
   - *(c) when $x \perp y < 1$, then $a \otimes (x \perp y) = (a \otimes x) \underline{\perp} (a \otimes y)$ for all $a \in [0,1]$*
   - *(d) when $a \Delta b < 1$, then $(a \Delta b) \otimes x = (a \otimes x) \underline{\perp} (b \otimes x)$, for all $x \in [0,1]$.*

**Definition 6** *[9] For a given t-conorm $\Delta$, the operation $-_\Delta$ on $[0,1]^2$ is defined by: $a -_\Delta b := inf\{c | b \Delta c \geq a\}$.*

**Definition 7** *[9] Let $\mu$ be a fuzzy measure on $X$, let $\mathcal{F} = (\Delta, \perp, \underline{\perp}, \otimes)$ be a t-system for integration. Then, the fuzzy t-conorm integral (or fuzzy t-integral)* *of a function* $f : X \to [0,1]$ *based on* $(\Delta, \perp, \underline{\perp}, \otimes)$ *with respect to* $\mu$ *is defined by:*

$$(\mathcal{F}) \int f \otimes d\mu = \underline{\perp}_{i=1}^{N} (a_i -_\Delta a_{i-1}) \otimes \mu(A_{s(i)})$$

*where* $a_i = f(x_{s(i)})$, $a_0 = f(x_{s(0)}) = 0$ *and* $A_{s(i)} = \{x_{s(i)}, \cdots, x_{s(N)}\}$.

This definition reduces to Choquet and to Sugeno integral with appropriate definitions of the t-conorm system for integration. This is stated in the next proposition.

**Proposition 3** *The t-conorm integral satisfies the following properties:*

1. *It is equivalent to the Choquet integral when the t-conorm system equals to* $(\Delta, \perp, \underline{\perp}, \otimes) = (\hat{+}, \hat{+}, \hat{+}, \cdot)$ *where* $\hat{+}(x, y) = min(1, x + y)$ *and where* $\cdot$ *stands for the product.*

2. *It is equivalent to the Sugeno integral when the t-conorm system equals to* $(\Delta, \perp, \underline{\perp}, \otimes) = (\max, \max, \max, \min)$.

# 3 Twofold integral

We have reviewed in Section 2 Choquet and Sugeno integrals and their generalization defined by Murofushi and Sugeno: the fuzzy t-conorm integral. As can be observed in its definition and in Proposition 3, this generalization is built through the unification of the operations addition and maximum in terms of t-conorms $\Delta, \perp, \underline{\perp}$ and product and minimum in terms of a product-like operation $\otimes$. Additionally, it is important to underline that the fuzzy measure present in both integrals is considered the same measure and, therefore, the t-conorm integral considers a single fuzzy measure.

Our approach for generalization is completely different. The cornerstone of our construction is to consider fuzzy measures in Sugeno and Choquet integrals as completely different from a semantics point of view. In particular, we consider Sugeno's measure as denoting fuzziness and Choquet's measure as denoting randomness (see Section 2.1). From this assumption, it is natural to infer that the generalization, we call it the twofold integral, has to be an expression that contains the two measures. Then, naturally, particularizations of the measures should lead either to the Choquet integral or the Sugeno integral.

For reducing the twofold integral into Choquet or Sugeno ones, we consider the measure corresponding to complete ignorance. This is the measure $\mu^*$ in Definition 2.

Let $\mu_S$ be the fuzzy measure used in a Sugeno integral and let $\mu_C$ be the fuzzy measure used in a Choquet integral. Let

$$TI_{\mu_S,\mu_C}(a_1,\ldots,a_N)$$

denote our generalized integral called twofold integral. Then, this expression reduces to $CI_{\mu_C}(a_1,\ldots,a_N)$ when $\mu_C = \mu^*$ and reduces to $SI_{\mu_S}(a_1,\ldots,a_N)$ when $\mu_S = \mu^*$. This is, when nothing is known about the membership of $x$ (i.e., $\mu_S = \mu^*$) but only information about randomness is known, then the integral to be used is Choquet's one. Instead, when nothing is known about the probability distribution (i.e., $\mu_C = \mu^*$) but only the fuzziness then the integral to be used is Sugeno's one.

