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On two fragments with negation and without implication of the logic of residuated lattices

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Abstract. The logic of (commutative integral bounded) residuated lattices is known under different names in the literature: monoidal logic [26], intuitionistic logic without contraction [1], H_{BCK} [36] (nowadays called $\mathbf{FL_{ew}}$ by Ono), etc. In this paper we study the $\langle \vee, *, \neg, 0, 1\rangle$-fragment and the $\langle \vee, \wedge, *, \neg, 0, 1\rangle$-fragment of the logical systems associated with residuated lattices, both from the perspective of Gentzen systems and from that of deductive systems. We stress that our notion of fragment considers the full consequence relation admitting hypotheses. It results that this notion of fragment is axiomatized by the rules of the sequent calculus $\mathbf{FL_{ew}}$ for the connectives involved. We also prove that these deductive systems are non-protoalgebraic, while the Gentzen systems are algebraizable with equivalent algebraic semantics the varieties of pseudocomplemented (commutative integral bounded) semilatticed and latticed monoids, respectively. All the logical systems considered are decidable.

1. Introduction

This paper[1] is a contribution to the study of two implication-less fragments of the logic of (commutative integral bounded) residuated lattices. We stress that logical systems associated with residuated lattices have been studied several times in the literature, both from the perspective of Gentzen systems and from that of deductive systems. We consider these notions under the agreement that we have hypotheses in our formal proofs, i.e., we do not restrict ourselves to formal proofs without hypotheses. It is known [1] that the Gentzen system and the deductive system

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[1] We notice that [8] is an extended version of the present paper. There the reader can find more detailed preliminaries and proofs. We also stress that a preliminary summary of the present paper was published in [7].

associated with residuated lattices are equivalent (here the term equivalent has a formal meaning that will be explained in detail later). Therefore, in the case of residuated lattices if we concentrate on deductive systems we are not missing anything. The deductive systems associated with residuated lattices are known under different names in the literature: monoidal logic [26], intuitionistic logic without contraction [1], H_{BCK} [36] (corresponding to what Ono now calls $\mathbf{FL_{ew}}$), etc. The reason why we can say that all these deductive systems correspond to residuated lattices is that all previous systems are algebraizable in the sense of [6] with equivalent algebraic semantics the variety of residuated lattices. It is also known that all of them are indeed definitionally equivalent in the sense of [42]. Hence, from a naive point of view they are the same except for the primitive connectives chosen. Throughout this paper we will take $\langle \vee, \wedge, *, \rightarrow, \neg, 0, 1\rangle$ as the canonical language for residuated lattices and their logical systems.

The study of the logic of residuated lattices is also important in the context of the studies of t-norm based fuzzy logics [24] because it is a subsystem of the logic of left-continuous t-norms MTL [16] and so it is a subsystem of t-norm based fuzzy logics (for a survey of residuated t-norm based fuzzy logics see [17]).

The aim of this paper is to study two fragments[2], with negation and without implication, of the logical systems associated with residuated lattices. The languages involved are $\langle \vee, *, \neg, 0, 1\rangle$ and $\langle \vee, \wedge, *, \neg, 0, 1\rangle$. For each of the languages we consider two fragments, one as a Gentzen system and the other as a deductive system. On this occasion these two approaches will not be equivalent, so it is really necessary to consider both systems.

Some earlier publications, related to this one, studied the $\langle \vee, \wedge, \neg, 0, 1\rangle$-fragment of intuitionistic logic. Blok and Pigozzi proved in [6] that this fragment, as a deductive system, is not algebraizable (in fact it is not even protoalgebraic) and that the variety of pseudocomplemented distributive lattices is an algebraic semantics for it, with defining equation $p \approx 1$. However, Rebagliato and Verdú proved in [37] that the $\langle \vee, \wedge, \neg, 0, 1\rangle$-fragment given by the sequent calculus LJ for intuitionistic logic is indeed algebraizable (with equivalent algebraic semantics the variety of pseudocomplemented distributive lattices), and that the deductive system considered by Blok and Pigozzi is exactly the external one associated with the Gentzen system considered by Rebagliato and Verdú. Since intuitionistic logic is obtained by adding contraction to the logic of residuated lattices we can say that this paper analyzes the statements of this paragraph when contraction is removed. We will prove that they remain valid. Indeed, the known results for the $\langle \vee, \wedge, \neg, 0, 1\rangle$-fragment of intuitionistic logic can be considered as a motivation for our research.

The present paper is structured as follows. In Section 2 we recall the basic definitions and results about Gentzen systems and also some concepts about deductive systems which will be used in this paper. We stress that the main difference vis-à-vis the common approach in the literature is that we use the full consequence relation admitting hypotheses in the formal proofs.

[2] We again stress that we consider the full consequence relation, and not just the formal proofs without hypotheses.

In Section 3 we introduce the four logical systems that we study in the paper. Strictly speaking they are not defined as the fragments mentioned above. We need to wait until Section 5 to prove that they are really fragments.

Section 4 is devoted to studying the algebraic structures that will be used in the semantical analysis of the four logical systems that we are interested in. First of all, in Section 4.1 we recall some known results about the variety of residuated lattices and we discuss the two methods used in the literature to obtain completions [33] for residuated lattices: the Dedekind-MacNeille completion and the ideal completion. Then, in Section 4.2 we introduce the classes of pseudocomplemented (commutative integral bounded) semilatticed and latticed monoids, $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ for short. We stress that the notion of pseudocomplementation introduced here is with respect to the monoidal operation $*$. We will prove that these classes of algebras are varieties whose quasiequational theories are decidable. Their members are exactly the subreducts of the variety of residuated lattices, i.e., every $\mathbb{PM}^{s\ell}$-algebra and every $\mathbb{PM}^{\ell}$-algebra is embeddable into a (complete) residuated lattice. This can be proved with the ideal completion. Here we single out that it is impossible to build this embedding in such a way that all existing (infinite) joins are preserved, contrary to what happens in the case of residuated lattices. Therefore, it is false that the Dedekind-MacNeille completion of a $\mathbb{PM}^{s\ell}$-algebra is a residuated lattice. As regards the properties of congruences of $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$, we have that these varieties are neither congruence modular nor congruence permutable nor 1-regular, unlikethe variety of residuated lattices, which is arithmetical and 1-regular. Finally we also prove that if the reduct of a residuated lattice is subdirectly irreducible in $\mathbb{PM}^{s\ell}$ or in $\mathbb{PM}^{\ell}$ then this residuated lattice is subdirectly irreducible, while the reverse implication is false.

In Section 5 we study the connection between the logical systems and the varieties of algebras introduced in the previous section. We do this for the Gentzen systems in Section 5.1, and for the deductive systems in Section 5.2. While we obtain that the Gentzen systems are algebraizable, it turns out that the deductive systems are not even protoalgebraic. Although the deductive systems are non-protoalgebraic we manage to give an algebraic semantics, which is based precisely on the classes of algebras previously discussed. We also prove that these fragments are decidable and that the deductive systems are not selfextensional.

2. Basic concepts

The logical systems that we consider in this paper are Gentzen systems and deductive systems, the latter being a particular case of the former. Most of the literature on Gentzen systems, and on deductive systems, focusses only on their derivable sequents, i.e., on the sequents derivable without any hypothesis. Our approach is completely different since we analyze the full consequence relation admitting hypotheses in the proofs. The reader should bear in mind this difference between our approach and the one commonly considered in the literature. In this section we introduce and clarify, from this more general perspective, the notions (and notation) that we will need later. For the sake of brevity, this section contains neither proofs[3]

[3] The reader interested in proofs can check [37, 39, 22].

nor explanations on all concepts involved in this paper: we only concentrate in more relevant ones[4].

2.1. Gentzen systems and their algebraization

Gentzen systems. By a *propositional language* we mean an algebraic signature. Given a propositional language $\mathcal{L}$ we will denote by $Fm_{\mathcal{L}}$ the set of $\mathcal{L}$-formulas and by $\mathbf{Fm}_{\mathcal{L}}$ the algebra of $\mathcal{L}$-formulas. Throughout the paper, we will follow the convention of using boldface for algebras. We will use the lowercase letters $\varphi, \psi, \ldots$ for $\mathcal{L}$-formulas, and the uppercase $\Gamma, \Delta, \ldots$ for finite (maybe empty) sequences of $\mathcal{L}$-formulas. Given $m, n \in \omega$, an *$\mathcal{L}$-sequent of type $\langle m, n\rangle$* is a pair $\varsigma = \langle \Gamma, \Delta\rangle$ of finite sequences of $\mathcal{L}$-formulas such that the length of Γ is m and the length of Δ is n. While ς will refer to a $\mathcal{L}$-sequent, we will use the metavariable Φ for sets of $\mathcal{L}$-sequents. We will write $\emptyset$ for the empty sequence[5], φ for $\langle\varphi\rangle$, $\Gamma \Rightarrow \Delta$ for the sequent $\langle \Gamma, \Delta\rangle$, and $\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \psi_0, \ldots, \psi_{n-1}$ instead of $\langle\varphi_0, \ldots, \varphi_{m-1}\rangle \Rightarrow \langle\psi_0, \ldots, \psi_{n-1}\rangle$. Given a set $\mathcal{T} \subseteq \omega \times \omega$ we will denote by $Seq_{\mathcal{L}}^{\mathcal{T}}$ the set of all $\mathcal{L}$-sequents with type belonging to $\mathcal{T}$.

A *Gentzen system* is a triple $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash\rangle$ where $\mathcal{L}$ is a propositional language, $\mathcal{T}$ is a non-empty set of pairs of natural numbers, and $\vdash$ is a relation between subsets of $Seq_{\mathcal{L}}^{\mathcal{T}}$ and elements of $Seq_{\mathcal{L}}^{\mathcal{T}}$ satisfying the following conditions.

1) If $\varsigma \in \Phi$, then $\Phi \vdash \varsigma$.
2) If $\Phi \vdash \varsigma$ and for every $\varsigma' \in \Phi$, $\Phi' \vdash \varsigma'$, then $\Phi' \vdash \varsigma$.
3) If $\Phi \vdash \varsigma$ and $\Phi \subseteq \Phi'$, then $\Phi' \vdash \varsigma$.
4) If $\Phi \vdash \varsigma$, then $e[\Phi] \vdash e(\varsigma)$ for any substitution e (i.e., for any endomorphism of the algebra $\mathbf{Fm}_{\mathcal{L}}$)[6].

The first three conditions say that $\vdash$ is a *consequence relation* or a *closure operator* on the set $Seq_{\mathcal{L}}^{\mathcal{T}}$, and the last one is called *invariance under substitutions*. The Gentzen system is *finitary* if, moreover, it satisfies the following condition:

5) If $\Phi \vdash \varsigma$, then there is a finite subset Φ' of Φ with $\Phi' \vdash \varsigma$.

For the sake of simplicity, we will only consider finitary Gentzen systems. Thus, we will refer to finitary Gentzen systems simply as Gentzen systems. A well known way to define a Gentzen system is through derivations in a *sequent calculus*. As usual, we will write $\Phi, \varsigma \vdash \varsigma'$ instead of $\Phi \cup \{\varsigma\} \vdash \varsigma'$. The set $\mathcal{T}$ is called the *type* of $\mathcal{G}$. The components of a Gentzen system $\mathcal{G}$ sometimes will be written respectively as $\mathcal{L}(\mathcal{G})$, $\mathcal{T}(\mathcal{G})$ and $\vdash_{\mathcal{G}}$ since this avoids any ambiguity. Two sequents ς and ς' are *$\mathcal{G}$-equivalent* (notation: $\varsigma \dashv\vdash_{\mathcal{G}} \varsigma'$ or simply $\varsigma \dashv\vdash \varsigma'$) if it holds at the same time that $\varsigma \vdash_{\mathcal{G}} \varsigma'$ and that $\varsigma' \vdash_{\mathcal{G}} \varsigma$. A sequent ς is said *$\mathcal{G}$-derivable* if $\emptyset \vdash_{\mathcal{G}} \varsigma$.

The definition of Gentzen system generalizes the notion of deductive system defined by Blok and Pigozzi in [6]. It turns out that a *deductive system $\mathcal{S}$* is no less than a Gentzen system with type $\{0\} \times \{1\}$ where the formula φ is identified with the sequent $\emptyset \Rightarrow \varphi$.

[4] For an analysis of all concepts involved in the paper see [8].

[5] The context will tell us if this symbol denotes the empty set or the empty sequence.

[6] Here $e(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \psi_0, \ldots, \psi_{n-1})$ is obviously defined as the sequent $e(\varphi_0), \ldots, e(\varphi_{m-1}) \Rightarrow e(\psi_0), \ldots, e(\psi_{n-1})$.

Fragments. Let $\mathcal{G}$ be a Gentzen system $\langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$, and let $\mathcal{L}'$ be a sublanguage of $\mathcal{L}$. The $\mathcal{L}'$*-fragment* of $\mathcal{G}$ is the Gentzen system $\mathcal{G}' = \langle \mathcal{L}', \mathcal{T}, \vdash_{\mathcal{G}'} \rangle$ defined by the fact that for all $\Phi \cup \{\varsigma\} \subseteq Seq^{\mathcal{T}}_{\mathcal{L}'}$,

$$\Phi \vdash_{\mathcal{G}'} \varsigma \quad \text{iff} \quad \Phi \vdash_{\mathcal{G}} \varsigma.$$

In this case it is said that $\mathcal{G}$ is a *conservative expansion* of $\mathcal{G}'$. We stress that this notion of fragment considers the full consequence relation and not just the derivable sequents.

Algebraization of Gentzen systems. A class K of $\mathcal{L}$-algebras is an *algebraic semantics* for a Gentzen system $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash \rangle$ in the case that there is a translation $\tau : Seq^{\mathcal{T}}_{\mathcal{L}} \longrightarrow \mathcal{P}_{fin}(Eq_{\mathcal{L}})$ such that for all $\Phi \cup \{\varsigma\} \subseteq Seq^{\mathcal{T}}_{\mathcal{L}}$,

$$\Phi \vdash \varsigma \quad iff \quad \tau[\Phi] \models_{\mathsf{K}} \tau(\varsigma).^{7}$$

If moreover there is a kind of inverse translation then what we obtain is the notion of algebraization. To be more precise, a Gentzen system $\mathcal{G}$ is said to be *algebraizable with equivalent algebraic semantics* K if there is a translation $\tau : Seq^{\mathcal{T}}_{\mathcal{L}} \longrightarrow \mathcal{P}_{fin}(Eq_{\mathcal{L}})$ and a translation $\rho : Eq_{\mathcal{L}} \longrightarrow \mathcal{P}_{fin}(Seq^{\mathcal{T}}_{\mathcal{L}})$ such that

1) for all $\Phi \cup \{\varsigma\} \subseteq Seq^{\mathcal{T}}_{\mathcal{L}}$, it holds that $\Phi \vdash \varsigma$ iff $\tau[\Phi] \models_{\mathsf{K}} \tau(\varsigma)$,
2) for all $\Phi \cup \{\varsigma\} \subseteq Eq_{\mathcal{L}}$, it holds that $\Phi \models_{\mathsf{K}} \varsigma$ iff $\rho[\Phi] \vdash \rho(\varsigma)$,
3) for all $\varsigma \in Eq_{\mathcal{L}}$, it holds that $\varsigma \;=\!\!\mid\models_{\mathsf{K}} \tau\rho(\varsigma)$,
4) for all $\varsigma \in Seq^{\mathcal{T}}_{\mathcal{L}}$, it holds that $\varsigma \dashv\vdash \rho\tau(\varsigma)$.

If we replace $Eq_{\mathcal{L}}$ with $Seq^{\mathcal{T}}_{\mathcal{L}}$ in the previous definition what we obtain is the more general notion of *equivalence between Gentzen systems*. It is well known that the definition of equivalence is redundant because the conjunction of 1) and 3) is equivalent to the conjunction of 2) and 4) [39, Proposition 2.1].

