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Generalized transformed t-conorm integral and multifold integral

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Abstract

Fuzzy integrals are commonly used as aggregation operators. In this paper we present new composite models based on fuzzy integrals with several t-conorms. These models permit to further increase the computational power of the previous models. Basic properties of new composite models are studied. In particular, it is shown that for composite models, the only meaningful case is when not all the t-conorms are the same.

Keywords: Fuzzy measure; Choquet integral; Fuzzy integrals; t-conorm integral; Twofold integral

1. Introduction

Aggregation operators are used for a large variety of purposes [8,27,28]. Due to these needs, different families of aggregation operators have been defined (see e.g., [3] for a review of some of them). The weighted mean, the OWA operator and the fuzzy integrals are some examples of operators that can be used to aggregate numerical values. Fuzzy integrals [1,2,6,16,22], in general, are useful tools for this purpose as they allow for some additional flexibility in relation to other operators as the weighted mean or the OWA. This additional flexibility is due to the fact that data is fused with respect to a fuzzy measure. This fuzzy measure can be used to represent some knowledge on the sources. For example, they can be used to express that the sources are not independent. See [7,10] for recent results on fuzzy measures and integrals.

To further increase the computational power of such operators and to augment their modeling capabilities, composite models have recently been developed. Some initial steps in this direction include multi-step Choquet integrals [11,12,23]. The twofold integral [19,26] and the new construction method developed by Calvo et al. [4,5] also follow this approach.

In this work we study new composite models, based on fuzzy integrals. In [14,12] t-conorm systems (or t-systems) and t-conorm integrals were introduced. T-conorm systems are a generalization of addition and multiplication. Then, the t-conorm integral results into a generalization of both Choquet and Sugeno integrals. Nevertheless, it was proven that from a purely mathematical point of view, the only essential fuzzy integrals are the Choquet and Sugeno integrals. Narukawa and Murofushi [17,18] define the Choquet Stieltjes integral and show the basic properties and some examples demonstrating its use in practical situations.

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In this paper, we define first a generalized transformed t-conorm (GTT) integral and, then, a generalized t-conorm multifold integral (GTM) based on t-conorm systems (or t-system)[14] for integration, that is a generalization of both t-conorm integral and the Choquet Stieltjes integral. Finally, we define a multifold Choquet integral and a multifold Sugeno integral, and we show that a multifold Choquet integral can be expressed as a Choquet integral and that a multifold Sugeno integral is equivalent to the Sugeno integral. This fact shows that the only meaningful multifold integrals are the ones whose t-system is different.

The structure of the paper is as follows. First, in Section 2 we review some results that are needed in the rest of the paper. Then, in Section 3 we introduce the generalized transformed t-conorm integral and we show its relation with a generalized Losonczi's mean. Section 4 is devoted to the generalized t-conorm multifold integral. Some properties are studied. The paper finishes with some conclusions.

2. Preliminaries

This section reviews main results on fuzzy integrals. In particular, we focus on the Choquet [6] and Sugeno [22] integrals and the t-conorm system for integration.

2.1. Choquet integral

In this subsection, the well known definition of the Choquet integral and some of its basic properties will be given (see [6] or [3] for details).

Throughout this paper we assume that X is a finite set, that is, $X := \{1, 2, \dots, n\}$. Let $[0, 1]^n$ be the space of all non-negative real functions x from X to the unit interval [0, 1].

The ith order statistic $x^{(i)}$ [29] is a functional on $[0, 1]^n$ which is defined by arranging the components of $x = (x_1, \dots, x_n) \in [0, 1]^n$ in increasing order

$$x^{(1)} \leqslant \cdots \leqslant x^{(i)} \leqslant \cdots \leqslant x^{(n)}.$$

Now, we present the basic definition of a fuzzy measure.

Definition 1 (*Sugeno [22]*). A fuzzy measure μ on $(X, 2^X)$ is a real valued set function, $\mu : 2^X \longrightarrow [0, 1]$ with the following properties;

(1) $\mu(\emptyset) = 0$, $\mu(X) = 1$.
(2) $\mu(A) \leqslant \mu(B)$ whenever $A \subset B$, $A, B \in 2^X$.