Now, we define below the twofold integral.

**Definition 8** *Let $\mu_C$ and $\mu_S$ be two fuzzy measures on X, then the* twofold integral *of a function $f : X \to [0,1]$ with respect to the fuzzy measure $\mu$ is* *defined by:*

$$TI_{\mu_S,\mu_C}(f) = \sum_{i=1}^{N} \Big( \big( \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \big) \big( \mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)}) \big) \Big) \quad (3)$$

*where* $f(x_{s(i)})$ *indicates that the indices have been permuted so that* $0 \leq f(x_{s(1)}) \leq \cdots \leq f(x_{s(N)}) \leq 1$, $A_{s(i)} = \{x_{s(i)}, \cdots, x_{s(N)}\}$, $A_{s(N+1)} = \emptyset$.

We will also use the notation $TI_{\mu_S,\mu_C}(a_1, \ldots, a_N)$ to denote the integral $TI_{\mu_S,\mu_C}(f)$ with $a_i = f(x_i)$.

In the next proposition, we consider whether the integration is well defined when there are several elements in $X$ such that $f(x_i) = f(x_j)$ for $i \neq j$. Note that in such case, there exist several permutations $s$ satisfying:

$$0 \leq f(x_{s(1)}) \leq ... \leq f(x_{s(N)}) \leq 1$$

The proposition proves that the result of the twofold integral is independent of selecting $f(x_i)$ before or after $f(x_j)$ in the permutation. In other words, all permutations lead to the same value. In fact, the proposition below is established considering only two $x_i$ with the same value $f(x_i)$. The same applies for more coincident values. This latter case is described below.

**Proposition 4** *Let* $X$ *be a finite set, let* $f$ *be a function from* $X$ *to* $[0, 1]$ *such that there exist* $i_1$ *and* $i_2$ *with* $i_1 \neq i_2$ *such that* $f(x_{i_1}) = f(x_{i_2})$ *but otherwise* $f(x_i) \neq f(x_j)$ *when* $i \neq j$, *and let* $s_1$ *and* $s_2$ *be the two possible permutations on* $\{1, \ldots, |X|\}$ *that satisfy:*

$$0 \leq f(x_{s(1)}) \leq ... \leq f(x_{s(N)}) \leq 1$$

*Then, the following holds:*

$$\sum_{i=1}^{N} \Big( \big( \bigvee_{j=1}^{i} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) \big) \big( \mu_C(A_{s_1(i)}) - \mu_C(A_{s_1(i+1)}) \big) \Big) =$$

$$\sum_{i=1}^{N}\Big(\big(\bigvee_{j=1}^{i} f(x_{s_2(j)}) \wedge \mu_S(A_{s_2(j)})\big)\big(\mu_C(A_{s_2(i)}) - \mu_C(A_{s_2(i+1)})\big)\Big)$$

*Proof.* First, let us consider the two possible permutations $s_1$ and $s_2$. Whithout loss of generality, we define $s_1$ and $s_2$ as follows:

$$f(x_{s_1(1)}) < \cdots < f(x_{s_1(r)}) = f(x_{s_1(r+1)}) < \cdots < f(x_{s_1(N)})$$

$$\text{with } s_1(r) = i_1, s_2(r+1) = i_2$$

$$f(x_{s_2(1)}) < \cdots < f(x_{s_2(r)}) = f(x_{s_2(r+1)}) < \cdots < f(x_{s_2(N)})$$

$$\text{with } s_1(r) = i_2, s_2(r+1) = i_1$$

Now, it is clear that we can decompose the addition in the twofold integral expression on four parts as follows (we consider here the expression for $s_1$):

$$\sum_{i=1}^{N}\Big(\big(\bigvee_{j=1}^{i} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\big)\big(\mu_C(A_{s_1(i)}) - \mu_C(A_{s_1(i+1)})\big)\Big) = \tag{4}$$

$$\sum_{i=1}^{r-1}\Big(\big(\bigvee_{j=1}^{i} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\big)\big(\mu_C(A_{s_1(i)}) - \mu_C(A_{s_1(i+1)})\big)\Big) + \tag{5}$$

