On expansions of t-norm based logics with truth-constants

Francesc Esteva
IIIA - CSIC
esteva@iiia.csic.es

Lluís Godo
IIIA - CSIC
godo@iiia.csic.es

Carles Noguera
IIIA-CSIC
cnoguera@iiia.csic.es

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Abstract

This paper focuses on completeness results about generic expansions of logics of both continuous t-norms and Weak Nilpotent Minimum (WNM) with truth-constants. Indeed, we consider algebraic semantics for expansions of these logics with a set of truth-constants $\{\overline{r} \mid r \in C\}$, for a suitable countable $C \subseteq [0,1]$, and provide a full description of completeness results when (i) either the t-norm is a finite ordinal sum of Łukasiewicz, Gödel and Product components (and hence continuous) or the t-norm is a Weak Nilpotent Minimum with a finite partition and (ii) the set of truth-constants *covers* all the unit interval in the sense that each component (in case of continuous t-norm) or each interval of the partition (in the WNM case) contains values of C in its interior. Results on expansions of the logic of a continuous t-norm were already published, while many of the results about WNM are presented here for the first time.

Keywords: Monoidal t-norm based Logic (MTL), Basic Fuzzy logic BL, Gödel, Łukasiewicz and Product Logics, Nilpotent minimum Logic (NM), Weak nilpotent minimum logics (WNM), t-norm-based logic, Rational Pavelka Logic, Rational Gödel logic, Rational Product logic, Rational t-norm based logic, completeness results.

1 Introduction

T-norm based fuzzy logics are basically logics of *comparative truth*. In fact, the residuum $\Rightarrow$ of a (left-continuous) t-norm $*$ satisfies the condition $x \Rightarrow y = 1$ if, and only if, $x \leq y$ for all $x, y \in [0,1]$. This means that a formula $\varphi \to \psi$ is a logical consequence of a theory if the truth degree of ψ is at least as high as the truth degree of φ in any interpretation which is a model of the theory. Indeed, the logic of continuous t-norms as it is presented in Hájek's seminal book [22], only deals with valid formulae and deductions using 1 as the only designated truth-value. This line is followed by the majority of logical papers written from then in the setting of many-valued fuzzy logics.

But, in general, these systems do not exploit in depth neither the idea of comparative truth nor the potentiality of dealing with explicit partial truth that a many-valued logic setting offers. On the one hand, for instance, a logic which is based exclusively on the idea of comparative truth is the system $Ł_\infty^\leq$ (see [19]) where a deduction is valid if, and only if, the degre of truth of the premises is less or equal than the degree of truth of conclusion. The system developed there is based on Łukasiewicz infinitely-valued logic Ł but it could be defined over any other t-norm based logic. Actually, since Gödel logic G is the only t-norm based logic enjoying the classical deduction-detachment theorem, it is the only case in which the usual G logic coincides with $G_\infty^\leq$.

On the other hand, in some situations one might be also interested to explicitly represent and reason with partial degrees of truth. To do so, one convenient and elegant way is introducing truth-constants into the language. This approach actually goes back to Pavelka [35] who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz Logic Ł by adding into the language a truth-constant $\overline{r}$ for each *real* $r \in (0, 1)$, together with a number of additional axioms. Although the resulting logic is not strongly complete with respect to the intended semantics defined by the Łukasiewicz t-norm (like the original Łukasiewicz logic), Pavelka proved that his logic, denoted here PL, is complete in a different sense. Namely, he defined the *truth degree* of a formula φ in a theory T as $\|\varphi\|_T = \inf\{e(\varphi) \mid e$ is a PL-evaluation model of $T\}$, and the *provability degree* of φ in T as $|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{PL}} \overline{r} \rightarrow \varphi\}$ and proved that these two degrees coincide. This kind of completeness is usually known as Pavelka-style completeness, and strongly relies on the continuity of Łukasiewicz truth functions. Novák extended Pavelka's approach to a first order logic [32]. Furthermore, Łukasiewicz logic extended with truth-constants has been extensively developed by Nóvak and colleagues in the frame of the so-called fuzzy logic with evaluated syntax [33].

Later, Hájek [22] showed that the logic PL could be significantly simplified while keeping the Pavelka-style completeness results. Indeed he showed it is enough to extend the language only by a countable number of truth-constants, one constant $\overline{r}$ for each *rational* in $r \in (0, 1)$, and by adding to Łukasiewicz Logic the two following additional axiom schemata, called book-keeping axioms:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r *_{Ł} s}$$
$$(\overline{r} \rightarrow \overline{s}) \leftrightarrow \overline{r \Rightarrow_{Ł} s}$$

where $*_Ł$ and $\Rightarrow_Ł$ are the Łukasiewicz t-norm and its residuum respectively. He called this new system Rational Pavelka Logic, RPL for short. Moreover, he proved that RPL is strongly complete (in the usual sense) for finite theories.

Similar *rational* expansions for other continuous t-norm based fuzzy logics can be analogously defined, but Pavelka-style completeness cannot be obtained since Łukasiewicz Logic is the only fuzzy logic whose truth-functions are continuous. [1]

[1]An easy argument shows that for logics based on other continuous t-norms Pavelka-style completeness does not hold. Let L_* be the logic of a continuous t-norm $*$ (not isomorphic to Łukasiewicz t-norm) and its residuum $\Rightarrow$ (as defined in [16]). Then it is known that the induced negation $\neg x = x \Rightarrow 0$ is not continuous in $x = 0$, i.e. $\sup\{\neg x \mid x > 0\} < \neg 0 = 1$.

Let p be a propositional variable and let $T = \{p \rightarrow \overline{r} \mid r > 0\}$. One can show that $\|p \rightarrow \overline{0}\|_T \neq |p \rightarrow \overline{0}|_T$. Indeed, $\|p \rightarrow \overline{0}\|_T = \inf\{e(p) \Rightarrow 0 \mid e(p) \leq r$ for all $r > 0\} = 0 \Rightarrow 0 = 1$, and we show that $|p \rightarrow \overline{0}|_T < 1$. For this, it is enough to prove that $T \not\vdash \overline{r_0} \rightarrow (p \rightarrow \overline{0})$ for any $r_0 < 1$ such that $r_0 > \sup\{\neg x \mid x > 0\}$ (such an element exists because $*$ is not isomorphic to

However, several expansions with truth-constants of fuzzy logics different from Łukasiewicz have been studied, mainly related to the other two outstanding continuous t-norm based logics, namely Gödel and Product logic. We may cite [22] where an expansion of G_Δ (the expansion of Gödel Logic G with Baaz's projection connective Δ) with a finite number of rational truth-constants, [15] where the authors define logical systems obtained by adding (rational) truth-constants to $G_\sim$ (Gödel Logic with an involutive negation) and to Π (Product Logic) and $\Pi_\sim$ (Product Logic with an involutive negation). In the case of the rational expansions of Π and $\Pi_\sim$ an infinitary inference rule (from $\{\varphi \to \overline{r} : r \in \mathbb{Q} \cap (0, 1]\}$ infer $\varphi \to \overline{0}$) is introduced in order to get Pavelka-style completeness. Rational truth-constants have been also considered in some stronger logics like in the logic $\text{Ł}\Pi\frac{1}{2}$ [16], a logic that combines the connectives from both Łukasiewicz and Product logics plus the truth-constant $\overline{1/2}$, and in the logic PŁ [26], a logic which combines Łukasiewicz Logic connectives plus the Product Logic conjunction (but not implication), as well as in some closely related logics.

Following this line, Cintula gives in [8] a definition of what he calls *Pavelka-style extension* of a particular fuzzy logic. He considers the Pavelka-style extensions of the most popular fuzzy logics, and for each one of them he defines an axiomatic system with infinitary rules (to overcome discontinuities like in the case of Π explained above) which is proved to be Pavelka-style complete. Moreover he also considers the first order versions of these extensions and provides necessary conditions for them to satisfy Pavelka-style completeness.

Recently, the approach based on traditional algebraic semantics has been considered to study completeness results (in the usual sense) for expansions of t-norm based logics with truth-constants. Indeed, as already mentioned, only the case of Łukasiewicz logic was known after [22]. Using this algebraic approach, the expansion of Gödel (and of some t-norm based logic related to the Nilpotent Minimum t-norm) with rational truth-constants and the expansion of Product logic with countable sets of truth-constants have been respectively studied in [17] and in [36]. Very recently, the basic cases of Łukasiewicz, Gödel and Product logics have been extended to the more general case of logics of continuous t-norms which are finite ordinal sums of the three basic components [13]. In these papers, the issue of canonical standard completeness (that is, completeness with respect to the standard algebra where the truth-constants are interpreted as their own values) for these logics has been determined. Also, special attention has been paid to the fragment of formulae of the kind $\overline{r} \to \varphi$, where φ is a formula without additional truth-constants. Actually, this kind of formulae have been extensively considered in other frameworks for reasoning with partial degrees of truth, like in Novák's evaluated syntax formalism based on Łukasiewicz Logic (see e.g. [34]), in Gerla's framework of abstract fuzzy logics [20] or in fuzzy logic programming (see e.g. [38]).

In this paper, within the algebraic semantics approach, we survey completeness results for expansions with truth-constants of logics of continuous t-norms and of weak nilpotent minimum t-norms[2] (WNM t-norms for short) in a general

Łukasiewicz t-norm). Suppose not. In such a case, there would exist a finite theory $T_0 \subseteq T$ such that $T_0 \vdash \overline{r_0} \to (p \to \overline{0})$. Then, by soundness, it should be $r_0 \leq \neg e(p)$ for any evaluation e such that $e(p) \leq s$, where $s = \min\{r \mid \overline{r} \to p \in T_0\}$, which is a contradiction (e.g. take $e(p) = s$).

[2]A weak nilpotent t-norm $*$ is a left-continuous t-norm satisfying $x * y = \min(x, y)$ if

setting. More specifically, we provide a full description of completeness results for the expansions of logics of these t-norms with a set of truth-constants $\{\overline{r} \mid r \in C\}$, for a suitable countable $C \subseteq [0,1]$, when (i) the t-norm is either a finite ordinal sum of Łukasiewicz, Gödel and Product components or a WNM that has a finite partition and (ii) the set of truth-constants *covers* all the unit interval in the sense that each component (for continuous case) or interval of the partition (for the WNM case) contains at least one value of C in its interior. Results on expansions of logics of continuous t-norms are already published, while many of the results about WNM logics are presented here for the first time.

The paper is structured as follows. After this introduction, we provide the necessary background in the next section. We give the general definitions of t-norm based logics we will deal with in the paper, the notion of standard completeness and general results for axiomatic extensions of these logics, the equivalence between different kinds of standard completeness and properties of the corresponding algebraic varieties (the partial embeddability property playing an important role), and finally, completeness results for logics of continuous and WNM t-norms. In Section 3 we introduce the expanded logics with truth-constants and their algebraic counterpart. In Section 4 we study the structure and relevant algebraic properties of the expanded linearly ordered algebras, which are needed to obtain the completeness results described in Section 5. Section 6 deals with completeness results when restricting the language to evaluated formulae. Section 7 summarizes and generalizes results about complexity issues for expanded t-norm based logics with truth-constants. We finish with some concluding remarks and open problems.

2 Preliminaries

The weakest logic that we will consider in this survey is the Monoidal T-norm based Logic (MTL). It is defined by Esteva and Godo in [14] by means of a Hilbert style calculus in the language $\mathcal{L} = \{\&, \to, \wedge, \overline{0}\}$ of type $\langle 2, 2, 2, 0 \rangle$. The only inference rule is *modus ponens* and the axiom schemata are the following (taking $\to$ as the least binding connective):

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \& \psi \to \varphi$
(A3)	$\varphi \& \psi \to \psi \& \varphi$
(A4)	$\varphi \wedge \psi \to \varphi$
(A5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(A6)	$\varphi \& (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
(A7b)	$(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$\overline{0} \to \varphi$

Cintula has shown in [9] that (A3) is in fact redundant. The usual defined connectives are introduced as follows:

$x * y > 0$ for all $x, y \in [0, 1]$.

$$
\begin{array}{rcl}
\varphi \vee \psi & := & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi); \\
\varphi \leftrightarrow \psi & := & (\varphi \to \psi) \& (\psi \to \varphi); \\
\neg \varphi & := & \varphi \to \overline{0}; \\
\overline{1} & := & \neg \overline{0}.
\end{array}
$$

Axiom schema	Name
$\neg\neg\varphi \to \varphi$	Involution (Inv)
$\neg\varphi \vee ((\varphi \to \varphi\&\psi) \to \psi)$	Cancellation (C)
$\varphi \to \varphi\&\varphi$	Contraction (Con)
$\varphi \wedge \psi \to \varphi\&(\varphi \to \psi)$	Divisibility (Div)
$\varphi \wedge \neg\varphi \to \overline{0}$	Pseudocomplementation (PC)
$\varphi \vee \neg\varphi$	Excluded Middle (EM)
$(\varphi\&\psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi\&\psi)$	Weak Nilpotent Minimum (WNM)

Table 1: Some usual axiom schemata in fuzzy logics.

Tables 1 and 2 collect some axiom schemata and the axiomatic extensions of MTL that they define.[3]

Logic	Additional axiom schemata
SMTL	(PC)
ΠMTL	(C)
IMTL	(Inv)
WNM	(WNM)
NM	(Inv) and (WNM)
BL	(Div)
SBL	(Div) and (PC)
Ł	(Div) and (Inv)
Π	(Div) and (C)
G	(Con)
CPC	(EM)

Table 2: Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.

The algebraic counterpart[4] of MTL logic is the class of the so-called *MTL-algebras*. They are defined as follows.

Definition 1 ([14]) *An* MTL-*algebra is an algebra* $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *of type* $\langle 2,2,2,2,0,0\rangle$ *such that:*

[3]Of course, some of these logics were known well before MTL was introduced. We only want to point out that it is possible to present them as the axiomatic extensions of MTL obtained by adding the corresponding axioms to the Hilbert style calculus for MTL given above.