It holds that if K is an equivalent algebraic semantics for $\mathcal{G}$, then so is the quasivariety $\mathsf{K}^{\mathcal{Q}}$ generated by K [22, Corollary 4.2]. It is also known that if K and K' are equivalent algebraic semantics for $\mathcal{G}$, then K and K' generates the same quasivariety [22, Corollary 4.4]. This quasivariety is called *the equivalent quasivariety semantics* for $\mathcal{G}$.

We notice that if $\mathcal{S}$ is a deductive system then the fact that it is algebraizable in the sense of [6] with the set of equivalence formulas $\Delta(p, q)$ and the set of defining equations $\Theta(p)$ coincides precisely with the fact of being algebraizable in the above sense under the translations $\tau(p) := \Theta(p)$ and $\rho(p \approx q) := \Delta(p, q)$. Hence, the algebraization of Gentzen systems generalizes the algebraization of deductive systems introduced in [6].

Now we state a result that we will need in Section 5.1. It gives a sufficient condition to prove the algebraization of a Gentzen system [39, Lemma 2.5] (see [22, Lemma 4.5] for a more accessible proof). In fact, it is also known that this condition is necessary [39, Lemma 2.24].

[7] Whenever we have a function, namely e, we use a different notation, as is done in [14], to talk about the image $e(x)$ of an element x in the domain and the image $e[X]$ of a subset X of the domain.

Lemma 2.1. *Let $\mathcal{G}$ be a Gentzen system $\langle \mathcal{L}, \mathcal{T}, \vdash \rangle$ and let K be a quasivariety. Suppose that there is a translation $\tau : Seq^{\mathcal{T}}_{\mathcal{L}} \longrightarrow \mathcal{P}_{fin}(Eq_{\mathcal{L}})$ and a translation $\rho : Eq_{\mathcal{L}} \longrightarrow \mathcal{P}_{fin}(Seq^{\mathcal{T}}_{\mathcal{L}})$ such that*

1) *for all $\varsigma \in Seq^{\mathcal{T}}_{\mathcal{L}}$, $\varsigma \dashv\vdash_{\mathcal{G}} \rho\tau(\varsigma)$,*
2) *for all $\varphi \approx \psi \in Eq_{\mathcal{L}}$, $\varphi \approx \psi \;\; \dashv\models_{\mathsf{K}} \;\; \tau\rho(\varphi \approx \psi)$,*
3) *for all $\mathbf{A} \in \mathsf{K}$, the set R defined by*

$$\{\langle \bar{a}, \bar{b}\rangle \in A^m \times A^n : \langle m, n\rangle \in \mathcal{T},\ \mathbf{A} \models \tau(p_0, \dots, p_{m-1} \Rightarrow q_0, \dots, q_{n-1})[\![\bar{a}, \bar{b}]\!]\}$$

is a $\mathcal{G}$-filter, i.e., is closed under the interpretations of the rules of $\mathcal{G}$,
4) *for every $\Phi \in Th\,\mathcal{G}$, the relation*

$$\theta_\Phi := \{\langle \varphi, \psi\rangle \in Fm^2_{\mathcal{L}} : \rho(\varphi \approx \psi) \subseteq \Phi\},$$

is a congruence relative *to the quasivariety* K*, i.e.,* $\mathbf{Fm}_{\mathcal{L}}/\theta_\Phi \in \mathsf{K}$.

Then $\mathcal{G}$ is algebraizable with equivalent algebraic semantics K.

The Leibniz operator. One interesting property of algebraizable Gentzen systems with respect to quasivarieties is the existence of a characterization of congruences relative to the quasivariety. To describe this characterization we need the notion of Leibniz operator. Let $\mathbf{A}$ be an $\mathcal{L}$-algebra, and let $\mathcal{T}$ be a set of types. If $m, n \in \omega$, $\langle \bar{x}, \bar{y}\rangle \in A^m \times A^n$ and $a, b \in A$, then $\langle \bar{x}, \bar{y}\rangle(b/a)$ will denote the result of replacing one occurrence (if it exists) of a in $\langle \bar{x}, \bar{y}\rangle$ with b. Given a $\langle \mathcal{L}, \mathcal{T}\rangle$-matrix $\langle \mathbf{A}, R\rangle$, the *Leibniz congruence* $\Omega_{\mathbf{A}}(R)$ of the matrix $\langle \mathbf{A}, R\rangle$ is the equivalence relation on A defined in the following way: $\langle a, b\rangle \in \Omega_{\mathbf{A}}(R)$ if, and only if, for every $\langle m, n\rangle \in \mathcal{T}$, $\langle \bar{x}, \bar{y}\rangle \in A^m \times A^n$, $k \in \omega$, $\varphi(p, q_0, \dots, q_{k-1}) \in Fm_{\mathcal{L}}$ and $c, c_0, \dots, c_{k-1} \in A$,

$$\langle \bar{x}, \bar{y}\rangle(\varphi^{\mathbf{A}}(a, c_0, \dots, c_{k-1})/c) \in R \quad \Leftrightarrow \quad \langle \bar{x}, \bar{y}\rangle(\varphi^{\mathbf{A}}(b, c_0, \dots, c_{k-1})/c) \in R.$$

We emphasize that the previous definition does not depend on any Gentzen system. It holds that $\Omega_{\mathbf{A}} : \bigcup\{A^m \times A^n : \langle m, n\rangle \in \mathcal{T}\} \longrightarrow Con(\mathbf{A})$. This map is known as the *Leibniz operator* on $\mathbf{A}$. It is easy to show that $\Omega_{\mathbf{A}} R$ is characterized by the fact that it is the largest congruence of $\mathbf{A}$ that is compatible with R (i.e., if $\langle \bar{x}, \bar{y}\rangle \in R$ and $\langle a, b\rangle \in \theta$, then $\langle \bar{x}, \bar{y}\rangle(b/a) \in R$). Now we state the promised result characterizing the congruences [37, Theorem 2.23] (see [22, Theorem 4.7] for a more accessible proof).

Theorem 2.2. *Let $\mathcal{G}$ be a Gentzen system and K a quasivariety. The following statements are equivalent.*

1) *$\mathcal{G}$ is algebraizable with equivalent algebraic semantics* K.
2) *For every $\mathcal{L}$-algebra $\mathbf{A}$, the Leibniz operator $\Omega_{\mathbf{A}}$ is an isomorphism between the lattice of $\mathcal{G}$-filters of $\mathbf{A}$ and the lattice of K-congruences of $\mathbf{A}$.*
3) *The Leibniz operator $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$ is a lattice isomorphism between $Th\,\mathcal{G}$ and Con_{K} $\mathbf{Fm}_{\mathcal{L}}$.*

External deductive system associated with a Gentzen system. Let $\mathcal{G}$ be a Gentzen system $\langle \mathcal{L}, \mathcal{T}, \vdash \rangle$. There are at least two methods in the literature used to associate a deductive system with $\mathcal{G}$. The common method is based on considering the derivable sequents. Specifically, $\Sigma \vdash_{\mathcal{S}_i(\mathcal{G})} \varphi$ holds when

there is a finite subset $\{\varphi_0, \dots, \varphi_{n-1}\}$ of Σ such that $\emptyset \vdash \varphi_0, \dots, \varphi_{n-1} \Rightarrow \varphi$.

We notice that this approach yields a deductive system, called *internal*, only if the Gentzen system satisfies some of the structural rules. Another method, which works for all Gentzen systems such that $\langle 0, 1 \rangle \in \mathcal{T}$ (even when structural rules are not satisfied), yields the external deductive system[8]. The *external deductive system* associated with $\mathcal{G}$ is defined as the deductive system $\mathcal{S}_e(\mathcal{G})$ such that $\Sigma \vdash_{\mathcal{S}_e(\mathcal{G})} \varphi$ iff

$$\{\, \emptyset \Rightarrow \psi : \psi \in \Sigma\} \vdash \ \emptyset \Rightarrow \varphi.$$

Since we have restricted ourselves to finitary Gentzen systems it is clear that $\mathcal{S}_e(\mathcal{G})$ is finitary.

2.2. *Some concepts concerning deductive systems*

This section is devoted to recalling several notions for deductive systems that have already been developed in the literature.

Definability of connectives. Definitional extension and definitional equivalence. The concepts of definability, definitional extension and definitional equivalence that we now consider come from [42]. Let $\mathcal{S}$ be a deductive system with language $\mathcal{L}$, and let Σ be an $\mathcal{S}$-theory. Then, $\langle \mathbf{Fm}_{\mathcal{L}}, \Sigma \rangle$ is an $\langle \mathcal{L}, \{0\} \times \{1\} \rangle$-matrix that is an $\mathcal{S}$-model. In the previous section we introduced Leibniz congruences; but for this particular kind of matrices the Leibniz congruence can also be characterized by a simpler method, since

$$\langle \varphi, \psi \rangle \in \Omega_{\mathbf{Fm}_{\mathcal{L}}}(\Sigma) \ \text{ iff } \begin{cases} \text{for every } k \in \omega \text{ and } \varphi(p, q_0, \dots, q_{k-1}), \psi, \psi' \in Fm_{\mathcal{L}} : \\ \varphi(\psi, q_0, \dots, q_{k-1}) \in \Sigma \ \Leftrightarrow \ \varphi(\psi', q_0, \dots, q_{k-1}) \in \Sigma. \end{cases}$$

Given $\iota \in \mathcal{L}$ a connective of arity k and $\mathcal{L}'$ a sublanguage of $\mathcal{L}$, the connective ι is *definable on $\mathcal{S}$ in terms of the connectives of $\mathcal{L}'$* if there is a formula $\varphi(p_1, \dots, p_k) \in Fm_{\mathcal{L}'}$ such that for every $\Sigma \in Th\, S$ and for every $\alpha_1, \dots, \alpha_k \in Fm_{\mathcal{L}}$,

$$\langle \iota(\alpha_1, \dots, \alpha_k)\, , \varphi(\alpha_1/p_1, \dots, \alpha_k/p_k) \rangle \in \Omega_{Fm_{\mathcal{L}}}(\Sigma). \tag{2.1}$$

Using the fact that $\Omega_{Fm_{\mathcal{L}}}(\Sigma)$ is a fully invariant congruence it follows that in order to have (2.1) it is enough to check that $\langle \iota(p_1, \dots, p_k)\, , \varphi \rangle \in \Omega_{Fm_{\mathcal{L}}}(\Sigma)$. $\mathcal{S}$ is a *definitional extension* of a deductive system S' if $\mathcal{S}'$ is a fragment of $\mathcal{S}$ and every connective of $\mathcal{L}$ is definable on $\mathcal{S}$ in terms of the connectives of the language of $\mathcal{S}'$. Two deductive systems are *definitionally equivalent* if there is a deductive system which is a definitional extension of both deductive systems.

[8] We use the words ‘internal’ and ‘external’ following Avron (see [2]).

Protoalgebraicity. Given a deductive system there are several notions that measure the closeness to an equational logic. One example of this, introduced above, is the presence or absence of an algebraic semantics. Another example is the hierarchy developed in the Abstract Algebraic Logic framework [6, 13, 18]. This hierarchy classifies at different levels the deductive systems that enjoy a certain good correspondence with respect to equational logics. While algebraizability corresponds to the strongest relationship between the logical side and the algebraic side, protoalgebraicity corresponds to the weakest relationship (inside this hierarchy). A deductive system $\mathcal{S}$ is *protoalgebraic* when for every algebra $\mathbf{A}$ the Leibniz operator $\Omega_{\mathbf{A}}$ is monotone on the set of all $\mathcal{S}$-filters of $\mathbf{A}$, i.e., if F and G are $\mathcal{S}$-filters and $F \subseteq G$ then $\Omega_{\mathbf{A}}(F) \subseteq \Omega_{\mathbf{A}}(G)$. It is known that a deductive system is protoalgebraic iff there is a set of formulas $\Delta(p, q)$ in at most two variables such that

1) for every formula $\delta(p, q) \in \Delta, \emptyset \vdash_{\mathcal{S}} \delta(p, p)$,
2) $p, \Delta(p, q) \vdash_{\mathcal{S}} q$.

From this follows that protoalgebraicity is preserved under extensions and conservative expansions (*monotonicity*). Another interesting property is that all algebraizable deductive systems are protoalgebraic. Finally, we notice that if a deductive system $\mathcal{S}$ is not protoalgebraic, then there is no binary connective $\to$ such that

1) $\emptyset \vdash_{\mathcal{S}} p \to p$ (Identity),
2) $p, p \to q \vdash_{\mathcal{S}} q$ (Modus Ponens).

Therefore, the protoalgebraicity of a deductive system, roughly speaking, means that there is no way of obtaining a natural implication inside it.

Selfextensional, extensional and intensional deductive systems. Given a deductive system $\mathcal{S}$ and a set Σ of formulas, the *Frege relation of* Σ *relative to* $\mathcal{S}$, in symbols $\Lambda_{\mathcal{S}}\Sigma$, is the equivalence relation on $\mathbf{Fm}_{\mathcal{L}}$ defined as follows:

$$\Lambda_{\mathcal{S}}\Sigma := \{\langle\varphi, \psi\rangle : \Sigma, \varphi \vdash_{\mathcal{S}} \psi \text{ and } \Sigma, \psi \vdash_{\mathcal{S}} \varphi\}.$$

Thus, $\langle\varphi, \psi\rangle \in \Lambda_{\mathcal{S}}\Sigma$ if and only if φ and ψ belong to the same $\mathcal{S}$-theories that extend Σ. $\mathcal{S}$ is a *selfextensional* deductive system if $\Lambda_{\mathcal{S}}\emptyset$ is a congruence of the formula algebra. If additionally it holds that $\Lambda_{\mathcal{S}}\Sigma$ is a congruence of the formula algebra for every set Σ of formulas, then $\mathcal{S}$ is an *extensional* (or *Fregean*) deductive system. The deductive systems that are not extensional are called *intensional* or *non Fregean*. The interest in selfextensional deductive systems comes from the work of Wójcicki [42, 43], where they are characterized as referential (i.e., the deductive systems admitting a certain kind of Kripke semantics). For additional information on the notions of this paragraph see [18, Section 2.1] and the references therein.

3. The logical systems that we study

In this section we introduce the logical systems that we will study in this paper, which are related to intuitionistic logic without contraction. They will be introduced at the end of this section. First, we recall two logical systems, related to

Table 3.1. Inference Rules of $\mathbf{FL_{ew}}$

$$\frac{}{\varphi \Rightarrow \varphi}\ (\text{Ax } 1) \qquad \frac{}{0 \Rightarrow \emptyset}\ (\text{Ax } 2) \qquad \frac{}{\emptyset \Rightarrow 1}\ (\text{Ax } 3)$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \varphi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Delta}\ (\text{Cut}) \qquad \frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta}\ (e \Rightarrow)$$

$$\frac{\Sigma, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \Gamma \Rightarrow \Delta}\ (w \Rightarrow) \qquad \frac{\Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \varphi}\ (\Rightarrow w)$$

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \Delta \qquad \Sigma, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ (\Rightarrow \wedge)$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee_2)$$

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge_1 \Rightarrow) \qquad \frac{\Sigma, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge_2 \Rightarrow)$$

$$\frac{\Sigma, \varphi, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi * \psi, \Gamma \Rightarrow \Delta}\ (* \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi * \psi}\ (\Rightarrow *)$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \varphi \rightarrow \psi, \Pi \Rightarrow \Delta}\ (\rightarrow\Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi}\ (\Rightarrow\rightarrow)$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma, \neg\varphi \Rightarrow \emptyset}\ (\neg \Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \neg\varphi}\ (\Rightarrow \neg)$$

them, already studied in the literature: the sequent calculus $\mathbf{FL_{ew}}$ and the deductive system $IPC^*\backslash c$.

