Contributions to the Development of Possibilistic Decision Theory

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Abstract

Here we summarise the work done by the authors in the recent years at IIIA[1] on Possibilistic Qualitative Decision Theory, some of them also in collaboration with Didier Dubois and Henri Prade.

Keywords: Decision Theory under uncertainty, Possibilistic model, Axiomatic Characterization

1 Introduction

As it is known, taking a decision amounts to choose, according to some criteria, the "best" of a set of available alternatives *taking into account the available knowledge*. There are many approaches to rational decison making under uncertainty, however, many of them agree on the fact that the selection of decisions is determined by two factors: the decision maker's *preference on consequences* and the *information or belief about the current state of affairs* the decision maker (DM for short) has.

Usual assumptions in the different proposals for decision theories are:

- *rationality hypothesis*: the DM is interested in maximising his utilities.
- *the feasibility of representing DM's preference relation* $\preccurlyeq$ *on consequences by a preference function on them*, i.e. the existence of a function $u : X \to (U, \leq_U)$, X being *the set of consequences* and $(U, \leq_U)$ the *preference valuation set*, such that $x \preccurlyeq y$ iff $u(x) \leq_U u(y)$, is assumed [2].

If S is the *set of states or situations*, a decision d is represented as a mapping $d : S \to X$ providing the consequence $d(s)$ of the decision in each posible situation s. Hence, a decision making problem may be represented by a 4-tuple $< S,\ X,\ D,\ u >$ with D being the *set of available decisions or alternatives.*

If we precisely know which is the current situation s_0 we might apply this simple decision criteria:

> Given a situation s_0 and a set of available decisions D, *a best decision* will be a maximal element of D with respect to the order $\preccurlyeq_{s_0}$ induced by preferences on the consequences, $\preccurlyeq_{s_0}$ being defined as
>
> $$d \preccurlyeq_{s_0} d' \quad \text{iff} \quad u(d(s_0)) \leq_U u(d'(s_0)). \quad (1)$$

But usually, there is only partial information about what the current situation is, and hence we cannot apply on (1) to define an order in D. In such a case we are faced with a decision problem under uncertainty. Further, a representation of uncertainty may be given or not. If no uncertainty representation is given, we may consider different criteria like those that evaluate a decision in terms of its worst possible consequence, in terms of its best one, or in terms of some weighted aggregation of them (for more details of some of these criteria you may see, for example, [15, 17, 20]).

Other alternatives emerge when considering that some kind of uncertainty measure is used to model the uncertainty [14] (see Figure 1). In this case, another component is added to the 4-tuple modelling the problem. In such a case, a decision problem is reprsented by a 5-tuple $< S,\ X,\ D,\ u,\ \mu >$, where $\mu : 2^S \to V$ is an uncertainty measure, V being an uncertainty scale. Some particular kinds of measures are probability or possibility and necessity measures [21]. The basic references in classical *Decision Theory under Uncertainty* are Von Neumann and Morgenstern's *Expected Utility Theory* (1944), and the version of Savage [18],

[1]Most of the work was done while the first author was at IIIA with a research fellowship.

[2]Usually, it is supposed $U = \mathbb{R}$.

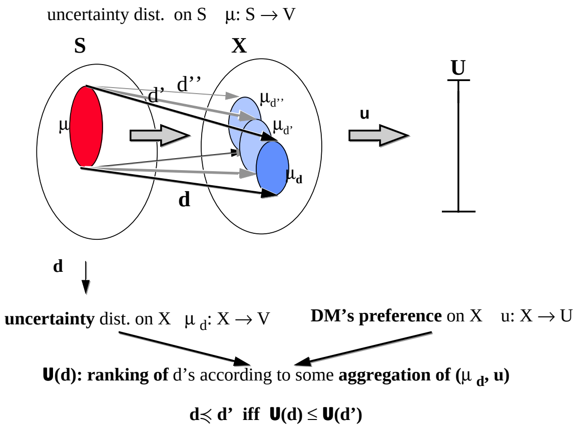

Figure 1: Decision Model with Uncertainty Representation.

which provide a set of postulates characterising preference relations under probabilistic uncertainty and the rationality hypothesis. In particular, Von Neumann and Morgenstern assume a probability distribution $p : X \rightarrow [0, 1]$ encodes the uncertainty on the actual situation s_0. Then, each decision induces a probability distribution p_d on X defined as

$$p_d(x) = \sum_{s \in S : d(s) = x} p(s).$$

In this framework each decision d is identified with its associated probability distribution p_d on X. Distributions are ranked in terms of their expected value with respect to the decision maker's preferences on consequences $EU(p_d) = \sum_{x \in X} p_d(x)u(x)$. So, to rank decisions they consider the following criterion

$$d \preccurlyeq_p d' \text{ (iff } p_d \preccurlyeq p_{d'}) \text{ iff } EU(p_d) \leq EU(p_{d'})$$

In the other hand, if the uncertainty is represented by a possibility distribution we are in the context of Possibilistic Decision Theory, introduced by Dubois and Prade in [5]. There, decisions are ranked, via their associated possibility distributions on consequences, using two qualitative utility functionals, which are related to Sugeno integrals, expressed in terms of the uncertainty on situations and of the Decision Maker's preferences on consequences. The preference relations on decisions induced by these utility functionals are also characterised by a set of rational postulates, someones different form the Expected Utility Theory and which describe a more qualititative behavious.