[4]We assume some basic knowledge on Universal Algebra. All the undefined notions can be found in [3].

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *is a bounded lattice.*

2. $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *is a commutative monoid with unit* $\overline{1}^{\mathcal{A}}$.

3. *The operations* $\&^{\mathcal{A}}$ *and* $\rightarrow^{\mathcal{A}}$ *form an adjoint pair:*
$$\forall a, b, c \in A,\ a \&^{\mathcal{A}} b \leq c \text{ iff } b \leq a \rightarrow^{\mathcal{A}} c.$$

4. *It satisfies the prelinearity equation:*
$$(x \rightarrow^{\mathcal{A}} y) \vee^{\mathcal{A}} (y \rightarrow^{\mathcal{A}} x) \approx \overline{1}^{\mathcal{A}}$$

An additional (unary) negation operation is defined as $\neg^{\mathcal{A}} a := a \rightarrow^{\mathcal{A}} \overline{0}^{\mathcal{A}}$, *for every* $a \in A$.

If the lattice order is total we will say that $\mathcal{A}$ *is an* MTL*-chain.*

For the sake of a simpler notation, superscripts in the operations of the algebras will be omitted when they are clear from the context.

The class of all MTL-algebras is a variety which will be denoted as $\mathbb{MTL}$.

Given an MTL-algebra $\mathcal{A}$ and an element $a \in A$, we say that a is the (negation) *fixpoint* of $\mathcal{A}$ if, and only if, $a = \neg a$. It is easy to prove that there exists at most one fixpoint (see, for example, [25]). The sets of positive and negative elements of $\mathcal{A}$ are respectively defined as:

$$A_+ := \{a \in A : a > \neg a\}$$

$$A_- := \{a \in A : a \leq \neg a\}$$

Consider the terms $p(x) := x \vee \neg x$ and $n(x) := x \wedge \neg x$. The next proposition is an easy but useful result describing these sets.

Proposition 2 ([30]) *Let* $\mathcal{A}$ *be an* MTL*-algebra. Then:*

- $A_+ = \{p(a) : a \in A, \neg a \neq \neg\neg a\}$.
- $A_- = \{n(a) : a \in A\}$.

Notice that $p(a)$ *is the fixpoint if, and only if,* $\neg a = \neg\neg a$.

Given an MTL-algebra $\mathcal{A}$, a *filter* is any set $F \subseteq A$ such that:

- $\overline{1}^{\mathcal{A}} \in F$,
- If $a \in F$ and $a \leq b$, then $b \in F$, and
- If $a, b \in F$, then $a \& b \in F$.

In the rest of the paper we will use the following notations:

1. $Fi(\mathcal{A})$ denotes the set of proper filters of $\mathcal{A}$.
2. Given a filter $F \in Fi(\mathcal{A})$, $\overline{F}$ denotes the set $\{a \in A \mid \neg a \in F\}$

3. For each element $a \in A$, F_a denotes the filter generated by a, i.e. the minimum filter containing a.

Next proposition states the usual one-to-one correspondence between filters and congruences.

Proposition 3 *Let $\mathcal{A}$ be an* MTL*-algebra. For every filter $F \subseteq A$ we define $\Theta(F) := \{\langle a,b\rangle \in A^2 : a \leftrightarrow b \in F\}$, and for every congruence θ of $\mathcal{A}$ we define $Fi(\theta) := \{a \in A : \langle a,1\rangle \in \theta\}$. Then, Θ is an order isomorphism from the set of filters onto the set of congruences and Fi is its inverse.*

By virtue of this correspondence, we will do a notational abuse by writing $\mathcal{A}/F$ instead of $\mathcal{A}/\Theta(F)$, and for each $a \in A$, $[a]_F$ will denote the class of a in $\mathcal{A}/F$.

Given any class $\mathbb{K}$ of MTL-algebras, we denote its equational consequence as $\models_{\mathbb{K}}$, i.e. given a set of equations Λ and an equation $\varphi \approx \psi$ in the language $\mathcal{L}$, $\Lambda \models_{\mathbb{K}} \varphi \approx \psi$ means that for every $\mathcal{A} \in \mathbb{K}$ and every evaluation e of the formulae in $\mathcal{A}$, $e(\varphi) = e(\psi)$ whenever $e(\alpha) = e(\beta)$ for every $\alpha \approx \beta \in \Lambda$. If $\Lambda = \emptyset$, then we write $\models_{\mathbb{K}} \varphi \approx \psi$, instead of $\emptyset \models_{\mathbb{K}} \varphi \approx \psi$. When there is only one algebra in $\mathbb{K}$, say $\mathcal{A}$, we write $\Lambda \models_{\mathcal{A}} \varphi \approx \psi$ instead of $\Lambda \models_{\{\mathcal{A}\}} \varphi \approx \psi$.

Theorem 4 ([14]) *For every set of formulae $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ if, and only, $\{\psi \approx \overline{1} : \psi \in \Gamma\} \models_{\mathbb{MTL}} \varphi \approx \overline{1}$.*

This completeness result can be refined by means of the following property of MTL-algebras.

Proposition 5 ([14]) *Every* MTL*-algebra is representable as a subdirect product of* MTL*-chains.*

Corollary 6 *For every set of formulae $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{MTL}} \varphi$ if, and only, $\{\psi \approx \overline{1} : \psi \in \Gamma\} \models_{\{\mathrm{MTL}-chains\}} \varphi \approx \overline{1}$.*

Moreover, MTL is actually an algebraizable logic in the sense of Blok and Pigozzi (see [2]) and $\mathbb{MTL}$ is its equivalent algebraic semantics. This implies that all axiomatic extensions of MTL are also algebraizable and their equivalent algebraic semantics are the subvarieties of $\mathbb{MTL}$ defined by the translations of the axioms into equations. In particular, there is an order-reversing isomorphism between axiomatic extensions of MTL and subvarieties of $\mathbb{MTL}$:

1. If $\Sigma \subseteq Fm_{\mathcal{L}}$ and L is the extension of MTL obtained by adding the formulae of Σ as schemata, then the equivalent algebraic semantics of L is the subvariety of $\mathbb{MTL}$ axiomatized by the equations $\{\varphi \approx \overline{1} : \varphi \in \Sigma\}$. We denote this variety by $\mathbb{L}$ and we call its members *L-algebras*. There are two exceptions to that rule: the algebras associated to Ł are called *MV-algebras* following the terminology of Chang in [4], and the algebras associated to the Classical Propositional Calculus (CPC for short) are called, of course, *Boolean algebras*. Moreover, since L-algebras are representable as subdirect product of L-chains, the completeness of MTL with respect to chains is inherited by L.

2. Let $\mathbb{L} \subseteq \mathbb{MTL}$ be the subvariety axiomatized by a set of equations Λ. Then the logic associated to $\mathbb{L}$ is the axiomatic extension L of MTL given by the axiom schemata $\{\varphi \leftrightarrow \psi : \varphi \approx \psi \in \Lambda\}$.

Moreover, a lot of expansions of MTL are also algebraizable. Indeed, let L be an axiomatic extension of MTL, let $\mathcal{L}'$ be a language extending $\mathcal{L}$, consider a set $\Sigma \subseteq Fm_{\mathcal{L}'}$ and let L' be the expansion of L obtained by adding the formulae of Σ as axiom schemata. Assume that for every new n-ary connective λ in the language $\mathcal{L}'$,

$$\{p_1 \leftrightarrow q_1, \dots, p_n \leftrightarrow q_n\} \vdash_{\mathrm{L}'} \lambda(p_1, \dots, p_n) \leftrightarrow \lambda(q_1, \dots, q_n)$$

Then, L' is algebraizable and its equivalent algebraic semantics is the variety of algebras in the language $\mathcal{L}'$ axiomatized by the axioms of $\mathbb{L}$ plus the equations $\{\varphi \approx 1 : \varphi \in \Sigma\}$. We call the members of this variety L'-*algebras*. In general, L' needs not be a conservative expansion of L; in fact, we can extract from [2] the following criterion.

Proposition 7 ([2]) *Under the previous hypothesis,* L' *is a conservative expansion of* L *if, and only if, every* L*-algebra is a subreduct of some* L'*-algebra.*

Some algebraizable expansions of the so far mentioned logics have been introduced in the literature. Among them, a remarkable set of expansions are those obtained by enriching the language with the projection connective Δ (see [1]). Namely, given any axiomatic extension L of MTL, the expansion L_Δ is defined by adding to the language a unary connective Δ, and adding to the Hilbert-style system of L the following axiom schemata:

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$

(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$

(Δ3) $\Delta\varphi \to \varphi$

(Δ4) $\Delta\varphi \to \Delta\Delta\varphi$

(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

and the rule of *necessitation*:
$$\frac{\varphi}{\Delta\varphi}$$

This logic is algebraizable and its equivalent algebraic semantics is the variety of L_Δ-algebras, i. e. expansions with Δ of L-algebras satisfying the translation of the axioms $(\Delta 1), \dots, (\Delta 5)$ and the equation $\Delta\overline{1} \approx \overline{1}$. It is easy to prove that all L_Δ-algebras are representable as subdirect products of L_Δ-chains. The interpretation of the Δ connective in these chains is very simple, namely if $\mathcal{A}$ is an L_Δ-chain, then $\Delta^{\mathcal{A}}(\overline{1}^{\mathcal{A}}) = \overline{1}^{\mathcal{A}}$ and $\Delta^{\mathcal{A}}(a) = \overline{0}^{\mathcal{A}}$ for every $a \in A \setminus \{\overline{1}^{\mathcal{A}}\}$.

Proposition 8 *For every axiomatic extension* L *of* MTL, L_Δ *is a conservative expansion of* L.

Proof: It is obvious that every L-chain is the reduct of an L_Δ-chain (just take the same chain and define Δ in the only possible way for chains), thus we can apply Proposition 7. $\Box$

2.1 Standard completeness properties and equivalencies

Fuzzy Logic has always been interested in semantics defined over the real unit interval. Such kind of semantics can be found inside the class of MTL-algebras. Indeed, given a left-continuous t-norm $*$ and its residuum $\Rightarrow$ (defined as $a \Rightarrow b = \max\{c : a * c \leq b\}$), the algebra

$$[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1\rangle$$

is an MTL-chain. Notice that $[0,1]_*$ is completely determined by the t-norm. Moreover, it is obvious that in every MTL-chain $\mathcal{A}$ over $[0,1]$, the operation $\&^{\mathcal{A}}$ is a left-continuous t-norm. We will call these chains *standard algebras*.

Examples: It is well known that a standard algebra $[0,1]_*$ is a BL-chain if, and only if, $*$ is continuous. Prominent examples of continuous t-norms are the Łukasiewicz, the product and the minimum t-norms. We will denote their corresponding standard algebras by $[0,1]_{\text{Ł}}$, $[0,1]_\Pi$ and $[0,1]_G$, respectively. In [28] and [29] it is proved that every standard BL-algebra is decomposable as an ordinal sum of isomorphic copies of these three basic components.

For some expansions of MTL their completeness with respect to chains can be improved to completeness with respect to standard algebras. This leads to the following *standard completeness* properties.

Definition 9 (SC, FSSC, SSC) *If a logic* L *is an algebraizable expansion of* MTL *in a language $\mathcal{L}'$, we say that* L *has the* (finitely) strong standard completeness property*, (F)SSC for short, when for every (finite) set of formulae $T \subseteq Fm_{\mathcal{L}'}$ and every formula φ it holds that $T \vdash_{\text{L}} \varphi$ iff $\{\psi \approx \overline{1} : \psi \in T\} \models_{\mathcal{A}} \varphi \approx \overline{1}$ for every standard* L*-algebra $\mathcal{A}$. We say that* L *has the* standard completeness property*, SC for short, when the equivalence is true for $T = \emptyset$.*

Of course, the SSC implies the FSSC, and the FSSC implies the SC. These completeness properties are preserved when taking fragments of the logics.

Proposition 10 ([13]) *Suppose that* L$'$ *is a conservative expansion of* L*. Then:*

- *If* L$'$ *enjoys the SC, then* L *enjoys the SC.*
- *If* L$'$ *enjoys the FSSC, then* L *enjoys the FSSC.*
- *If* L$'$ *enjoys the SSC, then* L *enjoys the SSC.*

On the scope of algebraizable logics, these properties have their equivalent algebraic properties.

Theorem 11 *Let* L *be an algebraizable axiomatic expansion of* MTL *(in particular an axiomatic extension of* MTL*), and let $\mathbb{L}$ be its equivalent variety semantics. Then:*

1. L *has the SC if, and only if,* $\mathbb{L} = \mathbf{V}(Stand_{\mathrm{L}})$*,*

2. L *has the FSSC if, and only if,* $\mathbb{L} = \mathbf{Q}(Stand_{\mathrm{L}})$*,*

3. L *has the SSC if, and only if, every countable chain of* $\mathbb{L}$ *belongs to* $\mathbf{ISP}(Stand_{\mathrm{L}})$

where $Stand_{\mathrm{L}}$ *is the class of all standard algebras in* $\mathbb{L}$*,* $\mathbf{V}(Stand_{\mathrm{L}})$ *is the variety generated by* $Stand_{\mathrm{L}}$*, and* $\mathbf{Q}(Stand_{\mathrm{L}})$ *is the quasivariety generated by* $Stand_{\mathrm{L}}$*.*

Items 1 and 2 are well-known results in algebraic logic, while item 3 is proved in [31].

Nevertheless, these completeness properties have not usually been proved using the equivalencies above, but by means of some forms of embeddings of L-chains into standard L-chains. Actually, the SSC has been proved for the following logics by showing that all countable chains are embebbable into a standard one: MTL (in [27]), IMTL and SMTL (in [12]), G (in [10]) and WNM and NM (in [14]). In fact, as stated in next theorem, SCC is equivalent to the embeddability of the subdirectly irreducible countable chains. As regards to the FSSC, in some cases (see for instance [24, 22, 6] for Product, Łukasiewicz and BL logics), rather than using the equivalencies stated in Theorem 11, the result has been obtained by proving first that every chain of the equivalent variety semantics is partially embeddable into a standard algebra. For a long time, this condition was only known to be sufficient, but in [13] it has been proved that it is actually equivalent to the FSSC.