First of all we consider the sequent calculus $\mathbf{FL_{ew}}$ (cf. [32, 35]), which is given in the language $\mathcal{L} = \langle \vee, \wedge, *, \rightarrow, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0 \rangle$. $\mathbf{FL_{ew}}$ is the calculus of $\mathcal{L}$-sequents of type $\omega \times \{0, 1\}$ defined by the axioms and rules in Table 3.1 (φ, ψ are $\mathcal{L}$-formulas, $\Gamma, \Delta, \Pi, \Sigma$ are finite (possibly empty) sequences of $\mathcal{L}$-formulas and Δ is a sequence of at most one $\mathcal{L}$-formula).

When we add the structural rule of contraction

$$\frac{\Sigma, \varphi, \varphi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \Gamma \Rightarrow \Delta}(c \Rightarrow)$$

to the previous calculus what we obtain is $\mathbf{FL_{ewc}}$ [32, 35], which is a redundant version of the Gentzen's calculus LJ for the intuitionistic propositional logic since the multiplicative conjunction $*$ behaves as the additive conjunction $\wedge$. Notice that since the calculus $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ewc}}$ have both the structural rule of exchange, we can consider without loss of generality that the sequence Σ is empty, because taking $\Sigma = \emptyset$ in the formulation of $\mathbf{FL_{ew}}$ or $\mathbf{FL_{ewc}}$, we obtain exactly the same associated Gentzen systems.

Table 3.2. Hilbert-style axioms of $IPC^*\backslash c$

(A1) $(\varphi \to \psi) \to ((\gamma \to \varphi) \to (\gamma \to \psi))$	(A10) $(\gamma \to \varphi) \wedge (\gamma \to \psi) \to (\gamma \to \varphi \wedge \psi)$
(A2) $(\varphi \to (\psi \to \gamma)) \to (\psi \to (\varphi \to \gamma))$	(A11) $\varphi \to (\psi \to \varphi * \psi)$
(A3) $\varphi \to (\psi \to \varphi)$	(A12) $(\varphi \to (\psi \to \gamma)) \to (\varphi * \psi \to \gamma)$
(A4) $(\varphi \to \gamma) \to ((\psi \to \gamma) \to (\varphi \vee \psi \to \gamma))$	
(A5) $\varphi \to \varphi \vee \psi$	(A13) $\neg\varphi \to (\varphi \to \psi)$
(A6) $\psi \to \varphi \vee \psi$	(A14) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
(A7) $\varphi \wedge \psi \to \varphi$	(A15) $\varphi \to \neg\neg\varphi$
(A8) $\varphi \wedge \psi \to \psi$	(A16) $0 \to \varphi$
(A9) $\varphi \to (\psi \to \varphi \wedge \psi)$	(A17) $\varphi \to 1$

Theorem 3.1. [32, Theorem 6] *Cut elimination holds for* $\mathbf{FL_{ew}}$ *and* $\mathbf{FL_{ewc}}$.

Next we introduce the deductive system $IPC^*\backslash c$. This system has been studied in slightly different languages from the one that we take, since $IPC^*\backslash c$ is definitionally equivalent to H_{BCK} [36], to the monoidal logic [26] and to the system introduced with the same name in [1]. $IPC^*\backslash c$ is the deductive system in the language $\mathcal{L} = \langle \vee, \wedge, *, \to, \neg, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ defined by the Modus Ponens rule and the axioms in Table 3.2 (using implication as the least binding connective).

Theorem 3.2. $\mathcal{G}_{\mathbf{FL_{ew}}}$ *and* $IPC^*\backslash c$ *are equivalent, with translations* τ *and* ρ *defined as follows:*

$$\tau(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \varphi) := \begin{cases} \{\varphi_0 \to (\varphi_1 \to (\cdots \to (\varphi_{m-1} \to \varphi)\cdots))\}, & \text{if } m \geq 1 \\ \{\varphi\}, & \text{if } m = 0 \end{cases}$$

$$\tau(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \emptyset) := \begin{cases} \{\varphi_0 \to (\varphi_1 \to (\cdots \to (\varphi_{m-1} \to 0)\cdots))\}, & \text{if } m \geq 1 \\ \{0\}, & \text{if } m = 0 \end{cases}$$

$$\rho(\varphi) := \{\emptyset \Rightarrow \varphi\}.$$

That is, the following conditions are satisfied:

1) *For every* $\Sigma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Sigma \vdash_{IPC^*\backslash c} \varphi$ *iff* $\{\rho(\sigma) : \sigma \in \Sigma\} \vdash_{\mathbf{FL_{ew}}} \rho(\varphi)$.
2) *For every* $\Gamma \Rightarrow \Delta \in Seq_{\mathcal{L}}^{\omega\times\{0,1\}}$, $\Gamma \Rightarrow \Delta \dashv\vdash_{\mathbf{FL_{ew}}} \rho\tau(\Gamma \Rightarrow \Delta)$.

Proof. The reader can straightforwardly check that the above deductive system $IPC^*\backslash c$ is definitionally equivalent to the one presented under the same name in [1]: the essential difference is simply that the language of $IPC^*\backslash c$ includes negation as a primitive connective, whereas in the version considered in [1] the negation is definable but not primitive. The same distinction is found between $\mathcal{G}_{\mathbf{FL_{ew}}}$ and the Gentzen system $\mathcal{G}_{LJ^*\backslash c}$ considered in [1]. Therefore, we conclude our theorem by [1, Theorem 11]. □

Corollary 3.3. $IPC^*\backslash c$ *is the external deductive system of* $\mathcal{G}_{\mathbf{FL_{ew}}}$.

Proof. It is a consequence of Theorem 3.2(1), the definition of the translation ρ and the definition of the external deductive system. □

Now we introduce the four logical systems that are the aim of this paper. They are given in the languages $\langle \vee, *, \neg, 0, 1\rangle$ and $\langle \vee, \wedge, *, \neg, 0, 1\rangle$. In the first language we will have the Gentzen system $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ and the deductive system $\mathcal{S}_e[\vee, *, \neg]$, while in the second language what we will consider is the Gentzen system $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$ and the deductive system $\mathcal{S}_e[\vee, \wedge, *, \neg]$.

Definition 3.4.

- $\mathbf{FL_{ew}}[\vee, *, \neg]$ *is the sequent calculus in the language* $\langle \vee, *, \neg, 0, 1\rangle$ *obtained by deleting from* $\mathbf{FL_{ew}}$ *the rules of introduction of the additive conjunction and the implication, i.e., we consider all their axioms, all their structural rules and their introduction rules simply for the connectives* $\vee, *, \neg$*. The external deductive system associated with the Gentzen system* $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ *is denoted by* $\mathcal{S}_e[\vee, *, \neg]$.
- $\mathbf{FL_{ew}}[\vee, \wedge, *, \neg]$ *is the sequent calculus in the language* $\langle \vee, \wedge, *, \neg, 0, 1\rangle$ *obtained as before except for the fact that we also consider the introduction rules for* $\wedge$*, and* $\mathcal{S}_e[\vee, \wedge, *, \neg]$ *is the external deductive system associated with the Gentzen system* $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$.

In Section 5 we will prove that these Gentzen systems are fragments of $\mathcal{G}_{\mathbf{FL_{ew}}}$, and that these deductive systems are fragments of $IPC^*\backslash c$. Again we stress that our notion of fragment also considers the proofs admitting hypotheses, and not just the proofs without hypotheses. For the case of $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ this means that

$$\Phi \vdash_{\mathbf{FL_{ew}}[\vee,*,\neg]} \varsigma \quad \text{iff} \quad \Phi \vdash_{\mathbf{FL_{ew}}} \varsigma. \tag{3.1}$$

for every set $\Phi \cup \{\varsigma\}$ of sequents in the language $\langle \vee, *, \neg, 0, 1\rangle$. Indeed, from the following theorem it trivially follows that (3.1) holds for the previous two Gentzen systems when $\Phi = \emptyset$.

Theorem 3.5. *Cut elimination holds for* $\mathbf{FL_{ew}}[\vee, *, \neg]$ *and* $\mathbf{FL_{ew}}[\vee, \wedge, *, \neg]$.

Proof. It is an immediate consequence of Theorem 3.1. □

4. The associated algebraic counterpart

In Section 4.2 we introduce the algebraic structures involved in the study of the four logical systems analyzed in this paper and prove several facts about these algebras. Before introducing these algebras, in Section 4.1 we will recall several well-known results about residuated lattices, the algebraic counterpart of logics without contraction. In Section 4.2 we will use these results since our main interest is in relating the algebras that we will introduce with residuated lattices. Section 4's main result, which will play a crucial role in Section 5, is Theorem 4.16, which states that the algebras that we will introduce are exactly subreducts of residuated lattices.

4.1. Residuated lattices

The class $\mathbb{RL}$ of *residuated lattices* is the class of algebras $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ satisfying the following conditions:

1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice with associated order $\leq$,
2) $\langle A, *, 1\rangle$ is a commutative monoid with the unit 1,
3) $x * z \leq y \Leftrightarrow z \leq x \rightarrow y$ (the law of *residuation*),
4) $\neg x \approx x \rightarrow 0$.

Therefore, in residuated lattices it holds that for every $a, b \in A$,

$$a \rightarrow b = \max\{c \in A : a * c \leq b\} \quad \text{and} \quad \neg a = \max\{c \in A : a * c \leq 0\}.$$

Another immediate remark is that we can replace 4) in the previous definition with

4') $x * z \leq 0 \Leftrightarrow z \leq \neg x$ (the law of *pseudocomplementation*).

The term pseudocomplement is used in the literature, e.g. [3, 23], to refer to the previous condition replacing $*$ with $\wedge$. Hence, we stress that the notion of pseudocomplementation that we consider is not the standard one, but it is a quite natural generalization.

Next we present several results on residuated lattices. The reader can find the omitted proofs and more detailed explanations in [28, 29, 5, 33, 34]. It is also notable that sometimes in the more recent literature, e.g. [27, 33, 35], these algebras have been called *commutative integral bounded residuated lattices* to distinguish them from the non-commutative and non-integral case. A slight difference between our presentation of residuated lattices and the common one is the presence of negation $\neg$ in the similarity type, which we include to ensure that the algebras that we will introduce in Section 4.2 are subreducts of residuated lattices.

The law of residuation implies the following distributivity of $*$ over $\vee$:

$$a * \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a * b_i). \tag{4.1}$$

This law must be read as saying that if $\{a, b_i\}_{i \in I} \subseteq A$ and $\bigvee_{i \in I} b_i$ exists, then $\bigvee_{i \in I} (a * b_i)$ also exists and the previous equality holds.

It is also known that $\mathbb{RL}$ can be axiomatized using only equations, that is, $\mathbb{RL}$ is a variety. In fact, it is an arithmetical variety (i.e., congruence distributive and congruence permutable) that is generated by its finite simple algebras. It is also known that $\mathbb{RL}$ has the finite embeddability property, i.e., for a given partial subalgebra $\mathbf{B}$ of a residuated lattice $\mathbf{A}$, there exists a finite residuated lattice $\mathbf{D}$ into which $\mathbf{B}$ can be embedded. In particular this implies that $\mathbb{RL}$ is generated as a quasivariety by its finite members. Note that their congruences behave rather well: residuated lattices are 1-regular and there is an isomorphism between their congruences and their lattice filters closed under $*$. This makes it possible to obtain a characterization of subdirectly irreducible residuated lattices. In the case of finite algebras this characterization becomes quite simple: a finite residuated lattice is subdirectly irreducible if and only if it has a penultimate element.

Among residuated lattices the complete ones are particularly interesting, because of the following theorem. We recall that a residuated lattice is *complete* if it is complete as a lattice.

Theorem 4.1. *Every residuated lattice is embeddable into a complete residuated lattice.*

There are at least two well-known methods in the literature for obtaining these completions, the *Dedekind-MacNeille completion* and the *ideal completion*. Before explaining how these two methods work, we recall a characterization obtained by Ono (see [33, Section 1][9] and the references therein) of the complete residuated lattices.

Proposition 4.2. *If* $\mathbf{M} = \langle M, *, 1\rangle$ *is a commutative monoid and the map* $C : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ *is a closure operator satisfying that for every* $a, b \in M$ *and* $X, Y \subseteq M$,

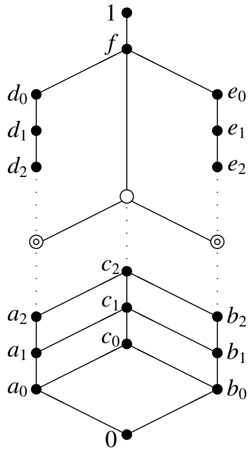
$$\text{if } a \in C(X) \text{ and } b \in C(Y), \text{ then } a * b \in C(\{x * y : x \in X, y \in Y\}), \quad (4.2)$$

then $\mathbf{C_M} = \langle C_\mathbf{M}, \vee_C, \cap, *_C, \rightarrow, \neg, C(\emptyset), M\rangle$ *is a complete residuated lattice where* $C_\mathbf{M} = \{X \subseteq M : C(X) = X\}$, $X \vee_C Y = C(X \cup Y)$, $X *_C Y = C(\{x * y : x \in X, y \in Y\})$, $X \rightarrow Y = \{z \in M : x * z \in Y \text{ for each } x \in X\}$ *and* $\neg X = \{z \in M : x * z \in C(\emptyset) \text{ for each } x \in X\}$. *Any complete residuated lattice is isomorphic to* $\mathbf{C_M}$ *for a certain commutative monoid* $\mathbf{M}$ *and a certain closure operator* C *satisfying the above property.*

Let $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1\rangle$ be a residuated lattice. The *Dedekind-MacNeille completion of* $\mathbf{A}$ is defined as the complete residuated lattice obtained through the method of the last proposition using the monoid reduct of $\mathbf{A}$ and the closure operator defined by $C^{DM}(X) = (X^{\rightarrow})^{\leftarrow}$ for each $X \subseteq A$, where $Y^{\rightarrow}$ and $Z^{\leftarrow}$ denote the set of upper bounds and of lower bounds of Y and Z, respectively. It is not hard to check that C^{DM} satisfies the condition (4.2) of Proposition 4.2 since residuated lattices satisfy the infinitary law (4.1). We denote the corresponding completion by $\mathbf{A^{DM}}$. This gives us a concrete representation of the Dedekind-MacNeille completion. It is also possible to characterize it from an abstract point of view, in the case of lattices without further structure this was done in [4, 41] (see [21] for a more accessible presentation).

