Towards the definition of fuzzy measures for fuzzy systems

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Abstract Standard (one stage or flat) fuzzy inference systems compute an output value applying all the rules at once to a set of input values, and then combining the results obtained by all the rules. However, in such approach independence on the rules play a central role. In this paper we point out the difficulties of building such systems when rules are not independent and, then, we propose an approach based on the Sugeno integral to solve the problem.

Keywords: Fuzzy inference systems, fuzzy rules, Sugeno integral.

1 Introduction

Fuzzy inference systems have been proven to be powerful tools for solving complex problems. Such systems, developed after Zadeh's Fuzzy Sets theory [12], are rule based systems in which rules are described using linguistic terms with an underlying semantics based on fuzzy sets. Detailed description of fuzzy systems can be found in *e.g.* [1, 11]. Some applications using fuzzy inference systems can be found in *e.g.* [10, 9].

Recently, research has been focused on the construction of systems for complex domains. This corresponds to applications with a large number of variables and changing environments. These problems cannot be solved by standard techniques. This is so because, on the one hand, the so-called course of dimensionality appears [5, 3] (the number of rules increases exponentially with the number of variables). On the other, the performance of the system declines as soon as the properties of the environment move away from the foreseen ones (see *e.g.* [2]). To solve these problems hierarchical fuzzy systems have been considered (see *e.g.* [5, 7, 10, 4].

Nevertheless, a difficulty arises in both hierarchical and standard (flat or one-stage) fuzzy systems. Namely, rules have to be independent. Otherwise, when this does not hold, rules that compete for the same subdomain increase their global influence in the output. In other words, rules *cooperate* to bias the output. This problem, that does not usually occur in fuzzy systems constructed using grid-like structures, arises in complex domains when rules are developed by domain experts. In this case, when grid-like structures do not apply (as in *e.g.* [9]), handy-crafted rules might have interactions that are difficult to avoid.

In this paper we illustrate this problem in detail and propose a solution to the problem. A toy example is also given.

The structure of the paper is as follows. In Section 2 we briefly outline fuzzy inference systems. In Section 3 we illustrate the problem of non-independence with an example. In Section 4 we propose a solution based on fuzzy integrals and introduce a family of fuzzy measures to be used with the fuzzy integrals to tackle the problem. The paper finishes in Section 5 with some conclusions and outlining some future work.

2 Fuzzy inference systems

In this section we briefly recall the operational procedure for standard (flat or one-stage) fuzzy inference systems. Details are only given on those aspects needed latter on. For more details see *e.g.* [1, 11].

Fuzzy inference systems are defined by sets of rules, where each rule R_i (for i in $\{1, \ldots, N\}$) has the following structure:

$$R_i\text{: } \mathbf{IF} \;\; X^1 \text{ is } A_i^1 \;\mathbf{and}\; \ldots \mathbf{and} \;\; X^m \text{ is } A_i^m \;\mathbf{THEN}\;\; Y \text{ is } B_i$$

Here, A_i^j and B_i denote fuzzy terms (defined in terms of fuzzy sets).

For simplicity, we will consider systems that have a single input variable X. Therefore, the structure of the rules is as follows:

$$R_i\text{: } \mathbf{IF} \;\; X \text{ is } A_i \;\mathbf{THEN}\;\; Y \text{ is } B_i$$

The application of a set of rules $\{R_i\}_i$ (for i in $\{1, \ldots, N\}$) to some particular input x is as follows:

(i) The application of each rule R_i to the particular input x. This leads to a fuzzy set (on the domain of Y) for each rule. The computation of such fuzzy set requires the degree in which the antecedent is satisfied (usually $A_i(x)$) and then modify the output B_i according to such degree. This is, $B_i \wedge A_i(x)$. Here $\wedge$ stands for the minimum.

(ii) The output of all rules are combined. As each rule computes the fuzzy set $B_i \wedge A_i(x)$, the combination of all outputs also leads to a fuzzy set. This is usually computed using the maximum ($\vee$ stands for the maximum): $\tilde{B} = \vee_{i=1}^{N} \big(B_i \wedge A_i(x)\big)$. As $\tilde{B}$ is a fuzzy set, for all y in the domain of Y we have:

$$\tilde{B}(y) = \vee_{i=1}^{N} \big(B_i(y) \wedge A_i(x)\big) \tag{1}$$

(iii) The fuzzy set $\tilde{B}$ is defuzzified. In this process a value y_0 in the domain of Y is selected on the basis of the information in $\tilde{B}$.