We have extended the possibilistic model in several aspects to be applicable in broader contexts than the original one considered in [5]. For instance, generalised utilities functions involving other t-norms than minimum have been introduced, uncertainty and preferences have been allowed to be measured in non-linear structures, and partially inconsistent belief states have been also considered. Moreover, the feasibility of refining these orderings is considered and the commensurability hypothesis between preference and uncertainty measure sets is weaken. After a description of the basic possibilistic model in the next Section, these extensions are described in Section 3, while a brief summary of their correspoinding axiomatic characterizations is given in Section 4. Finally, we conclude by commenting on our ongoing work.

2 Possibilistic Qualitative Decision: the basic framework

The necessity of dealing with qualitative, non-numerical knowledge –since qualitative representations are closer to the way people reason– has led in the recent years to a considerable development of Qualitative Decision Theory. Main qualitative approaches up to 1999 were summarised by Doyle and Thomason in [1].

Among the qualitative approaches, we find the model representing uncertainty by possibility distributions. In [5], a qualitative counterpart of Von Neumann and Morgenstern's Expected Utility Theory [19] was proposed by Dubois and Prade. In this model uncertainty on situations is assumed to be of possibilistic nature, that is, a belief states is represented by a normalised possibility distribution $\pi_0 : S \rightarrow V$ on the finite set of possible situations S and with values on a finite linearly ordered scale V . As usual, decisions are modelled as mappings $d : S \rightarrow X$ from situations to the finite set of possible consequences X. Analogously to the probabilistic case, each decision d is associated with a (normalised) possibility distribution on consequences[3] $\pi_d : X \rightarrow V$ defined as

$$\pi_d(x) = \bigvee \{\pi_0(s) \mid s \in S, d(s) = x\},$$

therefore, the idea is to rank decisions ranking the associated possibility distributions.

Before analysing the model, let us first observe that some decision problems in which uncertainty is involved may be seen as a problem of ranking possibility distributions on consequences as well.

As it has been mentioned, the Decision Maker may be faced with different cases of incomplete or ill specified

[3]Obviously, each π_d depends on the actual situation s_0, we omit the reference to the actual situation for a simpler notation.

decision problems. Three of the different cases that result in possibility distributions on X are the following (for more details, e.g. [22]):

- *the situation is uncertain*: s_0 is represented by a normalised possibility distribution on S, $\pi_{s_0}:S \to V$, representing the belief state about which is the real situation. Then, each decision $d:S \to X$ induces a corresponding possibility distribution π_{d,s_0}, on the set of consequences, defined as
$$\pi_{d,s_0}(x) = \max\{\pi_{s_0}(s) | d(s) = x\}, \qquad (2)$$
with $\max \emptyset = 0$. $\pi_{d,s_0}(x)$ *represents the plausibility of x being the consequence of d.* As π_{s_0} is normalised, π_{d,s_0} is normalised as well.
- *the situation is precisely known but the decision is not precisely defined:* in each situation we do not have a precise consequence but a possibility distribution on the consequences. So, d is modelled by a possibility distribution π_d on the set of consequences.
- *the decision (as a mapping) is partially unknown*: we know the decision in some other situations but not in the actual one. Thus, we have partial information about decisions by having stored the performance of decisions taken in different past situations. This leads to a case-based decision problem, in which the similarity function involved determines that each decision may also be identified with a possibility distribution on consequences.

Therefore, we include these cases in our framework assuming as working hypothesis that *uncertainty may be modelled by possibility distributions on consequences*, that is,

> For an actual situation s_0, we may identify each decision with a normalised possibility distribution on X, therefore, choosing the "best" decision is equivalent to choosing its associated possibility distribution.

Hence, in order to select the best decision in a possibilistic à la Von Neumann and Morgerstern 's[4] approach [3], we are looking for possibility distributions on consequences that maximise a utility function $\mathcal{U}$ on $\Pi(X)$, i.e. we consider

$$d \preccurlyeq_{s_0} d' \quad \text{iff} \quad \pi_d \sqsubseteq \pi_{d'} \quad \text{iff} \quad \mathcal{U}(\pi_d) \le \mathcal{U}(\pi_{d'}).$$

Therefore, we now focus on preference relations in the set of possibility distributions on consequences. Let $\mathbf{\Pi(X)} = \{\pi : X \to V \mid \exists x \in X,\ \pi(x) = 1\}$ denote the set of *normalised* possibility distributions -also called the possibilistic lotteries- on consequences[5].

The qualitative possibilistic model also shares the usual assumption that the decision maker's preferences on the set of possible consequences are representable by a utility function $u : X \to U$, i.e. a consequence x is preferred to x' whenever $u(x) >_U u(x')$, where $(U, \ge_U)$ is another finite linearly ordered scale[6]. The utility scale U is assumed to be *commensurate* with V, which amounts to assume the existence of an *onto*, order-preserving mapping $h : V \to U$ linking both scales.

