

RATIONAL COMPLETENESS RESULTS FOR PROMINENT PROPOSITIONAL FUZZY LOGICS WITH TRUTH-CONSTANTS

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Abstract

In this paper we consider expansions of Łukasiewicz, Product, Gödel and Nilpotent Minimum logics with truth-constants for an algebra of rational truth-values. We study the semantics for these logics given by chains defined over the rational unit interval and the completeness properties they provide, with a special attention to the completeness with respect to the canonical chain (i.e. the algebra where each truth-constant is interpreted in its corresponding truth-value).

Keywords: t-norm based fuzzy logics, truth-constants, rational semantics, completeness.

1 INTRODUCTION

Fuzzy logics are the logics corresponding to the notion of comparative truth, i.e. the paradigm where the classical truth-values are extended by adding intermediate ones which are comparable by using some total ordering. The expansions of fuzzy logics with truth-constants have been proposed as a means to deal explicitly in the language with the intermediate truth-values. They first appeared already in the 70s when Pavelka [13] introduced a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz logic by adding into the language a truth-constant \bar{r} for each real $r \in [0, 1]$, together with a number of additional

axioms. Hájek in [11] simplified in significant form Pavelka's system by showing that the same results could be obtained by adding just a truth-constant for each rational $r \in [0, 1]$. Expansions of other t-norm based fuzzy logics with countable sets of truth-values and their completeness properties with respect to the standard semantics over the real unit interval have been studied in [4, 6, 8, 9, 14]. Nevertheless, although the power of rational-chain semantics for fuzzy logics was shown in [3, 7, 2], it had not been considered for logics with truth-constants yet. Thus, in this paper we will study rational completeness properties for fuzzy logics with truth-constants.

2 PRELIMINARIES

The basic logic in this framework is the (propositional) Monoidal t-norm based logic MTL [5], with primitive connectives $\&$ (multiplicative conjunction), \rightarrow (implication), \wedge (additive conjunction) and the truth-constant $\bar{0}$. MTL is in fact the logic of left-continuous t-norms and their residua [12], in the sense that the set of its theorems is exactly $\bigcap\{Taut(*) \mid * \text{ is a left-continuous t-norm}\}$, where $Taut(*)$ denotes the set of tautologies when interpreting respectively $\&$, \rightarrow and \wedge by $*$, its residuum \Rightarrow and the min operation. In this setting, we denote by $[0, 1]_*$ the standard MTL-chain defined by the left-continuous t-norm $*$ and its residuum \Rightarrow , i. e. $[0, 1]_* = \langle [0, 1], *, \Rightarrow, \min, \max, 0, 1 \rangle$, and by L_* the axiomatic extension of MTL whose equivalent algebraic semantics is the variety generated by $[0, 1]_*$.

In this paper we will mainly focus on four prominent examples of these logics, namely Gödel (G),

Lukasiewicz (L), Product (Π), and Nilpotent Minimum (NM) logics, corresponding respectively to the cases when $*$ is the minimum t-norm, the Lukasiewicz t-norm, the product t-norm, or the nilpotent minimum t-norm (see [11, 5] for their axiomatics and further details).

Given a logic L_* and a class \mathbb{K} of L_* -chains, one defines three completeness properties:

- L_* has the property of *strong* \mathbb{K} -completeness, $S\mathbb{K}C$ for short, if for every set of formulae Γ and every formula φ , $\Gamma \vdash_{L_*} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- L_* has the property of *finite strong* \mathbb{K} -completeness, $FS\mathbb{K}C$ for short, if for every *finite* set of formulae Γ and every formula φ , $\Gamma \vdash_{L_*} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- L_* has the property of \mathbb{K} -completeness, $\mathbb{K}C$ for short, if for every formula φ , $\vdash_{L_*} \varphi$ iff $\models_{\mathbb{K}} \varphi$.

If \mathbb{K} is the class of all chains over the real unit interval $[0, 1]$ we use the notation \mathcal{RC} and call the properties *standard completeness*, while if it is the class of all chains over the rational unit interval $[0, 1]^{\mathbb{Q}} = [0, 1] \cap \mathbb{Q}$ we use the notation \mathcal{QC} . The standard and rational completeness properties for the above considered logics are in Table 1.

Theorem 2.1 ([2]). *Let $*$ be a left-continuous t-norm. If L_* has the FSRC, then it has the SQC.*

	L	Π	G	NM
\mathcal{RC}	Yes	Yes	Yes	Yes
FSRC	Yes	Yes	Yes	Yes
SRC	No	No	Yes	Yes
\mathcal{QC}	Yes	Yes	Yes	Yes
FSQC	Yes	Yes	Yes	Yes
SQC	Yes	Yes	Yes	Yes

Table 1: Standard and rational completeness properties for propositional fuzzy logics.