The *ideal completion of* $\mathbf{A}$ is defined in the same way but now using the closure operator C^{Id} defined as follows: for every $X \subseteq A$, $C^{Id}(X)$ is the lattice ideal generated by X, i.e., the smallest lattice ideal containing X. C^{Id} also satisfies (4.2) of the last proposition, but this time it can be proved using only the finitary version of (4.1), i.e., to settle that C^{Id} satisfies (4.2) we do not need the infinitary version of (4.1). This fact will be used for the algebras introduced in Section 4.2. We denote the ideal completion of $\mathbf{A}$ by $\mathbf{A^{Id}}$. Thus, the elements of A^{Id} are the subsets I of

[9] Our statement of the result is adapted to the notion of closure operator that we use (see p. 618 or [9, Definition 5.1 (Chapter I)]), which is slightly different from the one used by Ono.



$$A_1 := \{0, f, 1\} \cup \{x_n : x \in \{a, b, c, d, e\},\ n \in \omega\}$$

$$x * y \quad := \begin{cases} y, & \text{if } x = 1 \\ x, & \text{if } y = 1 \\ 0, & \text{if } x, y \neq 1 \end{cases}$$

$$x \to y := \begin{cases} 1, & \text{if } x \leq y \\ f, & \text{if } x \not\leq y \end{cases}$$

Fig. 4.1. *A residuated lattice* $\mathbf{A_1}$ *such that* $\mathbf{A_1^{DM}} \not\subseteq \mathbf{A_1^{Id}}$

A such that i) $0 \in I$, ii) $y \leq x \in I$ implies $y \in I$, iii) $x, y \in I$ implies $x \vee y \in I$. A simple check shows that given lattice ideals I, I_1, I_2 the operations of $\mathbf{A^{Id}}$ are defined in the following way:

$$\begin{aligned} I_1 \vee I_2 &= \{a \in A : a \leq i_1 \vee i_2 \text{ for some } i_1 \in I_1, i_2 \in I_2\} \\ I_1 \wedge I_2 &= \{a \in A : a \leq i_1 \wedge i_2 \text{ for some } i_1 \in I_1, i_2 \in I_2\} = I_1 \cap I_2 \\ I_1 * I_2 &= \{a \in A : a \leq i_1 * i_2 \text{ for some } i_1 \in I_1, i_2 \in I_2\} \\ I_1 \to I_2 &= \{a \in A : i_1 * a \in I_2 \text{ for each } i_1 \in I_1\} \\ \neg I &= \{a \in A : a \leq \neg i \text{ for each } i \in I\}. \end{aligned} \tag{4.3}$$

What it is interesting in the above constructions is that $\mathbf{A}$ is embeddable into $\mathbf{A^{DM}}$ and also into $\mathbf{A^{Id}}$. In fact, the map $i_{\mathbf{A}} : a \in A \mapsto \{b \in A : b \leq a\}$ is at the same time an embedding from $\mathbf{A}$ into $\mathbf{A^{DM}}$ and from $\mathbf{A}$ into $\mathbf{A^{Id}}$. This map takes values over the principal lattice ideals. It is clear that if I is a principal lattice ideal then $I = (I^{\to})^{\leftarrow}$. We also have that $I = (I^{\to})^{\leftarrow}$ implies that I is a lattice ideal; this means that $A^{DM} \subseteq A^{Id}$. However, it is not true that $\mathbf{A^{DM}} \subseteq \mathbf{A^{Id}}$ in general, i.e., the first one may not be a subalgebra of the other one. In order to give a counterexample we can consider the residuated lattice given in Figure 4.1. In the picture we adopt the convention that the points depicted as i) $\bullet$ are the ones in the algebra, ii) $\circledcirc$ are not in the algebra but correspond[10] to points in the Dedekind-MacNeille completion, and iii) $\circ$ correspond to points that are in the ideal completion but not in the Dedekind-MacNeille completion. Let us consider the ideals $I_1 = \{0\} \cup \{a_n : n \in \omega\}$ and $I_2 = \{0\} \cup \{b_n : n \in \omega\}$, i.e., the ones correponding to the two $\circledcirc$ points. Then, $I_1, I_2 \in A_1^{DM} \subseteq A_1^{Id}$ while $I_1 \vee_{\mathbf{A^{Id}}} I_2 = \{0\} \cup \{x_n : x \in \{a, b, c\}, n \in \omega\} \notin A_1^{DM}$. We notice that in order to obtain $\mathbf{A_1^{DM}}$ from $\mathbf{A_1}$ we add two points, while in the

[10] Strictly speaking this means that the set of points in the algebra that are below the point $\circledcirc$ is a member of the Dedekind-MacNeille completion.

case of $\mathbf{A}_1^{\mathbf{Id}}$ we need to add to $\mathbf{A}_1^{\mathbf{DM}}$ an additional point. What is obviously true in general is the fact that if $\mathbf{A}$ is a finite residuated lattice, then $\mathbf{A} \cong \mathbf{A}^{\mathbf{DM}} = \mathbf{A}^{\mathbf{Id}}$.

Depending on our interests, one construction is more appropriate than the other. The main property of the Dedekind-MacNeille completion is that it preserves all existing infinite joins and infinite meets, i.e., $i_{\mathbf{A}}$ is a complete (also called regular) embedding of $\mathbf{A}$ into $\mathbf{A}^{\mathbf{DM}}$. Therefore, Theorem 4.1 can be strengthened saying that every residuated lattice is completely embeddable into a complete residuated lattice. However, this completion does not preserve lattice equations, e.g., the Dedekind-MacNeille completion of a distributive lattice is not always distributive [20, 11, 15, 12] (see Figure 4.2). On the other hand, the ideal completion preserves all lattice equations[11], while it does not preserve infinite joins in general. This is shown by the example in Figure 4.1 since $i_{\mathbf{A}_1}(\bigvee_{\mathbf{A}_1}\{c_n : n \in \omega\}) = i_{\mathbf{A}_1}(f)$ while $\bigvee_{\mathbf{A}_1^{\mathbf{Id}}}\{i_{\mathbf{A}_1}(c_n) : n \in \omega\}) = \{0\} \cup \{x_n : x \in \{a, b, c\}, n \in \omega\} \neq i_{\mathbf{A}_1}(f)$.

Another interesting property of the Dedekind-MacNeille completion is that it is minimal. This means that if $\mathbf{A}$ is a complete residuated lattice then $i_{\mathbf{A}}$ is an isomorphism between $\mathbf{A}$ and $\mathbf{A}^{\mathbf{DM}}$. It is also possible to consider the word minimal in other senses. As a first case we can ask if for every embedding ϕ from a residuated lattice $\mathbf{A}$ into a complete residuated lattice $\mathbf{B}$ there is an embedding ϕ^* from $\mathbf{A}^{\mathbf{DM}}$ into $\mathbf{B}$ that extends ϕ (i.e., $\phi = \phi^* \circ i_{\mathbf{A}}$). The algebra of the Figure 4.1 shows that the Dedekind-MacNeille completion of residuated lattices is not minimal in this sense[12] (take $\mathbf{B}$ as the ideal completion of $\mathbf{A}$). Another possibility to consider is whether for every complete embedding ϕ from a residuated lattice $\mathbf{A}$ into a complete residuated lattice $\mathbf{B}$ there is a complete embedding ϕ^* from $\mathbf{A}^{\mathbf{DM}}$ into $\mathbf{B}$ extending ϕ. It is quite simple to see that the Dedekind-MacNeille completion of residuated lattices is not minimal in this new sense either[13]. Lastly, we can consider the word minimal as meaning that for every complete embedding ϕ from a residuated lattice $\mathbf{A}$ into a complete residuated lattice $\mathbf{B}$ there is an embedding (maybe not complete) ϕ^* from $\mathbf{A}^{\mathbf{DM}}$ into $\mathbf{B}$ that extends ϕ. Neither in this sense the Dedekind-MacNeille completion of a residuated lattice is minimal. This easily follows from the fact that the Dedekind-MacNeille completion of a lattice (without further structure) is not minimal in this very sense. A counterexample to this last statement is given in Figure 4.2 (take $\mathbf{A}$ as the lattice $\mathbf{L}$ and $\mathbf{B}$ as the ideal completion of $\mathbf{L}$), where we adopt the same convention than in page 628.

4.2. *Pseudocomplemented (semi)latticed monoids*

Now it is time to define the two classes of algebras that we are interested in to analyze our four logical systems.

[11] Using the equalities given in (4.3) it is easy to check that all equations that only use $\vee, \wedge, *, 0, 1$ are preserved since for all terms t using only $\vee, \wedge, *, 0, 1$ it holds that $i_{\mathbf{A}}(t^{\mathbf{A}}(\bar{a})) = \{b \in A : b \leq t^{\mathbf{A}}(\bar{a})\}$. However, the involutive law $x \approx \neg\neg x$ is not preserved in general: e.g., take the Łukasiewicz-algebra over $[0, 1]_{\mathbb{R}}$ (see [10]). Fortunately this particular equation is preserved under the Dedekind-MacNeille completion [31, Theorem 5.1].

[12] In this sense it is known that the Dedekind-MacNeille completions of partial orders are minimal [40, pp. 72–74].

[13] The Dedekind-MacNeille completions of Boolean algebras are minimal in this sense [25, Theorem 11 (Chapter 21)].

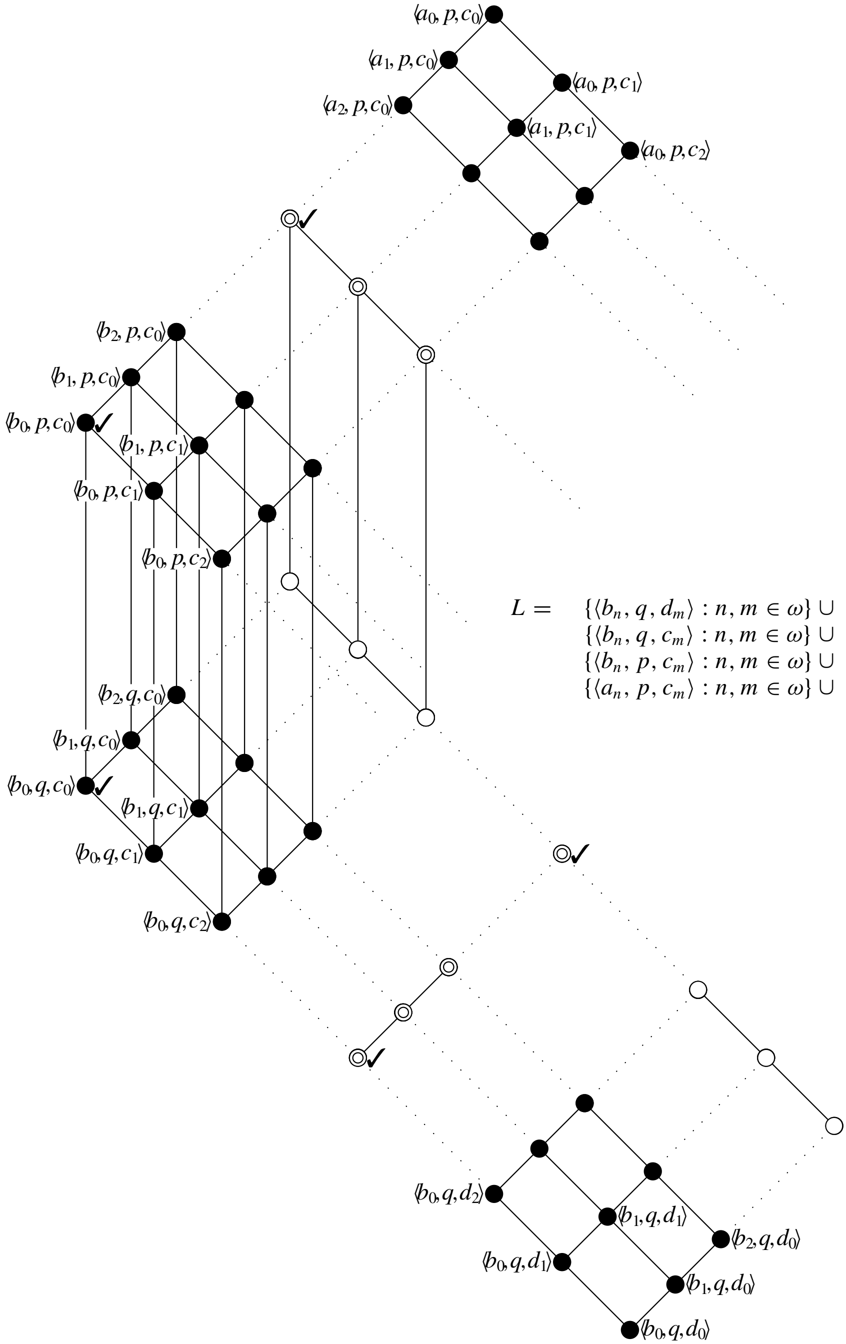


Fig. 4.2. Funayama's [20] distributive lattice **L** such that $\mathbf{L}^{\mathbf{DM}}$ is not modular (this is shown by the sublattice given by the five points marked with ✓)

Definition 4.3. *An algebra* $\mathbf{A} = \langle A, \vee, *, \neg, 0, 1\rangle$ *of type* $\langle 2, 2, 1, 0, 0\rangle$ *is a* pseudocomplemented semilatticed monoid *if it satisfies:*

1) $\langle A, \vee, 0, 1\rangle$ *is a bounded semilattice,*
2) $\langle A, *, 1\rangle$ *is a commutative monoid with unit* 1*,*
3) $x * (y \vee z) \approx (x * y) \vee (x * z)$*,*
4) $x * z \leq 0 \Leftrightarrow z \leq \neg x$ *(the law of pseudocomplementation).*

The class of pseudocomplemented semilatticed monoids is denoted by $\mathbb{PM}^{s\ell}$*. An algebra* $\mathbf{A} = \langle A, \vee, \wedge, *, \neg, 0, 1\rangle$ *of type* $\langle 2, 2, 2, 1, 0, 0\rangle$ *such that the reduct is a pseudocomplemented semilatticed monoid and* $\langle A, \vee, \wedge\rangle$ *is a lattice is called a* pseudocomplemented latticed monoid*. The class of these algebras is denoted by* $\mathbb{PM}^{\ell}$*.*

Remark 4.4. If $\mathbf{A}$ is a residuated lattice, then the reducts of $\mathbf{A}$ to the adequate languages are, respectively, in $\mathbb{PM}^{s\ell}$ and in $\mathbb{PM}^{\ell}$.

A more accurate name for these algebras should include at the beginning the words *commutative integral bounded*, but for the sake of simplicity we adopt the nomenclature given in the definition. We notice that the first three conditions of the previous definition corresponds to (commutative integral bounded) *semilatticed monoids* [33]. In them, and of course also in the pseudocomplemented ones, it holds that i) $*$ is monotone in both arguments, ii) $x * y \leq x$ and $x * y \leq y$, and iii) $x * 0 \approx 0$. It is easy to check the following statement.

Proposition 4.5. *The variety of bounded distributive lattices is the subvariety of semilatticed monoids defined by the equation* $x * x \approx x$*, i.e.,*

$$\{\mathbf{A} : \mathbf{A} = \langle A, \vee, \wedge, 0, 1\rangle \text{ distributive lattice}\}$$
$$= \{\mathbf{A} : \mathbf{A} = \langle A, \vee, *, 0, 1\rangle \text{ semilatticed monoid and } \mathbf{A} \models x * x \approx x\}.$$

And the variety of pseudocomplemented distributive lattices is the subvariety of $\mathbb{PM}^{s\ell}$ *defined by* $x * x \approx x$*.*

From the above statement we can consider the pseudocomplemented semilatticed monoids as generalizations of pseudocomplemented distributive lattices [3, 30]. In the rest of the section our presentation is motivated by wondering if the behaviour of pseudocomplemented distributive lattices with respect to Heyting algebras is the same one as the behaviour of pseudocomplemented semilatticed monoids with respect to residuated lattices, which can be summarized by the question

$$\frac{\text{pseudocomp. distributive lattices}}{\text{Heyting algebras}} = \frac{\text{pseudocomp. semilatticed monoids}}{\text{residuated lattices}}?$$

For instance, Theorem 4.16 generalizes the well-known result that every pseudocomplemented distributive lattice is the subreduct of a Heyting algebra [6, Theorem 2.6], which is a particular case of Theorem 4.16 using footnote 11.

I. Next we will prove that the classes of pseudocomplemented semilatticed and latticed monoids are varieties. From their definition it is obvious that both classes of algebras are quasi-equational. In fact, all the conditions are equations except for the law of pseudocomplementation, which corresponds to the pair of quasi-equations[14]

$$\text{4a)}\ \ x * z \leq 0 \ \Rightarrow\ z \leq \neg x \qquad \text{and} \qquad \text{4b)}\ \ z \leq \neg x \ \Rightarrow\ x * z \leq 0.$$

However, as announced above, it is possible to characterize these classes using only equations. We thank Roberto Cignoli for a personal communication stating this result.