Therefore, given U, u and h, the basic possibilistic model introduced by Dubois and Prade [5] propose to rank these distributions according to an optimistic or a pessimistic criterion represented by Sugeno-like integrals. Namely,

$$d \preceq^- d' \text{ iff } \mathcal{U}^-(d) \le \mathcal{U}^-(d'),$$
$$d \preceq^+ d' \text{ iff } \mathcal{U}^+(d) \le \mathcal{U}^+(d'),$$

where

$$\mathcal{U}^-(d) = QU^-(\pi_d \mid u) = \min_{x\in X} \max(n(\pi_d(x)), u(x)),$$
$$\mathcal{U}^+(d) = QU^+(\pi_d \mid u) = \max_{x\in X} \min(h(\pi_d(x)), u(x)),$$

where $n = n_U \circ h$, n_U being the reversing involution on U. Since $\mathcal{U}^-(d)$ evaluates to what extent all possible consequences of d are good, $\mathcal{U}^-$ models a pessimistic criterion, while $\mathcal{U}^+(d)$ represents an optimistic behavior by evaluating to what extent at least one possible consequence is good.

Notice that both criteria are qualitative in the sense that they only involve the minimum, the maximum and an order-reversing operators.

3 Extensions of the Possibilistic Qualitative Decision Model

As it is said, sometimes, this basic framework might result a bit restricted since we may have decision making

[4] An alternative approach of the possibilistic model à la Savage has been also developed, see e.g. [6].

[5] For a simpler notation we will use x for denoting both an element belonging to X and the possibility distribution on X such that $\pi(z) = 1_V$ if $z = x$, and 0_V otherwise. Similarly, we shall also denote by A both a subset $A \subseteq X$ and the possibility distribution on X such that $\pi(z) = 1_V$ if $z \in A$ and 0_V otherwise. With this convention, we can consider X as included in $\Pi(X)$.

[6] The infimum and the supremum of each set will be denoted by 0 and 1 respectively, although they might be different for each set, for a simpler notation, we use the same notation for them assuming they may be distinguished by the context.

problems that are out of the scope of the basic model, for being able of facing these problems, we have proposed several extensions of the decision model.

Generalising the Qualitative Utilities Functions and enabling possibly non-normalised distributions

Generalised utility functionals involving general t-norm operations in the uncertainty scale were considered in [3]. For each t-norm-like operation $\top$ on V such that h is coherent w.r.t. $\top$, i.e. h satisfies

$$h(\alpha) = h(\beta) \Rightarrow h(\alpha \top \lambda) = h(\beta \top \lambda) \quad \forall \lambda \in V,$$

we proposed the following generalised qualitative utility functionals:

$$GQU^-_\top(\pi_d \mid u) = \min_{x \in X} n(\pi_d(x) \top \lambda_x), \tag{3}$$

$$GQU^+_\top(\pi_d \mid u) = \max_{x \in X} h(\pi_d(x) \top \mu_x), \tag{4}$$

with $n(\lambda_x) = u(x) = h(\mu_x)$, and n is as above. In particular, when $\top = \min$ the above original functionals are recovered, but -as it might be seen in Section 5.2 in [22], these orderings may result different if we consider for example the t-norm of Lukasiewicz, sometimes more discriminating.

Comparing these utility functions with the pure ordinal ones, we have that, for any decision d: $QU^+(\pi_d|u) \geq GQU^+(\pi_d|u) \geq GQU^-(\pi_d|u) \geq QU^-(\pi_d|u)$. Moreover, if GQU^+ and GQU^- are considered in terms of the t-norm $\top_V$ involved, GQU^- is non-increasing with respect to $\top_V$; while GQU^+ is non-decreasing. That is, if $\top \leq \top_1$ are t-norms in V; then $GQU^-_\top \geq GQU^-_{\top_1}$ and $GQU^+_\top \leq GQU^+_{\top_1}$.

In some decision problems, like in case-based decision one or when partially conflicting belief states are considered, non-normalised possibility distributions can be involved. A direct application of (3) or (4) in these problems may result in contra-intuitive results. Hence, to properly handle these decision problems, another extension of the model is proposed in [11], where we defined adapted utility functionals in terms of a ponderation of the original distributions by their levels of inconsistency. That is, they are defined as follows:

$$\underline{GQU^-_\top}(\pi_d \mid u) = \min(GQU^-_\top(\mathcal{N}(\pi_d) \mid u), n(\mathcal{I}(\pi_d))), \tag{5}$$

$$\underline{GQU^+_\top}(\pi_d \mid u) = \max(GQU^+_\top(\mathcal{N}(\pi_d) \mid u), h(\mathcal{I}(\pi_d))), \tag{6}$$

where $\mathcal{N}(\pi_d)$ denotes the normalisation of π_d, defined as

$$\mathcal{N}(\pi)(x) = \begin{cases} 1, & \text{if } \pi(x) = \mathcal{H}(\pi) \\ \pi(x), & \text{otherwise}^7 \end{cases}$$

with the *height* of a distribution being defined as $\mathcal{H}(\pi) = \max\{\pi(x) \mid x \in X\}$, and $\mathcal{I}(\pi) = n_V(\mathcal{H}(\pi))$ is called the *inconsistency level* of π, where n_V is the reversing involution on V.