Definition 12 *Given two algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *of the same language we say that* $\mathcal{A}$ *is* partially embeddable *into* $\mathcal{B}$ *when every finite partial subalgebra of* $\mathcal{A}$ *is embeddable into* $\mathcal{B}$*. Generalizing this notion to classes of algebras, we say that a class* $\mathbb{K}$ *of algebras is* partially embeddable into a class $\mathbb{M}$ *if every finite partial subalgebra of a member of* $\mathbb{K}$ *is embeddable into a member of* $\mathbb{M}$*.*

Theorem 13 ([13]) *If* L *is an algebraizable axiomatic expansion of* MTL *(in particular if it is an axiomatic extension of* MTL*), then*

(i) L *has the FSSC if, and only if, the class of* L*-chains is partially embeddable into the class of standard* L*-algebras* $Stand_{\mathrm{L}}$*, whenever the language of* L *is finite.*

(ii) L *has the SSC if, and only if, every countable subdirectly irreducible chain of* $\mathbb{L}$ *is embeddable into a standard* L*-chain.*

Notice that in (i) the implication from right to left is true even if the language is infinite.

Sometimes standard completeness properties can be refined with respect to some subclass of standard algebras; sometimes it is even enough to consider only one standard algebra. When the standard completeness can be proved with respect to a particular standard algebra which is the intended semantics for the logic, we call it *canonical* standard completeness. As a matter of fact, the equivalencies in Theorems 11 and 13 remain true when restricted to some subclass of standard algebras.

2.2 About the logics L_*

The canonical standard completeness is a matter of special interest when one considers the logic of the variety generated by the algebra defined by one particular t-norm, because then this t-norm gives the intended semantics for the logic.

Definition 14 *Let $*$ be a left-continuous t-norm.* L_* *will denote the axiomatic extension of* MTL *whose equivalent algebraic semantics is* $\mathbf{V}([0,1]_*)$*, the variety generated by* $[0,1]_*$.

By definition, for every left-continuous t-norm $*$, the logic L_* enjoys the SC restricted to $[0,1]_*$, i.e. the canonical SC. But what about (canonical) FSSC and SSC properties for the logics L_*?

We start with the case of continuous t-norms. Throughout the rest of this paper we will use the following notation:

$$\mathbf{CONT} = \{* \text{ is a continuous t-norm}\}$$

$$\mathbf{CONT\text{-}fin} = \{* \in \mathbf{CONT} \mid * \text{ is an ordinal sum of finitely many basic components}\}$$

Recall that Ł, Π and G are actually important examples of logics L_*, namely the logics L_* for $*$ being the Łukasiewicz, the product and the minimum t-norm respectively (proved in [5], [24] and [10], respectively). Besides, in [16] the authors provide an algorithm that produces a finite axiomatization Hilbert-style calculus with finitely many axiom schemata (with modus ponens as the only inference rule) for every logic L_* with $*$ being a continuous t-norm. In these cases, canonical SC results can be improved to canonical FSSC by showing that given any continuous t-norm $*$, all chains in $\mathbf{V}([0,1]_*)$ are partially embeddable into $[0,1]_*$. The result is obvious for G-chains, and taking into account the relation of both MV-chains and Π-chains to lattice ordered Abelian groups and Gurevich-Kokorin Theorem, it is also true for Ł and Π (see [7]). This was finally generalized to any logic L_* for $*$ being a continuous t-norm in [13]. Therefore, after Theorem 13, we obtain the following result.

Theorem 15 *For every $* \in$ **CONT**, the logic L_* has the canonical FSSC.*

Moreover, this result cannot be improved to SSC with one exception.

Proposition 16 ([13]) *For every $* \in$ **CONT** such that $* \neq \min$, the logic L_* does not enjoy the SSC.*

Besides, in [16] the authors provide an algorithm that produces a Hilbert-style calculus for L_* for every continuous t-norm $*$.

Now we move to the case of logics L_* for $*$ being a left-continuous non-continuous t-norms, which is much more difficult, mainly because, unlike the continuous case, there is no general representation theorem for left-continuous t-norms.

However, some particular families of left-continuous non-continuous t-norms are well studied and even some finite axiomatizations are known for their corresponding logics. Namely, the standard WNM-algebras are studied in [21] and in [31]. Next, we summarize the main results.

The operations in WNM-chains are easily described. Let $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ be a WNM-chain. Then for every $a, b \in A$:

$$a \& b = \begin{cases} a \wedge b & \text{if } a > \neg b, \\ \overline{0}^{\mathcal{A}} & \text{otherwise.} \end{cases}$$

$$a \rightarrow b = \begin{cases} \overline{1}^{\mathcal{A}} & \text{if } a \leq b, \\ \neg a \vee b & \text{otherwise.} \end{cases}$$

In [14] it is shown that the operation $*$ in WNM-chains defined over the real unit interval $[0,1]$ is given by a special kind of left-continuous t-norm. These t-norms are defined in the following way. If n is a negation function[5] and $a, b \in [0,1]$, the operation $*_n$ defined as:

$$a *_n b = \begin{cases} \min\{a, b\} & \text{if } a > n(b), \\ 0 & \text{otherwise,} \end{cases}$$

is a left-continuous t-norm and its residuum is given by:

$$a \Rightarrow_n b = \begin{cases} 1 & \text{if } a \leq b, \\ \max\{n(a), b\} & \text{otherwise.} \end{cases}$$

for every $a, b \in [0,1]$. Moreover, it fulfills $a \Rightarrow_n 0 = n(a)$. It is straightforward that $[0,1]_{*_n} := \langle [0,1], *_n, \Rightarrow_n, \min, \max, 0, 1 \rangle$ is a WNM-chain, and all WNM-chains over $[0,1]$ are of this form.

Notice that a standard WNM-chain given by a negation function n is an NM-chain if, and only if, n is involutive, i.e. $n(n(a)) = a$ for every $a \in [0,1]$. It follows from the study of such negations in [37] that there is only one standard NM-chain up to isomorphism, namely the one given by the negation $n(x) = 1-x$. We will refer to it as $[0,1]_{\mathrm{NM}}$. The left-continuous t-norm corresponding to this algebra was introduced by Fodor in [18].

Since standard WNM-chains are completely determined by their negation functions, the study of L_* logics when $[0,1]_*$ is a WNM-chain, requires some knowlegde on the properties of such negations functions.

Lemma 17 ([11]) *Let $\mathcal{A}$ be a* MTL*-chain. Then for every $a \in A$:*

(i) $\neg a = \neg\neg\neg a$,

(ii) $a \leq \neg\neg a$,

(iii) $a = \neg\neg a$ *if, and only if, there is* $b \in A$ *such that* $a = \neg b$, *and*

[5] A non-increasing function $n : [0,1] \rightarrow [0,1]$ is a negation function if $x \leq n(n(x))$ for any $x \in [0,1]$ and $n(1) = 0$, see [11].

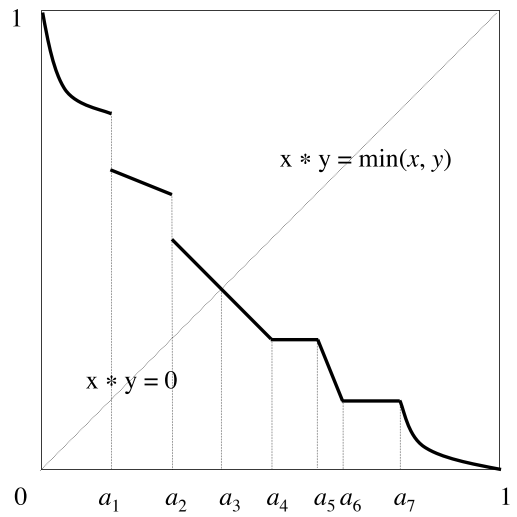

Figure 1: An example of WNM t-norm with a finite partition.

(iv) $\neg\neg a = \min\{b \in A : a \le b$ *and* $b = \neg\neg b\}$,

(v) *when* $A = [0, 1]$, $\neg$ *is a left-continuous function.*

The last one gives rise to the following definition:

Definition 18 *Let $\mathcal{A}$ be a* WNM*-chain and let $a \in A$ be an involutive element. We define $I_a := \{b \in A : \neg b = \neg a\}$ and we call it* the interval associated to *a, where the negation function is constant with value $\neg a$. We say that a has a trivial associated interval when $I_a = \{a\}$.*

A weak negation function has a form of symmetry; roughly speaking: if we complete its graph by drawing vertical lines in the jumps, then the obtained graph is symmetric with respect to the diagonal $x = y$. Therefore, the constant intervals I_a in the positive part of the chain symmetrically correspond to jumps in the negative parts (and viceversa).

Definition 19 *We say that the standard* WNM*-chain defined by a weak negation function $n : [0, 1] \to [0, 1]$ has a finite partition if n is constant in a finite number of intervals. In such a case we define the associated finite partition by considering the set $X = \{0, 1\} \cup \{a \in (0, 1) : a$ is either the maximum or infimum of a non-trivial interval associated to an involutive element, or a discontinuity of n, or the fixpoint$\}$. The family of intervals determined by X is called the partition of the* WNM *t-norm $*_n$.*

Notice that this partition yields two kinds of intervals: those where the negation takes a constant value, and those where all the elements are involutive. As a

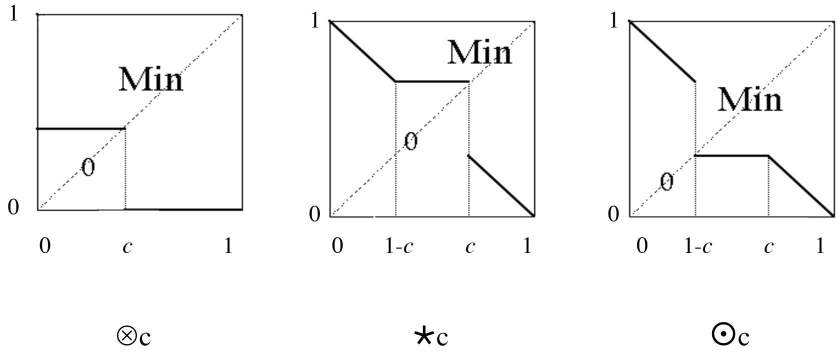

Figure 2: Three parametric families of WNM t-norms with finite partition.

matter of nomenclature, we call them *constant intervals* and *involutive intervals*, respectively. Figure 1 shows an example of a WNM t-norm with a fixpoint, a_3, and with a finite partition where the constant intervals are $[a_4, a_5]$ and $[a_6, a_7]$, while the involutive intervals are $[0, a_1]$, $[a_1, a_2]$, $[a_2, a_3]$, $[a_3, a_4]$, $[a_5, a_6]$ and $[a_7, 1]$.

Figure 2 shows three families of WNM t-norms with finite partition parametrized with a real number c: $c \in [0, 1)$ for $\otimes_c$, $c \in [1/2, 1)$ for $\star_c$ and $c \in [1/2, 1]$ for $\odot_c$. Notice that $\otimes_0 = \odot_1 = \min$ and $\star_{1/2} = \odot_{1/2}$ is the Nilpotent Minimum t-norm. These families are actually the only WNM t-norms with a finite partition of at most three intervals.

An interesting observation is that in any standard WNM-chain $[0, 1]_{*_n}$, if a is positive element then $F_a = [a, 1]$ and the elements of the quotient algebra $[0, 1]_*/F_a$ are such that

$$[x]_{F_a} = \begin{cases} [1]_{F_a}, & \text{if } x \in F_a \ \text{ (i.e. if } x \geq a) \\ [0]_{F_a}, & \text{if } x \in \overline{F_a} \ \text{ (i.e. if } x \leq n(a)) \\ \{x\}, & \text{otherwise} \end{cases}$$

Therefore, the quotient algebra $[0, 1]_*/F_a$ is isomorphic to another standard WNM-chain. If a belongs to a constant interval, then this standard chain has $I_1 \neq \{1\}$, see Figure 3.

To refer to the class of WNM t-norms and those with a finite partition we will use from now on the following notation:

$$\mathbf{WNM} = \{* \text{ is a weak nilpotent minimum t-norm}\}$$

$$\mathbf{WNM\text{-}fin} = \{* \in \mathbf{WNM} \mid * \text{ has a finite partition}\}$$

In [31] the following results have been proved.

Theorem 20 ([31]) *In the context of* L_* *logics for* $* \in \mathbf{WNM}$*, the following statements hold:*

1. *If $* \in$ **WNM-fin** , then the logic L_* is finitely axiomatizable (and an algorithm for finding the axiomatization has been given).*
2. *If $* \in$ **WNM-fin** and $I_1 \neq \{1\}$, then L_* has the canonical SSC, i.e. with respect to the class $\{[0,1]_*\}$.*
3. *If $* \in$ **WNM-fin** and $I_1 = \{1\}$, then L_* has the SSC with respect to the class $\{[0,1]_*, [0,1]_*/F_a\}$, where a is the maximum involutive element such that $I_a \neq \{a\}$. Moreover, this result cannot be improved, i. e. L_* does not enjoy the SSC with respect to only one of these two algebras.*
4. *If $* \notin$ **WNM-fin** and $I_1 \neq \{1\}$, then L_* has the canonical FSSC, i.e. with respect to the class $\{[0,1]_*\}$.*
5. *If $* \notin$ **WNM-fin** and $I_1 = \{1\}$, then L_* has the FSSC with respect to the class $\{[0,1]_*\} \cup \{[0,1]_*/F_a$: a positive involutive element such that $I_a \neq \{a\}\}$.*
6. *There are $* \notin$ **WNM-fin** for which L_* has not the SSC.*

Figure 3 shows on the left an example of $* \in$ **WNM-fin** falling under item 3 of the last theorem, where $I_1 = \{1\}$ and a is the maximum involutive element such that $I_a \neq \{a\}$, while on the right it shows the t-norm of a standard algebra isomorphic to the quotient algebra $[0,1]_*/F_a$.