3 Fuzzy inference systems with non-independent rules

Fuzzy systems are often constructed in such a way that rules follow a simple geometric structure. In this way, the number of rules that can be fired in each subregion of the application domain is the same.

A typical case is when rules follow a grid-like structure. This situation corresponds to the case in which for each input variable we have a fuzzy partition (of its domain) and rules are defined over the product of the elements in such fuzzy partitions. This is the case of fuzzy systems defined in a tabular form.

When the set of rules follow such a grid-like structure, all rules can be considered as independent as there are no major interactions between their outcomes. Each rule has its own *area of influence*. However, there are situations that do not follow this pattern. In this latter case, difficulties can arise because when a region accumulates several rules, their outcome can bias the outcome of the whole system. To illustrate the difficulties due to non-independent rules, we consider below a toy example with these four rules:

R_1**: IF** X is 1 **THEN** Y is 1

R_2**: IF** X is 2 **THEN** Y is 3.7

R_3**: IF** X is 1.95 **THEN** Y is 4.04

R_4**: IF** X is 1.9 **THEN** Y is 4.4

We have used triangular fuzzy sets for representing the fuzzy numbers on the consequent. In particular, we considered (0.2, 1.00, 2.3), (1.6, 3.7, 6.3), (1.7, 4.04, 6.6) and (1.8, 4.4, 6.8).

This example is solely used for illustration. The rules are intended to model the relation (x, x^2). It can be easily observed that rules R_2, R_3 and R_4 are redundant as they try to give information about the same region on the X domain (*i.e.* the region around the value 2).

Now, we consider the application of the rules to $x = 1.6$. In this case, if rules are redundant all four rules will be fired and the fuzzy set $\tilde{B}$ will include the outcomes of the four rules. Let $\alpha_i = A_i(x)$ be the degree of satisfaction of rule R_i, we take α = (0.25,0.625,0.6875,0.75). Figure 1 shows the result of firing such rules.

This figure shows that as rules R_2, R_3 and R_4 conclude all about a value near 4, such region has a larger influence in the output (a larger dark region) than the one would be obtained by a single rule. It should be underlined that such influence can be positive (*i.e.*, increasing the output value) or negative (*i.e.*, decreasing the output value). The sign depends on the shape and position of the membership functions.

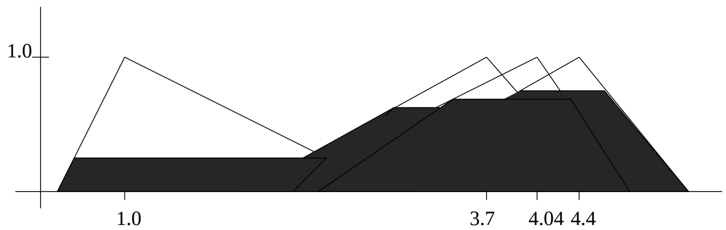

Figure 1: Outcome of the rules

The influence of the rules in the output can be quantified applying a defuzzification method to the fuzzy set. We have selected below the center of gravity for defuzzification. The example described above with four rules leads to a defuzzified value equal to 2.8356. Instead, if only rules R_1 and R_3 were applied (with α_1 and α_2), the final output would be 2.5491. Alternatively, if we replace rules R_2, R_3 and R_4 by a rule with an average consequent fired with the average $(\alpha_2 + \alpha_3 + \alpha_4)/3$ we get a defuzzified value of 2.5521. Note that the ideal output for $x = 1.6$ equals to $1.6^2 = 2.56$. Thus, in this example, the redundancy of the rules bias the output towards larger values.

4 Fuzzy measures and fuzzy integrals for fuzzy systems

To solve the problem illustrated in Section 3 about non-independent rules, we can take advantage of the fact, proven in [8] (see also [6]), that Expression 1 is equivalent to the weighted maximum, and, thus, it is equal to a Sugeno integral with an appropriate fuzzy measure.

This is, a fuzzy measure μ on the set of rules R_i can be defined so that Expression 1 is equivalent to the Sugeno integral of the values $B_1(y), \ldots, B_N(y)$:

$$\tilde{B}(y) = \vee_{i=1}^{N} \big(B_i(y) \wedge A_i(x)\big) = SI_\mu(B_1(y), \ldots, B_N(y)) \qquad (2)$$

This fuzzy measure is defined on $Z \subseteq \{R_i\}$ as $\mu(Z) = \max_{R_i \in Z} A_i(x)$.

