T-norm based fuzzy logics: Hilbert-style axiomatizations and hypersequent calculi
(Simposio C@lculus)

Francesc Esteva, Lluís Godo
IIIA - CSIC, 08193 Bellaterra

In this paper we survey recent results about complete axiomatizations Hilbert-style for t-norm based fuzzy logics logics as well as analytic hypersequent calculi for some of them.

1 Introduction

Fuzzy logic has become a subject of increasing interest as the basis for reasoning with vague knowledge. According to Zadeh [15], fuzzy logic in narrow sense is a logical system, which is an extension of many-valued logic, aiming at formalizing approximate reasoning. Intermediate truth values are understood as partial degrees of truth of fuzzy propositions. This many-valued semantics underlying fuzzy logic involve various binary operations on the unit real interval $[0, 1]$, generalizing the classical Boolean truth functions on $\{0, 1\}$. Original Zadeh's functional definition for the intersection and union of fuzzy sets used min and max respectively to combine the membership degrees (belonging to the unit real interval [0, 1]). But very soon these operations were generalized by t-norms and t-conorms operations[1] respectively (see, for example [1]), which are widely used today in Fuzzy Set theory.

Building on these ideas, logicians like Hájek (see the introductory chapter in [9]), Nóvak, Gottwald and others consider the core of fuzzy logic as a family of residuated many-valued logical calculi with truth values on the real unit interval [0, 1], and with min, max, a t-norm $*$ and its residuum $\rightarrow$ as basic truth functions interpreting lattice meet and joint connectives (additive "and" and "or"), a strong conjunction (multiplicative "and") and the implication, respectively. The fact that a t-norm has residuum if and only if the t-norm is left-continuous lead the authors to define a logic called **MTL** [7] with the idea of obtaining the most general residuated logic based on t-norm calculi, i.e. the logic of *left-continuous t-norms* and their residua. This was proved later in [13]. **MTL** is axiomatically defined in [7] as a weakening of Hájek's Basic Fuzzy logic **BL**, which captures the logic of *continuous t-norms*. The name **MTL** is an abbreviation for monoidal t-norm based logic, stressing the fact that **MTL** can also be obtained by extending Höhle's Monoidal logic **ML** [12] (the logic corresponding to residuated lattices) with the so-called pre-linearity axiom, characteristic of t-norm-based calculi. Also **MTL** has been shown to be equivalent to Ono's FL_{ew}[Lin] logic [14], an extension of the Full Lambek calculus FL with exchange, weakening and prelinearity axioms, and hence it also establishes links with the class of substructural logics.

[1]By a t-norm one means some binary operation in the real unit interval $[0, 1]$ which is associative, commutative, non-decreasing in both arguments and which has 1 as a neutral element. A t-conorm is similarly defined but having 0 as neutral element.

In the first part of this paper, we overview recent results by the authors and others about Hilbert-style axiomatizations of propositional t-norm based residuated calculi. These systems have been called *t-norm based residuated fuzzy logics* and can be suitably placed in a hierarchy of logics depending on their characteristic axioms, all of them being extensions of $\mathbf{MTL}$ and having Classical logic as common extension (see later Figure 1). From a proof-theoretic point of view, it is well known that Hylbert-style calculi are not a suitable basis for efficient proof search (by humans or computers). For the latter task one has to develop proof methods that are "analytic"; i.e., the proof search proceeds by step-wise decomposition of the formula to be proved. Sequent calculi, together with natural deduction systems, tableaux or resolution methods, yield suitable formalisms to deal with the above task. In the second part of the paper we survey some analytic calculi that have been recently proposed for $\mathbf{MTL}$ and some of its extensions (e.g. [3, 8]) using *hypersequents*, a natural generalization of Gentzen's sequents introduced by Avron [2].