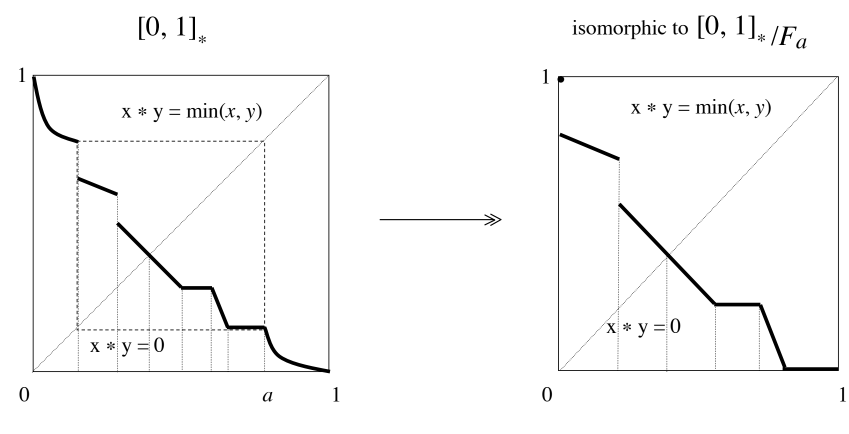

Figure 3: A WNM t-norm with a finite partition such that $I_1 = \{1\}$ (left) and its corresponding t-norm of the quotient algebra $[0,1]_*/F_a$.

3 Adding truth-constants

In this section we introduce the basic definitions and first general results regarding the expansions with truth-constants for those extensions of MTL which are the logic of a given left-continuous t-norm.

Definition 21 (logic $\mathrm{L}_*(\mathcal{C})$) *Let $*$ be a left-continuous t-norm, and let $\mathcal{C} = \langle C, *, \Rightarrow, \min, \max, 0, 1\rangle \subseteq [0,1]_*$ be a* countable *subalgebra. Consider the expanded language $\mathcal{L}_C = \mathcal{L} \cup \{\overline{r} : r \in C \setminus \{0,1\}\}$ where we introduce a new constant for every element in $C \setminus \{0,1\}$. We define $\mathrm{L}_*(\mathcal{C})$ as the expansion of L_* in the language $\mathcal{L}_C$ obtained by adding the so-called* book-keeping axioms*:*

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r * s}$$
$$(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$$

for every $r, s \in C$.

Notice that in this definition the book-keeping axioms $\overline{r} \wedge \overline{s} \leftrightarrow \overline{\min\{r,s\}}$ that would correspond to the other primitive connective in MTL, $\wedge$, are not present, since they are easily derivable in $\mathrm{L}_*(\mathcal{C})$ as actually defined.

The algebraic counterparts of the $\mathrm{L}_*(\mathcal{C})$ logics are defined in the natural way.

Definition 22 *Let $*$ be a left-continuous t-norm and let $\mathcal{C}$ be a countable subalgebra of $[0,1]_*$. An $\mathrm{L}_*(\mathcal{C})$-algebra is a structure*

$$\mathcal{A} = \langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$$

such that:

1. $\langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ *is an L_*-algebra, and*
2. *for every $r, s \in C$ the following identities hold:*

$$\overline{r}^{\mathcal{A}} \&^{\mathcal{A}} \overline{s}^{\mathcal{A}} = \overline{r * s}^{\mathcal{A}}$$
$$\overline{r}^{\mathcal{A}} \to^{\mathcal{A}} \overline{s}^{\mathcal{A}} = \overline{r \Rightarrow s}^{\mathcal{A}}.$$

The canonical standard $\mathrm{L}_*(\mathcal{C})$*-chain is the algebra*

$$[0,1]_{\mathrm{L}_*(\mathcal{C})} = \langle [0,1], *, \Rightarrow, \min, \max, \{r : r \in C\}\rangle,$$

i. e. the $\mathcal{L}_C$-expansion of $[0,1]_*$ where the truth-constants are interpreted by themselves.

Since the additional symbols added to the language are 0-ary, the condition of algebraizability given in the prelimininaries is trivially fulfilled. Therefore, $\mathrm{L}_*(\mathcal{C})$ is also an algebraizable logic and its equivalent algebraic semantics is the variety of $\mathrm{L}_*(\mathcal{C})$-algebras, denoted as $\mathbb{L}_*(\mathcal{C})$. In particular this means that the logics $\mathrm{L}_*(\mathcal{C})$ are strongly complete with respect to the variety of $\mathrm{L}_*(\mathcal{C})$-algebras. Furthermore, reasoning as in the MTL case, we can prove that all $\mathrm{L}_*(\mathcal{C})$-algebras are representable as a subdirect product of $\mathrm{L}_*(\mathcal{C})$-chains, hence we also have completeness of $\mathrm{L}_*(\mathcal{C})$ with respect to $\mathrm{L}_*(\mathcal{C})$-chains.

Theorem 23 *For any $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_C}$, $\Gamma \vdash_{\mathrm{L}_*(\mathcal{C})} \varphi$ if, and only if, $\{\psi \approx \overline{1} : \psi \in \Gamma\} \models_{\{\mathrm{L}_*(\mathcal{C})-chains\}} \varphi \approx \overline{1}$.*

This general completeness with respect to chains, can be refined by using [8, Lemma 3.4.4], where Cintula proves a very general result for expansions of fuzzy logics with rational truth-constants. Adapted to our framework, it reads as follows.

Theorem 24 ([8]) *Let $*$ be a left-continuous t-norm such that L_* is strongly complete with respect a class $\mathbb{K}$ of L_*-chains. Then $\mathrm{L}_*(\mathcal{C})$ is strongly complete with respect to the class of $\mathrm{L}_*(\mathcal{C})$-chains whose $\mathcal{L}$-reducts are in $\mathbb{K}$.*

Notice that when $\mathbb{K}$ is the class of all L_*-chains, then this theorem does not provide anything new other than the result of Theorem 23. If $\mathbb{K}$ is the class of standard L_*-chains, the condition that L_* should be strongly complete is very demanding. For instance if we restrict ourselves to continuous t-norm based logics, then only Gödel logic G satisfies this condition (SSC). If we consider logics of genuine left-continuous t-norms, then so far we can only additionally consider the NM logic and some WNM logics (see previous section).

Since all the logics $\mathrm{L}_*(\mathcal{C})$ are expansions of MTL, sharing Modus Ponens as the only inference rule, they have the same local deduction-detachment theorem as MTL has. In fact, the proof for MTL or BL also applies here.

Theorem 25 *For every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}_C}$, $\Gamma, \varphi \vdash_{\mathrm{L}_*(\mathcal{C})} \psi$ if, and only if, there is a natural $k \geq 1$ such that $\Gamma \vdash_{\mathrm{L}_*(\mathcal{C})} \varphi^k \to \psi$.*

One can also show the following general result about the conservativity of $\mathrm{L}_*(\mathcal{C})$ w.r.t. L_*.

Proposition 26 *$\mathrm{L}_*(\mathcal{C})$ is a conservative expansion of L_*.*

Proof: Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ be arbitrary formulae and suppose that $\Gamma \vdash_{\mathrm{L}_*(\mathcal{C})} \varphi$. Then, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}_*(\mathcal{C})} \varphi$. By the above deduction theorem, there exists a natural k such that $\vdash_{\mathrm{L}_*(\mathcal{C})} (\Gamma_0)^k \to \varphi$, identifying the set Γ_0 with the strong conjunction of all its formulae. By soundness, this implies that $\models_{[0,1]_{\mathrm{L}_*(\mathcal{C})}} (\Gamma_0)^k \to \varphi$. Since the new truth-constants do not occur in $\Gamma_0 \cup \{\varphi\}$, we have $\models_{[0,1]_*} (\Gamma_0)^k \to \varphi$, and by SC of L_*, $\vdash_{\mathrm{L}_*} (\Gamma_0)^k \to \varphi$, and hence $\Gamma \vdash_{\mathrm{L}_*} \varphi$ as well. □

In the rest of the paper we will study the SC, FSSC and SSC properties for the logics with truth-constants $\mathrm{L}_*(\mathcal{C})$, and also canonical standard completeness properties, i.e. SC, FSSC and SSC restricted to the canonical standard algebra.

4 Structure of $\mathrm{L}_*(\mathcal{C})$-chains

We have seen in Theorem 25 that the logics $\mathrm{L}_*(\mathcal{C})$ are complete with respect to the $\mathrm{L}_*(\mathcal{C})$-chains. To study standard completeness results for $\mathrm{L}_*(\mathcal{C})$ we need to get a deeper insight into $\mathrm{L}_*(\mathcal{C})$-chains. This is done in this section.

Next we assume $*$ is a left-continuous t-norm and $\mathcal{C}$ is a countable subalgebra of $[0,1]_*$.

Lemma 27 *For any $\mathrm{L}_*(\mathcal{C})$-chain $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$, let $F_{\mathcal{C}}(\mathcal{A}) = \{r \in C : \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}\}$ and $\overline{F_{\mathcal{C}}(\mathcal{A})} = \{r \in C : \neg r \in F_{\mathcal{C}}(\mathcal{A})\}$. Then:*

(i) $F_{\mathcal{C}}(\mathcal{A})$ is a filter of $\mathcal{C}$.

(ii) The set $\{\overline{r}^{\mathcal{A}} : r \in C\}$ forms an L_*-subalgebra of $\mathcal{A}$ isomorphic to $\mathcal{C}/F_{\mathcal{C}}(\mathcal{A})$, through the mapping $\overline{r}^{\mathcal{A}} \mapsto [r]_{\mathcal{A}}$, in such a way that

$$[1]_{\mathcal{A}} = F_{\mathcal{C}}(\mathcal{A}) \text{ and } [0]_{\mathcal{A}} = \overline{F_{\mathcal{C}}(\mathcal{A})},$$

where $[r]_{\mathcal{A}}$ denotes the equivalence class of $r \in C$ w.r.t. to the congruence defined by the filter $F_{\mathcal{C}}(\mathcal{A})$.

Proof: (i) If $r \in F_{\mathcal{C}}(\mathcal{A})$ and $s \in C$ with $s > r$, then $s \in F_{\mathcal{C}}(\mathcal{A})$ because by the book-keeping axioms we have $\overline{s}^{\mathcal{A}} = \overline{\max(r,s)}^{\mathcal{A}} = \overline{r}^{\mathcal{A}} \vee \overline{s}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$. Moreover if $r, s \in F_{\mathcal{C}}(\mathcal{A})$ then $r * s \in F_{\mathcal{C}}(\mathcal{A})$ since $\overline{r * s}^{\mathcal{A}} = \overline{r}^{\mathcal{A}} \& \overline{s}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$. Therefore $F_{\mathcal{C}}(\mathcal{A})$ is a filter.
(ii) An easy computation shows that $\overline{s}^{\mathcal{A}} = \overline{r}^{\mathcal{A}}$ iff $\overline{(r \Rightarrow s) * (s \Rightarrow r)}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$, i.e. elements of the same class have to be interpreted by the same element of A while elements of different classes have to be interpreted by different elements of A. □

In general, the equivalence classes of $\mathcal{C}$ with respect to a filter F, i.e. the elements of $\mathcal{C}/F$, are difficult to describe, but some interesting cases can be indeed fully described. The next lemma refers to these cases.

Lemma 28 *Let $* \in$ **CONT** $\cup$ **WNM** and let $\mathcal{C}$ be a (countable) subalgebra of $[0,1]_*$. For any $F \in Fi(\mathcal{C})$ we have:*

(i) for any $r, s \notin F \cup \overline{F}$, $[r]_F = [s]_F$ iff $r = s$;

*(ii) moreover, if $* \in$ **CONT** then $\overline{F} = \{0\}$.*

Proof: For $*$ being a continuous t-norm the proofs of (i) and (ii) can be found in [13]. If $*$ is a WNM t-norm the proof of (i) is an easy generalization of the proof for NM and some particular WNM t-norm logics given in [17]. □

This lemma shows that the interpretation of the constants over a $\mathrm{L}_*(\mathcal{C})$-chain $\mathcal{A}$ depends only on the filter $F_{\mathcal{C}}(\mathcal{A})$. Indeed, if $i : C \to \{\overline{r}^{\mathcal{A}} : r \in C\}$ denotes that interpretation, i.e. $i(r) = \overline{r}^{\mathcal{A}}$ for all $r \in C$, then i maps truth-values r to $\overline{1}^{\mathcal{A}}$ or $\overline{0}^{\mathcal{A}}$ depending on whether $r \in F_{\mathcal{C}}(\mathcal{A})$ or $r \in \overline{F_{\mathcal{C}}(\mathcal{A})}$ respectively, and over the rest of the elements of C, i.e. those in $C \setminus (F_{\mathcal{C}}(\mathcal{A}) \cup \overline{F_{\mathcal{C}}(\mathcal{A})})$, i is a one-to-one mapping.

The standard chains of the variety $\mathbb{L}_*(\mathcal{C})$, i.e. the $\mathrm{L}_*(\mathcal{C})$-algebras over $[0,1]$, are the key to get standard completeness results for the logic $\mathrm{L}_*(\mathcal{C})$ when using the technique of partially embedding $\mathrm{L}_*(\mathcal{C})$-chains into standard ones. In order to know when such embeddings are possible, it is necessary to study the standard $\mathrm{L}_*(\mathcal{C})$-chains in more detail. This question is in fact related to describe the ways the truth-constants from C can be interpreted in $[0,1]$ respecting the book-keeping axioms. We have seen in Lemmas 27 and 28 some necessary conditions showing the preeminent role the set $Fi(\mathcal{C})$ of proper filters of $\mathcal{C}$ plays in this question. Observe that each proper filter of $\mathcal{C}$ is either of type $F_a = \{x \in C : x \geq a\}$ or of type $F_{>a} = \{x \in C : x > a\}$ for some $a \in C$.