Theorem 4.6. *The classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ are the varieties axiomatized by the equations involved in their definitions plus the equations (which replace the law of pseudocomplementation)*

$$\neg 1 \approx 0, \qquad \neg 0 \approx 1 \qquad \textit{and} \qquad x * \neg(y * x) \leq \neg y.$$

Proof. First of all we prove that these equations hold in these classes. The pseudocomplementation law says that $x * \neg x \approx 0$ holds. Therefore, using the fact that 1 is the unit of the monoid we have that $\neg 1 = 1 * \neg 1 = 0$. As $0 * 1 = 0 \leq 0$, the pseudocomplementation law says that $1 \leq \neg 0$. The other inequality is clear, so $1 = \neg 0$. Let a, b be two elements in an $\mathbb{PM}^{s\ell}$-algebra. Then, $b * (a * \neg(b * a)) = (b * a) * \neg(b * a) \leq 0$. Thus, the pseudocompletation law allows us to conclude that $a * \neg(b * a) \leq \neg b$.

Now it is time to prove that using these equations the two quasi-equations involved in the pseudocomplementation law hold. Let a, b be two elements in an algebra satisfying these equations. If $a * b \leq 0$, then $a \vee \neg b = (a * 1) \vee \neg b = (a * \neg 0) \vee \neg b = (a * \neg(b * a)) \vee \neg b = \neg b$. And suppose now that $b \leq \neg a$. Then, $a * b \leq (a * b) \vee (a * \neg a) = a * (b \vee \neg a) = a * \neg a = (a * \neg a) \vee 0 = (a * \neg a) \vee \neg 1 = (a * \neg(1 * a)) \vee \neg 1 = \neg 1 = 0$. □

II. Now we seek when a $\mathbb{PM}^{s\ell}$-algebra is the reduct of a residuated lattice, and the same for a $\mathbb{PM}^{\ell}$-algebra. It is said that a $\mathbb{PM}^{s\ell}$-algebra is *complete* if it is a complete semilattice as an ordered set, and that a $\mathbb{PM}^{\ell}$-algebra is *complete* if it is a complete lattice as an ordered set. We start by proving that every complete $\mathbb{PM}^{s\ell}$-algebra is the reduct of a complete $\mathbb{PM}^{\ell}$-algebra, and that a complete $\mathbb{PM}^{\ell}$-algebra is the reduct of a residuated lattice if, and only if, it satisfies the infinitary distributive law.

Proposition 4.7. *Every complete $\mathbb{PM}^{s\ell}$-algebra is the $\langle \vee, *, \neg, 0, 1\rangle$-reduct of a complete $\mathbb{PM}^{\ell}$-algebra.*

Proof. Let $\mathbf{A}$ be a complete $\mathbb{PM}^{s\ell}$-algebra. Since A has a minimum element, then we have that the ordered set $\langle A, \leq\rangle$ associated to the complete semilattice is a complete lattice such that, for every subset $X \subseteq A$, it holds that $\bigwedge X = \bigvee X^{\leftarrow}$, i.e., $\max X^{\leftarrow} = \bigvee X^{\leftarrow}$. So, $\mathbf{A}$ is the $\langle \vee, *, \neg, 0, 1\rangle$-reduct of the complete $\mathbb{PM}^{\ell}$-algebra of universe A in such a way that the operation $\wedge$ is defined by $a \wedge b =: \bigvee\{x \in A : x \leq a \text{ and } x \leq b\}$ and the rest of operations are the ones of $\mathbf{A}$. □

[14] We use an inequality $t_1 \leq t_2$ as an abbreviation for the equation $t_1 \vee t_2 \approx t_2$.

Proposition 4.8. *Let* **A** *be a complete* $\mathbb{PM}^\ell$*-algebra. The following conditions are equivalent:*

1) **A** *satisfies the above infinitary law* (4.1) *of distributivity of* $*$ *over* $\vee$,
2) *There is a (unique) operation* $\rightarrow$ *defined on A satisfying the law of residuation.*

Proof. One direction is proved taking the definition $a \rightarrow b = \bigvee\{c \in A : a*c \leq b\}$ for every $a, b \in A$. The other direction is a straightforward check. We stress that this proof is based on very few hypotheses on **A**; indeed the associativity of $*$ is not needed. □

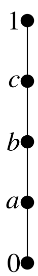
Proposition 4.9. *A complete* $\mathbb{PM}^\ell$*-algebra is the* $\langle\vee, \wedge, *, \neg, 0, 1\rangle$*-reduct of a residuated lattice if, and only if, it satisfies the infinitary distributive law* (4.1).

Proof. Let **A** be a complete $\mathbb{PM}^\ell$-algebra. If **A** is the reduct of a residuated lattice $\mathbf{A}'$, then **A** satisfies (4.1) since $\mathbf{A}'$ satisfies (4.1). Conversely, if **A** satisfies (4.1) then the algebra $\langle\mathbf{A}, \rightarrow\rangle$, where $\rightarrow$ is the operation given by Proposition 4.8, is easily checked to be a residuated lattice. □

Corollary 4.10. *Every finite member of* $\mathbb{PM}^{s\ell}$ *or* $\mathbb{PM}^\ell$ *is the reduct of a residuated lattice.*

Proof. All the algebras that the statement dealt with are complete and satisfy (4.1). This last part is proved by induction from Definition 4.3(3). Therefore, as a consequence of Propositions 4.7 and 4.9 we finish the proof. □

Remark 4.11. We have just seen that the finite algebras in the classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^\ell$ are essentially the same as finite residuated lattices. It is also possible to find some differences in the behaviour of these classes even in the finite case. For instance, let us consider the finite $\mathbb{PM}^\ell$-algebras $\mathbf{A_2}$ and $\mathbf{A_3}$ defined by



$A_2 = \{0, a, b, 1\}$ $\qquad$ $A_3 = \{0, a, b, c, 1\}$

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	0	a	b
c	0	a	a	c	c
1	0	a	b	c	1

	$\neg$
0	1
a	b
b	b
c	0
1	0

Then the inclusion map is an embedding. But it is not an embedding as far as we consider the residuation operation because $b \rightarrow_{\mathbf{A_2}} a = b$ while $b \rightarrow_{\mathbf{A_3}} a = c$.

Proposition 4.12. *There are two finite residuated lattices that satisfy the same* $\langle\vee, \wedge, *, \neg, 0, 1\rangle$*-equations but not the same* $\langle\vee, \wedge, *, \rightarrow, \neg, 0, 1\rangle$*-equations.*

Proof. We consider the residuated lattices $\langle\mathbf{A_3}, \rightarrow\rangle$ and $\langle\mathbf{A_2}, \rightarrow\rangle \times \langle\mathbf{A_3}, \rightarrow\rangle$ where the algebras involved are the ones defined in Remark 4.11. Since $\mathbf{A_2} \subseteq \mathbf{A_3}$ it is clear that $\mathbf{A_3}$ and $\mathbf{A_2} \times \mathbf{A_3}$ satisfy exactly the same $\langle\vee, \wedge, *, \neg, 0, 1\rangle$-equations, but the equation $x \wedge \neg(\neg x \rightarrow x) \approx 0$ holds in $\langle\mathbf{A_3}, \rightarrow\rangle$ while not in $\langle\mathbf{A_2}, \rightarrow\rangle \times \langle\mathbf{A_3}, \rightarrow\rangle$.
□

Specifically, the last proposition implies that $\to$ is not definable in residuated lattices using $\langle \vee, \wedge, *, \neg, 0, 1\rangle$. Another way to obtain this is the following interesting result.

Proposition 4.13. *There are complete* $\mathbb{PM}^{\ell}$*-algebras which are not the reduct of any residuated lattice.*

Proof. We consider the residuated lattice $\mathbf{A_5} = \langle A_5, \vee, \wedge, *, \to, \neg, 0, 1\rangle$ where $A_5 = \{0, 1\} \cup \{x \in \mathbb{R} : \frac{1}{4} \leq x \leq \frac{3}{4}\}$, the lattice operations corresponds to the standard order over the real numbers and the other operations are defined by the following tables (where $a, b, c \in [\frac{1}{4}, \frac{3}{4}]_{\mathbb{R}}$ and $a < c$):

$*$	0	b	1
0	0	0	0
a	0	$\frac{1}{4}$	a
1	0	b	1

$\to$	0	a	c	1
0	1	1	1	1
a	0	1	1	1
c	0	$\frac{3}{4}$	1	1
1	0	a	c	1

	$\neg$
0	1
a	0
1	0

And now we consider the algebra $\mathbf{A_4} = \langle A_4, \vee, \wedge, *, \neg, 0, 1\rangle$ where $A_4 = A_5 \setminus \{\frac{3}{4}\}$ and the operations are the restrictions of the ones defined over A_5. It is clear that $\mathbf{A_4}$ is a complete algebra and it is easy to check that it is a $\mathbb{PM}^{\ell}$-algebra. However, there is no possibility of defining a residuation $\to$ over A_4 in such a way that its expansion becomes a residuated lattice. This is an immediate consequence of the fact that the infinitary distributive law does not hold in $\mathbf{A_4}$, e.g.,

$$\left(\bigvee^{\mathbf{A_4}}[\frac{1}{4}, \frac{3}{4})_{\mathbb{R}}\right) * \frac{1}{2} = 1 * \frac{1}{2} = \frac{1}{2} \text{ while } \bigvee^{\mathbf{A_4}}\{x * \frac{1}{2} : x \in [\frac{1}{4}, \frac{3}{4})_{\mathbb{R}}\} = \frac{1}{4}.$$

□

We have already seen that there are (complete) $\mathbb{PM}^{s\ell}$-algebras and $\mathbb{PM}^{\ell}$-algebras that are not the reduct of any residuated lattice. But are they the subreduct of a certain residuated lattice? That is, are they, up to isomorphisms, equal to the class of the subalgebras of the reducts in the adequate languages of a residuated lattice? In the counterexample of the above proposition it is clear that $\mathbf{A_4}$ is the subreduct of the residuated lattice $\mathbf{A_5}$. In fact, there is a positive answer in general. This is our next aim, that is, we want to prove that every $\mathbb{PM}^{s\ell}$-algebra is embeddable into a complete residuated lattice, and the same for a $\mathbb{PM}^{\ell}$-algebra. As we explained above there are two main constructions to obtain complete residuated lattices: the Dedekind-MacNeille completion and the ideal completion. To perform these constructions all what we need is a commutative monoid. So, they can also be developed if we start with an algebra in $\mathbb{PM}^{s\ell}$ or $\mathbb{PM}^{\ell}$. Unfortunately, the Dedekind-MacNeille construction breaks down for $\mathbb{PM}^{\ell}$-algebras (and so, for $\mathbb{PM}^{s\ell}$-algebras) because it does not give us a residuated lattice. The algebra $\mathbf{A_4}$ used in the last proof illustrates the fact that there are $\mathbb{PM}^{\ell}$-algebras such that the closure operator C^{DM} associated with its monoidal reduct does not verify condition (4.2) in Proposition 4.2 because $1 \in C^{DM}([0, \frac{3}{4})_{\mathbb{R}})$ while $1 * 1 \notin C^{DM}(\{x * y : x, y < \frac{3}{4}\})$. We can also prove the following statement.

Proposition 4.14. *There are $\mathbb{PM}^{\ell}$-algebras (and so, $\mathbb{PM}^{s\ell}$-algebras) that cannot be embedded in any complete residuated lattice in such a way that all existing infinite joins are preserved.*

Proof. As a counterexample we again take the algebra $\mathbf{A_4}$ of the Proposition 4.13. Suppose that there is a complete residuated lattice $\mathbf{B}$ and an embedding $h : \mathbf{A_4} \hookrightarrow \mathbf{B}$ that preserves all existing infinite joins. Then, using the fact that $\mathbf{B}$ satisfies the infinitary distributive law we have $h(\frac{1}{2}) = h(1 * \frac{1}{2}) = h(1) * h(\frac{1}{2}) = h(\bigvee^{\mathbf{A_4}}[\frac{1}{4}, \frac{3}{4})_{\mathbb{R}}) * h(\frac{1}{2}) = (\bigvee^{\mathbf{B}}\{h(x) : x \in [\frac{1}{4}, \frac{3}{4})_{\mathbb{R}})\}) * h(\frac{1}{2}) = \bigvee^{\mathbf{B}}\{h(x) * h(\frac{1}{2}) : x \in [\frac{1}{4}, \frac{3}{4})_{\mathbb{R}})\} = \bigvee^{\mathbf{B}}\{h(x * \frac{1}{2}) : x \in [\frac{1}{4}, \frac{3}{4})_{\mathbb{R}})\} = \bigvee^{\mathbf{B}}\{h(\frac{1}{4})\} = h(\frac{1}{4})$. And this means that h is not an injective map. □

Next we will see that the ideal completion works well in order to obtain the embedding theorem for our classes of algebras. When $\mathbf{A}$ is an algebra in $\mathbb{PM}^{s\ell}$ or $\mathbb{PM}^{\ell}$, we will denote by $\mathbf{A^{Id}}$ the complete residuated lattice constructed in Section 4.1 using the monoidal reduct of $\mathbf{A}$ by the method of Proposition 4.2 applied to the closure operator C^{Id}. On this occasion it is easily verified that condition (4.2) holds, so $\mathbf{A^{Id}}$ is really a residuated lattice. We will to prove that every algebra $\mathbf{A}$ in $\mathbb{PM}^{s\ell}$ or $\mathbb{PM}^{\ell}$ can be embedded in the complete residuated lattice $\mathbf{A^{Id}}$. Using the characterization (4.3) the reader can directly check that our last statement holds, but the proof that we present below is based on the following result obtained by Ono (cf. [33, Theorem 7]).

Theorem 4.15.

1) *For each (commutative integral bounded) semilatticed monoid $\mathbf{A}$, the map $i_{\mathbf{A}} : a \in A \mapsto (a]$ is an embedding, preserving all existing residuals and meets, from $\mathbf{A}$ into the $\langle \vee, *, 0, 1\rangle$-reduct of the complete residuated lattice $\mathbf{A^{Id}}$.*
2) *For each residuated lattice $\mathbf{A}$, the map $i_{\mathbf{A}}$ is an embedding, preserving all existing meets, from $\mathbf{A}$ into the complete residuated lattice $\mathbf{A^{Id}}$.*

Note that in the above result it is not claimed that all existing joins are preserved; indeed, this is false. We also note that the first part of this theorem implies that the class of (commutative integral bounded) semilatticed monoids is the class of $\langle \vee, *, 0, 1\rangle$-subreducts of residuated lattices. Note also that the second part of the theorem, already obtained by Ono and Komori in [36, Theorem 8.12] by a method other than the one involved in the above reference, is an immediate consequence of the first part since the monomorphism $i_{\mathbf{A}}$ preserves all existing meets and residuals. Using the same procedure we can achieve our aim.

Theorem 4.16. *Every $\mathbb{PM}^{s\ell}$-algebra is embeddable into a complete residuated lattice. Every $\mathbb{PM}^{\ell}$-algebra is embeddable into a complete residuated lattice. Therefore, $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ are respectively the classes of $\langle \vee, *, \neg, 0, 1\rangle$-subreducts and $\langle \vee, \wedge, *, \neg, 0, 1\rangle$-subreducts of residuated lattices.*

Proof. Let $\mathbf{A}$ be a $\mathbb{PM}^{s\ell}$-algebra or a $\mathbb{PM}^{\ell}$-algebra. By Theorem 4.15, we have that the map $i_{\mathbf{A}} : a \in A \mapsto (a]$ is an embedding between the $\langle \vee, *, 0, 1\rangle$-reducts of $\mathbf{A}$ and $\mathbf{A^{Id}}$ preserving existing residuals and meets. Since the pseudocomplement of an element is the residual of the element by 0, it follows that $i_{\mathbf{A}}$ preserves pseudo-complements. So, the map $i_{\mathbf{A}}$ is an embedding from $\mathbf{A}$ into the corresponding reduct of the residuated lattice $\mathbf{A}^{\mathbf{Id}}$. □

Now we state some trivial corollaries from the last theorem. The first two state that the equational systems associated with $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ are fragments (again we stress admitting hypotheses) of the equational system associated with $\mathbb{RL}$; and the last corollary relates the free algebras of these varieties.