2 Axiomatizing $\mathbf{MTL}$ and related t-norm based logics

In propositional t-norm based fuzzy logics, formulas are built in the usual way from a countable set of propositional variables, binary connectives $\wedge$, $\&$, $\rightarrow$ and truth constant $\overline{0}$. Other derived connectives can be defined: $\neg\varphi$ is $\varphi \rightarrow \overline{0}$, $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, $\varphi \equiv \psi$ is $(\varphi \rightarrow \psi)\&(\psi \rightarrow \varphi)$, and $\overline{1}$ is $\neg\overline{0}$. The distinguishing feature of t-norm based logics is their semantics. As already mentioned, truth-values are taken in real unit interval [0, 1] and min, max, a (left-continuous) t-norm $*$ and its residuum $\Rightarrow$ are taken as basic truth functions interpreting the lattice meet and joint connectives $\wedge$ and $\vee$ (additive "and" and "or"), the strong conjunction $\&$ (multiplicative "and") and the implication $\rightarrow$, respectively. Hence, each (left-continuous) t-norm $*$ actually determines a many-valued calculus $PC(*)$ with a set of tautologies $Taut(*)$.

Chronologically, the first axiomatized logics within the class of t-norm based fuzzy logics above described were Lukasiewicz's infinite-valued logic and Gödel's many-valued logic (also known as Dummet's logic), of course many years before the introduction of fuzzy logic, by Rose et Rosser and Dummet respectively. These logics were proved to be complete respectively with respect to the many-valued calculi defined by the so-called Lukasiewicz t-norm ($x * y = \max(x + y - 1, 0)$) and Gödel t-norm ($x * y = \min(x, y)$) and their residua. Later, the so-called Product logic was introduced in [11] and proved to be complete in [0, 1] with respect to the calculus defined by the product t-norm ($x * y = x \cdot y$) and its residuum. In [9] Hájek introduced the so-called Basic Fuzzy Logic, $\mathbf{BL}$ for short, as a common fragment of the above three logics, indeed, he proves that Lukasiewicz, Gödel and Product logic can be obtained as extensions of $\mathbf{BL}$:

Lukasiewicz = $\mathbf{BL}$ + (*Inv*) $\neg\neg\varphi \rightarrow \varphi$
Gödel = $\mathbf{BL}$ + (*Con*) $\varphi \rightarrow \varphi\&\varphi$
Product = $\mathbf{BL}$ + (*Weak-con*) $\neg(\varphi \wedge \neg\varphi)$
+ $(\Pi 1)$ $(\neg\neg\varphi)\&(\varphi\&\psi \rightarrow \varphi\&\chi) \rightarrow (\psi \rightarrow \chi)$

The idea in mind was to define a logic to capture with the 1-tautologies common to all many-valued calculi in [0, 1] defined by a continous t-norm and its residuum. Hájek's conjecture was proved soon later [10, 5]. On the other hand, monoidal t-norm based logic $\mathbf{MTL}$ was introduced in [7] as the weakening of $\mathbf{BL}$ by not requiring the divisibility axiom

$$\varphi \wedge \psi \to \varphi \& (\varphi \to \psi) \qquad (Div)$$

which amounts to the continuity of the t-norm. Here we list the axioms of $\mathbf{MTL}$:

$A1$	$\overline{0} \to \varphi$	$A6$	$(\varphi \& (\varphi \to \psi)) \to (\varphi \wedge \psi)$
$A2$	$(\varphi \& \psi) \to \varphi$	$A7a$	$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi))$
$A3$	$(\varphi \& \psi) \to (\psi \& \varphi)$	$A7b$	$((\varphi \& \psi) \to \chi)) \to (\varphi \to (\psi \to \chi))$
$A4$	$(\varphi \wedge \psi) \to \varphi$	$A9$	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
$A5$	$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$	$A10$	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$