One can wonder whether, given a filter $F \in Fi(\mathcal{C})$, there always exists a standard $\mathrm{L}_*(\mathcal{C})$-chain $\mathcal{A}$ such that $F_{\mathcal{C}}(\mathcal{A}) = F$. Obviously, the simplest thing to look at is whether the algebra

$$[0,1]^F_{\mathrm{L}_*(\mathcal{C})} = \langle [0,1], *, \Rightarrow_*, \min, \max, \{i_F(r) : r \in C\}\rangle,$$

where the mapping $i_F : C \to [0,1]$ is defined as

$$i_F(r) = \begin{cases} 1, & \text{if } r \in F \\ 0, & \text{if } r \in \overline{F} \\ r, & \text{otherwise} \end{cases} \tag{1}$$

is always an $\mathrm{L}_*(\mathcal{C})$-algebra over $[0,1]_*$, or in other words, whether the mapping i_F is always a proper interpretation of the truth-constants, in the sense of satisfying the book-keeping axioms.

It is easy to check that this is actually the case when $* \in$ **CONT**, and in such a case $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ will be called *standard algebra of type* F. Moreover, when $* \in$ **CONT-fin** does not contain any Gödel component and $\mathcal{C}$ *covers* all components of $*$, one can show that these are all the standard chains one can define, in the sense that there are as many $\mathrm{L}_*(\mathcal{C})$-algebras over $[0,1]_*$ (up to isomorphism) as proper filters of $\mathcal{C}$.

Proposition 29 ([13]) *Let* $* \in$ **CONT-fin** *be such that it does not contain any Gödel component, and let* $\mathcal{C} \subseteq [0,1]_*$ *be a countable subalgebra such that each component of* $[0,1]_*$ *contains at least one value of* C *different from the bounds of the component. Then:*

(i) For any $F \in Fi(\mathcal{C})$*, the algebra* $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ *is an* $\mathrm{L}_*(\mathcal{C})$*-algebra. Conversely, any standard* $\mathrm{L}_*(\mathcal{C})$*-chain whose* $\mathcal{L}$*-reduct is* $[0,1]_*$ *is (up to isomorphism) an algebra* $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$*, for some* $F \in Fi(\mathcal{C})$*.*

(ii) Let $X = \{[\mathcal{A}] : \mathcal{A}$ *standard* $\mathrm{L}_*(\mathcal{C})$*-algebra over* $[0,1]_*\}$ *be the set of isomorphism classes of* $\mathrm{L}_*(\mathcal{C})$*-algebras over* $[0,1]_*$*. Then, the function* $\Phi : X \to Fi(\mathcal{C})$ *defined by* $\Phi([\mathcal{A}]) = F_{\mathcal{C}}(\mathcal{A})$ *for every* $[\mathcal{A}] \in X$ *is a bijection.*

The case of $\mathrm{L}_*(\mathcal{C})$ logics when $* \in$ **WNM-fin** is not so simple. We illustrate the problem with an example. Let $*$ be the WNM t-norm depicted in the left hand side of Figure 3 and take $C = \mathbb{Q} \cap [0,1]$. Let a be a positive involutive element such that $I^*_a \neq \{a\}$ and let F_a be the principal filter generated by a. Then the mapping $i_{F_a} : C \to [0,1]$, defined as in expression (1), is not a proper interpretation of the truth-constants since for each $b \in I^*_a$, $\neg i(b) = \neg b = \neg a$ and $i(\neg b) = i(\neg a) = 0$, i.e. the book-keeping axioms are not satisfied and hence the algebra $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ is not an $\mathrm{L}_*(\mathcal{C})$-algebra. Thus the mapping (1) used to interpret the truth-constants in the case of continuous t-norms does not always work in the case of a WNM t-norm.

In fact, for the case $* \in$ **WNM-fin**, if we want to associate to each filter $F \in Fi(\mathcal{C})$ a standard chain of $\mathbb{L}_*(\mathcal{C})$ such that $F_{\mathcal{C}}(\mathcal{A}) = F$, we need to proceed in a different way. We will divide the job by cases.

1. If the classes of $\mathcal{C}/F$ satisfy the condition that $\neg[r]_F = [0]_F$ implies $[r]_F = [1]_F$, then the interpretation used in the case of continuous t-norms works well and the chain $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ is an $\mathrm{L}_*(\mathcal{C})$-chain like in the continuous case.

2. If in $\mathcal{C}/F$ there are classes such that

$$[r]_F \neq [1]_F \text{ (that is, } r \notin F\text{) and } \neg[r]_F = [0]_F,$$

then the mapping $i_F : C \to [0,1]$ defined by expression (1) is not, in general, an interpretation as the example above proves.

Thus in this case, we consider two further subcases:

(a) If $[0,1]_*$ is such that $I^*_1 \neq \{1\}$ (i.e. $\neg x = 0$ for some $x < 1$), then the mapping $i'_F : C \to [0,1]$ defined by,

$$i'_F(r) = \begin{cases} 1, & \text{if } r \in F \\ 0, & \text{if } r \in \overline{F} \\ f(r), & \text{if } \neg r = 0 \text{ and } r \notin (F \cup \overline{F}) \\ r, & \text{otherwise} \end{cases} \tag{2}$$

where $f : \{r \in C \mid \neg r = 0, r \notin (F \cup \overline{F})\} \to I^*_1$ is an (arbitrary) one-to-one increasing mapping, is an interpretation which satisfies the book-keeping axioms. Then the algebra

$$[0,1]^F_{\mathrm{L}_*(\mathcal{C})} := \langle [0,1], *, \Rightarrow_*, \min, \max, \{i'_F(r) : r \in C\}\rangle$$

is an $\mathrm{L}_*(\mathcal{C})$-chain over $[0,1]_*$.

(b) If $[0,1]_*$ is such that $I^*_1 = \{1\}$ (i.e. $\neg x = 0$ implies $x = 1$), then the mapping $i'_F : C \to [0,1]$ defined in the previous case does not apply here since having $I^*_1 = \{1\}$ makes impossible to define a one-to-one mapping f as required there. In this case we take as initial chain, not the standard chain $[0,1]_*$, but the chain $([0,1]_*)/F_a$ (which still belongs to the variety $\mathbb{L}_*$) where $a \in C$ is the greatest element in the constant intervals of $[0,1]_*$. Notice that $[1]_{F_a} = [a,1]$, $[0]_{F_a} = [0,\neg a]$ and $[r]_{F_a} = \{r\}$ for any $r \in (\neg a, a)$. Hence, $([0,1]_*)/F_a$ is isomorphic to an L_*-chain $[\neg a, a]_{*'}$ by identifying $[1]_{F_a}$ with a, $[0]_{F_a}$ with $\neg a$, and $[r]_{F_a}$ with r for all $r \in (\neg a, a)$, and by taking $*'$ as the obvious adaptation to the interval $[\neg a, a]$ of the original $*$. Now it is clear that $[\neg a, a]_{*'}$ is such that $I^{*'}_1 \neq \{1\}$ and therefore we can define a mapping $i''_F : C \to [\neg a, a]$ analagously to (2) which makes the algebra

$$\langle [\neg a, a], *', \Rightarrow_{*'}, \min, \max, \{i''_F(r) : r \in C\}\rangle$$

an $\mathrm{L}_*(\mathcal{C})$-chain. Finally, by means of an increasing linear transformation $h : [\neg a, a] \to [0,1]$, it is easy to get an isomorphic $\mathrm{L}_*(\mathcal{C})$-chain over $[0,1]$

$$[0,1]^F_{\mathrm{L}_*(\mathcal{C})} := \langle [0,1], \circ, \Rightarrow_\circ, \min, \max, \{j_F(r) : r \in C\}\rangle$$

where $x \circ y = h(h^{-1}(x) *' h^{-1}(y))$ and $j_F(r) = h(i''_F(r))$ for all $r \in C$. Notice that $\circ$ needs not coincide with $*$.

Notice that the algebra $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ built in case (a) and in case (b) is not univocally defined since its definition depends on the choice of some mappings, but all possible choices would yield isomorphic algebras.

Thus, we have the following corollary.

Corollary 30 *Let* $* \in$ **CONT-fin**$\cup$**WNM-fin** *and let* $\mathcal{C}$ *be a countable subalgebra of* $[0,1]_*$*. Then, for any filter* $F \in Fi(\mathcal{C})$*, there exists a standard* $\mathrm{L}_*(\mathcal{C})$*-chain* $\mathcal{A}$ *such that* $F_{\mathcal{C}}(\mathcal{A}) = F$*, namely* $\mathcal{A} = [0,1]^F_{\mathrm{L}_*(\mathcal{C})}$.

Any standard $\mathrm{L}_*(\mathcal{C})$-chain $\mathcal{A}$ such that $F_{\mathcal{C}}(\mathcal{A}) = F$ will be called from now on *standard* $\mathrm{L}_*(\mathcal{C})$*-chain of type* F.

5 Completeness results

In this section we will give completeness results for the logics $\mathrm{L}_*(\mathcal{C})$ in the following particular cases:

1. When $* \in$ **CONT-fin** and $\mathcal{C}$ is a countable subalgebra of $[0,1]_*$ such that C has elements in the interior of each component of the t-norm $*$, and in addition every $r \in C$ belonging to a Łukasiewicz component generates a finite MV-chain.

2. When $* \in$ **WNM-fin** and $\mathcal{C}$ is a countable subalgebra of $[0,1]_*$ such that has elements in the interior of each interval of the partition.

Thus, from now on we will assume that the algebra $\mathcal{C}$ satisfies these conditions.

In the following subsection we will focus on strong and finite strong standard completeness results while in the second subsection we will focus on the issue of canonical standard completeness.

5.1 About SSC and FSSC results

We start with a general result on strong standard completeness when $* \in$ **WNM-fin** which is a consequence of Cintula's Theorem 24 and the SSC results given in statements 2 and 3 of Theorem 20.

Theorem 31 *For every* $* \in$ **WNM-fin** *and every suitable* $\mathcal{C}$*, the logic* $\mathrm{L}_*(\mathcal{C})$ *enjoys the SSC restricted to the family* $\{[0,1]^F_{\mathrm{L}_*(\mathcal{C})} : F \in Fi(\mathcal{C})\}$.

As particular cases of the above theorem we obtain that the logics $\mathrm{G}(\mathcal{C})$ and $\mathrm{NM}(\mathcal{C})$ enjoy the SSC restricted to the corresponding family $\{[0,1]^F_{\mathrm{L}_*(\mathcal{C})} : F \in Fi(\mathcal{C})\}$.

Notice that these results can never be improved to canonical SSC, as the following example shows.

Example 1 *For every non-trivial filter F (that exists in all these cases) and every $r \in F \setminus \{1\}$, the derivation*

$$(p \to q) \to \overline{r} \models q \to p$$

is valid in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ *but not in* $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$.

Observe that by Proposition 16 and Theorem 10 and being $\mathrm{L}_*(\mathcal{C})$ a conservative expansion of L_*, the SSC is false for the logics $\mathrm{L}_*(\mathcal{C})$ for each $* \in$ **CONT-fin** when $* \neq \min$.

Since there is no result relating the FSSC for logics without truth-constants to the FSSC for the corresponding expanded logics with truth-constants, in order to study the FSSC we need to use the bridge result given in Theorem 13, i.e. we have to study partial embeddability property for algebras with truth-constants. Next we summarize some results which can be found in [17, 36, 13].

Definition 32 *The logic* $\mathrm{L}_*(\mathcal{C})$ *has the* partial embeddability property *if, and only if, for every filter* $F \in Fi(\mathcal{C})$ *and every subdirectly irreducible* $\mathrm{L}_*(\mathcal{C})$*-chain* $\mathcal{A}$ *of type* F*,* $\mathcal{A}$ *is partially embeddable into* $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$.

Obviously, the logics with truth-constants that enjoy the SSC restricted to the family of standard chains of type F, being F a proper filter of $\mathcal{C}$, enjoy the partial embeddability property as well. Thus in the next theorem we consider cases that in general do not enjoy the SSC.

Theorem 33 ([13]) *For every $* \in$ **CONT-fin** *and every suitable $\mathcal{C}$, the logic* $\mathrm{L}_*(\mathcal{C})$ *enjoys the partial embeddability property, and therefore it has the FSSC restricted to the family* $\{[0,1]^F_{\mathrm{L}_*(\mathcal{C})} : F \in Fi(\mathcal{C})\}$.

Notice that the following interesting cases are included in the previous theorem:

(1) $*$ is the product t-norm.

(2) $*$ is the Łukasiewicz t-norm and C is contained in the rationals of $[0,1]$.

The proofs for the expansion of Product logic can be found in [36] respectively. The proof for expansions of Łukasiewicz logic can be found in [22] and for the other cases of continuous t-norms in [13].

Observe that in the Łukasiewicz case the subalgebra of constants $\mathcal{C}$ has a unique proper filter $F = \{1\}$ and thus the logic enjoys the canonical FSSC. Moreover, Example 1 also shows that the rest of the logics do not enjoy the canonical FSSC. Furthermore, observe that in the case of the product t-norm the family of standard chains associated to proper filters of C contains only two chains (since any product chain has only two proper filters).

All these results are collected in Tables 3 and 4.

	$\mathrm{G}(\mathcal{C})$	$\Pi(\mathcal{C})$	$Ł(\mathcal{C})$	$\mathrm{L}_*(\mathcal{C})$, for other $* \in$ **CONT-fin**
SC	Yes	Yes	Yes	Yes
FSSC	Yes	Yes	Yes	Yes
SSC	Yes	No	No	No
Canonical FSSC	No	No	Yes	No
Canonical SSC	No	No	No	No

Table 3: (Finite) strong standard completeness results for logics of a t-norm from **CONT-fin**.