A triplet $(X, 2^X, \mu)$ is said to be a fuzzy measure space.

Definition 2 (*Choquet [6], Murofushi and Sugeno [13]*). Let μ be a fuzzy measure on $(X, 2^X)$. Then, the Choquet integral of $x \in [0, 1]^n$ with respect to μ is defined by

$$C_\mu(x) := \int_0^1 \mu_x(r)\,\mathrm{d}r,$$

where $\mu_x(r) = \mu(\{i \mid x_i \geqslant r\})$.

Using the ith order statistics, the Choquet integral can be rewritten as

$$C_\mu(x) = \sum_{i=1}^{n} (x^{(i)} - x^{(i-1)})\mu(\{(i) \cdots (n)\}),$$

where we define $x^{(0)} := 0$.

Definition 3 (*Sugeno [22]*). Let μ be a fuzzy measure on $(X, 2^X)$. The Sugeno integral of $x \in [0,1]^n$ with respect to μ is defined by

$$S_\mu(x) := \sup_{r \in [0,1]} [r \wedge \mu_x(r)],$$

where $\mu_x(r) := \mu(\{i \mid x_i \geqslant r\})$.

Using the ith order statistics, the Sugeno integral can be rewritten as

$$S_\mu(x) = \bigvee_{i=1}^{n} (x^{(i)} \wedge \mu(\{(i) \cdots (n)\})).$$

2.2. *Generalized* t-*conorm integral*

In this section, we use t-conorms and t-norms. They are binary operators that generalize addition and multiplication, and also max and min.

A triangular conorm (t-conorm) $\perp$ is a binary operation on [0, 1] fulfilling the conditions:

(T1) $x \perp 0 = x$.
(T2) $x \perp y \leqslant u \perp v$ whenever $x \leqslant u$ and $y \leqslant v$.
(T3) $x \perp y = y \perp x$.
(T4) $(x \perp y) \perp z = x \perp (y \perp z)$.

A t-conorm is said to be strict if and only if it is continuous on [0, 1] and strictly increasing on $[0, 1)^2$. A continuous t-conorm $\perp$ is said to be Archimedean if and only if $x \perp x > x$ for all $x \in (0, 1)$.

Example 1. (1) The maximum operator $x \vee y$ is a non-Archimedean t-conorm.

(2) The bounded sum $x \hat{+} y := 1 \wedge (x + y)$ is an Archimedean t-conorm. This is non-strict t-conorm.

(3) The Sugeno operator $x +_\lambda y := 1 \wedge (x + y + \lambda xy)$ $(-1 \leqslant \lambda < \infty)$ is an Archimedean t-conorm, which is strict only if $\lambda = -1$.

Let m be a fuzzy measure on $(X, 2^X)$. We say that m is $\perp$-decomposable if $m(A \cup B) = m(A) \perp m(B)$ whenever $A \cap B = \emptyset$ for $A, B \in 2^X$.

The continuous t-conorms we consider in this chapter are restricted to Archimedean ones or to the maximum operator. The first reason is that any non-Archimedean continuous t-conorm is represented by Archimedean continuous t-conorms and $\vee$. This fact is mentioned by Weber [30]. The second reason is the decomposition theorem by Sugeno and Murofushi [24]: if $\perp$ is a non-Archimedean t-conorm which is not $\vee$ and m is a $\perp$-decomposable fuzzy measure, the fuzzy measure space $(X, 2^X, m)$ can be decomposed into fuzzy measure spaces $\{(X_i, \mathcal{X}_i, m_i)\}$ where $\{X_i\}$ is a partition of X, $\mathcal{X}_i := \{A \cap X_i \mid A \in 2^X\}$, m_i is $\perp_i$ decomposable measure and $\perp_i$ can be regarded as an Archimedean t-conorm or $\vee$.