Corollary 4.17. *If* $\Pi \cup \{\varphi \approx \psi\}$ *is a set of equations in the language* $\langle \vee, *, \neg, 0, 1\rangle$, *then*

$$\Pi \models_{\mathbb{RL}} \varphi \approx \psi \quad \textit{iff} \quad \Pi \models_{\mathbb{PM}^{s\ell}} \varphi \approx \psi$$

In particular, $\mathbb{PM}^{s\ell}$ *and* $\mathbb{RL}$ *satisfy the same quasi-equations in the previous language.*

Corollary 4.18. *If* $\Pi \cup \{\varphi \approx \psi\}$ *is a set of equations in the language* $\langle \vee, \wedge, *, \neg, 0, 1\rangle$, *then*

$$\Pi \models_{\mathbb{RL}} \varphi \approx \psi \quad \textit{iff} \quad \Pi \models_{\mathbb{PM}^{\ell}} \varphi \approx \psi$$

In particular, $\mathbb{PM}^{\ell}$ *and* $\mathbb{RL}$ *satisfy the same quasi-equations in the previous language.*

Corollary 4.19. *Let* X *be a set of an arbitrary cardinality. Then,*

– $\mathbf{F}_{\mathbb{PM}^{s\ell}}(X)$*, the free algebra over* $\mathbb{PM}^{s\ell}$ *generated by* X*, is a subreduct of* $\mathbf{F}_{\mathbb{PM}^{\ell}}(X)$,
– $\mathbf{F}_{\mathbb{PM}^{\ell}}(X)$ *is a subreduct of* $\mathbf{F}_{\mathbb{RL}}(X)$.

Proof. It follows from the last two corollaries, using the fact that the free algebras over a certain set of generators of these varieties can be represented as Lindenbaum-Tarski algebras. □

III. Now we discuss finite embeddability property and decidability. First of all, let us recall that given an algebra $\mathbf{A} = \langle A, \langle f_i^{\mathbf{A}} : i \in I\rangle\rangle$ of any type, and any non-empty subset $B \subseteq A$, the *partial subalgebra* $\mathbf{B}$ of $\mathbf{A}$ is the structure[15] $\langle B, \langle f_i^{\mathbf{B}} : i \in I\rangle\rangle$, where for every k-ary functional f_i, and $b_1, \ldots, b_k \in B$,

$$f_i^{\mathbf{B}}(b_1 \ldots, b_k) = \begin{cases} f_i^{\mathbf{A}}(b_1 \ldots, b_k), & \text{if } f_i^{\mathbf{A}}(b_1 \ldots, b_k) \in B, \\ \text{undefined}, & \text{if } f_i^{\mathbf{A}}(b_1 \ldots, b_k) \notin B. \end{cases}$$

A class K of algebras has the *finite embeddability property*, FEP for short, if every finite partial subalgebra of each member of K can be embedded in a finite member of K. The next result is an easy consequence of the known fact that residuated lattices have the FEP [5, Theorem 5.9].

[15] We notice that it is not an algebra since the operations may not be defined around all the universe. These structures have sometimes been called *partial algebras*.

Theorem 4.20. $\mathbb{PM}^{s\ell}$ *and* $\mathbb{PM}^{\ell}$ *have the finite embeddability property. Therefore, their quasi-equational (and universal) theory are decidable.*

Proof. Let $\mathbf{A}$ be an $\mathbb{PM}^{s\ell}$-algebra and let $\mathbf{B}$ be a finite partial subalgebra of $\mathbf{A}$. By Theorem 4.16, $\mathbf{A}$ is embeddable in the residuated lattice $\mathbf{A}^{\mathbf{Id}}$ by the map $i_{\mathbf{A}}$. Now we have that $i_{\mathbf{A}}[\mathbf{B}]$ is a finite partial subalgebra of $\mathbf{A}^{\mathbf{Id}}$ and thus, since $\mathbb{RL}$ has the FEP, it can be embedded in a finite residuated lattice $\mathbf{D}$. Let h be this embedding and let $\mathbf{D}'$ be the $\langle \vee, *, \neg, 0, 1\rangle$-reduct of $\mathbf{D}$. Then $\mathbf{D}'$ is a finite $\mathbb{PM}^{s\ell}$-algebra and the map $h \circ i_{\mathbf{A}}$ is an embedding from $\mathbf{B}$ into $\mathbf{D}'$. A similar argument works for $\mathbb{PM}^{\ell}$.

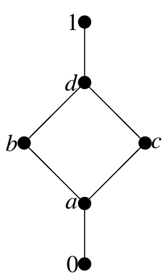
The first part implies that the sets of universal formulas that fail in our classes of algebras are recursively enumerable. Using the well-known fact that first-order logic is recursively axiomatizable we also have that the sets of universal formulas that hold in our classes of algebras are recursively enumerable. Combining the two procedures we obtain the desired decision procedure. □

IV. We will now say several things about the congruence lattice of the algebras in the varieties $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$. As $\mathbb{PM}^{\ell}$ is a variety of lattices, then it is congruence distributive [9, Section §12 (Chapter II)]. On the other hand, it is well known that the variety of semilattices is not congruence distributive, from which it is easy to see that $\mathbb{PM}^{s\ell}$ is not congruence distributive either. Indeed, the same applies for congruence modularity (we recall that all distributive lattices are modular). For instance, in the algebra $\mathbf{A_6}$ given in Figure 4.3 the modularity of its congruence lattice fails due to the fact that $\psi_1 \subseteq \psi_2$ while $\psi_1 \vee (\psi_2 \cap \psi_3) \neq \psi_2 \cap (\psi_1 \vee \psi_3)$. What is known about the congruence lattices of semilattices [19] is that $\theta_1 \cap \theta_2 = \theta_1 \cap \theta_3$ implies $\theta_1 \cap \theta_2 = \theta_1 \cap (\theta_2 \vee \theta_3)$ (meet semidistributivity). Of course this remains true over the algebras in $\mathbb{PM}^{s\ell}$.

It is not difficult to see that neither $\mathbb{PM}^{s\ell}$ nor $\mathbb{PM}^{\ell}$ are congruence permutable. In fact, they are not congruence permutable because $\langle 1, a\rangle \in \delta_1 \circ \delta_2$ but $\langle 1, a\rangle \notin \delta_2 \circ \delta_1$ in the algebra of the Figure 4.4.

The same algebra shows that these varieties are not 1-regular. Thus, we are in a very different situation from the one in the case of residuated lattices. Later on, in Theorem 5.11 we will obtain a characterization of the notion of congruence over $\mathbb{PM}^{s\ell}$-algebras and $\mathbb{PM}^{\ell}$-algebras.

We now analyze which algebras are subdirectly irreducible in these classes. In general if a class of algebras is not 1-regular this is a difficult problem. One of the



$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	a	a	a
b	0	a	a	a	a	b
c	0	a	a	a	a	c
d	0	a	a	a	a	d
1	0	a	b	c	d	1

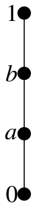
	0	a	b	c	d	1
$\neg$	1	0	0	0	0	0

ψ_1-classes $= \{\{0\}, \{a\}, \{c\}, \{b, d\}, \{1\}\}$

ψ_2-classes $= \{\{0\}, \{a, c\}, \{b, d\}, \{1\}\}$

ψ_3-classes $= \{\{0\}, \{a, b\}, \{c, d\}, \{1\}\}$

Fig. 4.3. A non-congruence modular $\mathbb{PM}^{s\ell}$-algebra $\mathbf{A_6}$



$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

	$\neg$
0	1
a	0
b	0
1	0

δ_1-classes $= \{\{0\}, \{a, b\}, \{1\}\}$

δ_2-classes $= \{\{0\}, \{a\}, \{b, 1\}\}$

δ_3-classes $= \{\{0\}, \{a\}, \{b\}, \{1\}\}$

Fig. 4.4. A non-congruence permutable $\mathbb{PM}^{\ell}$-algebra $\mathbf{A_7}$

few known solutions to problems of this kind was given by Lakser for the case of pseudocomplemented distributive lattices [30]. He proved there that the pseudocomplemented distributive lattices that are subdirectly irreducible are exactly the result of adjoining a new largest element over a Boolean algebra. For the cases of $\mathbb{PM}^{s\ell}$-algebras and $\mathbb{PM}^{\ell}$-algebras we have not been able to obtain characterization of this kind.

Next we will give some necessary conditions for subdirectly irreducible algebras of $\mathbb{PM}^{s\ell}$ ($\mathbb{PM}^{\ell}$). We will try to relate our problem to the problem of determining if a residuated lattice is subdirectly irreducible.

First of all we observe that, even in the finite case, there are residuated lattices that are subdirectly irreducible as residuated lattices but not as $\mathbb{PM}^{s\ell}$-algebras (neither as $\mathbb{PM}^{\ell}$-algebras), e.g., the expansion to a residuated lattice of the algebra $\mathbf{A_7}$ given in Figure 4.4. Now, we will see that the other direction holds in finite algebras. In fact, it holds over the reducts of residuated lattices and so in particular, by Corollary 4.10, over finite algebras of $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$.

Theorem 4.21.

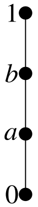
1) *Let* $\mathbf{A}$ *be a* $\mathbb{PM}^{s\ell}$*-algebra that is the reduct of a residuated lattice* $\mathbf{A}'$*. If* $\mathbf{A}$ *is subdirectly irreducible then* $\mathbf{A}'$ *is a subdirectly irreducible residuated lattice.*
2) *Let* $\mathbf{A}$ *be a* $\mathbb{PM}^{\ell}$*-algebra that is the reduct of a residuated lattice* $\mathbf{A}'$*. If* $\mathbf{A}$ *is subdirectly irreducible then* $\mathbf{A}'$ *is a subdirectly irreducible residuated lattice.*

In particular, every finite member of $\mathbb{PM}^{s\ell}$ *or* $\mathbb{PM}^{\ell}$ *which is subdirectly irreducible is a subdirectly irreducible residuated lattice.*

Proof. The same proof works in both cases. Suppose that $\mathbf{A}' = \langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1\rangle$ and that $\mathbf{A}$ is subdirectly irreducible in the corresponding variety. Let $a, b \in A$ with $a \neq b$ be such that $\theta(a, b)$ is the smallest non-trivial congruence of $\mathbf{A}$. It is enough to see that there is a smallest non-trivial lattice filter of $\mathbf{A}'$ closed under $*$.
Claim I: For each $x < 1$ there exists $k \in \omega$ such that $x^k \leq (a \rightarrow b) \wedge (b \rightarrow a)$.
Let $x < 1$ and let $F(x)$ be $\{y \in A : x^k \leq y$ for certain $k \in \omega\}$. This set is a lattice filter closed under $*$. Then, by the isomorphism into congruences of $\mathbf{A}'$ we know that $\theta := \{\langle y, z\rangle \in A^2 : (y \rightarrow z) \wedge (z \rightarrow y) \in F(x)\} \in Con(\mathbf{A}') \subseteq Con(\mathbf{A})$. As $\langle 1, x\rangle \in \theta$ and $1 \neq x$ the minimality of $\theta(a, b)$ says that $\theta(a, b) \subseteq \theta$, i.e., $\langle a, b\rangle \in \theta$. Therefore, there exists $k \in \omega$ such that $x^k \leq (a \rightarrow b) \wedge (b \rightarrow a)$.
Claim II: Let F be $\{x \in A : ((a \rightarrow b) \wedge (b \rightarrow a))^k \leq x$ for certain $k \in \omega\}$. Then, F is the smallest non-trivial lattice filter of $\mathbf{A}'$ closed under $*$.

Let I be a lattice filter of $\mathbf{A}'$ closed under $*$ such that $I \neq \{1\}$, i.e., there exists $x \in I$ with $x < 1$. The above claim says that there exists $k \in \omega$ such that $x^k \leq (a \to b) \wedge (b \to a)$. Thus, $(a \to b) \wedge (b \to a) \in I$, i.e., $F \subseteq I$. □

Remark 4.22. If we look at the steps of the above proof we see that we have not proved that $\theta(a, b)$ is also a congruence of $\mathbf{A}'$. In general this is false. For instance, consider the $\mathbb{PM}^{s\ell}$-algebra $\mathbf{A_8}$ defined by the following diagram and tables:



$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

	$\neg$
0	1
a	0
b	0
1	0

ϕ_1-classes $= \{\{0\}, \{a, b\}, \{1\}\}$

ϕ_2-classes $= \{\{0\}, \{a, b, 1\}\}$

We have that $\mathbf{A_8}$ is subdirectly irreducible being ϕ_1 its smallest non-trivial congruence. And its expansion to a residuated lattice is also subdirectly irreducible, but ϕ_2 is its smallest non-trivial congruence.

Remark 4.23. By Theorem 4.21 we conclude that subdirectly irreducible $\mathbb{PM}^{s\ell}$-algebras (and $\mathbb{PM}^{\ell}$-algebras) which are the reduct of a residuated lattice satisfy all the properties that hold in subdirectly irreducible residuated lattices. For instance, if $x, y < 1$ then $x \vee y < 1$ [29, Proposition 1.4]. In particular the subdirectly irreducible finite algebras of $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{s\ell}$ must have a penultimate element.

Lastly we give a theorem that relates subdirectly irreducible algebras with respect to its ideal completion.

Theorem 4.24.

1) *If* $\mathbf{A}$ *is a subdirectly irreducible* $\mathbb{PM}^{s\ell}$*-algebra then* $\mathbf{A^{Id}}$ *is a subdirectly irreducible residuated lattice.*
2) *If* $\mathbf{A}$ *is a subdirectly irreducible* $\mathbb{PM}^{\ell}$*-algebra then* $\mathbf{A^{Id}}$ *is a subdirectly irreducible residuated lattice.*

Proof. The proof works in both cases. Let $a, b \in A$ with $a \neq b$ be such that $\theta(a, b)$ is the smallest non-trivial congruence of $\mathbf{A}$. It is enough to see that there is a smallest non-trivial lattice filter of $\mathbf{A^{Id}}$ closed under $*$.
Claim I: For each $x < 1$ there exists $k \in \omega$ such that $x^k * a \leq b$ and $x^k * b \leq a$.
Let $x < 1$ and let θ be $\{\langle y, z\rangle \in A^2 : x^k * y \leq z$ and $x^k * z \leq y$ for certain $k \in \omega\}$. It is easy to check that θ is a congruence of $\mathbf{A}$. As $\langle 1, x\rangle \in \theta$ and $1 \neq x$ the minimality of $\theta(a, b)$ says that $\theta(a, b) \subseteq \theta$, i.e., $\langle a, b\rangle \in \theta$. Therefore, there exists $k \in \omega$ such that $x^k * a \leq b$ and $x^k * b \leq a$.
Claim II: Let I be $\{x \in A : x^k * a \leq b$ and $x^k * b \leq a$ for certain $k \in \omega\}$ and let $\mathcal{F}$ be $\{J \in A^{Id} : I^k \subseteq J\}$. Then, $\mathcal{F}$ is the smallest non-trivial lattice filter of $\mathbf{A^{Id}}$ closed under $*$.
Let $\mathcal{F}'$ be a lattice filter of $\mathbf{A^{Id}}$ closed under $*$ such that $\mathcal{F}' \neq \{A\}$, i.e., exists $J \in \mathcal{F}'$ with $1 \notin J$. Then, $J \subseteq I$ by the previous claim. As $J \in \mathcal{F}'$ since $\mathcal{F}'$ is a lattice filter we obtain that $I \in \mathcal{F}'$. Therefore, $\mathcal{F} \subseteq \mathcal{F}'$. □

5. Connecting the logical systems and the algebras

This section studies our four logical systems establishing connections between them and the algebras in Section 4.2. In Section 5.1 we do this for our two Gentzen systems, and in section 5.2 for our two deductive systems; but first of all we recall what happens when we also have the implication $\rightarrow$ in the language.