	$\mathrm{G}(\mathcal{C})$	$\mathrm{NM}(\mathcal{C})$	$\mathrm{L}_*(\mathcal{C})$, for other $* \in$ **WNM-fin**
SC	Yes	Yes	Yes
FSSC	Yes	Yes	Yes
SSC	Yes	Yes	Yes
Canonical FSSC	No	No	No
Canonical SSC	No	No	No

Table 4: (Finite) strong standard completeness results for logics of a t-norm from **WNM-fin**.

5.2 About canonical standard completeness

From the results of the last sections, we already know that all the considered logics enjoy the SC restricted to the family of standard chains associated to proper filters of $\mathcal{C}$, i.e, their theorems are exactly the common tautologies of the chains of the family $\{[0,1]^F_{\mathrm{L}_*(\mathcal{C})} : F \in Fi(\mathcal{C})\}$. But although the logics considered in the last sections have not in general the canonical SSC or the canonical FSSC (only $Ł(\mathcal{C})$ enjoys it when $C \subseteq \mathbb{Q} \cap [0,1]$), some of them still enjoy the canonical SC, i. e. their theorems are exactly the tautologies of their corresponding canonical standard algebra. In order to prove it, we need to show that tautologies of the canonical standard chain are a subset of the tautologies of each one of the standard chains associated to each proper filter of $\mathcal{C}$. We will study this problem by cases in next subsections.

5.2.1 The case of continuous t-norms

For the case of expansions of Łukasiewicz logic with truth-constants, since $Ł(\mathcal{C})$ has the canonical FSSC it trivially has the canonical SC as well.

Theorem 34 ([22]) *$Ł(\mathcal{C})$ has the canonical SC.*

For the case of the expansions of Gödel or product logics with truth-constants, in [17] and in [36] the following results were proved.

Theorem 35 ([17]) *$\mathrm{G}(\mathcal{C})$ has the canonical SC.*

Theorem 36 ([36]) *$\Pi(\mathcal{C})$ has the canonical SC.*

But the canonical SC is not valid in general for any logic $L_*(\mathcal{C})$ with $* \in$ **CONT-fin**. In [13] this is shown by providing counterexamples, i. e. by exhibiting in each case a suitable formula φ that is a tautology of the canonical standard algebra $[0,1]_{L_*(\mathcal{C})}$ but not of the algebra $[0,1]^F_{L_*(\mathcal{C})}$ for some proper filter F of $\mathcal{C}$. Suppose that the first component of $[0,1]_*$ is defined on the interval $[0,a]$.

1. If $[0,1]_* = [0,a]_{Ł} \oplus \mathcal{A}$ and $a \in C$, then an easy computation shows that the formula
$$\overline{a} \to (\neg\neg x \to x)$$
is valid in the canonical standard algebra but it is not valid in the standard chain $[0,1]^F_{L_*(\mathcal{C})}$ defined by the filter $F = [a,1] \cap C$ (where $\overline{a}$ is interpreted as 1).

2. If $[0,1]_* = [0,a]_{\Pi} \oplus \mathcal{A}$, take $b \in C \cap (0,a)$. Then an easy computation shows that the formula
$$\overline{b} \to \neg x \vee ((x \to x\&x) \to x)$$
is valid in the canonical standard algebra but it is not valid in the standard chain $[0,1]^F_{L_*(\mathcal{C})}$ defined by the filter $F = (0,1] \cap C$ (where $\overline{b}$ is interpreted as 1).

3. If $[0,1]_* = [0,a]_{G} \oplus \mathcal{A}$, take b as any element of $C \cap (0,a)$. Then the formula
$$\overline{b} \to (x \to x\&x)$$
is valid in the canonical standard algebra but it is not valid in the standard chain $[0,1]^F_{L_*(\mathcal{C})}$ defined by the filter $F = [b,1] \cap C$ (where $\overline{b}$ is interpreted as 1).

Observe that for a t-norm whose decomposition begins with two copies of Łukasiewicz t-norm, the idempotent element a separating them has to belong to the truth-constants subalgebra $\mathcal{C}$. Indeed, take into account that, by assumption, C must contain a non idempotent element c of the second component and for this element there exists a natural number n such that $c^n = a$ and thus $a \in C$. Hence this case is subsumed in the above first item.

The remaining cases (when the first component is Łukasiewicz but its upper bound a does not belong to C) are studied by cases:

(1) If $[0,1]_* = [0,a]_{Ł} \oplus [a,1]_{G}$ or $[0,1]_* = [0,a]_{Ł} \oplus [a,1]_{\Pi}$, then the logic $L_*(\mathcal{C})$ has the canonical SC. Actually, in that case the filters of $\mathcal{C}$ are the same as the filters of $C \cap [0,1]_{G}$ or $C \cap [0,1]_{\Pi}$ respectively, and thus a modified version of the proof of the canonical SC for $G(\mathcal{C})$ and $\Pi(\mathcal{C})$ applies (see [13] for the complete proofs).

(2) If $[0,1]_*$ is an ordinal sum of three or more components, then $L_*(\mathcal{C})$ has not the canonical SC as the following examples show:

2.1.- If $[0,1]_* = [0,a]_Ł \oplus [a,b]_G \oplus \mathcal{A}$, take $d \in F = (a,b] \cap C$ in the second component. Then the formula,

$$\overline{d} \to (\neg\neg x \to x) \vee (x \to x\&x)$$

is a tautology of the canonical standard algebra but not of $[0,1]^F_{Ł_*(\mathcal{C})}$.

2.2.- If $[0,1]_* = [0,a]_Ł \oplus [a,b]_\Pi \oplus \mathcal{A}$, take $d \in F = (a,b] \cap C$ in the second component. Then the formula,

$$\overline{d} \to (\neg\neg x\&\neg\neg y\&((x \to x\&y) \to y)\&(y \to x)\&(x \to x\&x) \to x)$$

is a tautology of the canonical standard algebra and not of $[0,1]^F_{Ł_*(\mathcal{C})}$.

Summarizing (see Table 5) the canonical SC holds for the expansion of the logic of a continuous t-norm $*$ which is a finite ordinal sum of the three basic ones by a set of truth-constants if, and only if, $[0,1]_*$ is either one of the three basic algebras ($[0,1]_Ł$, $[0,1]_G$ or $[0,1]_\Pi$) or $[0,1]_* = [0,a]_Ł \oplus [a,1]_\Pi$ or $[0,1]_* = [0,a]_Ł \oplus [a,1]_G$ (with $a \notin C$).

5.2.2 The case of WNM-fin t-norms

The question canonical SC for **WNM-fin** t-norms is fully solved. Some cases have been proved to be canonical standard complete and in the other cases we provide a counterexample showing that they are not canonical standard complete. In fact in [17] it is proved that the expansions of Gödel logic, NM logic and the logics corresponding to the t-norms $\otimes_c$ and $\star_c$ depicted in Figure 2 enjoy the canonical SC[6]. Here we give a new unified and simpler proof.

Theorem 37 *If* $* \in$ **WNM-fin** *such that its negation on the set of positive elements is either both involutive and continuous, or is identically* 0*, then* $Ł_*(\mathcal{C})$ *enjoys the canonical SC*

Proof: Suppose φ is a tautology with respect to $[0,1]_{Ł_*(\mathcal{C})}$. We will prove that φ is also a tautology with respect to $[0,1]^F_{Ł_*(\mathcal{C})}$ for each $F \in Fi(\mathcal{C})$, which implies that $\vdash_{Ł_*(\mathcal{C})} \varphi$. Let e be an interpretation over the chain $[0,1]^F_{Ł_*(\mathcal{C})}$. Suppose that $\mathcal{A}$ is the finite algebra generated by $\{e(\psi) \mid \psi$ subformula of $\varphi\}$ and $\alpha = \min\{r \in F \mid \overline{r}$ occurs in $\varphi\}$. Suppose that $f : (\neg\alpha, \alpha) \to (0,1)$ is a bijection such that $f(r) = r$ for all $r \notin F \cup \overline{F}$ such that $\overline{r}$ in φ and f is a homomorphism from $\mathcal{A}$ to the canonical standard chain. Then define an evaluation e' on the canonical standard chain defined by $e'(p) = f^{-1}(e(p))$ if p is a propositional variable that appears in φ and $e'(p) = 1$ otherwise. Since φ is a tautology for the canonical standard chain, $e'(\varphi) = 1$. Take the algebra $[0,1]_*/F_\alpha$ where F_α is the principal filter generated by α. By hypothesis this algebra is isomorphic to $[0,1]_*$. Define the evaluation e'' on the quotient algebra obtained from e' and it obviously satisfies $e''(\varphi) = [1]_{F_\alpha}$. But a simple computation shows that the algebra $\mathcal{B}$ generated by $\{e''(\psi) \mid \psi$ subformula of $\varphi\}$ is isomorphic to $\mathcal{A}$ and $e''(\varphi)$ over the quotient algebra corresponds to $e(\varphi)$ over the chain $[0,1]^F_{Ł_*(\mathcal{C})}$ and thus $e(\varphi) = 1$. □

[6] In [17] it is wrongly claimed that the expansions $Ł_*(\mathcal{C})$ for $* = \odot_c$ (see Figure 2) were also canonical standard complete, in Example 2 we provide a counter-example.

Actually, the only expansions of logics L_* with $* \in$ **WNM-fin** that enjoy the canonical SC are those falling under the hypotheses of last theorem. This is proved below by showing counterexamples for the remaining cases, where $p(x)$ and $n(x)$ denote the terms $x \vee \neg x$ and $x \wedge \neg x$ respectively.

Example 2 *Let* $* \in$ **WNM-fin** *not falling under the hypotheses of last theorem. We distinguish the following three cases:*

- *Suppose the negation is continuous on the set of positive elements and the only constant interval formed by positive elements is* I_1*. In such a case, there is an interval* I *of involutive positive elements, followed by* I_1*. Take a truth-constant* b *in the interior of* I*. Then the formula,*

$$(\neg\neg p(x) \to p(x)) \vee (\bar{b} \to p(x))$$

  is a tautology for $[0,1]_{L_*(\mathcal{C})}$ *and it is not a tautology for* $[0,1]^F_{L_*(\mathcal{C})}$ *for any* F *containing* b*. Take into account that in* $[0,1]_{L_*(C)}$ *a positive element is either involutive or greater than* b*.*

- *Suppose the negation is continuous on the set of positive elements and there is some constant interval formed by positive elements different from* I_1 *(this is the case of the family of t-norms* $\odot_c$ *in Figure 2). Let* b *be the minimum involutive positive element with a non-trivial associated interval. Then the formula,*

$$(\neg\neg p(x) \to p(x)) \vee (\neg p(x) \to \neg\bar{b})$$

  is a tautology for $[0,1]_{L_*(\mathcal{C})}$ *and it is not a tautology for* $[0,1]^F_{L_*(\mathcal{C})}$ *for any* F *containing* b*. Notice that in this case* $[0,1]^F_{L_*(C)}$ *is such that either a positive element is involutive or its negation is not greater than* $\neg b$*.*

- *Suppose the negation is continuous on the set of positive elements. Let* b *be the minimum discontinuity point of the negation function in the set of positive elements. Then* $I_{\neg b}$ *is the greatest constant interval in the negative part with biggest element* $\neg b$ *and not containing the fixpoint. Then take*

$$(\neg\neg n(x) \to n(x)) \vee (\neg n(x) \to \neg\neg n(x)) \vee (n(x) \to \neg\bar{b})$$

  is a tautology for $[0,1]_{L_*(\mathcal{C})}$ *and it is not a tautology for* $[0,1]^F_{L_*(\mathcal{C})}$ *for any* F *containing* b*. Notice that in* $[0,1]_{L_*(\mathcal{C})}$ *a negative element is either involutive or belongs to a constant interval whose greatest element is the fixpoint (if it exists) or it is less or equal than* $\neg b$*.*

These three examples prove that a rather large family of expansions of the logic of a t-norm from **WNM-fin** with truth-constants do not enjoy the canonical SC. In fact, only the following cases, proved to enjoy the canonical SC in [17], are not included in the previous examples:

- when the set of positive elements define an involutive interval of the partition (NM, $\star_c$ of Figure 2).
- when the set of positive elements define a constant interval of the partition (G, $\otimes_c$ of Figure 2).

All the results about canonical SC are gathered in Table 5.

$[0,1]_*$		Canonical SC for $L_*(\mathcal{C})$
$[0,1]_Ł$		Yes
$[0,1]_G$		Yes
$[0,1]_\Pi$		Yes
$[0,a]_G \oplus \mathcal{A}$		No
$[0,a]_\Pi \oplus \mathcal{A}$		No
$[0,a]_Ł \oplus \mathcal{A}$,	$a \in C$	No
$[0,a]_Ł \oplus [a,1]_G$,	$a \notin C$	Yes
$[0,a]_Ł \oplus [a,1]_\Pi$,	$a \notin C$	Yes
$[0,a]_Ł \oplus [a,b]_G \oplus \mathcal{A}$,	$a \notin C$	No
$[0,a]_Ł \oplus [a,b]_\Pi \oplus \mathcal{A}$,	$a \notin C$	No
$[0,1]_{NM}$		Yes
$[0,1]_{\otimes_c}$		Yes
$[0,1]_{\star_c}$		Yes
$[0,1]_*$, for other $* \in$ **WNM-fin**		No

Table 5: Canonical standard completeness results for logics $L_*(\mathcal{C})$ when $* \in$ **CONT-fin** $\cup$ **WNM-fin**. Recall that $\otimes_c$, and $\star_c$ are those WNM t-norms depicted in Figure 2.