Definition 4 (*Murofushi and Sugeno [14]*). A t-conorm system for integration (or t-system for short) is a quadruplet $(F, M, I, \boxdot)$ consisting of $F := ([0,1], \perp_1)$, $M = ([0,1], \perp_2)$, $I = ([0,1], \perp_3)$, where $\perp_i,\ i = 1, 2, 3$ are continuous t-conorms which are $\vee$ or Archimedean, and a non-decreasing operator $\boxdot : F \times M \to I$ satisfying:

(M1) $\boxdot$ is left continuous on (0, 1].
(M2) $a \boxdot x = 0$ if and only if $a = 0$ or $x = 0$.
(M3) if $x \perp_2 y < 1$ then $a \boxdot (x \perp_2 y) = (a \boxdot x) \perp_3 (a \boxdot y)$.
(M4) if $a \perp_1 b < 1$ then $(a \perp_1 b) \boxdot x = (a \boxdot x) \perp_3 (b \boxdot x)$.

The t-system is expressed by $(\perp_1, \perp_2, \perp_3, \boxdot)$ instead of $(F, M, I, \boxdot)$. For example, a t-system is expressed by $(+_\lambda, \vee, \hat{+}, \cdot)$ in the case where $\perp_1 = +_\lambda$, $\perp_2 = \vee$, $\perp_3 = \hat{+}$, $\boxdot = \cdot$ (the ordinary multiplication), and by $(\vee, \vee, \vee, \wedge)$ in the case where $\perp_1 = \perp_2 = \perp_3 = \vee$ and $\boxdot = \wedge$.

For a given t-conorm $\perp$, we define an operation $-_\perp$ by

$$a -_\perp b := \inf\{c \mid b \perp c \geqslant a\}$$

for all $(a, b) \in [0, 1]^2$. Note that if $\perp = \vee$ (maximum operator) we have

$$a -_\perp b = \begin{cases} a & \text{if } a \geqslant b, \\ 0 & \text{if } a < b. \end{cases}$$

Using the operator $-_\perp$, a function $x : X \to [0, 1]$ can be expressed as

$$x = {\perp_1}_{i=1}^{n} (x^{(i)} -_\perp x^{(i-1)}) 1_{A_i},$$

where $A_i := \{(i), \ldots, (n)\}$ for $i = 1, \ldots, n$ and $x^{(0)} := 0$.

Definition 5 (*Murofushi and Sugeno [14], Narukawa and Murofushi [15]*). Let $(X, 2^X, m)$ be a fuzzy measure space and $(\perp_1, \perp_2, \perp_3, \boxdot)$ be a t-system. For a function $x : X \to [0, 1]$ $(x = {\perp_1}_{i=1}^{n} (x^{(i)} -_\perp x^{(i-1)}) 1_{A_i})$ the generalized t-conorm integral is defined as follows:

$$(GT) \int x \boxdot dm := {\perp_3}_{i=1}^{n} (x^{(i)} -_{\perp_1} x^{(i-1)}) \boxdot m(A_i).$$

If m is a normal $\perp_2$ decomposable fuzzy measure, the generalized t-conorm integral coincides with the t-conorm integral [14].

Example 2. (1) In the case of $\perp_1 = \perp_2 = \perp_3 = +$ and $\boxdot = \cdot$, the generalized t-conorm integral coincides with Choquet integral.

(2) If $\perp_1 = \perp_2 = \perp_3 = \vee$ and $\boxdot = \wedge$, the generalized t-conorm integral is the Sugeno integral.

In the definition of generalized t-conorm integral, we do not need the $\perp_2$ decomposability for the fuzzy measure m. Almost all important fuzzy integrals assume that $\perp_1 = \perp_3$. Therefore, in the following, we address a t-system $(\perp, \boxdot)$, that means $\perp_1 = \perp_2 = \perp_3 = \perp$, although we may assume that $\perp_1$, $\perp_2$ and $\perp_3$ are different.

The t-system can be generalized by a pseude-addition $\oplus$ and a $\oplus$-fitting pseudo multiplication $\odot$ [1,2].

3. Generalized transformed t-conorm integral

Definition 6. Let $(X, 2^X, m)$ be a fuzzy measure space, $(\perp, \boxdot)$ be a t-system and $\varphi : [0, 1] \to [0, 1]$ be a non-decreasing function.