Measuring preferences and uncertainty on non-linear scales

There are certain kind of decision problems where we are not able to measure uncertainty and/or preferences in such linearly ordered sets, but only in partially ordered ones. For example:

- When there are several sources of uncertainty, each one being measured in a linear scale, the set of values for uncertainty, $(\overline{V}, \leq_{\overline{V}})$, is a product of scales, that is, $\overline{V} = \Pi_{j=1,..,k} V_j$; each V_j being a finite linearly ordered set.
- In a similar way, we may have that DM's preferences on consequences are only partially ordered. Indeed, a preference relation among consequences is usually modelled by a preference function $u : X \to U$; where U is a finite preference scale, frequently a (numerical or a qualitative) linear scale. However, in many cases, this preference function may be vectorial. Indeed, suppose that consequences are evaluated with respect to k different criteria or attributes, each one represented by a preference function $u_j : X \to U$. Then, the global preference on consequences can be evaluated in terms of the vectorial function $\overline{u} : X \to U \times^{k)} \times ... \times U$; with $\overline{u}(x) = (u_1(x), ..., u_k(x))$. Considering in $U \times^{k)} \times ... \times U$ the usual product ordering (Pareto ordering), we are outside of the linear models.
- Also we may have partially ordered uncertainty in case-based decision problems when the degrees of similarity on problems are only partially ordered. In this case, if we are not provided with an aggregation criterion for similarity vectors that summarises the criteria on an ordinal linear scale, we are not able to apply the previously mentioned models.

Hence, we are also interested in a possibilistic qualitative decision model that let us make decisions in cases where the DM's preferences on consequences are only partially ordered or when the uncertainty on the consequences is measured on a lattice. In order to cope with some of these situations, we have proposed an extension of the model in three steps [22]:

- First, we considered preferences and/or uncertainty were measured on finite Cartesian product of (finite) linear scales.

[7] Obviously, if π is normalised, $\mathcal{N}(\pi) = \pi$, and (5) and (6) coincide with (3) and (4) respectively.

- Second, we considered both preferences and uncertainty were graded on distributive lattices, in particular, when both are non-linear distributive lattices.
- Finally, we considered a particular case of allowing different type of measurement lattices, indeed we measured preferences on a linear one, while uncertainty was measured on a residuated distributive lattice.

These extensions of the model are described in [12, 24]. In particular, for finite distributive lattices with involution we consider generalised utility functionals analogous to (3) and (4) but with minimum and maximum replaced by lattice meet and joint operators, and h required to be an epimorphism. In that framework, the normalisation is defined as

$$\mathcal{N}(\pi)(x) = \begin{cases} 1, & \text{if } \pi(x) \text{ is maximal} \\ \pi(x), & \text{otherwise,} \end{cases}$$

and the height of a distribution is defined as $\mathcal{H}(\pi) = \bigvee\{\pi(x) \mid x \in X\}$.

Refining orderings

One of the drawbacks of the possibilistic decision criteria (both the optimistic and pessimistic) is their lack of discrimination power in some situations. In those situations, in order to break the indifference status among alternative decisions, it is natural to consider possible refinements of the preference ordering. In [4], several possibilities were analysed and the refinements of the ordering induced by one criterion with the orderings induced by other criteria were characterised.

Recently, we extended [10] this approach by allowing to work with non-normalised distributions and to measure uncertainty and preferences on finite distributive lattices.

In order to try to break some indifference status established taking into account the rankings representable[8] by the QU or the GQU, we may sequentially apply some of the qualitative criteria previously mentioned. A first option, very simple and easy to justify, is to use the optimistic criterion to refine the pessimistic one, i.e.

$$\begin{aligned} \pi \sqsubseteq \pi' \iff & \{GQU^-(\pi \mid u) <_U GQU^-(\pi' \mid u)\} \text{ or} \\ & \{[GQU^-(\pi \mid u) = GQU^-(\pi' \mid u)] \text{ and} \\ & [GQU^+(\pi \mid u) \leq_U GQU^+(\pi' \mid u)]\}, \end{aligned}$$

where both generalised utility functions are defined over the same set U and with the same preference function u.

[8] A relation $\preccurlyeq$ on a set E is representable by a function $f : W \to (C, \leq_C) iff (e \preccurlyeq e'$ iff $f(e) \leq_C f(e'))$.

There are several alternative combinations for refining. For example, the lexicographic ordering of an arbitrary set of generalised pessimistic/optimistic criteria that might include: (i) different t-norms, and (ii) different preference sets and assignments, may be seen as an easy generalisation of this idea. Indeed, given a set of preference mappings $u_j : X \to U_j$, a set $\top_j$ of t-norms on V, and a set of coherent commensurability mappings $h_j : V \to U_j$, we define :

$$\begin{aligned} &\pi \sqsubseteq \pi' \iff \\ &\quad (\mathcal{U}_1(\pi), \ldots, \mathcal{U}_m(\pi)) \leq_{LEX} (\mathcal{U}_1(\pi'), \ldots, \mathcal{U}_m(\pi')) \quad (7) \end{aligned}$$

with either $\mathcal{U}_j = GQU^-_{\top_j}(\cdot \mid u_j, h_j)$ or $\mathcal{U}_j = GQU^+_{\top_j}(\cdot \mid u_j, h_j)$, where $\leq_{LEX}$ denotes the *lexicographic ordering* on the cartesian product $U_1 \times \ldots \times U_m$: $(u_1, \ldots, u_m) \leq_{LEX} (u'_1, \ldots, u'_m)$ if either $(u_1, \ldots, u_m) = (u'_1, \ldots, u'_m)$ or there is $j \leq m$ such that $u_i = u'_i$ for all $i < j$ and $u_j < u'_j$.