6 Completeness results for evaluated formulae

This section deals with completeness results when we restrict to what we call *evaluated formulae*, formulae of type $\overline{r} \to \varphi$, where φ is a formula without new truth-constants (different from 0 and 1). These formulae can be seen as a special kind of Novák's *evaluated formulae*, which are expressions a/A where a is a truth value (from a given algebra) and A is a formula that may contain truth-constants again, and whose interpretation is that the truth-value of A is at least a. Hence our formulae $\overline{r} \to \varphi$ would be expressed as r/φ in Novák's evaluated syntax. On the other hand, formulae $\overline{r} \to \varphi$ when φ is a Horn-like rule of the form $b_1\&...\&b_n \to h$ also correspond to typical fuzzy logic programming rules $(b_1\&...\&b_n \to h, r)$, where r specifies a lower bound for the validity of the rule.

From the previous sections we know that the FSSC is true for the expansion of L_* with a suitable subalgebra of truth-constants (not only for evaluated formulae), but the canonical FSSC is only true for expansions of Łukasiewicz logic. Restricting the language to evaluated formulae these results can be improved. To describe them we divide the subject by cases.

6.1 The case of continuous t-norms

Next theorem[7] states the canonical FSSC restricted to evaluated formulae for the expansions of Gödel and Product logics with truth-constants.

[7]The proof can be found in [17] for the case of $G(\mathcal{C})$ and in [36] for $\Pi(\mathcal{C})$.

Theorem 38 ([17, 36]) $\mathrm{G}(\mathcal{C})$ *and* $\Pi(\mathcal{C})$ *have the canonical FSSC if we restrict the language to evaluated formulae, i.e. for any finite index set* I *we have:*

- $\{\overline{r}_i \to \varphi_i\}_{i\in I} \vdash_{\mathrm{G}(\mathcal{C})} \overline{s} \to \psi$ *iff* $\{\overline{r}_i \to \varphi_i\}_{i\in I} \models_{[0,1]_{\mathrm{G}(\mathcal{C})}} \overline{s} \to \psi$.
- $\{\overline{r}_i \to \varphi_i\}_{i\in I} \vdash_{\Pi(\mathcal{C})} \overline{s} \to \psi$ *iff* $\{\overline{r}_i \to \varphi_i\}_{i\in I} \models_{[0,1]_{\Pi(\mathcal{C})}} \overline{s} \to \psi$.

where $\psi, \varphi_i \in Fm_{\mathcal{L}}$.

Now, following [13], we will study the canonical SC and the canonical FSSC restricted to evaluated formulae for other logics. Take any $* \in$ **CONT-fin** which is an ordinal sum of more than one basic component and suppose that the first component is defined on the interval $[0, a]$. In the following cases we can refute the canonical SC (and hence the canonical FSSC as well):

1. The first component of the t-norm $*$ is a copy of Łukasiewicz t-norm and $a \in C$.
2. The first component of the t-norm $*$ is a copy of product t-norm.
3. The first component of the t-norm $*$ is a copy of minimum t-norm.
4. There are more than two components and the second component is a copy of minimum t-norm.
5. There are more than two components and the second component is a copy of product t-norm.

Indeed, for all these cases we can use the same counterexample that was given in the previous section to show that the corresponding logics do not enjoy the canonical SC, because the counterexamples were actually evaluated formulae.

The following theorem deals with the remaining case of ordinal sums of two basic components. The case $[0, 1]_* = [0, a]_{\mathrm{Ł}} \oplus [a, 1]_{\mathrm{Ł}}$ is not considered here since in such a situation, under the working hypothesis that there exists $b \in (a, 1]$ such that $b \in C$, necessarily $a \in C$ as well.

Theorem 39 ([13]) *The restriction to evaluated formulae of the logic* $\mathrm{L}_*(\mathcal{C})$ *when either* $[0, 1]_* = [0, a]_{Ł} \oplus [a, 1]_{\mathrm{G}}$ *or* $[0, 1]_* = [0, a]_{Ł} \oplus [a, 1]_{\Pi}$ *and the minimum element of the second component does not belong to* C *has the canonical FSSC.*

All these results are summarized in Table 6, where interestingly enough it turns out that both standard completeness properties (SC and FSSC) restricted to evaluated formulae are equivalent, for each $* \in$ **CONT-fin**.

6.2 The case of WNM-fin t-norms

In this case, the only available results are those from [17] for evaluated formulae of the kind $\overline{r} \to \varphi$ where r is a positive constant. We will call them *positively evaluated formulae*. For general evaluated formulae there are no results so far.

Theorem 40 ([17]) *If $*$ is one of the three* WNM *t-norms depicted in Fig. 2 ($\otimes_c, \star_c, \odot_c$)then* $\mathrm{L}_*(\mathcal{C})$ *has the following canonical FSSC if we restrict the language to evaluated formulae:*

$$\{\overline{r}_i \to \varphi_i\}_{i\in I} \vdash_{\mathrm{L}_*(\mathcal{C})} \overline{s} \to \psi \text{ iff } \{\overline{r}_i \to \varphi_i\}_{i\in I} \models_{[0,1]_{\mathrm{L}_*(\mathcal{C})}} \overline{s} \to \psi.$$

where I is a finite index set, $\psi, \varphi_i \in Fm_{\mathcal{L}}$ and $r_i \in (c, 1]$.

Notice that in all these logics, the positive constants coincide with the interval $(c, 1] \cap C$, except for the logics corresponding to $\odot_c$ with $c > 1/2$ where the positive constants are those in $(1-c, 1] \cap C$. The case $\mathrm{L}_* = \mathrm{NM}$ appears above when $* = \star_{1/2} = \odot_{1/2}$, and then the condition for the constants is $r_i \in (\frac{1}{2}, 1]$).

For $* \in$ **WNM-fin** other than $\otimes_c, \star_c$ the canonical FSSC restricted to positively evaluated formulae does not hold as the following counterexamples show.

Example 3 *Let $* = \odot_c$ with $c > 1/2$. Let $r \in C$ such that $1 - c < r \leq c$. Then the semantical deduction*

$$\neg\neg p(x) \to p(x) \models \overline{r} \to p(x)$$

is valid in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ but not in $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ for any F containing r. Obviously, in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ any involutive and positive element is greater than r.

Example 4 *Let $* \in$* **WNM-fin** *be such that the first interval I of the partition associated to $*$ formed by positive elements is involutive and there is a constant interval on the right of it. In such a case, take a truth-constant r in the interior of I. Then the semantical deduction,*

$$(\neg\neg p(x) \to p(x)) \to p(x) \models \overline{r} \to p(x)$$

is valid in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ but not in $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ for any F containing r. Observe that in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ the premise is true if, and only if, $p(x)$ is not involutive or 1, and for these cases $p(x)$ is greater than r.

Example 5 *Let $* \in$* **WNM-fin** *such that the first interval of the partition associated to $*$ formed by positive elements is a constant interval with respect to the negation (I_c being c the biggest element of the interval). Additionally suppose that there is another interval of positive elements that is also a constant interval with respect to the negation. In such a case, take a truth-constant $r \in I_c$. Then the formula,*

$$\overline{r} \to \neg\neg p(x)$$

is a tautology for $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ and it is not a tautology for $[0,1]^F_{\mathrm{L}_*(\mathcal{C})}$ for any F containing r. Obviously in $[0,1]_{\mathrm{L}_*(\mathcal{C})}$ any involutive and positive element is greater than r.

Example 6 *Let $* \in$* **WNM-fin** *be such that there is a positive element which is a discontinuity point of the negation function. Then, due to symmetry of negation functions, there is a constant interval whose elements are negative and whose greatest element is not the fixpoint. Denote by I the greatest constant* *interval formed by negative elements whose greatest element is different from the fixpoint and take r as the greatest element of I, i.e. $I = I_r$. Then the semantical deduction,*

$$\left\{ \begin{array}{c} \neg\neg n(x) \rightarrow \neg(\neg\neg n(x) \rightarrow n(x)), \\ \neg n(x) \rightarrow \neg(\neg n(x) \rightarrow \neg\neg n(x)) \end{array} \right\} \models \neg\overline{r} \rightarrow \neg n(x)$$

is valid deduction in $[0,1]_{\mathrm{L}_*(C)}$ but it is not in $[0,1]^F_{\mathrm{L}_*(C)}$ for any F containing r. Observe that the first premise is true if, and only if, $n(x)$ is either not involutive or $n(x) = 0$ and the second premise is true if and only if $n(x)$ does not belong to a constant interval whose greatest element is the fixpoint. Thus, if x satisfies the premises, it is clear that $n(x)$ belongs to a constant interval which does not contain the fixpoint, thus it is less or equal to r, and hence the conclusion is also satisfied.

This four examples, as in the case of general SC studied in the last section, prove that a rather large family of expansions of the logic of a WNM t-norm with truth constants do not enjoy canonical FSSC even when we restrict the language to positively evaluated formulae.

The reader can see a summary of all these completeness results in Table 6. Notice that the canonical SC restricted to positively evaluated formulae remains an open problem when $* \in$ **WNM-fin** is not one the t-norms $\otimes_c$ or $\star_c$ in Figure 2. In fact, in the cases considered in Example 5, the canonical SC does not hold, but we still do not know whether it is true in other cases.

Furthermore, comparing this table with Table 5 we realise that for a logic $\mathrm{L}_*(\mathcal{C})$ where $* \in$ **CONT-fin** $\cup$ **WNM-fin** (except for the case which remains open), the canonical SC turns out to be equivalent to the canonical SC (and to the canonical FSSC) restricted to positively evaluated formulae.

Open problem: Are these equivalencies valid for wider classes of $\mathrm{L}_*(\mathcal{C})$ logics?

7 Adding truth-constants to expansions with Δ connective

For every left-continuous t-norm $*$, consider the logic $\mathrm{L}_{*\Delta}$, the expansion of the logic L_* by adding to the language the unary connective Δ as introduced in Section 2.

Since there is a one-to-one correspondence between L_*-chains and $\mathrm{L}_{*\Delta}$-chains, Theorem 13 leads to the next statement about the SSC and FSSC of logics $\mathrm{L}_{*\Delta}$.

Theorem 41 *For any left-continuous t-norm $*$, L_* has the SSC (resp. FSSC) with respect to a class of standard L_*-chains $\mathbb{K}$ if, and only if, $\mathrm{L}_{*\Delta}$ has the SSC (resp. FSSC) with respect to the class of standard $\mathrm{L}_{*\Delta}$-chains $\mathbb{K}_\Delta$, where $\mathbb{K}_\Delta$ denotes the class of Δ-expansions of chains in $\mathbb{K}$.*

Now we will consider expansions with truth-constants for these logics with Δ. Given a left-continuous t-norm $*$ and a countable subalgebra $\mathcal{C} \subseteq [0,1]_*$, we

	Restricted to pos. evaluated formulae of $Ł_*(\mathcal{C})$	
$[0,1]_*$	Canonical SC	Canonical FSSC
$[0,1]_Ł$	Yes	Yes
$[0,1]_G$	Yes	Yes
$[0,1]_\Pi$	Yes	Yes
$[0,a]_G \oplus \mathcal{A}$	No	No
$[0,a]_\Pi \oplus \mathcal{A}$	No	No
$[0,a]_Ł \oplus \mathcal{A}$, $a \in C$	No	No
$[0,a]_Ł \oplus [a,1]_G$, $a \notin C$	Yes	Yes
$[0,a]_Ł \oplus [a,1]_\Pi$, $a \notin C$	Yes	Yes
$[0,a]_Ł \oplus [a,b]_G \oplus \mathcal{A}$, $a \notin C$	No	No
$[0,a]_Ł \oplus [a,b]_\Pi \oplus \mathcal{A}$, $a \notin C$	No	No
$[0,1]_{NM}$	Yes	Yes
$[0,1]_{\otimes_c}$	Yes	Yes
$[0,1]_{\star_c}$	Yes	Yes
$[0,1]_*$, for other $* \in$ **WNM-fin**	?	No

Table 6: Canonical SC and FSSC results restricted to posiitvely evaluated formulae for logics $Ł_*(\mathcal{C})$ when $* \in$ **CONT-fin** $\cup$ **WNM-fin**.

define the logic $Ł_{*\Delta}(\mathcal{C})$ as the expansion of $Ł_{*\Delta}$ in the language $\mathcal{L}_C$ obtained by adding the following book-keeping axioms:

$$\begin{array}{l} \overline{r}\&\overline{s} \leftrightarrow \overline{r * s} \\ (\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s} \\ \Delta\overline{r} \leftrightarrow \overline{\Delta r} \end{array}$$

for every $r, s \in C$.

Again, using the general facts mentioned in the preliminaries we know that $Ł_{*\Delta}(\mathcal{C})$ is an algebraizable logic and we can axiomatize its equivalent algebraic semantics, the variety of $Ł_{*\Delta}(\mathcal{C})$-algebras. Moreover, it can be easily checked that $Ł_{*\Delta}(\mathcal{C})$-algebras are representable as subdirect product of chains.

Proposition 42 *For every left-continuous t-norm $*$ and every countable subalgebra $\mathcal{C} \subseteq [0,1]_*$, the logic $Ł_{*\Delta}(\mathcal{C})$ is a conservative expansion of $Ł_{*\Delta}$, whenever $Ł_{*\Delta}$ has the FSSC.*

Proof: Let us denote by $\mathbb{S}$ is the class of standard $Ł_{*\Delta}$-chains and by $\mathbb{S}(\mathcal{C})$ is the class of standard $Ł_{*\Delta}(\mathcal{C})$-chains. Let $\Gamma \cup \{\varphi\}$ be arbitrary formulae of $Ł_{*\Delta}$ and suppose that $\Gamma \vdash_{Ł_{*\Delta}(\mathcal{C})} \varphi$. Then, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{Ł_{*\Delta}(\mathcal{C})} \varphi$, and this implies that $\Gamma_0 \models_{\mathbb{S}(\mathcal{C})} \varphi$. Since the new truth-constants do not occur in $\Gamma_0 \cup \{\varphi\}$, we have $\Gamma_0 \models_{\mathbb{S}} \varphi$, and by FSSC of $Ł_{*\Delta}$, $\Gamma_0 \vdash_{Ł_{*\Delta}} \varphi$, and hence $\Gamma \vdash_{Ł_{*\Delta}} \varphi$. $\square$

Hence, for all $* \in$ **CONT** $\cup$ **WNM**, $Ł_{*\Delta}(\mathcal{C})$ is a conservative expansion of $Ł_{*\Delta}$.