For a function $x : X \to [0, 1]$ $(x = \perp_{i=1}^{n} (x^{(i)} -_\perp x^{(i-1)}) 1_{A_i})$, a generalized transformed t-conorm integral (GTT) with respect to m and φ is defined as follows:

$$(GTT)_{m,\varphi}(x) := \perp_{i=1}^{n} (\varphi(x^{(i)}) -_\perp \varphi(x^{(i-1)})) \boxdot m(A_i).$$

Since φ is non-decreasing, we have

$$m(\{j \mid x_j \geqslant x_i\}) = m(\{j \mid \varphi(x_j) \geqslant \varphi(x_i)\}).$$

Therefore we have the next proposition.

Proposition 7. *Let $(X, 2^X, m)$ be a fuzzy measure space, $(\perp, \boxdot)$ be a* t-*system and $\varphi : [0, 1] \to [0, 1]$ be a non-decreasing function.*

A GTT integral of $x \in [0, 1]^n$ with respect to m and φ is equal to the generalized t-conorm integral of $\varphi(x)$ with respect to m, that is,

$$(GTT)_{m,\varphi}(x) = (GT) \int \varphi(x) \boxdot \mathrm{d}m$$

Example 3. (1) Suppose that $\perp = +$ and $\boxdot = \cdot$, the GTS integral is a Choquet Stieltjes integral [17], that is,

$$\begin{aligned}(GTS)_{m,\varphi}(x) &= (C)\int \varphi(x)\,\mathrm{d}m \\ &:= CS_{m,\varphi}(x)\end{aligned}$$

(2) Suppose that $\perp = \vee$ and $\boxdot = \wedge$, then we call the GTS integral transformed Sugeno integral (TS), that is,

$$(TS)_{m,\varphi}(x) = \bigvee_{i=1}^{n} (\varphi(x_i) \wedge m(\{j|x_j > x_i\})).$$

That is, the transformed Sugeno integral of $x \in [0,1]^n$ is Sugeno integral of $\varphi(x)$.

Applying Proposition 7, we can characterize a Choquet Stieltjes integral as a generalized Losonczi mean [9].

Definition 8. Let $w : [0,1]^n \to [0,1]^n$ be weighting functions, that is, $w(x) := (w_1(x), w_2(x), \ldots, w_n(x))$ for $x \in [0,1]^n$, let ϕ be a non-decreasing function. The generalized Losonczi mean of $x \in [0,1]^n$ for $i \in \{1, \ldots, N\}$ is defined as:

$$LM_w(x) = \phi^{-1}\left(\frac{\sum_i w_i(x)\phi(x_i)}{\sum_i w_i(x)}\right)$$

where $x := (x_1, \ldots, x_n)$.

In the original definition of Losonczi [9], the weighting function w is defined as $w(x) := (w_1(x_1), w_2(x_2), \ldots, w_n(x_n))$ for $x = (x_1, x_2, \ldots, x_n)$. The definition above is one of the generalizations of the original definition.

Theorem 9. *Let $(X, 2^X, m)$ be a fuzzy measure space, and $\varphi : [0,1] \to [0,1]$ be a non-decreasing function and let $x \in [0,1]^n$.*

Then, there exists a weighting function w such that the Choquet Stieltjes integral of x with respect to m and φ can be represented as a generalized Losonczi mean. That is,

$$(CS)_{m,\varphi}(x) = \varphi(LM_w(x)).$$

Proof. Applying Proposition 7, we have

$$(CS)_{m,\varphi}(x) = (C)\int \varphi(x)\,\mathrm{d}m.$$

It is proved in [25] that there exists a probability $P_{m,\varphi(x)}$ on $(X, 2^X)$ such that

$$(C)\int \varphi(x)\,\mathrm{d}m = \int \varphi(x)\,\mathrm{d}P_{m,\varphi(x)}.$$