The only inference rule is modus ponens. An alternative axiomatization of $\mathbf{MTL}$ is obtained by extending Ono's system FL_{ew}, or equivalently Höhle's Monoidal logic $\mathbf{ML}$, by the pre-linearity axiom

$$(\varphi \to \psi) \vee (\psi \to \varphi)\ . \qquad (Lin)$$

FL_{ew} is an extension of Full Lambek calculus with the *weakening* and *exchange* rules. In [13] *standard* completeness for $\mathbf{MTL}$ is proved, i.e. theorems of $\mathbf{MTL}$ are exactly the common tautologies of many-valued calculi PC(*) defined by a left-continuous t-norm *. Therefore $\mathbf{MTL}$ turns out to be the weakest logic of the family of residuated logics with semantics on [0, 1] given by t-norms and their residua.

Several extensions of $\mathbf{MTL}$, which are not extensions of $\mathbf{BL}$, have been investigated in different works. In particular, in [6] it has been proved standard completeness for $\mathbf{IMTL}$[2], the extension of $\mathbf{MTL}$ with the double negation axiom (*Inv*), and for $\mathbf{SMTL}$, the extension of $\mathbf{MTL}$with the pseudo-complementation axiom (or weak contraction) axiom (*Weak-con*). Horcik has recently proved standard completeness for $\Pi\mathbf{MTL}$, the extension of $\mathbf{MTL}$ with the two characteristic axioms of product logic, (*Weak-con*) and ($\Pi 1$). The lattice of these logics (and others), together with the well-known extensions of BL (Product, Gödel, Lukasiewicz and SBL) is depicted in Figure 1 in the framework of Monoidal Logic.

3 Analytic Calculi for $\mathbf{MTL}$ and its extensions

Cut-free sequent calculi provide suitable analytic proof methods. Sequents are well-known structures of the form

$$\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_m$$

which can be intuitively understoof as "φ_1 and ... and φ_n implies ψ_1 or ... ψ_m". Sequent calculi have been defined for many logics, however they have problems with fuzzy logics, namely to cope with the linear ordering of truth-values in [0, 1]. To overcome with this problem when devising a sequent calculus for Gödel logic, Avron [2]

[2]It can be equivalently seen as $\mathbf{aMALL}$, i.e. linear logic without the exponentials connectives, extended with weakening.

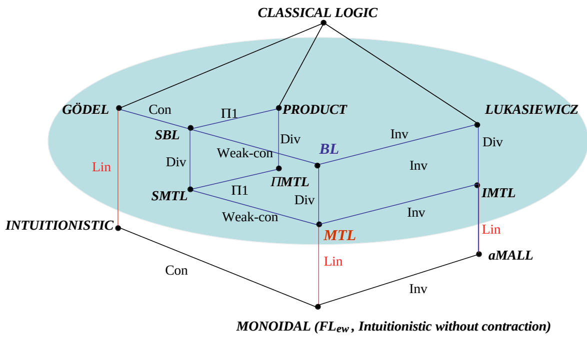

Figure 1: Main residuated many-valued logics, the shadowed part containing t-norm based logics.

introduced a natural generalization of sequents called *hyperquents*. A *hypersequent* is an expression of the form

$$\Gamma_1 \vdash \Delta_1 \mid \ \ldots \ \mid \Gamma_n \vdash \Delta_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \vdash \Delta_i$ is an ordinary sequent. $\Gamma_i \vdash \Delta_i$ is called a *component* of the hypersequent. The intended interpretation of the symbol "|" is disjunctive, so the above hypersequent can be read as stating that one of the ordinary sequents $\Gamma_1 \vdash \Delta_1$ holds.