This type of refinements may be useful in decision problems in which we have several utility criteria that can be ranked according to their importance. For example in a safety decision problem we can have a utility function u_1 that accounts for preference for high personal safety levels and another function u_2 that accounts for preference in having low costs. In such a case, the first criterion must be more important than the second one. Notice that lexicographic utility scales have been considered for a long in the framework of Expected Utility Theory by people like Fishburn and LaValle (see e.g. [8, 16]).

Let us introduce a concept that will be useful for simplifying notation and mainly to generalise the results. Given a finite set of binary relations $\mathcal{R} = \{\preccurlyeq_i\}_{i=1,\ldots,k}$ on a set E, each "Boolean" mapping $g : \{0,1\}^k \times \{0,1\}^k \to \{0,1\}$ induces a new binary relation $\preccurlyeq^g_{\mathcal{R}}$ on E by defining:

$$\begin{aligned} e \preccurlyeq^g_{\mathcal{R}} e' \iff \quad & g((\mu_{\preccurlyeq_1}(e, e'), \ldots, \mu_{\preccurlyeq_k}(e, e')), \\ & (\mu_{\preccurlyeq_1}(e', e), \ldots, \mu_{\preccurlyeq_k}(e', e))) = 1, \end{aligned}$$

where $\mu_{\preccurlyeq_i}$ is the membership of the binary relation $\preccurlyeq_i$. Notice that the *lexicographic ordering* is a particular case of the relations $\preccurlyeq^g_{\mathcal{R}}$. Indeed, if the $\preccurlyeq_i$'s are linear orderings, and let $g_*(\overline{x}, \overline{y}) = \max_{i=1,\ldots,k} z_i$, with

$$z_i = \begin{cases} \min(x_1, 1 - y_1), & i = 1 \\ \min[\min_{j=1,\ldots,i-1}\{\min(x_j, y_j)\}, & \\ \qquad \min(x_i, 1 - y_i)], & 1 < i < k \\ \min(\min_{j=1,\ldots,k-1}\{\min(x_j, y_j)\}, x_k), & i = k, \end{cases}$$

then $\preccurlyeq^{g_*}_{\mathcal{R}}$ is nothing but the lexicographic ordering on E induced by $\preccurlyeq_1, \ldots, \preccurlyeq_k$, taken them in this sequence. In particular, taking g_* and $E = \Pi(X)$ with $\preccurlyeq_i = \leq_{\mathcal{U}_i}$, the preference ordering (7) on possibility distributions can be simply written as $\pi \preccurlyeq^{g_*}_{\mathcal{R}} \pi'$. Moreover, the Pareto ordering is also a particular example of these $\preccurlyeq^{g}_{\mathcal{R}}$ orderings [9].

Weakening commensurability hypothesis: non onto linking mapping

In the models developed up to now, we have been assuming an hypothesis of commensurateness between the plausibility set V and the preference set U in order to define the criteria for ranking possibility distributions. Actually, the existence of an order-preserving mapping $h : V \to \mathrm{U}$ such that $h(1) = 1$ and $h(0) = 0$ is assumed to define the qualitative utility functions. However, to characterise the orderings, h is also required to be onto. In order to be able of deal with problems in which the cardinality of the preference valuation set may be greater than the cardinality of the uncertainty valuation set, we propose a weakening of the commensurability hypothesis by not-requiring h to be onto. Given h like this, i.e. h is an order-preserving mapping s.t. $h(0) = 0$ and $h(1) = 1$, for any $\pi \in \Pi(X)$, we consider the qualitative utility functions

$$QU^-_W(\pi|u) = \min_{x\in X} \max(n(\pi(x)), u(x))$$

where $n = n_U \circ h$, n_U being the reversing involution in U, and

$$QU^+_W(\pi|u) = \max_{x\in X} \min(h(\pi(x)), u(x)).$$

Notice that $QU^-_W(\cdot|u)$ and $QU^-_W(\cdot|u)$, restricted to X, coincide with the preference function u, i.e. $QU^-_W(x|u) = u(x) = QU^+_W(x|u)$, for all $x \in X$. It is interesting to notice that these functions still preserve the possibilistic mixture.