2.1 ADDING TRUTH-CONSTANTS

Given a left-continuous t-norm $*$ let \mathcal{C} be a *countable* subalgebra of $[0, 1]_*$. Then, $L_*(\mathcal{C})$ is the

propositional fuzzy logic defined as follows:

- the language of $L_*(\mathcal{C})$ is the one of L_* expanded with a new propositional constant \bar{r} for each $r \in \mathcal{C} \setminus \{0, 1\}$,
- the axioms and rules of $L_*(\mathcal{C})$ are those of L_* plus the book-keeping axioms:

$$\begin{aligned} \bar{r} \& \bar{s} &\leftrightarrow \overline{r * s} \\ (\bar{r} \rightarrow \bar{s}) &\leftrightarrow \overline{r \Rightarrow s} \end{aligned}$$

for each $r, s \in \mathcal{C}$.

Its algebraic counterpart, the $L_*(\mathcal{C})$ -algebras, are the expansions of L_* -algebras with nullary functions $\bar{r}^{\mathcal{A}}$ (one for each $r \in \mathcal{C}$) satisfying the book-keeping axioms, i.e. for every $r, s \in \mathcal{C}$ the following identities hold:

$$\begin{aligned} \bar{r}^{\mathcal{A}} \& \bar{s}^{\mathcal{A}} &= \overline{r * s^{\mathcal{A}}} \\ \bar{r}^{\mathcal{A}} \rightarrow \bar{s}^{\mathcal{A}} &= \overline{r \Rightarrow s^{\mathcal{A}}}. \end{aligned}$$

$L_*(\mathcal{C})$ -chains defined over the real unit interval $[0, 1]$ are called *standard*. Among them, there is one which reflects the intended standard semantics, the so-called *canonical standard* $L_*(\mathcal{C})$ -chain $[0, 1]_{L_*(\mathcal{C})}$ which is the standard chain where the truth-constants are interpreted by their defining values, i.e. one has $\bar{r}^{\mathcal{A}} = r$ for all $r \in \mathcal{C}$ whenever $\mathcal{A} = [0, 1]_{L_*(\mathcal{C})}$. It is worth to point out that for a logic $L_*(\mathcal{C})$ there may exist multiple standard chains as soon as there exist different ways of interpreting the truth-constants on $[0, 1]$ respecting the book-keeping axioms. Indeed, in the standard chains of L, Π, G and NM logics, the only possible interpretations are of the following type: for each proper filter F of \mathcal{C} (a non-empty upper subset closed under the t-norm and not containing 0), truth-constants admit the following interpretation:

$$\bar{r}^{\mathcal{A}} = \begin{cases} 1, & \text{if } r \in F \\ 0, & \text{if } \neg r \in F \\ r, & \text{otherwise} \end{cases}$$

The resulting standard chain is denoted $[0, 1]_{L_*(\mathcal{C})}^F$. In the case of L the only proper filter is the trivial one $\{1\}$. In the case of Π there are two: $(0, 1]$ and $\{1\}$. For G (resp. NM) there are many: $[c, 1]$ for

every $c > 0$ (resp. $c > \frac{1}{2}$) and $(c, 1]$ for every $c \geq 0$ (resp. $c \geq \frac{1}{2}$).

Completeness properties w.r.t. both the class of all standard chains and the canonical standard chain have been studied in the literature and solved for the logics we are considering (see Table 2). In the cases where the standard completeness fails one can try to improve the situation by restricting the logic to the so called *evaluated language*, i.e. formulae $\bar{r} \rightarrow \varphi$ where no additional truth-constant occurs in φ . In the case of NM one must require that the value r is *positive*, i.e. $r > \frac{1}{2}$. Table 3 summarizes these completeness results.

	$L(\mathcal{C})$	$\Pi(\mathcal{C})$	$G(\mathcal{C})$	$NM(\mathcal{C})$
\mathcal{RC}	Yes	Yes	Yes	Yes
\mathcal{FSRC}	Yes	Yes	Yes	Yes
\mathcal{SRC}	No	No	Yes	Yes
$\text{Can}\mathcal{RC}$	Yes	Yes	Yes	Yes
$\text{Can}\mathcal{FSRC}$	Yes	No	No	No
$\text{Can}\mathcal{SRC}$	No	No	No	No

Table 2: Standard completeness properties for propositional fuzzy logics with truth-constants.