Like in ordinary sequent calculi, in a hypersequent calculus there are axioms and rules which are divided into two groups: *logical* and *structural rules*. The logical rules are essentially the same as those in sequent calculi, the only difference is the presence of dummy contexts G and H, called *side hypersequents* which are used as variables for (possibly empty) hypersequents. The structural rules are divided into *internal* and *external rules*. The internal rules deal with formulas within components. If they are present, they are the usual weakening and contraction. The external rules manipulate whole components within a hypersequent. These are external weakening (EW) and external contraction (EC):

$$(EW) \quad \frac{H}{H \mid \Gamma \vdash A} \qquad\qquad (EC) \quad \frac{H \mid \Gamma \vdash A \mid \Gamma \vdash A}{H \mid \Gamma \vdash A}$$

In hypersequent calculi it is possible to define further structural rules which simultaneously act on several components of one or more hypersequents. It is this type of rule which increases the expressive power of hypersequent calculi with respect to ordinary sequent calculi. An example of such a kind of rule is Avron's communication rule:

$$(com) \quad \frac{H \mid \Pi_1, \Gamma_1 \vdash A \qquad G \mid \Pi_2, \Gamma_2 \vdash B}{H \mid G \mid \Pi_1, \Pi_2 \vdash A \mid \Gamma_1, \Gamma_2 \vdash B}$$

Indeed, by adding (com) to the hypersequent calculus for intuitionistic logic one gets a cut-free calculus for Gödel logic [2]. Following this approach, a proof theory for **MTL** has been investigated in [3], where an analytic hypersequent calculus has been introduced. This calculus, called **HMTL**, has been defined by adding the (com) rule to the hypersequent calculus for intuitionistic logic without contraction (**aMAILL**). More precisely, axioms and rules of **HMTL** are the following:

$$A \vdash A \qquad \overline{0} \vdash A \qquad\qquad (cut)\ \frac{H \mid \Gamma \vdash A \quad G \mid A, \Gamma' \vdash C}{H \mid G \mid \Gamma, \Gamma' \vdash C}$$

Internal and External Structural Rules :

$$(iw)\ \frac{H \mid \Gamma \vdash C}{H \mid \Gamma, B \vdash C} \qquad\qquad (EC), \quad (EW), \quad (com)$$

Multiplicative fragment :

$$(\&, l)\ \frac{H \mid \Gamma, A, B \vdash C}{H \mid \Gamma, A\&B \vdash C} \qquad\qquad (\&, r)\ \frac{H \mid \Gamma \vdash A \quad G \mid \Gamma' \vdash B}{H \mid G \mid \Gamma, \Gamma' \vdash A\&B}$$

$$(\rightarrow, l)\ \frac{G \mid \Gamma \vdash A \quad H \mid \Gamma', B \vdash C}{G \mid H \mid \Gamma, \Gamma', A \rightarrow B \vdash C} \qquad\qquad (\rightarrow, r)\ \frac{H \mid \Gamma, A \vdash B}{H \mid \Gamma \vdash A \rightarrow B}$$

Additive fragment :

$$(\wedge, l_i)_{i=1,2}\ \frac{H \mid \Gamma, A_i \vdash C}{H \mid \Gamma, A_1 \wedge A_2 \vdash C} \qquad\qquad (\wedge, r)\ \frac{G \mid \Gamma \vdash A \quad H \mid \Gamma \vdash B}{G \mid H \mid \Gamma \vdash A \wedge B}$$

$$(\vee, l)\ \frac{H \mid \Gamma, A \vdash C \quad G \mid \Gamma, B \vdash C}{H \mid G \mid \Gamma, A \vee B \vdash C} \qquad\qquad (\vee, r_i)_{i=1,2}\ \frac{H \mid \Gamma \vdash A_i}{H \mid \Gamma \vdash A_1 \vee A_2}$$

In fact, in [3] it is shown that **HMTL** is sound and complete for **MTL** and that **HMTL** admits cut-elimination. Cut-free hypersequent calculi have also been obtained by Ciabattoni et al. [4] for **IMTL** and **SMTL**.

Finally, it is worth mentioning that Metcalfe et al. have recently proposed (see e.g. [8]) hypersequent calculi also for Luaksiewicz and Product logics, but adopting different interpretrations for hypersequents. However, the problem of finding a suitable calculus for **BL** remains open.

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