Sugeno integrals are well known aggregation operators that combine data in numerical or ordinal scales with respect to a fuzzy measure. Fuzzy measures permits to represent some background knowledge about the information sources that are being aggregated. In particular, it permits to represent redundancy and complementariness.

Now, if we use the Sugeno integral with the fuzzy measure μ defined above for combining the membership functions, we get the same value 2.8356. This corresponds to the integral of $B_i(y)$ with $\mu(Z)$ defined from $\alpha = (0.25, 0.625, 0.6875, 0.75)$ by $\mu(Z) = \max_{R_i \in Z} A_i(x) = \max_{R_i \in Z} \alpha_i$. Table 1 gives the values of $\mu(Z)$ for all Z in $\{R_i\}$.

Table 1: Fuzzy measure μ_1

$\mu(\{R_1, R_2, R_3, R_4\}) = 0.75$	$\mu(\{R_1, R_2\}) = 0.625$
$\mu(\{R_1, R_2, R_3\}) = 0.6875$	$\mu(\{R_1, R_3\}) = 0.6875$
$\mu(\{R_1, R_2, R_4\}) = 0.75$	$\mu(\{R_1, R_4\}) = 0.75$
$\mu(\{R_1, R_3, R_4\}) = 0.75$	$\mu(\{R_2, R_3\}) = 0.6875$
$\mu(\{R_2, R_3, R_4\}) = 0.75$	$\mu(\{R_2, R_4\}) = 0.75$
$\mu(\{R_1\}) = 0.25$	$\mu(\{R_3, R_4\}) = 0.75$
$\mu(\{R_2\}) = 0.625$	$\mu(\{R_3\}) = 0.6875$
$\mu(\{R_4\}) = 0.75$	$\mu(\emptyset) = 0$

Rewritting the fuzzy system as a fuzzy integral instead of a simple weighted min permits to consider the integration of the values $B_i(y)$ using other fuzzy measures than $\mu(Z)$ as defined above. In particular, alternative fuzzy measures can be defined to reduce the effects of redundancy in rules. The fuzzy measure μ_1 defined in Table 2 has this effect. This measure has been defined so that the $\mu_1(Z)$ is lower than μ when Z contains some of the $\{R_2, R_3, R_4\}$ but not all of them (in fact, it is proportional to the number of redundant rules in Z). Note *e.g.* that $\mu(\{R_1, R_3\}) = 0.6875$ in the original fuzzy measure but that $\mu(\{R_1, R_3\}) = 0.25$ in the new one. Similarly, $\mu(\{R_1, R_3, R_4\}) = 0.75$ while for the new measure $\mu(\{R_1, R_3, R_4\}) = 0.5$.

With such new fuzzy measure μ_1, the defuzzified value (for the input $x = 1.6$) equals to 2.4827, that is more similar to the goal 2.56 than the original 2.8356. Note that this value is also similar to the outcome when only one of the redundant rules is used. Figure 2 illustrates the results of the combination using Sugeno integral with both the original measure μ and the alternative measure μ_1.

4.1 A fuzzy measure for fuzzy systems

Now, we propose a family of fuzzy measures to be used when there are redundant rules in a fuzzy system. As we will prove latter, this definition encompasses the fuzzy measure μ_1 introduced in the previous section.

Our construction is based on the assumption that there is some prior knowledge on which rules can cause problems due to their redundancy. More precisely, the prior knowledge is expressed in terms of a partition P of the set of rules $\{R_i\}$. Each set π of the partition P represents a set of redundant rules.