	$L(\mathcal{C})$	$\Pi(\mathcal{C})$	$G(\mathcal{C})$	$NM(\mathcal{C})$
\mathcal{RC}	Yes	Yes	Yes	Yes
\mathcal{FSRC}	Yes	Yes	Yes	Yes
\mathcal{SRC}	No	No	Yes	Yes
$\text{Can}\mathcal{RC}$	Yes	Yes	Yes	Yes
$\text{Can}\mathcal{FSRC}$	Yes	Yes	Yes	Yes
$\text{Can}\mathcal{SRC}$	No	No	No	No

Table 3: Standard completeness properties for propositional fuzzy logics with truth-constants restricted to (positively) evaluated formulae.

3 NEW RESULTS: THE GENERAL CASE

Let $*$ be a left-continuous t-norm and \Rightarrow its residuum such that the rational unit interval $[0, 1]^{\mathbb{Q}}$ is closed under the operations $*$ and \Rightarrow . Let $[0, 1]_*^{\mathbb{Q}}$ be the L_* -chain defined by the restriction of $*$ and \Rightarrow to $[0, 1]^{\mathbb{Q}}$. Let \mathcal{C} be a countable subalgebra of $[0, 1]_*^{\mathbb{Q}}$ and consider the logic $L_*(\mathcal{C})$. Now

$L_*(\mathcal{C})$ -chains defined over the rational unit interval are called *rational chains* and among them, the one which reflects the intended rational semantics is the so-called *canonical rational* $L_*(\mathcal{C})$ -chain

$$[0, 1]_{L_*(\mathcal{C})}^{\mathbb{Q}} = \langle [0, 1], *, \Rightarrow, \min, \max, \langle r : r \in \mathcal{C} \rangle \rangle,$$

i.e. the rational chain over $[0, 1]_*^{\mathbb{Q}}$ where the truth-constants are interpreted by their defining values.

As already mentioned, in this paper we restrict ourselves to the logics $L_*(\mathcal{C})$ defined by four prominent t-norms (Gödel, Łukasiewicz, Product and Nilpotent Minimum), and study for them completeness properties with respect to all rational $L_*(\mathcal{C})$ -chains (\mathcal{QC} , \mathcal{FSQC} and \mathcal{SQC}) and with respect to the canonical rational $L_*(\mathcal{C})$ -chain ($\text{Can}\mathcal{QC}$, $\text{Can}\mathcal{FSQC}$ and $\text{Can}\mathcal{SQC}$). In fact, all the logics we are considering obviously enjoy the \mathcal{QC} , \mathcal{FSQC} and \mathcal{SQC} by virtue of Theorem 2.1, since all of them enjoy the \mathcal{FSRC} (see Table 2). We start the study of canonical completeness properties for the Łukasiewicz-based logics $L(\mathcal{C})$.

Theorem 3.1. *For every countable $\mathcal{C} \subseteq [0, 1]_L^{\mathbb{Q}}$, the logic $L(\mathcal{C})$ enjoys the $\text{Can}\mathcal{FSQC}$.*

Proof. Suppose that for some arbitrary set of formulae we have $\varphi_1, \dots, \varphi_n \not\vdash_{L(\mathcal{C})} \psi$. We must prove that $\varphi_1, \dots, \varphi_n \not\models_{[0, 1]_{L(\mathcal{C})}^{\mathbb{Q}}} \psi$. On one hand, by the

\mathcal{FSRC} , there is an evaluation e over $[0, 1]_{L(\mathcal{C})}$ such that $e(\varphi_1) = \dots = e(\varphi_n) = 1$ and $e(\psi) < 1$. On the other hand, as proved in [1]¹, we know that $[0, 1]_{L(\mathcal{C})}$ is partially embeddable into $[0, 1]_{L(\mathcal{C})}^{\mathbb{Q}}$.

Therefore, by embedding into $[0, 1]_{L(\mathcal{C})}^{\mathbb{Q}}$ the values of all the subformulae of the formulae involved, we obtain an evaluation over the rational canonical chain such that is a model of $\{\varphi_1, \dots, \varphi_n\}$ while it is not a model of ψ . \square

However, this property fails for the remaining logics we are considering. Indeed, suppose that $*$ is the product, Gödel or Nilpotent Minimum t-norm and $\mathcal{C} \subseteq [0, 1]_*^{\mathbb{Q}}$, let F be a non-trivial proper filter of \mathcal{C} and take $r \in F \setminus \{1\}$. Then:

$$\bullet (p \rightarrow q) \rightarrow \bar{r} \models_{[0, 1]_{L_*(\mathcal{C})}^{\mathbb{Q}}} q \rightarrow p$$

¹Actually the result we use from [1] comes from a translation of the paper in Russian [10].