$$+\Big(\big(\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\big)\big(\mu_C(A_{s_1(r)}) - \mu_C(A_{s_1(r+1)})\big)\Big) + \tag{6}$$

$$+\Big(\big(\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\big)\big(\mu_C(A_{s_1(r+1)}) - \mu_C(A_{s_1(r+2)})\big)\Big) + \tag{7}$$

$$+\sum_{i=r+2}^{N}\Big(\big(\bigvee_{j=1}^{i} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\big)\big(\mu_C(A_{s_1(i)}) - \mu_C(A_{s_1(i+1)})\big)\Big) + \tag{8}$$

$$\tag{9}$$

As $s_1(j) = s_2(j)$ for $i \leq r-1$ and for $i \geq r+2$, differences between the integrals for $s_1$ and $s_2$ can only be due to the terms:

$$\Big(\big(\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\big)\big(\mu_C(A_{s_1(r)}) - \mu_C(A_{s_1(r+1)})\big)\Big)$$

$$+\Big(\big(\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\big)\big(\mu_C(A_{s_1(r+1)}) - \mu_C(A_{s_1(r+2)})\big)\Big)$$

Thus, proving the equivalence in these expressions for both permutations we prove the equivalence of both integrals. In fact, this is equivalent to prove that

$$\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})$$

equals to

$$\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})$$

because in this case, the expression above corresponds to:

$$\big(\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\big) \cdot \big(\mu_C(A_{s_1(r)}) - \mu_C(A_{s_1(r+2)})\big)$$

and this does not depend on the order in $s_1$ because $A_{s_1(r)}$ includes both $x_{s_1(r)}$ and $x_{s_2(r+1)}$, and $A_{s_1}(r+2)$ does not include neither $x_{s_1}(r)$ nor $x_{s_2}(r+1)$.

Let us proof the equality:

$$\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = \bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})$$

The proof of this equivalence is done by cases. In particular, we compare $\mu_S(A_{s_1(r)})$ and $f(x_{s_1(r)})$. Before, note that $\mu_S(A_{s_1(i)}) \geq \mu_S(A_{s_1(r)})$ for all $i \leq r$ and that $f(x_{s_1(i)}) < f(x_{s_1(r)})$ for all $i < r$.

**a)** $\mu_S(A_{s_1(r)}) > f(x_{s_1(r)})$**:** In this case, $\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = f(x_{s_1(r)})$

Three subcases are considered in this case according to the comparison between $\mu_S(A_{s_1(r+1)})$ and $f(x_{s_1(r)})$ (note that $f(x_{s_1(r)}) = f(x_{s_1(r+1)})$):

$\mu_S(A_{s_1(r+1)}) > f(x_{s_1(r)})$: In this case, $\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = f(x_{s_1(r)})$ because $\mu_S(A_{s_1(r+1)}) \wedge f(x_{s_1(r+1)}) = f(x_{s_1(r)})$.

$\mu_S(A_{s_1(r+1)}) = f(x_{s_1(r)})$: In this case, $\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = f(x_{s_1(r)})$ because $\mu_S(A_{s_1(r+1)}) \wedge f(x_{s_1(r+1)}) = f(x_{s_1(r)}) = \mu_S(A_{s_1(r+1)})$.

$\mu_S(A_{s_1(r+1)}) < f(x_{s_1(r)})$: In this case, $\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = f(x_{s_1(r)})$ because $\mu_S(A_{s_1(r+1)}) \wedge f(x_{s_1(r+1)}) = \mu_S(A_{s_1(r+1)})$ but $\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = f(x_{s_1(r)}) > \mu_S(A_{s_1(r+1)})$. Therefore,
$\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) =$
$\left[\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\right] \vee \left[f(x_{s_1(r+1)}) \wedge \mu_S(A_{s_1(r+1)})\right] =$
$\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = f(x_{s_1(r)})$

**b)** $\mu_S(A_{s_1(r)}) = f(x_{s_1(r)})$: In this case, $\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = f(x_{s_1(r)}) = \mu_S(A_{s_1(r)})$. Again we consider three cases according to the comparison between $\mu_S(A_{s_1(r+1)})$ and $f(x_{s_1(r)})$:

$\mu_S(A_{s_1(r+1)}) > f(x_{s_1(r)})$: This case is not possible, because $\mu_S(A_{s_1(r)}) \geq \mu_S(A_{s_1(r+1)})$ but $\mu_S(A_{s_1(r)}) = f(x_{s_1(r)}) < \mu_S(A_{s_1(r+1)})$

$\mu_S(A_{s_1(r+1)}) = f(x_{s_1(r)})$: In this case, $\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = f(x_{s_1(r)})$ because $\mu_S(A_{s_1(r+1)}) \wedge f(x_{s_1(r+1)}) = f(x_{s_1(r)}) = \mu_S(A_{s_1(r+1)})$.

$\mu_S(A_{s_1(r+1)}) < f(x_{s_1(r)})$: In this case, $\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = f(x_{s_1(r)})$. This is analogous to the third case above.

**c)** $\mu_S(A_{s_1(r)}) < f(x_{s_1(r)})$: In this case, $\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) \geq \mu_S(A_{s_1(r)})$. Moreover, $\mu_S(A_{s_1(r+1)}) \wedge f(x_{s_1(r+1)}) = \mu_S(A_{s_1(r+1)}) \leq \mu_S(A_{s_1(r)})$. Therefore, $\bigvee_{j=1}^{r+1} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)}) = \left[\bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})\right] \vee \left[\mu_S(A_{s_1(r+1)}) \wedge f(x_{s_1(r+1)})\right] = \bigvee_{j=1}^{r} f(x_{s_1(j)}) \wedge \mu_S(A_{s_1(j)})$.

The last case proves the equality of the two expressions above and, therefore, the proposition is proven.

□

In this proposition we have considered only two $x_i$ with the same $f(x_i)$. Nevertheless, the same applies to longer sequences of $x_i$ with the same value. In this case, for all $s$, $f(x_{s(r)}) = f(x_{s(r+1)}) = f(x_{s(r+2)}) = \ldots$ and, $\mu_S(A_{s(r)}) \geq \mu_S(A_{s(r+1)}) \geq \mu_S(A_{s(r+2)}) \geq \ldots$. Therefore, the larger contribution for the integral is obtained for the first pair $f(x_{s(r)}), \mu_S(A_{s(r)})$ as in the proposition above. In the case of several subsets of $X$ with equal values of $f$, the same approach used above applies to all subsets. For all this, the integral is well defined and all permutations lead to the same value.

**Proposition 5** *Let $X$ be a finite set, let $f$ be a function from $X$ into $[0,1]$, then all permutations $s$ that satisfy:*

$$0 \leq f(x_{s(1)}) \leq ... \leq f(x_{s(N)}) \leq 1$$

*lead to the same value for the twofold integral.*

Now, we prove that this integral is a proper generalization of the Sugeno and Choquet integrals. To do so, we consider the fuzzy measure $\mu^*$ defined in Definition 2 and corresponding to a complete ignorance on the set $X$. We start proving that the twofold integral with $\mu_C = \mu^*$ reduces to a Sugeno integral. Latter we prove the analogous result for the Choquet integral using $\mu_S = \mu^*$.

**Proposition 6** *When $\mu_C = \mu^*$, the twofold integral reduces to the Sugeno integral:*

$$TI_{\mu_S,\mu_C}(a_1, \ldots, a_N) = SI_{\mu_S}(a_1, \ldots, a_N)$$

*Proof.* Note that under the assumption that $\mu_C = \mu^*$, we have that for $i = 1, \ldots, N$, $\mu_C(A_{s(i)}) = 1$ but that for $i = N+1$, $\mu_C(A_{s(i)}) = 0$. Therefore, in the following expression:

$$\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big)$$

we can consider two cases:

**Case a:** For $i = 1, \ldots, N-1$, $\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big) = 1 - 1 = 0$

**Case b:** For $i = N$, $\big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big) = 1 - 0 = 1$

Therefore,

$$\begin{aligned} TI_{\mu_S,\mu_C}(f) &= \sum_{i=1}^{N} \Big( \big( \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \big) \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big) \Big) = \\ &= \sum_{i=1}^{N-1} \Big( \big( \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \big) \cdot 0 \Big) + \Big( \big( \bigvee_{j=1}^{N} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \big) \cdot 1 \Big) = \\ &= \bigvee_{j=1}^{N} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \end{aligned}$$