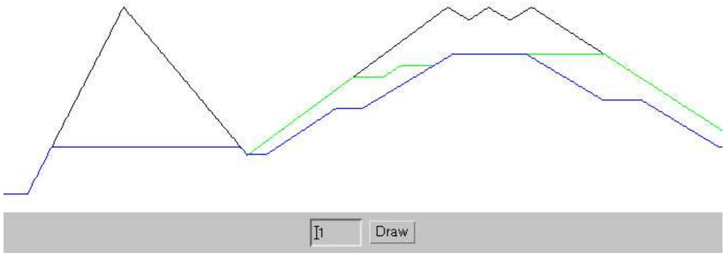

Figure 2: Using Sugeno integral for combining fuzzy rules

Table 2: Fuzzy measure μ_1

$\mu(\{R_1, R_2, R_3, R_4\}) = 0.75$	$\mu(\{R_1, R_2\}) = 0.25$
$\mu(\{R_1, R_2, R_3\}) = 0.4583$	$\mu(\{R_1, R_3\}) = 0.25$
$\mu(\{R_1, R_2, R_4\}) = 0.5$	$\mu(\{R_1, R_4\}) = 0.25$
$\mu(\{R_1, R_3, R_4\}) = 0.5$	$\mu(\{R_2, R_3\}) = 0.4583$
$\mu(\{R_2, R_3, R_4\}) = 0.75$	$\mu(\{R_2, R_4\}) = 0.5$
$\mu(\{R_1\}) = 0.25$	$\mu(\{R_3, R_4\}) = 0.5$
$\mu(\{R_2\}) = 0.2083$	$\mu(\{R_3\}) = 0.2292$
$\mu(\{R_4\}) = 0.25$	$\mu(\emptyset) = 0$

Definition 1. *Let P be a partition of the set of rules $\{R_i\}$, and let α_i be the degree of satisfaction of rule R_i, then*

$$\mu(Z) = \max_{\pi \in P} \left(\max_{R_i \in Z \cap \pi} \alpha_i \cdot \frac{|Z \cap \pi|}{|\pi|} \right)$$

The following proposition holds for the measure defined above:

Proposition 1. *The set function μ defined in Definition 1 satisfies $\mu(\emptyset) = 0$ and monotonicity (*i.e.*, $A \subset B$ implies $\mu(A) \leq \mu(B)$).*

Proof. We prove the two cases above:

1. $\mu(\emptyset) = \max_{\pi \in P} \left(\max_{R_i \in \emptyset} \alpha_i \cdot \frac{|\emptyset|}{|\pi|} \right) = 0$
2. If $Z_A \subset Z_B$ then $Z_A \cap \pi \subseteq Z_B \cap \pi$, and, thus, $|Z_A \cap \pi| \leq |Z_B \cap \pi|$. Therefore, for all π:

$$\max_{R_i \in Z_A \cap \pi} \alpha_i \cdot \frac{|Z_A \cap \pi|}{|\pi|} \leq \max_{R_i \in Z_A \cap \pi} \alpha_i \cdot \frac{|Z_B \cap \pi|}{|\pi|} \leq \max_{R_i \in Z_B \cap \pi} \alpha_i \cdot \frac{|Z_B \cap \pi|}{|\pi|}$$

$\square$

This proposition shows that μ satisfies the two basic properties of fuzzy measures and thus it can be used in a consistent way with the Sugeno integral.

Note that this measure satisfies the boundary condition concerning the empty set ($\mu(\emptyset) = 0$) but that does not satisfy $\mu(\{R_i\}) = 1$. However, this latter condition was neither satisfied by the original measure μ in Expression 2. Nevertheless, in both cases $\mu(\{R_i\}) = \max \alpha_i$.

Proposition 2. *The set function μ defined in 1 satisfies $\mu(\{R_i\}) = \max \alpha_i$.*

Proof. Considering Definition 1, we obtain:

$$\mu(\{R_i\}) = \max_{\pi \in P} \left(\max_{R_i \in \pi} \alpha_i \cdot \frac{|\pi|}{|\pi|} \right) = \max_{\pi \in P} \left(\max_{R_i \in \pi} \alpha_i \right) = \max_{R_i} \alpha_i$$

□

The fuzzy measure μ_1 defined in Section 4 is a particular case of the measure considered here. This is established in the next proposition.

Proposition 3. *Let us consider the set $\{R_1, R_2, R_3, R_4\}$ and its partition $\{P_1 = \{R_1\}, P_2 = \{R_2, R_3, R_4\}\}$, let $\alpha = (0.25, 0.625, 0.6875, 0.75)$, then the fuzzy measure μ defined according to Definition 1 corresponds to the fuzzy measure μ_1 defined in Table 2*

5 Conclusions and future work

In this paper we have shown that one-stage fuzzy systems do not deal in an appropriate way with redundant rules. While the situation might not be relevant in applications where the domain is naturally decomposed using a grid-like pattern, in other applications this problem is crucial. In particular, this is the case of applications in complex domains where rules have been designed by domain experts (as in *e.g.* [9]). Redundant rules implies that the outcome of the system is biased towards the result of such rules.

We have proposed the use of Fuzzy measures and Sugeno integrals to deal with this problem. We have also introduced a particular family of fuzzy measures that can be used to reduce the bias of redundant rules. As future work, we consider the use of such model in complex environments.

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