Since $Ł_{*\Delta}$-chains are simple, adding Δ to $Ł_*(\mathcal{C})$-chains simplifies significantly their structure as next lemma shows.

Lemma 43 *Let $\mathcal{A}$ be a non-trivial* $\mathrm{L}_{*\Delta}(\mathcal{C})$*-chain, $*$ be a left-continuous t-norm and $\mathcal{C} \subseteq [0,1]_*$ be a countable subalgebra. Then, for every $r, s \in C$ such that $r < s$, we have $\overline{r}^{\mathcal{A}} < \overline{s}^{\mathcal{A}}$.*

Proof: Suppose $\overline{r}^{\mathcal{A}} = \overline{s}^{\mathcal{A}}$. Then, $\overline{1}^{\mathcal{A}} = \Delta\overline{1}^{\mathcal{A}} = \Delta\overline{s \to r}^{\mathcal{A}} = \overline{\Delta(s \to t)}^{\mathcal{A}} = \overline{0}^{\mathcal{A}}$; a contradiction. □

Therefore, in the variety of $\mathrm{L}_{*\Delta}(\mathcal{C})$-algebras all chains $\mathcal{A}$ are such that $F_{\mathcal{C}}(\mathcal{A}) = \{1\}$, among them the canonical standard chain that we denote by $[0,1]_{\mathrm{L}_{*\Delta}(\mathcal{C})}$. This has several nice consequences, which generalize the results for the continuous case given in [13] that can be proved in an analogous way.

Theorem 44 *Let $* \in$ **CONT-fin** $\cup$ **WNM-fin** and let $\mathcal{C} \subseteq [0,1]_*$ be a suitable countable subalgebra. Then:*

1. $\mathrm{L}_{*\Delta}(\mathcal{C})$ *has the canonical FSSC.*
2. $\mathrm{L}_{*\Delta}(\mathcal{C})$ *is a conservative expansion of* $\mathrm{L}_*(\mathcal{C})$ *iff* $\mathrm{L}_*(\mathcal{C})$ *has the canonical FSSC, i.e. iff $*$ is Łukasiewicz t-norm.*
3. $\mathrm{L}_{*\Delta}(\mathcal{C})$ *has the canonical SSC iff $* \in$* **WNM-fin**.

In Figure 4 we show which of the considered expansions of L_* are always conservative (the ones represented by bold arrows).

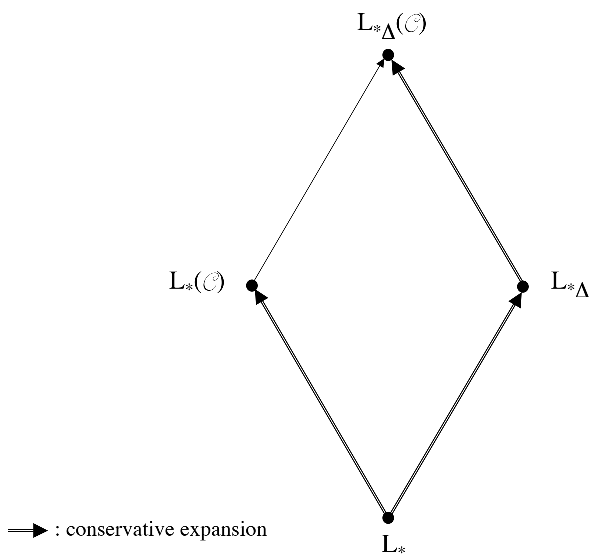

Figure 4: Diagram of expansions for $* \in$ **CONT-fin** $\cup$ **WNM-fin**.

8 Complexity results

In a recent paper [23], Hájek has studied the computational complexity of relevant subsets of formulae with rational truth-constants, i.e. formulae of the language $\mathcal{L}_C$ where $C = \mathbb{Q} \cap [0,1]$. His results will allow us to determine the computational complexity of some logics $\mathrm{L}_*(\mathcal{C})$ with $* \in$ **CONT-fin**.

In the following we will use $[0,1]^{\mathbb{Q}}$ to denote $\mathbb{Q} \cap [0,1]$. A left-continuous t-norm $*$ is called *r-admissible* when both $*$ and its residuum $\Rightarrow$ are closed operations on $[0,1]^{\mathbb{Q}}$. Notice that if $*$ is r-admissible, then $\mathcal{Q}_* = \langle [0,1]^{\mathbb{Q}}, *, \Rightarrow, \min, \max, 0, 1\rangle$ is a countable subalgebra of the standard algebra $[0,1]_*$, and hence it is meaningful to consider the logic $\mathrm{L}_*(\mathcal{Q}_*)$ and the canonical standard algebra $[0,1]_{\mathrm{L}_*(\mathcal{Q}_*)}$. To simplify a bit the notation we will denote the latter as $[0,1]_{\mathrm{L}_*(\mathcal{Q})}$.

We introduce the following three sets of formulae, namely the set of tautologies, the set of satisfiable formulae and the set of pairs of formulae in the semantical consequence relation, everything with respect to the canonical standard chain $[0,1]_{\mathrm{L}_*(\mathcal{Q})}$:

$$RTAUT(*) = \{\varphi \mid [0,1]_{\mathrm{L}_*(\mathcal{Q})} \models \varphi \approx \overline{1}\}$$

$$RSAT(*) = \{\varphi \mid [0,1]_{\mathrm{L}_*(\mathcal{Q})} \not\models \neg\varphi \approx \overline{1}\}$$

$$RSECON(*) = \{\langle \varphi, \psi\rangle \mid \varphi \approx \overline{1} \models_{[0,1]_{\mathrm{L}_*(\mathcal{Q})}} \psi \approx \overline{1}\}$$

Hájek's results in [23] can be summarized as follows. An r-admissible t-norm $* \in$ **CONT-fin** is called *strong r-admissible* when each Ł-component and Π-component is isomorphic to $[0,1]_{\text{Ł}}$ and $[0,1]_{\Pi}$ respectively via a bijection f mapping rationals into rationals such that both f and f^{-1} restricted to rationals are deterministically polynomially computable. Then, for a strong r-admissible t-norm $* \in$ **CONT-fin** with rational endpoints in all its basic components:

(i) when $[0,1]_*$ has no Π-component, $RTAUT(*)$ and $RSECON(*)$ are coNP-complete and $RSAT(*)$ is NP-complete;

(ii) otherwise, $RTAUT(*)$, $RSECON(*)$ and $RSAT(*)$ are in PSPACE.

Now, let us consider the set of theorems of $\mathrm{L}_*(\mathcal{Q}_*)$, the set of consistent formulae in $\mathrm{L}_*(\mathcal{Q}_*)$ and the set of pairs of formulae such that the second is derivable from the first in $\mathrm{L}_*(\mathcal{Q}_*)$:

$$RTHEO(*) = \{\varphi \mid \mathrm{L}_*(\mathcal{Q}) \vdash \varphi\}$$

$$RCONS(*) = \{\varphi \mid \varphi \nvdash_{\mathrm{L}_*(\mathcal{Q})} \overline{0}\}$$

$$RSYCON(*) = \{\langle \varphi, \psi\rangle \mid \varphi \vdash_{\mathrm{L}_*(\mathcal{Q})} \psi\}$$

Taking into account the canonical standard completeness results described in Section 5.2, we have the following cases for r-admissible t-norms $* \in$ **CONT-fin**:

1. $RTAUT(*) = RTHEO(*)$ and $RSAT(*) = RCONS(*)$ only when $[0,1]_*$ is isomorphic either to $[0,1]_{\text{Ł}}$, to $[0,1]_{\mathrm{G}}$, to $[0,1]_{\Pi}$, to $[0,a]_{\text{Ł}} \oplus [a,1]_{\mathrm{G}}$ (for $a \notin [0,1]^{\mathbb{Q}}$), or to $[0,a]_{\text{Ł}} \oplus [a,1]_{\Pi}$ (for $a \notin [0,1]^{\mathbb{Q}}$).

2. $RSECON(*) = RSYCON(*)$ only when $[0,1]_*$ is isomorphic to $[0,1]_{\text{Ł}}$.

Therefore, from the above discussion we can state the following complexity results.

Theorem 1 *Let* $* \in$ **CONT-fin** *be strong r-admissible. Then we have the following complexity results for* $\text{Ł}_*(\mathcal{Q}_*)$*:*

1. $RTHEO(*)$ *is coNP-complete and* $RCONS(*)$ *is NP-complete when* $[0,1]_*$ *is isomorphic to* $[0,1]_{Ł}$ *or to* $[0,1]_{\text{G}}$*;*
2. $RSYCON(*)$ *is coNP-complete when* $[0,1]_*$ *is isomorphic to* $[0,1]_{Ł}$*;*
3. $RTHEO(*)$*,* $RCONS(*)$ *and* $RSYCON(*)$ *are in PSPACE when* $[0,1]_*$ *is isomorphic to* $[0,1]_\Pi$*.*

Notice that, unfortunately, the cases $[0,a]_{\text{Ł}} \oplus [a,1]_{\text{G}}$ and $[0,a]_{\text{Ł}} \oplus [a,1]_\Pi$ are not covered by Hájek's results since the endpoint a must be assumed not belonging to $[0,1]^{\mathbb{Q}}$ in order to get canonical completeness.

Theorem 2 *Let* $* \in$ **WNM-fin** *be r-admissible such that all the endpoints of its partition and the negation fixpoint (if it exists) are rational. Then* $RTAUT(*)$ *and* $RSECON(*)$ *are coNP-complete and* $RSAT(*)$ *is NP-complete.*

Proof: The proof is a generalization of the one for $* = \min$ in [23, Theorem 2]. Given a formula φ, let $R(\varphi)$ be the universe of the WNM-subalgebra of $\mathcal{Q}_*$ generated by the set of truth-constants appearing in φ. It is clear that $R(\varphi)$ is finite. Let $Part(*)$ be the set $0 < s_1 < ... < s_m < 1$ of the endpoints of the partition associated to $*$ (including the negation fixpoint if it exists) and let $X = R(\varphi) \cup Part(*) = \{0 = t_0 < t_1 < ... < t_{k-1} < t_k = 1\}$ which forms another finite WNM-subalgebra of $\mathcal{Q}_*$. If φ contains n propositional variables, then we choose n rational elements $a_{i1}, ..., a_{in}$ in each open interval (t_i, t_{i+1}) such that $X \cup \bigcup_{i=0}^{k-1}\{a_{i1}, ..., a_{in}\}$ forms a WNM-subalgebra $\mathcal{A}_\varphi$ of $\mathcal{Q}_*$. Now, one can prove the following:

Claim: $\varphi \in RSAT(*)$ if and only if there exists an evaluation e on $\mathcal{A}_\varphi$ such that $e(\varphi) = 1$.

Proof: Let v be an evaluation on $[0,1]_{\text{Ł}_*(\mathcal{Q})}$ such that $v(\varphi) = 1$, and let $\mathcal{B}$ the $\text{Ł}_*(R(\varphi))$-algebra generated by the set $\{v(q) \mid q$ propositional variable in $\varphi\}$. Then one can check that $\mathcal{B}$ can be embedded in $\mathcal{A}_\varphi$. □

Therefore $\varphi \in RSAT(*)$ if and only if one can guess such an evaluation.

Analogously, one can prove that $\varphi \notin TAUT(*)$ iff one can guess an evaluation e on $\mathcal{A}_\varphi$ such that $e(\varphi) < 1$.

Finally, the case of checking $\langle \varphi, \psi \rangle \in RSECON(*)$ is reduced, due to the deduction theorem for WNM, to checking $\varphi^2 \to \psi \in TAUT(*)$. This ends the proof. □

Finally, taking into account the canonical standard completeness results for expansions of WNM logics (see Section 5.2), we can state the computational complexity of the following logics with rational truth-constants.

Theorem 3 *Let* $* \in$ **WNM-fin** *be r-admissible such that all the endpoints of its partition and the negation fixpoint (if it exists) are rational. For* L_* *being* G, NM *or* $*$ *belonging to one of three families depicted in Figure 2, we have that* $RTHEO(*)$ *and* $RSYCON(*)$ *are coNP-complete and* $RCONS(*)$ *is NP-complete.*

9 Conclusions

In the paper we have focused on an algebraic approach to study expansions of propositional logics of a left-continuous t-norm with truth-constants. Specially, we have surveyed in detail completeness results for the expansions of logics of left-continuous t-norms with a set of truth-constants $\{\overline{r} \mid r \in C\}$, for a suitable countable $C \subseteq [0,1]$, when (i) the t-norm is a finite ordinal sum of basic components or is WNM t-norm with finite partition, and (ii) the set of truth-constants *covers* all the unit interval in the sense that the interior of each basic component of the t-norm (in the case of continuous t-norms) or of each interval of the partition (in the case of the WNM t-norms) contains at least one value of C. From a practical point of view, this latter condition seems to correspond to the most interesting case for fuzzy logic-based systems, since they usually consider a set of truth values spread all over the real unit interval, and hence it is natural to assume there are elements of C in each component or partition of the t-norm.

Of course a lot of expansions with truth-contansts remain to be studied, among them:

- the case of a logic of a t-norm with a Łukasiewicz component containing some $r \in C$ which generates an infinite MV-chain (in other words, when r corresponds to an irrational value in the isomorphic copy of the component over $[0,1]$);
- the case when the set of truth-constants does not cover the unit interval;
- the case of continuous t-norms which are the ordinal sum of infinitely many components;
- the case of any other left-continuous t-norm, in particular WNM t-norms with infinite partition.

Another important issue to be addressed is the predicate calculi of these expanded logics. All these issues are matters for future research.

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