4 Axiomatic Characterisations

In this section we recall the axiomatic characterisations of the generalised utility functions, provided in [3] for linear structures of measurement, and in [12, 24] for lattices measurement sets. For each t-norm $\top$ in V, a possibilistic $\vee$-$\top$-mixture $M_\top$ is considered[10] it is an internal operation on $\Pi(X)$ defined as $M_\top(\pi, \pi'; \alpha, \beta) = (\alpha\top\pi) \vee (\beta\top\pi')$ $\alpha, \beta \in V$ with $\alpha = 1$ or $\beta = 1$. This mixture operation is a possibilistic counterpart of the convex linear combination of probability distributions, that combines two normal distributions into a new one. Given a preference relation $\sqsubseteq$ defined on $\Pi(X)$, there are different axiomatisation cases taking into account the structure of V and U ($\pi \sim \pi'$ means $\pi \sqsubseteq \pi'$ and $\pi' \sqsubseteq \pi$).

Linear scales In the case both scales V and U are linearly ordered, the following set of axioms $\mathbf{AX}_\top = \{A1, A2, A3_\top, A4_\top\}$ characterises the (pessimistic) preference orderings representable by a $GQU^-_\top$ utility functional, for each t-norm $\top$ on V:

A1 (*structure*): $\sqsubseteq$ is a total pre-order (i.e. $\sqsubseteq$ is reflexive, transitive, total)

A2 (*uncertainty aversion*): if $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.

A3$_\top$ (*independence*):
$\pi_1 \sim \pi_2 \Rightarrow M_\top(\pi_1, \pi; \alpha, \beta) \sim M_\top(\pi_2, \pi; \alpha, \beta)$.

A4$_\top$ (*continuity*): $\forall\pi \in \Pi(X)\ \exists\lambda \in V$ s.t. $\pi \sim \pi^-_\lambda$.

Assuming $A1$, we have that there exists $\overline{x}$ and $\underline{x}$ a maximal and a minimal element of $(X, \sqsubseteq)$ respectively, hence, for any $\lambda \in V$, π^-_λ denotes the distribution $M_\top(\overline{x}, \underline{x}; 1, \lambda)$. Obviously $A1$ is an structure axiom, it allows to represent the utility-based preference on a totally ordered scale and is the same as the first Von Neumann and Morgenstern's axiom, while $A2$ establishes that the less informative is the distribution, the less preferred it is, the worst distribution is which corresponds to total ignorance. $A3_\top$ obviously establishes that preferentially equivalent distributions may be replaced in mixtures with other distribution.

For representing an optimistic behaviour representable by a $GQU^+_\top$ functional, $A2$ and $A4_\top$ are respectively replaced by:

A2$^+$ (*uncertainty attraction*): if $\pi \leq \pi' \Rightarrow \pi \sqsubseteq \pi'$.

A4$^+_\top$: $\forall\pi \in \Pi(X)\ \exists\lambda \in V$ s.t. $\pi \sim \pi^+_\lambda$,

where $\pi^+_\lambda = M_\top(\overline{x}, \underline{x}; \lambda, 1)$. $\mathbf{AX}^+_\top$ denotes the set of axioms $\{\mathrm{A1}, \mathrm{A2}^+, \mathrm{A3}_\top, \mathrm{A4}^+_\top\}$.

In [3], it is shown that a preference relations on $\Pi(X)$ satisfies $AX^-_\top$ (resp. $AX^+_\top$) if and only if it is indeed representable by a $GQU^-_\top$ (resp. $GQU^+_\top$) functional.

Possibly non-normalised distributions These representations have been extended to also characterise preference relations on the set $\mathbf{\Pi^{ex}(X)}$ of non-necessarily normalised possibility distributions on X.

[9] It has to be noticed as well that not any g gives raise to a proper ordering, i.e. a reflexive and transitive relation.

[10] For each t-norm $\top$ and conorm $\perp$ on V; we will be interested in $\perp - \top$ mixtures that combine two possibility distributions π_1 and π_2 into a new one, denoted $M_{\perp-\top}(\pi_1, \pi_2; \alpha; \beta)$; with $\alpha, \beta \in V$ and $\alpha\perp\beta = 1$; defined as: $M_{\perp-\top}(\pi_1, \pi_2; \alpha; \beta)(x) = (\alpha\top\pi_1(x))\perp(\beta\top\pi_2(x))$. Since the mixtures are required to satisfy reduction of lotteries, hence, we need that $(a\top c)\perp(b\top c) = c\top(a\perp b)$ be satisfied. Therefore, we have to restrict ourselves to $\vee$-mixtures.

Then, one considers the extended sets of axioms $\underline{\mathbf{AX}}_\top = \mathbf{AX}_\top \cup \{\mathbf{AP7}\}$ and $\underline{\mathbf{AX}}^+_\top = \mathbf{AX}^+_\top \cup \{\mathbf{AP7}\}$, with

AP7(normalisation): $\pi \sim M_\top(\mathcal{N}(\pi), X; 1, \mathcal{I}(\pi))$,

with the understanding that $\mathbf{AX}_\top$ and $\mathbf{AX}^+_\top$ are assumed to be fulfilled only by the restriction of the preference relation on the subset $\Pi(X) \subset \Pi^{ex}(X)$ of normalised possibility distributions. It is shown in [11] that these axiomatics characterise those preference relations on $\Pi^{ex}(X)$ representable by the $\underline{GQU}^-_\top$ and $\underline{GQU}^+_\top$ functionals respectively.