Then defining a weighting function $w : [0,1]^n \to [0,1]^n$ by

$$w_i(x) := P_{m,\varphi(x)}(i)$$

for $i = 1, 2, \ldots, n$, we have that

$$\begin{aligned}\int \varphi(x)\,\mathrm{d}P_{m,\varphi(x)} &= \sum_{i=1}^{n} \varphi(x_i) P_{m,\varphi(x)}(i) \\ &= \frac{\sum_{i=1}^{n} w_i(x)\varphi(x_i)}{\sum_{i=1}^{n} w_i(x)} \\ &= \varphi(LM_w(x)),\end{aligned}$$

that is,

$$(CS)_{m,\varphi}(x) = \varphi(LM_w(x)). \qquad \square$$

Remark. The probability $P_{m,\varphi(x)}$ in the proof above is a weighting function w that depends on φ, x and m. This means that if a non-decreasing function ϕ is $\phi \neq \varphi$, we have, in general, that $P_{m,\phi(x)} \neq P_{m,\varphi(x)}$. Nevertheless, when ϕ and φ are strictly monotone, we have $P_{m,\phi(x)} = P_{m,\varphi(x)} = P_{m,x}$ since $\phi(x)$, $\varphi(x)$ and x are strongly comonotonic [25].

Since for $\varphi(x) = x$ the Choquet Stieltjes integral is a Choquet integral, the following corollary holds:

Corollary 10. *For every fuzzy measure m there exists a weighted function $w : [0, 1]^n \to [0, 1]^n$ such that*

$$(C)\int x\, \mathrm{d}m = \frac{\sum_{i=1}^n w_i(x)x_i}{\sum_{i=1}^n w_i(x)}.$$

4. Generalized t-conorm Multifold integral

Let m_k be fuzzy measures on $(X, 2^X)$, $\mathcal{T}_k := (\perp^k, \boxdot^k)$ be a set of t-systems for $k = 1, 2, \ldots$ and $x : X \to [0, 1]$. We define a function $\varphi_{k,x} : [0, 1] \to [0, 1]$ by:

$$\varphi_{k,x}(a) = {\perp^k}_{0 \leqslant x^{(i)} \leqslant a}(\varphi_{k-1,x}(x^{(i)}) -_{\perp^k} \varphi_{k-1,x}(x^{(i-1)})) \boxdot^k m_k(A_i),$$

for $k = 1, 2, \ldots$, where $\varphi_{0,x}(a) := a$ is a non-decreasing function for $a \in [0, 1]$. It is obvious that $\varphi_{k,x}$ is non-decreasing. Then we can define a generalized t-conorm multifold integral (GTM) of x with respect to m_i for $i = 1, 2, \ldots$ by

$$GTM_{(m_i)_{i=1,2,\ldots,k}}(x) := \varphi_{k,x}(1).$$

It is obvious that

$$GTM_{(m_i)_{i=1,2,\ldots,k}}(x) = GTT_{m_k,\varphi_k}(x).$$

Example 4. Let $\mathcal{T}_1, \mathcal{T}_2$ be t-systems such that $\mathcal{T}_1 := (\vee, \wedge)$ and $\mathcal{T}_2 := (+, \cdot)$. Then we have

$$\varphi_{1,x}(a) = \bigvee_{0 \leqslant x^{(i)} \leqslant a} (x^{(i)} \wedge m_1(\{j | x^{(j)} \geqslant x^{(i)}\})),$$

$$\begin{aligned} GTM_{(m_1,m_2)}(x) &= \sum_{k=1}^n (\varphi_{1,x}(a_k) - \varphi_{1,x}(a_{k-1}))m_2(\{j | a_j \geqslant a_k\}) \\ &= \sum_{k=1}^n \left(\bigvee_{0 \leqslant x^{(i)} \leqslant a_k} (x^{(i)} \wedge m_1(\{j | x^{(j)} \geqslant x^{(i)}\})) \right. \\ &\quad \left. - \bigvee_{0 \leqslant x^{(i)} \leqslant a_{k-1}} (x^{(i)} \wedge m_1(\{j | x^{(j)} \geqslant x^{(i)}\})) \right) m_2(\{j | a_j \geqslant a_k\}), \end{aligned}$$

where $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$, that is, in this case, $\mathrm{GTM}_{(m_1,m_2)}$ is a twofold integral [26].