As this latter expression is the Sugeno integral, the proposition is proven. □

**Proposition 7** *When $\mu_S = \mu^*$, the twofold integral reduces to the Choquet integral:*

$$TI_{\mu_S,\mu_C}(a_1, \ldots, a_N) = CI_{\mu_C}(a_1, \ldots, a_N)$$

*Proof.* When $\mu_S = \mu^*$, we have that $\mu_S(A_{s(i)}) = 1$ for $i = 1, \ldots, N$. Therefore, replacing $\mu_S(A_{s(i)})$ by its value in Expression 3 we get:

$$\sum_{i=1}^{N} \Big( \big( \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge 1 \big) \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big) \Big)$$

Now, as $\big(\bigvee_{j=1}^{i} f(x_{s(j)})\big) = f(x_{s(i)})$ because according to the permutation $s$, it holds $f(x_{s(i)}) \leq f(x_{s(i+1)})$, the integral reduces to:

$$\sum_{i=1}^{N} \Big( f(x_{s(i)}) \big(\mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)})\big) \Big)$$

As this expression corresponds to the Choquet integral, the proposition is proven.

□

An additional result that relates Sugeno, Choquet and twofold integrals is given below. The following proposition studies the case of twofold integrals with $\mu_C = \mu_S = \mu_*$.

**Proposition 8** *When $\mu_C = \mu_S = \mu^*$, the twofold integral reduces to the maximum:*

$$TI_{\mu_S,\mu_C}(a_1, \ldots, a_N) = \bigvee(a_1, \ldots, a_N)$$

*Proof.* Let us consider $\mu_S = \mu^*$ and use it in the Sugeno integral, it is clear that

$$\bigvee_{j=1}^{N} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) = \bigvee_{j=1}^{N} f(x_{s(j)}) \wedge 1 = \bigvee_{j=1}^{N} f(x_{s(j)}) = f(x_{s(N)})$$

Thus, the proposition is proven. □

## 3.1 Properties of twofold integrals

Aggregation operators are usually defined as monotonic functions that satisfy unanimity and with an output value in the interval defined by the minimum and the maximum of the values they aggregate. This is, the aggregation of $(a_1, \ldots, a_N)$ belongs to the interval $[\min a_i, \max a_i]$. In this section we study these properties for the twofold integral.

**Proposition 9** *Let $X$ be a finite set and let $\mu_C$ and $\mu_S$ be two fuzzy measures on $X$, then for all functions $f$ on $X$, the following expression holds*

$$f(x_{s(1)}) = \min f(x_{s(i)}) \leq TI_{\mu_S,\mu_C}(f) \leq \max f(x_{s(i)}) = f(x_{s(N)})$$

*with $s$ and $A$ defined as in Definition 8.*

*Proof.* First, note that for all $r < s$, the following holds:

$$\bigvee_{j=1}^{r} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq \bigvee_{j=1}^{s} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})$$

Therefore,

$$\bigvee_{j=1}^{1} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq \cdots \leq \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq \cdots \leq \bigvee_{j=1}^{N} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})$$

Now, we consider the first expression. We can show that:

$$\bigvee_{j=1}^{1} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) = f(x_{s(1)}) \wedge \mu_S(A_{s(1)}) = f(x_{s(1)}) \wedge \mu_S(\{x_1, \ldots, x_N\}) = f(x_{s(1)})$$

because $\mu_S(\{x_1, \ldots, x_N\}) = 1$. Considering the third expression above, we get:

$$\bigvee_{j=1}^{N} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq f(x_{s(N)}$$

This is based on the fact that the latter expression is exactly the Sugeno integral and, therefore, Proposition 2 applies (Sugeno integral leads to a result between the minimum and the maximum of the values it aggregates).