Non-linear Lattices As it is said, three cases of non-linear sets for measuring uncertainty and/or preferences have been considered. This is the case of V being a (finite) distributive residuated lattice with involution and U a (finite) distributive lattice with involution. Here, the mixture operation is defined analogously to the linear case, but considering now joint lattice operators and t-norm like operators. For a pessimistic behaviour of a preference relation $\sqsubseteq$ on $\Pi(X)$ we consider the set of axioms $\mathbf{AXP}_\top = \{$AP1, A2, A3$_\top$, A4$_\top$, AP5$_\top$, AP6$_\top\}$ with

AP1(*structure*): $(\Pi(X), \sqsubseteq)$ is a pre-lattice[11].

AP5$_\top$(*preference reversing*):
if $\pi^-_\lambda \sqsubseteq \pi^-_{\lambda'} \Rightarrow \pi^-_{n_V(\lambda)} \sqsupseteq \pi^-_{n_V(\lambda')}$,

AP6$_\top$(*incomparability preservation*):
if $\lambda <> \lambda' \Rightarrow \pi^-_\lambda \sqsubset\sqsupset \pi^-_{\lambda'}$,

where for any $\lambda \in V$, $\pi^-_\lambda = M_\top(\overline{\pi}, X; 1, \lambda)$ with $\overline{\pi}$ being a maximal element of $(\Pi(X), \sqsubseteq)$.

$AP1$ says that the quotient set $(\Pi(X)/\sim, \sqsubseteq)$ results a lattice, while $AP6$ establishes that two incomparable values of uncertainty, λ and λ', lead to two incomparable lotteries. Finally, $AP5$ says that the preference between lotteries with degrees of uncertainty λ and λ' with respect to a maximal $\overline{\pi}$ results reversed when the lotteries are considered with the respective opposite values of uncertainty.

For an optimistic behaviour we modify axiom $A2$ and the axioms involving π^-_λ. Indeed, we consider $\mathbf{AXP}^+_\top = \{$AP1, A2$^+$, A3$_\top$, A4$^+_\top$, AP5$^+_\top$, AP6$^+_\top$ $\}$, with

AP5$^+_\top$(*preference reversing*):
if $\pi^+_\lambda \sqsubseteq \pi^+_{\lambda'} \Rightarrow \pi^+_{n_V(\lambda)} \sqsupseteq \pi^+_{n_V(\lambda')}$,

AP6$^+_\top$(*incomparability preservation*):
if $\lambda <> \lambda' \Rightarrow \pi^+_\lambda \sqsubset\sqsupset \pi^+_{\lambda'}$,

[11] $(\Pi(X), \sqsubseteq)$ is a *pre-lattice* if $(\Pi(X)/\sim, \leq)$ is a lattice, where the lattice order $\leq$ is defined as $[\pi] \leq [\pi']$ iff $\pi \sqsubseteq \pi'$.

where $\pi^+_\lambda = M_\top(X, \underline{\pi}; \lambda, 1)$, with $\underline{\pi}$ a minimal element on $(\Pi(X), \sqsubseteq)$.

Preference relations on $\Pi(X)$ satisfying $\mathbf{AXP}_\top$ or $\mathbf{AXP}^+_\top$ are shown in [12] to be exactly those representable by lattice extensions of the $GQU^-_\top$ and $GQU^+_\top$ functionals.

Some refinements Now, we can proceed to the axiomatic characterisation of refinements of (partially ordered) preference relations involving arbitrary generalised utility functionals on lattice structures. The characterisation is a straightforward generalisation of the characterisations of the partial orderings previously described.

Theorem 1 (Representation Theorem) *Let V be a (finite) residuated distributive lattice with involution and let $\sqsubseteq$ be a preference relation on $\Pi(X)$ (normalised distributions on X over V) for which there exist a set $\mathcal{R} = \{\sqsubseteq_1, \ldots, \sqsubseteq_m\}$ of preference orderings also on $\Pi(X)$ and a boolean mapping g such that $\sqsubseteq = \preccurlyeq^g_{\mathcal{R}}$, i.e.*

$$\pi \sqsubseteq \pi' \iff g((\mu_{\sqsubseteq_1}(\pi, \pi'), \ldots, \mu_{\sqsubseteq_m}(\pi, \pi')), (\mu_{\sqsubseteq_1}(\pi', \pi), \ldots, \mu_{\sqsubseteq_m}(\pi', \pi))) = 1,$$

with each $\sqsubseteq_j$ satisfying either $\mathrm{AXP}_{\top_j}$ or $\mathrm{AXP}^+_{\top_j}$ (we will write $\sqsubseteq^-_j$ or $\sqsubseteq^+_j$ depending on whether $\sqsubseteq_j$ satisfies $\mathrm{AXP}_{\top_j}$ or $\mathrm{AXP}^+_{\top_j}$ respectively.). Then, there exist:

(i) *m utility finite distributive lattices with involution $(U_i, \wedge^i, \vee^i, n_{U^i}, 0, 1)$;*