- $(p \rightarrow q) \rightarrow \bar{r} \not\vdash_{([0,1]^\mathbb{Q})_{L^*(\mathcal{C})}^F} q \rightarrow p$

Therefore, there is an entailment which holds for the canonical rational chain, but not for all the chains. Thus none of these logics enjoy the CanFSQC. We turn now to the CanQC.

Theorem 3.2. *For every countable $\mathcal{C} \subseteq [0, 1]_{\Pi}^{\mathbb{Q}}$, the logic $\Pi(\mathcal{C})$ enjoys the CanQC.*

Proof. For this proof we need to introduce some notation. Given $x = \langle x_1, \dots, x_n \rangle \in \mathbb{R}^n$ and $\delta = \langle \delta_1, \dots, \delta_n \rangle \in (\mathbb{R}_+)^n$, we define the set $E_\delta(x) = \{ \langle y_1, \dots, y_n \rangle \in \mathbb{R}^n \mid x_i = y_i \text{ if } x_i \in \mathbb{Q}, \text{ and } y_i \in (x_i - \delta_i, x_i + \delta_i) \text{ if } x_i \notin \mathbb{Q} \}$.

Suppose that $\not\vdash_{\Pi(\mathcal{C})} \varphi$. Assume further that the variables of φ are among $\{p_1, \dots, p_n\}$. By the CanRC, there is an evaluation e on $[0, 1]_{\Pi(\mathcal{C})}$ such that $e(\varphi) < 1$. We prove by induction that for every subformula ψ of φ :

1. If $e(\psi) = 0$, then there is $E_\delta(e(p_1), \dots, e(p_n))$ such that for every evaluation v on $[0, 1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}$, if $\langle v(p_1), \dots, v(p_n) \rangle \in E_\delta(e(p_1), \dots, e(p_n))$ then $v(\psi) = 0$.
2. If $e(\psi) \neq 0$, then for every $\varepsilon > 0$ there is $E_\delta(e(p_1), \dots, e(p_n))$ such that for every evaluation v on $[0, 1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}$, if $\langle v(p_1), \dots, v(p_n) \rangle \in E_\delta(e(p_1), \dots, e(p_n))$ then $|v(\psi) - e(\psi)| < \varepsilon$.

From this it easily follows that there is an evaluation v on $[0, 1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}$ such that $v(\varphi) < 1$, and thus the theorem is proved. \square

Theorem 3.3. *For every \mathcal{C} , the logics $G(\mathcal{C})$ and $NM(\mathcal{C})$ enjoy the CanQC.*

Proof. It can be proved just in the same way as we proved in [8] that these logics enjoy the CanRC. We rewrite the proof here for the reader's convenience. Assume that $*$ is the Gödel or Nilpotent Minimum t-norm. Suppose φ is a tautology with respect to $[0, 1]_{L^*(\mathcal{C})}^{\mathbb{Q}}$. We will prove that φ is also a tautology with respect to $([0, 1]_{L^*(\mathcal{C})}^{\mathbb{Q}})^F$ for each filter F of \mathcal{C} , which, due to the QC, implies that $\vdash_{L^*(\mathcal{C})} \varphi$. Let e be an interpretation over the chain $([0, 1]_{L^*(\mathcal{C})}^{\mathbb{Q}})^F$. Suppose that \mathcal{A} is the finite algebra generated by $\{e(\psi) \mid \psi$

subformula of $\varphi\}$ and $\alpha = \min\{r \in F \mid \bar{r} \text{ occurs in } \varphi\}$. Take $f : [0, 1]_{L^*(\mathcal{C})}^{\mathbb{Q}} \rightarrow [0, 1]_{L^*(\mathcal{C})}^{\mathbb{Q}}$ such that $f : (-\alpha, \alpha)^{\mathbb{Q}} \rightarrow (0, 1)^{\mathbb{Q}}$ is a bijection, $f(r) = r$ for all $r \notin F$ such that $\neg r \notin F$ and \bar{r} occurs in φ . So defined, f is a homomorphism from \mathcal{A} into the canonical rational chain. Then define an evaluation e' over the canonical rational chain by $e'(p) = f^{-1}(e(p))$ if p is a propositional variable that appears in φ and $e'(p) = 1$ otherwise. Since φ is a tautology for the canonical rational chain, $e'(\varphi) = 1$. Take the algebra $[0, 1]_{L^*(\mathcal{C})}^{\mathbb{Q}}/F_\alpha$ where F_α is the filter $[\alpha, 1] \cap \mathcal{C}$. This algebra is isomorphic to $[0, 1]_{L^*(\mathcal{C})}^{\mathbb{Q}}$. Define the evaluation e'' on the quotient algebra obtained from e' and it obviously satisfies $e''(\varphi) = [1]_{F_\alpha}$. Now a simple computation shows that the algebra \mathcal{B} generated by $\{e''(\psi) \mid \psi \text{ subformula of } \varphi\}$ is isomorphic to \mathcal{A} and $e''(\varphi)$ over the quotient algebra corresponds to $e(\varphi)$ over the chain $([0, 1]_{L^*(\mathcal{C})}^{\mathbb{Q}})^F$, and hence $e(\varphi) = 1$. \square