Therefore,

$$f(x_{s(1)}) \leq \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)}) \leq f(x_{s(N)})$$

for all $i \in \{1, \ldots, N\}$

Due to the fact that the twofold integral aggregates all these values, say $k(i) = \bigvee_{j=1}^{i} f(x_{s(j)}) \wedge \mu_S(A_{s(j)})$ by means of a Choquet integral:

$$\sum_{i=1}^{N} \Big( k(i) \big( \mu_C(A_{s(i)}) - \mu_C(A_{s(i+1)}) \big) \Big)$$

and due to the fact that the Choquet integral, according to Proposition 2, returns a value in the interval $[\min k(i), \max k(i)]$ where $k(i) \in [f(x_{s(1)}), f(x_{s(N)})]$, the twofold integral also returns a value in $[f(x_{s(1)}), f(x_{s(N)})]$. Therefore, the proposition is proven. □

**Corollary 1** *The twofold integral satisfies unanimity.*

*Proof.* Trivial because when $f(x_i) = k$ then,

$$k = f(x_{s(1)}) \leq TI_{\mu_S,\mu_C}(f) \leq f(x_{s(N)}) = k$$

□

Now, we consider the monotonicity condition.

**Proposition 10** *Let $X$ be a finite set and let $\mu_C$ and $\mu_S$ be two fuzzy measures on $X$, then for all functions $f_1$ and $f_2$ over $X$ such that $f_2(x) \geq f_1(x)$ for all $x \in X$ it holds:*

$$TI_{\mu_S,\mu_C}(f_2) \geq TI_{\mu_S,\mu_C}(f_1)$$

*with $s$ and $A$ defined as in Definition 8.*

*Proof.* First, we have that $f_2(x_i) \geq f_1(x_i)$ implies that $f_2(x_{s_2(i)}) \geq f_1(x_{s_1(i)})$. Note that this is so although the permutations can be different for $f_1$ and $f_2$. Assume that this condition does not hold and, thus, $f_2(x_{s_2(i)}) < f_1(x_{s_1(i)})$. Then, this means that $f_2(x_{s_2(j)}) \leq f_2(x_{s_2(i)})$ for all $j < i$. Therefore, there are $i$ values in $f_2$ smaller than $f_1(x_{s_1}(i))$. However, in $f_1$ there are at most (inequality can hold) only $i-1$ values smaller than $f_1(x_{s_1(i)})$. Therefore, as $f_2 > f_1$, we have a contradiction and $f_2(x_{s_2(i)}) \geq f_1(x_{s_1(i)})$.

Now, let us consider an alternative expression for the Sugeno integral:

$$\bigvee_{i=1,N} \big(f(x_{s(i)}) \wedge \mu(A_{s(i)})\big) = sup_{\alpha \in [0,1]}[\alpha \wedge \mu(F_\alpha)]$$

where $F_\alpha = \{x | f(x) \geq \alpha\}$.

This expression can be used for computing:

$$\bigvee_{j=1,i} \big(f(x_{s(j)}) \wedge \mu(A_{s(j)})\big)$$

It corresponds to:

$$\bigvee_{j=1,i} \left(f(x_{s(j)}) \wedge \mu(A_{s(j)})\right) = sup_{\alpha \in [0, f(x_{s(i)})]}[\alpha \wedge \mu(F_\alpha)]$$

Now, as $f_2(x_{s_2(i)}) \geq f_1(x_{s_1(i)})$, it is clear that

$$sup_{\alpha \in [0, f_2(x_{s_2(i)})]}[\alpha \wedge \mu(F_{2,\alpha})] \geq sup_{\alpha \in [0, f_1(x_{s_1(i)})]}[\alpha \wedge \mu(F_{1,\alpha})]$$

where $F_{1,\alpha} = \{x | f_1(x) \geq \alpha\}$ and $F_{2,\alpha} = \{x | f_2(x) \geq \alpha\}$.

Therefore, as the Choquet integral is monotonic according to Proposition 1, the twofold integral is also monotonic. This proves the proposition. □

# 4 Conclusions.

In this paper, we have introduced the twofold integral, and proven some basic properties. We have seen that is satisfies the usual properties of aggregation operators.

Future work in this direction includes studying its relation with the 2-step Choquet integral and 2-step Sugeno integral [5] and the multi-step Choquet integral [7] (characterized in [6]).

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