Suppose that x is a characteristic function of $A \subset X$ that is, $x := 1_A$. Then, we have $\varphi_{1,1_A}(1) = 1 \boxdot_1 m_1(A)$ and $\varphi_{2,1_A}(1) = (1 \boxdot_1 m_1(A)) \boxdot_2 m_2(A)$. Therefore we have the next proposition by induction.

Proposition 11. *Let m_k be fuzzy measures on $(X, 2^X)$, $\mathcal{T}_k := (\perp^k, \boxdot^k)$ be a set of* t-*systems for $k = 1, 2, \ldots$ and 1_A is a characteristic function of $A \subset X$. Then we have*

$$GTM_{(m_i)_{i=1,2,\ldots,k}}(1_A) = (((1 \boxdot_1 m_1(A)) \boxdot_2 m_2(A)) \boxdot_3 \ldots) \boxdot_k m_k(A)$$

for $k = 1, 2, \ldots$.

Let f and g be functions on $X :\to [0, 1]$ with $f \leqslant g$. Since $\varphi_{1,f} \leqslant \varphi_{1,g}$, we have the next proposition by induction.

Proposition 12. *Let m_k be fuzzy measures on $(X, 2^X)$, $\mathcal{T}_k := (\perp^k, \boxdot^k)$ be a set of* t*-systems for $k = 1, 2, \ldots$ and f and g be a function $X \to [0, 1]$ with $f \leqslant g$. Then we have*

$$GTM_{(m_i)_{k=1,2,\ldots,k}}(f) \leqslant GTM_{(m_i)_{k=1,2,\ldots,k}}(g).$$

for $k = 1, 2, \ldots$.

In the following we will study a special class of GTM integral. In particular, we study the multifold Choquet integral (MC) and the multifold Sugeno integral (MS).

Definition 13. Let m_k be fuzzy measures on $(X, 2^X)$, $\mathcal{T}_k := (\perp^k, \boxdot^k)$ be a set of t-systems for $k = 1, 2, \ldots$.

(1) Suppose that $\mathcal{T}_1 = \mathcal{T}_2 = \cdots = \mathcal{T}_k = (+, \cdot)$. We say that a *GTM* integral is a multifold Choquet integral with respect to $(m_i)_{1 \leqslant i \leqslant k}$.
(2) Suppose that $\mathcal{T}_1 = \mathcal{T}_2 = \cdots = \mathcal{T}_k = (\vee, \wedge)$. We say that a *GTM* integral is a multifold Sugeno integral with respect to $(m_i)_{1 \leqslant i \leqslant k}$.

Theorem 14. *Let $(m_i)_{1 \leqslant i \leqslant k}$ be fuzzy measures on $(X, 2^X)$ then*:

(1) *A multifold Choquet integral with respect to $(m_i)_{1 \leqslant i \leqslant k}$ is a Choquet integral with respect to $\mu := m_1 \cdot m_2 \cdot \cdots \cdot m_k$.*
(2) *A multifold Sugeno integral with respect to $(m_i)_{1 \leqslant i \leqslant k}$ is a Sugeno integral with respect to $\mu := m_1 \wedge m_2 \wedge \cdots \wedge m_k$.*

Proof.

(1) Using ith order statistics, we have

$$\begin{aligned}&\varphi_{k,x}(x^{(i)}) - \varphi_{k,x}(x^{(i-1)})\\ &\quad = (\varphi_{k-1,x}(x^{(i)}) - \varphi_{k-1,x}(x^{(i-1)}))m_{k-1}(\{(i), (i+1), \ldots, (n)\}).\end{aligned}$$

By induction we have

$$\varphi_{k,x}(x^{(i)}) - \varphi_{k,x}(x^{(i-1)}) = (x^{(i)} - x^{(i-1)})(m_1 \cdot \cdots \cdot m_{k-1})(\{(i), (i+1), \ldots, (n)\}).$$

Therefore, we have

$$\begin{aligned}\mathrm{MC}_{(m_i)_{i=1,2,\ldots,k}}(x) &:= \sum_{i=1}^{n} \left(\varphi_{k,x}(x^{(i)}) - \varphi_{k,x}(x^{(i-1)})\right) m_k((i) \cdots (n))\\ &= \sum_{i=1}^{n} (x^{(i)} - x^{(i-1)})(m_1 \cdot \cdots \cdot m_{k-1} m_k)(\{(i), (i+1), \ldots, (n)\})\\ &= (C) \int x \, \mathrm{d} m_1 m_2 \cdots m_k.\end{aligned}$$