(ii) *m preference functions $u^i{:}X \to (U^i, \leq^i)$, each u^i satisfying either*

$$\{(u^i)^{-1}(1) \neq \emptyset \text{ and } \bigwedge_{x \in X} (u^i)(x) = 0\} \quad or$$

$$\{(u^i)^{-1}(0) \neq \emptyset \text{ and } \bigvee_{x \in X} (u^i)(x) = 1\},$$

depending whether $\sqsubseteq_i$ satisfies $\mathrm{AXP}_{\top_i}$ or $\mathrm{AXP}^+_{\top_i}$ respectively;

(iii) *m onto join-preserving mappings $h^i{:}V \to U^i$, all satisfying coherence w.r.t $\top_i$, also satisfying*

$$\text{if } \lambda <> \lambda' \text{ then } h^i(\lambda) <> h^i(\lambda'),$$

and $n_{U^i} \circ h^i \circ n_V = h^i$,

in such a way that it holds:

$$\pi \sqsubseteq \pi' \quad iff \quad \pi \preccurlyeq^g_{\{\preccurlyeq_{GQU^1}, \ldots, \preccurlyeq_{GQU^m}\}} \pi',$$

with $\pi \preccurlyeq_{GQU^i} \pi'$ iff

$$GQU^{sgn(i)}_{\top_i}(\pi | u^i, h^i) \leq_{U^i} GQU^{sgn(i)}_{\top_i}(\pi' | u^i, h^i),$$

where $sgn(i) = -$ *if* $\sqsubseteq_i = \sqsubseteq_i^-$, *and* $sgn(i) = +$ *if* $\sqsubseteq_i = \sqsubseteq_i^+$.

No onto linking mapping Let us remark again that the great difference with the previously analysed cases is that now h is ***not*** required to be onto. For this case, we propose this new set of axioms $AXM=\{A1, A2, A3, A4C, AxMix\}$ for preference relations on $\Pi(X)$, with the max-min mixture as the internal operation on $\Pi(X)$, that are representable by QU_W^-:

- $A4C$(*relaxed continuity*): There exists a subset[12] $X_{NM} \subseteq X$ such that all maximal elements of $(X, \sqsubseteq)$ and all minimal elements of $(X, \sqsubseteq)$ are in the complement of X_{NM}, and such that

$$(\forall \pi \in \Pi(X)) \quad \text{either} \quad (\exists \lambda \in V \;\; \text{s.t.} \;\; \pi \sim \pi_\lambda^-) \quad \text{or} \quad (\exists x \in X_{NM} \;\; \text{s.t.} \;\; \pi \sim x).$$

- $AxMix$:
  1. if $x, y \in X_{NM}$, $\beta \in V$ then

$$(1/x, \beta/y) \sim \begin{cases} x & \text{if } (x \sqsubseteq y) \\ & \quad \text{or } (x \sqsubset \pi_\beta^-) \\ \pi_\beta^- & \text{if } y \sqsubset \pi_\beta^- \sqsubset x \\ y & \text{if } \pi_\beta^- \sqsubset y \sqsubset x, \end{cases}$$

  2. if $x \in X_{NM}$ then

$$(1/\pi_\lambda^-, \beta/x) \sim \begin{cases} \pi_\lambda^- & \text{if } (\pi_\lambda^- \sqsubset x) \\ & \quad \text{or } (\pi_\lambda^- \sqsubseteq \pi_\beta^-) \\ \pi_\beta^- & \text{if } x \sqsubset \pi_\beta^- \sqsubset \pi_\lambda^- \\ x & \text{if } \pi_\beta^- \sqsubset x \sqsubset \pi_\lambda^-. \end{cases}$$

The underlying idea in $A4C$ is to relax the continuity of the preference. Now, we may say that there exists a subset on X such that either the distributions are preferentially equivalent to individual consequences on this set, or, the distributions are preferentially equivalent to having a λ-level of uncertainty with respect to $\overline{x}$. In [23], it is shown that QU_W^- represents the preference relations satisfying AXM, while if the preference relation represent an optimistic behaviour, that is , if $A4C$ and $AxMix$ are replaced by their "optimistic" versions, is representable by QU_W^+.

5 Ongoing work

Very recently, we have considered [10]two further steps in the foundations of Possibilistic Decision Theory à la Von Neumann and Morgenstern. In this framework, preference relations are defined on possibilistic distributions on the set of consequences. Namely, we have considered, as it is said, the definition and characterisation of refinements of partially ordered preference relations, moreover, two notions of conditional preference relations were considered. Refinements in the framework of possibilistic decision have also been recently considered by Fargier and Sabbadin in [7] where they show that both (pessimistic and optimistic) preference orderings representable by the possibilistic utility functionals QU^- and QU^+ always admit a refinement which is compatible with the Expected Utility model. On the other hand, Giang and Shenoy propose in [9] a model where both the possibilistic pessimistic and optimistic functionals are combined somehow in a single functional which behaves as a refinement. In future works we plan to establish links between these kind of refinements and those described in the paper. Finally, we think that the notions of conditional preference presented here will allow us to build tighter bridges between the two kinds of axiomatic approaches to possibilistic decision theory, à la Von Neumann and Morgenstern and à la Savage.

[12] Observe that $X_{NM} = \emptyset$ is possible, and then axiom $A4$ is recovered.

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