Theorem 5.1. (Cf. [1, Theorem 21]) *$IPC^*\backslash c$ and the equational system associated to $\mathbb{RL}$ are equivalent as Gentzen systems with translations τ and ρ defined as follows: $\tau(p) = \{p \approx 1\}$ and $\rho(p \approx q) = \{p \rightarrow q, q \rightarrow p\}$.*

By composing the translations of Theorem 3.2 (where the equivalence of $\mathcal{G}_{\mathbf{FL_{ew}}}$ and $IPC^*\backslash c$ is stated) and Theorem 5.1 the next known result trivially follows.

Theorem 5.2. [1, Theorem 22] *$\mathcal{G}_{\mathbf{FL_{ew}}}$ is algebraizable with equivalent variety semantics the variety $\mathbb{RL}$ and with translations τ and ρ defined as follows:*

$$\tau(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \varphi) := \begin{cases} \{\varphi_0 \rightarrow (\varphi_1 \rightarrow (\cdots \rightarrow (\varphi_{m-1} \rightarrow \varphi)\cdots)) \approx 1\}, & \text{if } m \geq 1 \\ \{\varphi \approx 1\}, & \text{if } m = 0 \end{cases}$$

$$\tau(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \emptyset) := \begin{cases} \varphi_0 \rightarrow (\varphi_1 \rightarrow (\cdots \rightarrow (\varphi_{m-1} \rightarrow 0)\cdots)) \approx 1\}, & \text{if } m \geq 1 \\ \{0 \approx 1\}, & \text{if } m = 0 \end{cases}$$

$$\rho(\varphi \approx \psi) := \{\varphi \Rightarrow \psi,\ \psi \Rightarrow \varphi\}.$$

Remark 5.3. Since all residuated lattices satisfy that

– $(x_0 * x_1 * \cdots * x_n) \rightarrow y \approx x_0 \rightarrow (x_1 \rightarrow (\cdots \rightarrow (x_n \rightarrow y)\cdots))$,
– $x \vee y \approx y$ iff $x \leq y$ iff $x \rightarrow y \approx 1$,

it holds that we can replace the translation τ in Theorem 5.2 with

– $\tau(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \varphi) = \{(\varphi_0 * \cdots * \varphi_{m-1}) \vee \varphi \approx \varphi\}$,
– $\tau(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \emptyset) = \{\varphi_0 * \cdots * \varphi_{m-1} \approx 0\}$.

Notice that this last translation only uses the connectives $*$, $\vee$, 0.

5.1. The algebraization of $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ and $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$

The algebraization results that we will obtain in this section can be seen, using the heuristic idea in Section 4.2, as a generalization of the next result.

Theorem 5.4. [37, Theorem 3.17] *The Gentzen system $\mathcal{G}_{\mathbf{FL_{ewc}}[\vee,*,\neg]}$ is algebraizable, with equivalent algebraic semantics the variety $\mathbb{PDL}$, with translations τ and ρ defined as follows:*

$$\tau(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \varphi) := \begin{cases} (\varphi_0 * \cdots * \varphi_{m-1}) \vee \varphi \approx \varphi\}, & \text{if } m \geq 1, \\ \{1 \approx \varphi\}, & \text{if } m = 0, \end{cases}$$

$$\tau(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \emptyset) := \begin{cases} \{\varphi_0 * \ldots * \varphi_{m-1} \approx 0\}, & \text{if } m \geq 1, \\ \{1 \approx 0\}, & \text{if } m = 0, \end{cases}$$

$$\rho(\varphi \approx \psi) := \{\varphi \Rightarrow \psi,\ \psi \Rightarrow \varphi\}.$$

Indeed, the above theorem could also be proved as an easy consequence from the algebraization of $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$. We notice that the above translations are motivated by Remark 5.3.

Theorem 5.5. *The Gentzen system* $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ *is algebraizable, with equivalent algebraic semantics the variety* $\mathbb{PM}^{s\ell}$*, with translations* τ *and* ρ *defined as in Theorem 5.4.*

Proof. To prove this theorem we will prove the four conditions in Lemma 2.1.

1) We have to show that if $\varsigma \in Seq^{\omega\times\{0,1\}}_{\langle\vee,*,\neg,0,1\rangle}$, then

$$\varsigma \quad \dashv\vdash_{\mathbf{FL_{ew}}[\vee,*,\neg]} \quad \rho\tau(\varsigma).$$

From now on we will write $\prod\Gamma$ as an abbreviation for $\varphi_0 * \cdots * \varphi_{m-1}$ if $\Gamma = \varphi_0, \ldots, \varphi_{m-1}$. Here we only deal with the case that ς is $\Gamma \Rightarrow \varphi$ with the length m of Γ equal to or greater than 1. Hence, what we have to prove is that $\Gamma \Rightarrow \varphi \ \dashv\vdash_{\mathbf{FL_{ew}}[\vee,*,\neg]} \{\prod\Gamma \vee \varphi \Rightarrow \varphi,\ \varphi \Rightarrow \prod\Gamma \vee \varphi\}$. The non-trivial parts of these formal proofs are

$$\dfrac{\dfrac{\Gamma \Rightarrow \varphi}{\prod\Gamma \Rightarrow \varphi}\,(*\Rightarrow)^{m-1} \qquad \varphi \Rightarrow \varphi}{(\prod\Gamma) \vee \varphi \Rightarrow \varphi}\,(\vee\Rightarrow)$$

and

$$\dfrac{\Gamma \Rightarrow \prod\Gamma \qquad \dfrac{\dfrac{\prod\Gamma \Rightarrow \prod\Gamma}{\prod\Gamma \Rightarrow (\prod\Gamma)\vee\varphi}\,(\Rightarrow\vee_1) \qquad (\prod\Gamma)\vee\varphi \Rightarrow \varphi}{\prod\Gamma \Rightarrow \varphi}\,(Cut)}{\Gamma \Rightarrow \varphi}\,(Cut)$$

2) This condition says that for every equation $\varphi \approx \psi \in Eq_{\langle\vee,*,\neg,0,1\rangle}$, it holds that $\varphi \approx \psi \ \dashv\models_{\mathbb{PM}^{s\ell}} \{\varphi \vee \psi \approx \psi,\ \psi \vee \varphi \approx \varphi\}$. This trivially holds.
3) We have to check that for every $\mathbf{A} \in \mathbb{PM}^{s\ell}$, the set R defined by

$$\{\langle\bar{x}, \bar{y}\rangle \in A^m \times A^n : m \in \omega, n \in \{0,1\}, \mathbf{A} \models \tau(p_0, \ldots, p_{m-1} \Rightarrow q_0, \ldots, q_{n-1})[\![\bar{x}, \bar{y}]\!]\}$$

contains the interpretations of the axioms of $\mathbf{FL_{ew}}[\vee,*,\neg]$ and is closed under the interpretations of the rules of $\mathbf{FL_{ew}}[\vee,*,\neg]$. We note that this set is

$$\{\langle\bar{x}, a\rangle \in A^m \times A : m \in \omega, \prod\bar{x} \le a\} \cup \{\langle\bar{x}, \emptyset\rangle \in A^m \times \{\emptyset\} : m \in \omega, \prod\bar{x} = 0\}$$

where $\prod\emptyset$ is an abbreviation for 1, and $\prod\bar{x}$ is an abbreviation for $x_0 * \cdots * x_{m-1}$ (when $m \ge 1$). Here we only deal with the rule $(\vee\Rightarrow)$. Let us suppose that $\langle\langle a, \bar{x}\rangle, \delta\rangle \in R$ and $\langle\langle b, \bar{x}\rangle, \delta\rangle \in R$. We define $c_\delta \in A$ as 0 if $\delta = \emptyset$ and as δ if $\delta \in A$. Then we have $a * \prod\bar{x} \le c_\delta$ and $b * \prod\bar{x} \le c_\delta$. Therefore $(a*\prod\bar{x}) \vee (b*\prod\bar{x}) \le c_\delta$ and by distributivity we have that $(a \vee b)*\prod\bar{x} \le c_\delta$. That is $\langle\langle a \vee b, \bar{x}\rangle, \delta\rangle \in R$.

4) Let us prove that for every $\Phi \in Th\,\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$,

$$\theta_\Phi := \{\langle\varphi,\psi\rangle \in Fm^2_{\langle\vee,*,\neg,0,1\rangle} : \rho(\varphi\approx\psi)\subseteq\Phi\}$$

is a congruence of $\mathbf{Fm}_{\langle\vee,*,\neg,0,1\rangle}$ relative to $\mathbb{PM}^{s\ell}$. The fact that θ_Φ is a congruence is easily proved. To see that $\mathbf{Fm}/\theta_\Phi \in \mathbb{PM}^{s\ell}$ we have to show that for every equation $\varphi\approx\psi$ defining $\mathbb{PM}^{s\ell}$ (see Theorem 4.6), then $\rho(\varphi\approx\psi)\subseteq\Phi$. It is enough to prove that for every equation $\varphi\approx\psi$ defining $\mathbb{PM}^{s\ell}$, $\emptyset \vdash_{\mathbf{FL_{ew}}[\vee,*,\neg]} \varphi\Rightarrow\psi$ and $\emptyset \vdash_{\mathbf{FL_{ew}}[\vee,*,\neg]} \psi\Rightarrow\varphi$. This is a simple checking.

This finishes the proof. □

Theorem 5.6. *The Gentzen system $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$ is algebraizable, with equivalent algebraic semantics the variety $\mathbb{PM}^{\ell}$, with translations τ and ρ defined as in Theorem 5.4.*

Proof. The proof is anologous to the one sketched for Theorem 5.5. □

New we state some easy consequences of the algebraization of the Gentzen systems $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ and $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$.

Corollary 5.7. *The Gentzen system $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ is the $\langle\vee,*,\neg,0,1\rangle$-fragment of $\mathcal{G}_{\mathbf{FL_{ew}}}$, and $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$ is the $\langle\vee,\wedge,*,\neg,0,1\rangle$-fragment of $\mathcal{G}_{\mathbf{FL_{ew}}}$.*

Proof. Since the two cases are analogous we simply prove the first one. We have to prove that for every $\Phi\cup\{\varsigma\}\subseteq Seq^{\omega\times\{0,1\}}_{\langle\vee,*,\neg,0,1\rangle}$,

$$\Phi \vdash_{\mathbf{FL_{ew}}} \varsigma \quad\text{iff}\quad \Phi \vdash_{\mathbf{FL_{ew}}[\vee,*,\neg]} \varsigma.$$

Let τ be the translation of $\mathcal{G}_{\mathbf{FL_{ew}}}$ in $\models_{\mathbb{RL}}$ stated in Theorem 5.2, and let τ' be the translation of $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ in $\models_{\mathbb{PM}^{s\ell}}$ stated in Theorem 5.5. Then we have the following chain of equivalences:

$$\begin{array}{ll}
\Phi \vdash_{\mathbf{FL_{ew}}} \varsigma & \text{iff}\\
\tau(\Phi) \models_{\mathbb{RL}} \tau(\varsigma) & \text{iff}\\
\tau'(\Phi) \models_{\mathbb{RL}} \tau'(\varsigma) & \text{iff}\\
\tau'(\Phi) \models_{\mathbb{PM}^{s\ell}} \tau'(\varsigma) & \text{iff}\\
\Phi \vdash_{\mathbf{FL_{ew}}[\vee,*,\neg]} \varsigma. &
\end{array}$$

The second equivalence is obtained by Remark 5.3, and the third one by Theorem 4.16. □

Corollary 5.8. *The contraction rule is admissible neither in $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ nor in $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$.*

Proof. We prove the theorem for $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ (the other case is analogous). It is obvious that $\emptyset \vdash_{\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}} p,p\Rightarrow p*p$. We will see that $\emptyset \not\vdash_{\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}} p\Rightarrow p*p$ with the help of Theorem 5.5. Take for example the algebra $\mathbf{A_8}$ of the Remark 4.22. This algebra obviously belongs to $\mathbb{PM}^{s\ell}$ but $b\not\leq b*b$. □

Corollary 5.9. *The Gentzen systems* $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ *and* $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$ *are not equivalent to any deductive system.*

Proof. By our previous algebraization results this is equivalent to the fact there is no deductive system equivalent to the equational system associated with $\mathbb{PM}^{s\ell}$ (and the same for the equational system associated with $\mathbb{PM}^{\ell}$). Let $\mathbf{A} = \langle\{0, a, b, c, 1\}, \vee, \wedge, \neg, 0, 1\}$ be the pseudocomplemented distributive lattice defined in the following way: $\wedge$ and $\vee$ are the supremum and the infimum corresponding to the order $0 < a < b < c < 1$ and $\neg$ is defined by $\neg 0 = 1$, $\neg a = \neg b = \neg c = \neg 1 = 0$. It is proved in [37, Theorem 3.1] that the Leibniz operator $\Omega_{\mathbf{A}}$ cannot be an isomorphism between the lattice $\mathcal{F}_{\mathcal{S}}(\mathbf{A})$ of $\mathcal{S}$-filters of $\mathbf{A}$ and the lattice $Con(\mathbf{A})$ of the congruences of $\mathbf{A}$ and so, by [6, Theorem 5.1], the variety of pseudocomplemented distributive lattices cannot be the equivalent algebraic semantics for any deductive system. Now if we consider the algebra $\mathbf{A}' = \langle \mathbf{A}, *\rangle$, with $* = \wedge$, we have that $\mathbf{A}' \in \mathbb{PM}^{\ell}$ and by using the argument given above, we have that $\mathbb{PM}^{\ell}$ is not the equivalent algebraic semantics for any deductive system. The same proof also works for $\mathbb{PM}^{s\ell}$. □

Corollary 5.10. *The Gentzen systems* $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$ *and* $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$ *are decidable, i.e., their entailments of the form* $\{\Gamma_i \Rightarrow \Delta_i : i \in I\} \vdash \Gamma \Rightarrow \Delta$, *with I finite, are decidable.*

Proof. It is a consequence of the algebraization and Theorem 4.20. □

Corollary 5.11. *For every* $\mathbf{A} \in \mathbb{PM}^{s\ell}$ *(*$\mathbb{PM}^{\ell}$*) the sequential Leibniz operator* $\Omega_{\mathbf{A}}$ *is an isomorphism between the lattices of* $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$*-filters (*$\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$*-filters) and* $\mathbb{PM}^{s\ell}$*(*$\mathbb{PM}^{\ell}$*)-congruences of* $\mathbf{A}$.

Proof. It is an application of Theorem 2.2. □

Hence, we have that the subdirectly irreducible algebras of $\mathbb{PM}^{s\ell}$ ($\mathbb{PM}^{\ell}$), i.e. those algebras having a smallest non-trivial congruence, are precisely the algebras with a smallest non-trivial $\mathcal{G}_{\mathbf{FL_{ew}}[\vee,*,\neg]}$-filter ($\mathcal{G}_{\mathbf{FL_{ew}}[\vee,\wedge,*,\neg]}$-filter).

5.2. Understanding $\mathcal{S}_e[\vee, *, \neg]$ *and* $\mathcal{S}_e[\vee, \wedge, *, \neg]$

In this section we study the two deductive systems that we are interested in. We now prove that they are fragments of $IPC^*\backslash c$.