(2) Using ith order statistics, we have

$$\begin{aligned}&\varphi_{k,x}(x^{(i)}) -_{\vee} \varphi_{k,x}(x^{(i-1)})\\ &\quad = (\varphi_{k-1,x}(x^{(i)}) -_{\vee} \varphi_{k-1,x}(x^{(i-1)})) \wedge m_{k-1}(\{(i), (i+1), \ldots, (n)\}).\end{aligned}$$

By induction we have

$$\begin{aligned}&\varphi_{k,x}(x^{(i)}) -_{\vee} \varphi_{k,x}(x^{(i-1)})\\ &\quad = (x^{(i)} -_{\vee} x^{(i-1)}) \wedge (m_1 \wedge \cdots \wedge m_{k-1})(\{(i), (i+1), \ldots, (n)\}).\end{aligned}$$

Therefore, we have

$$MS_{(m_i)_{i=1,2,\ldots,k}}(x) := \bigvee_{i=1}^{n} \left(\varphi_{k,x}(x^{(i)}) -_{\vee} \varphi_{k,x}(x^{(i-1)})\right) \wedge m_k((i), \ldots, (n))$$

$$= \bigvee_{i=1}^{n} (x^{(i)} -_{\vee} x^{(i-1)}) \wedge (m_1 \wedge \cdots \wedge m_{k-1} \wedge m_k)(\{(i), (i+1), \ldots, (n)\})$$
$$= (S) \int x d(m_1 \wedge m_2 \wedge m_k).$$

It follows from the Theorem above that if all the t-systems $\mathcal{T}_k$ are the same, the generalized t-conorm multifold integral is essentially represented by one generalized t-conorm integral. Therefore, the GTM integral is meaningful when the t-systems are different, as it is the case with the twofold integral.

5. Conclusions

In this paper we have studied several composite models. We have shown that the generalized transformed t-conorm integral can be expressed as a generalized Losonczi's mean. Then, we have defined the Generalized t-conorm Multifold integral in terms of t-conorm systems for integration. We have defined multifold Choquet integrals and multifold Sugeno integrals as examples of such integrals, and we show that a multifold Choquet integral is, essentially, a Choquet integral and that a multifold Sugeno integral is, essentially, a Sugeno integral. In a similar way, it follows that if all the t-systems $\mathcal{T}_k$ are the same, the generalized t-conorm multifold integral is essentially represented by one generalized t-conorm integral. Therefore the only meaningful multifold integrals are the ones in which the set of t-systems used for their definition are not all the same. This is the case of the twofold integral defined in [19,26].

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References

[1] P. Benvenuti, R. Mesiar, Integrals with respect to a general fuzzy measure, in: M. Grabisch, T. Murofushi, M. Sugeno (Eds.), Fuzzy Measures and Integrals, Theory and Applications, Physica-Verlag, Heidelberg, 2000Stud. Fuzziness Soft Comput. 40, 205–232.
[2] P. Benvenuti, R. Mesiar, D. Vivona, Monotone set functions-based integrals, in: E. Pap (Ed.), Handbook of Measure Theory, vol. I, II, North-Holland, Amsterdam, 2002, pp. 1329–1379.
[3] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation operators: properties classes and construction methods, in: T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators, Physica-Verlag, Wurzburg, 2002, pp. 3–104.
[4] T. Calvo, A. Mesiarová, L. Valášková, Composition of aggregation operators—one more new construction method, in: Proceedings of Agop, Alcala, 2003, pp. 51–53.
[5] T. Calvo, A. Mesiarová, L. Valášková, Composition of aggregation operators—one more new construction method, Kybernetika 39 (2003) 643–650.
[6] G. Choquet, Theory of capacities, Ann. Inst. Fourier, Grenoble 5 (1953/54) 131–295.
[7] E.P. Klement, R. Mesiar, E. Pap, A general approach to integration on finite sets, in: Proceedings of IPMU, 2004, pp. 61–66.
[8] I. Kojadinovic, Unsupervised aggregation by the Choquet integral based on entropy functionals: application to the evaluation of students, Modeling Decisions for Artificial Intelligence, Lecture Notes in Artificial Intelligence, vol. 3131, 2004, pp. 163–174.
[9] L. Losonczi, Uber eine neue Klasse von Mittelwerte, Acta Sci. Math. (Acta Univ. Szeged) 32 (1971) 71–78.
[10] R. Mesiar, A. Mesiarová, Fuzzy Integrals, Modeling Decisions for Artificial Intelligence, Lecture Notes in Artificial Intelligence, vol. 3131, 2004, pp. 7–14.
[11] R. Mesiar, D. Vivona, Two-step integral with respect to fuzzy measure, Tatra Mt. Math. Publ. 16 (1999) 359–368.
[12] T. Murofushi, Y. Narukawa, A characterization of multi-step discrete Choquet integral, Abstracts of the Sixth International Conference on Fuzzy Sets Theory and its Applications, 2002, p. 94.
[13] T. Murofushi, M. Sugeno, An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, Fuzzy Sets and Systems 29 (1989) 201–227.
[14] T. Murofushi, M. Sugeno, Fuzzy t-conorm integral with respect to fuzzy measures: generalization of Sugeno integral and Choquet integral, Fuzzy Sets and Systems 42 (1991) 57–71.
[15] Y. Narukawa, T. Murofushi, Choquet integral and Sugeno integral as aggregation functions, in: V. Torra (Ed.), Information Fusion in Data Mining, Springer, Berlin, 2003, pp. 27–39.

[16] Y. Narukawa, T. Murofushi, Decisions under risk and uncertainty through the use of Choquet integral, in: Proceedings of Fourth International Symposium on Advanced Intelligent Systems (ISIS 2003), 2003, pp. 555–558.
[17] Y. Narukawa, T. Murofushi, Decision modelling using the Choquet integral, in: V. Torra , Y. Narukawa (Eds.), Modeling Decisions for Artificial Intelligence (MDAI 2004), Lecture Notes in Artificial Intelligence, vol. 3131, 2004, pp. 183–193.
[18] Y. Narukawa, T. Murofushi, Choquet Stieltjes integral as a tool for decision modeling, Internat. J. Intell. Systems, to appear.
[19] Y. Narukawa, V. Torra, Twofold integral and multi-step Choquet integral, Kybernetika 40 (1) (2004) 39–50.
[22] M. Sugeno, Theory of fuzzy integrals and its applications, Doctoral Thesis, Tokyo Institute of Technology, 1974.
[23] M. Sugeno, K. Fujimoto, T. Murofushi, Hierarchical decomposition of Choquet integral models, Internat. J. Uncertainty, Fuzziness and Knowledge-Based Systems 3 (1995) 1–15.
[24] M. Sugeno, T. Murofushi, Pseudo additive measures and integrals, J. Math. Anal. Appl. 122 (1987) 197–222.
[25] M. Sugeno, Y. Narukawa, T. Murofushi, Choquet integral and fuzzy measures on locally compact spaces, Fuzzy Sets and Systems 99 (2) (1998) 205–211.
[26] V. Torra, Twofold integral: a Choquet integral and Sugeno integral generalization, Butlletí de l'Associació Catalana d'Intel · ligència Artificial 29 (2003) 13–19 (in Catalan) (Preliminary version: IIIA Research Report TR-2003-08) (in English).
[27] V. Torra, OWA operators in data modeling and reidentification, IEEE Trans. Fuzzy Systems 12 (5) (2004) 652–660.
[28] V. Torra, Y. Narukawa, S. Miyamoto, Modeling Decisions for Artificial Intelligence: Theory, Tools and Applications, Lecture Notes in Artificial Intelligence, vol. 3558, 2005, pp. 1–8.
[29] B.L. van der Waerden, Mathematical Statistics, Springer, Berlin, 1969.
[30] S. Weber, Two integrals and some modified versions—critical remarks, Fuzzy Sets and Systems 20 (1986) 97–105.