Theorem 5.12. *For all* $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle\vee,*,\neg,0,1\rangle}$, *it holds that*

$$\Sigma \vdash_{IPC^*\backslash c} \varphi \quad \textit{iff} \quad \Sigma \vdash_{\mathcal{S}_e[\vee,*,\neg]} \varphi.$$

Proof. By using the fact that $IPC^*\backslash c$ is the external deductive system of $\mathbf{FL_{ew}}$ (Corollary 3.3) and Corollary 5.7 we have that

$$\Sigma \vdash_{IPC^*\backslash c} \varphi \ \text{ iff } \ \{\emptyset \Rightarrow \psi : \psi \in \Sigma\} \vdash_{\mathbf{FL_{ew}}} \emptyset \Rightarrow \varphi \ \text{ iff}$$
$$\{\emptyset \Rightarrow \psi : \psi \in \Sigma\} \vdash_{\mathbf{FL_{ew}}[\vee,*,\neg]} \emptyset \Rightarrow \varphi.$$

And this is precisely what is claimed in the statement. □

Theorem 5.13. *For all* $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle \vee,\wedge,*,\neg,0,1\rangle}$*, it holds that*

$$\Sigma \vdash_{IPC^*\backslash c} \varphi \quad \textit{iff} \quad \Sigma \vdash_{\mathcal{S}_e[\vee,\wedge,*,\neg]} \varphi.$$

Proof. The proof is analogous to the previous one. □

Next we prove that $\mathcal{S}_e[\vee, *, \neg]$ and $\mathcal{S}_e[\vee, \wedge, *, \neg]$ are proper subsystems of the fragment without implication of intuitionistic logic. To see this we need to consider this logical system, intuitionistic logic, from a non-standard point of view. Let us denote by $IPC_{\langle \vee,*,\neg,0,1\rangle}$ the fragment in the language $\langle \vee, *, \neg, 0, 1\rangle$ of the intuitionistic propositional logic, where we use here the symbol $*$ for the additive conjunction (i.e., what is usually denoted by $\wedge$). And we will denote by $IPC^*_{\langle \vee,\wedge,*,\neg,0,1\rangle}$ the fragment in the language $\langle \vee, \wedge, *, \neg, 0, 1\rangle$ of the intuitionistic propositional logic, where the behaviour of $*$ is exactly the same as $\wedge$. We recall that [38] presents a finite axiomatization of these fragments: the non-triviality of this axiomatization comes from the fact that this fragment is not protoalgebraic [6] (in particular Modus Ponens does not hold). We notice that the same problem for $\mathcal{S}_e[\vee, *, \neg]$ and $\mathcal{S}_e[\vee, \wedge, *, \neg]$ remains open.

Theorem 5.14. $\mathcal{S}_e[\vee, *, \neg] \lneq IPC_{\langle \vee,*,\neg,0,1\rangle}$ *and* $\mathcal{S}_e[\vee, \wedge, *, \neg] \lneq IPC^*_{\langle \vee,\wedge,*,\neg,0,1\rangle}$.

Proof. We will restrict ourselves to proving the first statement.

1) First we check that $\mathcal{S}_e[\vee, *, \neg] \leq IPC_{\langle \vee,*,\neg,0,1\rangle}$. Since $\mathbf{FL_{ew}}[\vee, *, \neg] \leq \mathbf{FL_{ewc}}[\vee, *, \neg]$ it is clear that the inequality holds for the external deductive systems associated with these Gentzen systems. By definition of $\mathcal{S}_e[\vee, *, \neg]$ and the fact that $IPC_{\langle \vee,*,\neg,0,1\rangle}$ is the external deductive system of $\mathbf{FL_{ewc}}[\vee, *, \neg]$ (cf. the proof of [39, Corollary 4.6]) we conclude that $\mathcal{S}_e[\vee, *, \neg] \leq IPC_{\langle \vee,*,\neg,0,1\rangle}$.

2) To see that this inclusion is proper we consider the formula $\neg(\varphi * \neg(\varphi * \varphi))$, which clearly is valid in intuitionistic logic. Let us now prove that it does not hold in $\mathcal{S}_e[\vee, *, \neg]$. If $\neg(\varphi*\neg(\varphi*\varphi))$ is a theorem of $\mathcal{S}_e[\vee, *, \neg]$, then $\emptyset \Rightarrow \neg(\varphi*\neg(\varphi*\varphi))$ is derivable en $\mathbf{FL_{ew}}[\vee, *, \neg]$, so $\vDash_{\mathbb{PM}^{s\ell}} \neg(x * \neg(x * x)) \approx 1$ by the algebraization. To prove that this is false, we can take the Łukasiewicz three element algebra $\langle\{0, \frac{1}{2}, 1\}, \vee, *, \neg, 0, 1\rangle$ and the value $\frac{1}{2}$. □

In the rest of the section we will classify our deductive systems under the properties introduced in Section 2.2. The most interesting fact is that although they are non-protoalgebraic we know an algebraic semantics for these deductive systems.

Theorem 5.15. *The deductive systems* $\mathcal{S}_e[\vee, *, \neg]$ *and* $\mathcal{S}_e[\vee, \wedge, *, \neg]$ *are non-protoalgebraic.*

Proof. Since protoalgebraicity is monotonic, the result is an immediate consequence of Theorem 5.14 and the fact that the fragment without implication of the intuitionistic propositional logic is non-protoalgebraic [6, Theorem 3.5]. □

Theorem 5.16. *The variety* $\mathbb{PM}^{s\ell}$ *is an algebraic semantics for* $\mathcal{S}_e[\vee, *, \neg]$ *with defining equation* $p \approx 1$*. And the variety* $\mathbb{PM}^{\ell}$ *is an algebraic semantics for* $\mathcal{S}_e[\vee, \wedge, *, \neg]$ *with the same defining equation.*

Proof. We restrict ourselves to the first case. By Theorem 5.5 we know that for every $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle\vee,*,\neg,0,1\rangle}$,

$$\{\emptyset \Rightarrow \psi : \psi \in \Sigma\} \vdash_{FL_{ew}[\vee,*,\neg]} \emptyset \Rightarrow \varphi \quad \text{iff} \quad \{1 \approx \psi : \psi \in \Sigma\} \vDash_{\mathbb{PM}^{s\ell}} 1 \approx \varphi.$$

By the definition of $\mathcal{S}_e[\vee, *, \neg]$ it follows that for every $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle\vee,*,\neg,0,1\rangle}$,

$$\Sigma \vdash_{\mathcal{S}_e[\vee,*,\neg]} \varphi \quad \text{iff} \quad \{1 \approx \psi : \psi \in \Sigma\} \vDash_{\mathbb{PM}^{s\ell}} 1 \approx \varphi.$$

And this is precisely what is claimed in the statement. □

Theorem 5.17. *$\mathcal{S}_e[\vee, *, \neg]$ and $\mathcal{S}_e[\vee, \wedge, *, \neg]$ are decidable, i.e., their entailments of the form $\Gamma \vdash \varphi$, with Γ finite, are decidable.*

Proof. It follows from Theorem 5.16 and 4.20. □

Theorem 5.18. *$\mathcal{S}_e[\vee, *, \neg]$ and $\mathcal{S}_e[\vee, \wedge, *, \neg]$ are not selfextensional. Therefore, $IPC^*\backslash c$ is neither selfextensional logic.*

Proof. Let $\vdash$ be the consequence relation associated with $\mathcal{S}_e[\vee, *, \neg]$ or with $\mathcal{S}_e[\vee, \wedge, *, \neg]$. By Theorem 5.16 it is easily verified that $p \dashv\vdash p * p$. But by using the three element Łukasiewicz algebra it follows that $\neg(p * p) \nvdash \neg p$.

The second part follows from Theorem 5.12. □

Thus, these deductive systems are not extensional (Fregean), that is, they are intensional deductive systems.

6. Conclusions and future work

For the case of $\mathbf{FL_e}$ (i.e., removing boundedness in the algebras) we know that the situation is the same except for slight changes in the definition of the translations. For the case of **FL** (i.e., also removing commutativity in the algebras) it is also possible to develop these ideas, but this time the proof of the algebraization is more involved since we need to use simultaneously both pseudocomplementations (one for each of the residuals). We also plan to extend our research to all the fragments of the language $\langle\vee, \wedge, *, \neg, 0, 1\rangle$ that have the connectives $*, 0, 1$ appear.

To end the section we list what in the authors' opinion are the most important open problems. The main one was stated in the last section: the search for Hilbert-style axiomatizations for the deductive systems $\mathcal{S}_e[\vee, *, \neg]$ and $\mathcal{S}_e[\vee, \wedge, *, \neg]$. In Proposition 4.12 we saw that there are two different varieties of residuated lattices that have the same $\langle\vee, \wedge, *, \neg, 0, 1\rangle$-fragment, and it is natural to wonder if there is also a continuum. The last open problem is a characterization of the reduced matrices for the deductive systems that we considered (cf. [37, Theorem 3.15]).

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References

1. Adillon, R., Verdú, V.: On a contraction-less intuitionistic propositional logic with conjunction and fusion. Studia Logica, Special Issue on Abstract Algebraic Logic **65** (1), 11–30 (2000)
2. Avron, A.: The semantics and proof theory of linear logic. Theoretical Computer Science **570**, 161–184 (1988)
3. Balbes, R., Dwinger, P.: Distributive lattices. University of Missouri Press, Columbia, Missouri, 1974
4. Banaschewski, B.: Hüllensysteme und Erweiterung von Quasi-Ordnungen. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **2**, 117–130 (1956)
5. Blok, W.J., van Alten, C.J.: The finite embeddability property for residuated lattices, pocrims and BCK-algebras. Algebra Universalis **48** (3), 253–271 (2002)
6. Blok, W.J., Pigozzi, D.: Algebraizable logics. Mem. Amer. Math. Soc. vol. 396, A.M.S., Providence, 1989
7. Bou, F., García-Cerdaña, À., Verdú, V.: Analysis of two fragments of the logic of residuated lattices. In: W.A. Carnielli, F.M. Dionísio, P. Mateus (eds.) Proceedings of CombLog'04. Workshop on Combination of Logics: Theory and Applications, Lisbon, 2004, pp. 175–186
8. Bou, F., García-Cerdaña, À., Verdú, V.: On two fragments with negation and without implication of the logic of residuated lattices. Mathematics Preprint Series 369, IMUB University of Barcelona, 2005. Available on line at http://www.imub.es/publications.html.
9. Burris, S., Sankappanavar, H.P.: A course in universal algebra. The Millennium Edn. (2000)
10. Cignoli, R., D'Ottaviano, I.M.L., Mundici, D.: Algebraic foundations of many-valued reasoning, Trends in Logic—Studia Logica Library, vol. 7. Kluwer Academic Publishers, Dordrecht, 2000
11. Cotlar, M.: A method of construction of structures and its application to topological spaces and abstract arithmetic (Spanish). Univ. Nac. Tucumán Rev. Ser. A. **4**, 105–157 (1944)
12. Crawley, P.: Regular embeddings which preserve lattice structure. Proc. Amer. Math. Soc. **13**, 748–752 (1962)
13. Czelakowski, J.: Protoalgebraic logics. Trends in Logic—Studia Logica Library, vol. 10, Kluwer Academic Publishers, Dordrecht, 2001
14. Devlin, K.: The joy of sets. Fundamentals of Contemporary Set Theory, second edn. Undergraduate Texts in Mathematics. Springer-Verlag, 1991
15. Dilworth, R.P., McLaughlin, J.E.: Distributivity in lattices. Duke Math. J. **19**, 683–693 (1952)
16. Esteva, F., Godo, L.: Monoidal t-norm based logic: towards a logic for left-continuous t-norms. Fuzzy Sets and Systems **124**, 271–288 (2001)
17. Esteva, F., Godo, L., García-Cerdaña, À.: On the hierarchy of t-norm based residuated fuzzy logics. In: M. Fitting, E. Orłowska (eds.) Beyond two: theory and applications of multiple-valued logic. Studies in Fuzziness and Soft Computing, vol. 114, Physica, Heidelberg, 2003, pp. 251–272
18. Font, J.M., Jansana, R., Pigozzi, D.: A survey on abstract algebraic logic. Studia Logica, Special Issue on Abstract Algebraic Logic, Part II **74** (1/2), 13–97 (2003)
19. Freese, R., Nation, J.B.: Congruence lattices of semilattices. Pacific J. Math. **49**, 51–58 (1973). See also the manuscript Clarification to "Congruence Lattices of Semilattices", 1997
20. Funayama, N.: On the completion by cuts of distributive lattices. Proc. Imp. Acad. Tokyo **20**, 1–2 (1944)

21. Gehrke, M., Harding, J., Venema, Y.: MacNeille completions and canonical extensions. Research Report PP-2004-05, Institute for Logic, Language and Computation, University of Amsterdam, Amsterdam, 2004
22. Gil, A.J., Torrens, A., Verdú, V.: On Gentzen systems associated with the finite linear MV-algebras. Journal of Logic and Computation **7** (4), 473–500 (1997)
23. Grätzer, G.: General lattice theory, Second edition. Birkhäuser Verlag, Basel, 1998
24. Hájek, P.: Metamathematics of fuzzy logic, Trends in Logic—Studia Logica Library, vol. 4. Kluwer Academic Publishers, Dordrecht, 1998
25. Halmos, P.R.: Lectures on Boolean algebras. Springer-Verlag 1974. Reprint of the ed. published by Van Nostrand, New York, 1963
26. Höhle, U.: Commutative, residuated l-monoids. In: U. Höhle, E.P. Klement (eds.) Nonclassical logics and their applications to fuzzy subsets (Linz, 1992), Theory Decis. Lib. Ser. B Math. Statist. Methods, vol. 32, Kluwer Acad. Publ. Dordrecht, 1995, pp. 53–106
27. Jipsen, P., Tsinakis, C.: A survey of residuated lattices. In: J. Martinez (ed.) Ordered Algebraic Structures, Kluwer Academic Publishers, Dordrecht, 2002 pp. 19–56
28. Kowalski, T., Ono, H.: The variety of residuated lattices is generated by its finite simple members. Reports on Mathematical Logic (34), 59–77 (2000). Algebra & substructural logics, Tatsunokuchi, 1999
29. Kowalski, T., Ono, H.: Residuated lattices: an algebraic glimpse at logics without contraction (preliminary report) (2001)
30. Lakser, H.: Principal congruences of pseudocomplemented distributive lattices. Proceedings of the American Mathematical Society **37**, 32–37 (1973)
31. Ono, H.: Semantics for substructural logics. In: K. Došen, P. Schroeder-Heister (eds.) Substructural logics, Oxford University Press, 1993, pp. 259–291
32. Ono, H.: Proof-theoretic methods for nonclassical logic - an introduction. In: M. Takahashi, M. Okada, M. Dezani-Ciancaglini (eds.) Theories of Types and Proofs, MSJ Memoirs 2, Mathematical Society of Japan, 1998, pp. 207–254
33. Ono, H.: Closure operators and complete embeddings of residuated lattices. Studia Logica **74** (3), 427–440 (2003)
34. Ono, H.: Completions of algebras and completeness of modal and substructural logics. pp. 335–353 (2003)
35. Ono, H.: Substructural logics and residuated lattices - an introduction. In: V.F. Hendricks, J. Malinowski (eds.) 50 Years of Studia Logica Trends in Logic—Studia Logica Library. vol. 21, Dordrecht, 2003, pp. 193–228
36. Ono, H., Komori, Y.: Logics without the contraction rule. The Journal of Symbolic Logic **50** (1), 169–201 (1985)
37. Rebagliato, J., Verdú, V.: On the algebraization of some Gentzen systems. Fundamenta Informaticae, Special Issue on Algebraic Logic and its Applications **18**, 319–338 (1993)
38. Rebagliato, J., Verdú, V.: A finite Hilbert-style axiomatization of the implication-less fragment of the intuitionistic propositional calculus. Mathematical Logic Quarterly **40**, 61–68 (1994)
39. Rebagliato, J., Verdú, V.: Algebraizable Gentzen systems and the deduction theorem for Gentzen systems. Mathematics Preprint Series 175, University of Barcelona, 1995
40. Russell, R.: The large deviations of random time-changes. Ph. D. Dissertation, University of Dublin, 1998
41. Schmidt, J.: Zur Kennzeichnung der Dedekind-MacNeilleschen Hülle einer geordneten Hülle. Arch. Math. **7**, 241–249 (1956)
42. Wójcicki, R.: Theory of logical calculi. Basic theory of consequence operations, Synthese Library, vol. 199. Reidel, Dordrecht, 1988
43. Wójcicki, R.: A logic is referential iff it is selfextensional. Studia Logica **73** (3), 323–335 (2003)