The new results proved in this section are gathered in Table 4.

	$L(\mathcal{C})$	$\Pi(\mathcal{C})$	$G(\mathcal{C})$	$NM(\mathcal{C})$
QC	Yes	Yes	Yes	Yes
FSQC	Yes	Yes	Yes	Yes
SQC	Yes	Yes	Yes	Yes
CanQC	Yes	Yes	Yes	Yes
CanFSQC	Yes	No	No	No
CanSQC	No	No	No	No

Table 4: Rational completeness properties for propositional fuzzy logics with truth-constants.

4 NEW RESULTS: THE CASE OF EVALUATED FORMULAS

In this section we focus on completeness properties for the restriction of the logics to (positively) evaluated formulae. As regards to completeness properties with respect to the class of all rational chains there is nothing to say: all three properties hold for evaluated formulae because they hold in general for all formulae. Thus, we only need to examine the restricted canonical completeness properties.

All the logics under our scope fail to satisfy the

CanSQC restricted to (positively) evaluated formulae as it can be seen by the following counterexample (already used in [4] for standard semantics in the continuous t-norm case). Let $\Gamma = \{(\frac{n}{n+1}) \rightarrow \varphi \mid n \in \mathbb{N}\}$. For every logic $L_*(\mathcal{C})$ we have $\Gamma \models_{[0,1]_{L_*(\mathcal{C})}^{\mathbb{Q}}} \varphi$. But if $\Gamma \vdash_{L_*(\mathcal{C})} \varphi$, then, by finitariness, there would exist $n_0 \in \mathbb{N}$ such that $(\frac{n_0}{n_0+1}) \rightarrow \varphi \vdash_{L_*(\mathcal{C})} \varphi$, hence, we would have $(\frac{n_0}{n_0+1}) \rightarrow \varphi \models_{[0,1]_{L_*(\mathcal{C})}^{\mathbb{Q}}} \varphi$; a contradiction.

In spite of this negative result which does not improve the situation with respect to the unrestricted CanSQC, we can still show several positive results for the CanFSQC.

Theorem 4.1. *For every countable $\mathcal{C} \subseteq [0,1]_{\Pi}^{\mathbb{Q}}$, the logic $\Pi(\mathcal{C})$ enjoys the CanFSQC restricted to evaluated formulae.*

Proof. Assume that $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \cup \{\bar{s} \rightarrow \psi\}$ is a finite set of evaluated formulae such that $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \not\vdash_{\Pi(\mathcal{C})} \bar{s} \rightarrow \psi$. We must prove that $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \not\models_{[0,1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}} \bar{s} \rightarrow \psi$. By the CanFSQC restricted to evaluated formulae, there is an evaluation e on $[0,1]_{\Pi(\mathcal{C})}$ such that for every $i \in \{1, \dots, n\}$, $e(\bar{r}_i \rightarrow \varphi_i) = 1$ and $e(\bar{s} \rightarrow \psi) = 1$, i.e. $s > e(\psi)$ and $r_i \leq \varphi_i$ for every i . Without loss of generality we can assume that $r_i < e(\varphi_i)$ for every i (if it is not the case, we choose any positive real number α such that for every i , $r_i \leq e(\varphi_i) < e(\varphi_i)^\alpha$ and $s > e(\psi)^\alpha > e(\psi)$ and take instead of e the evaluation $e'(x) = e(x)^\alpha$). Then we use the same trick as in the proof of Theorem 3.2 showing by induction that for every subformula ψ of φ (we assume the variables in φ are among $\{p_1, \dots, p_n\}$):

1. If $e(\psi) = 0$ then there is $E_\delta(e(p_1), \dots, e(p_n))$ such that for every evaluation v on $[0,1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}$, if $\langle v(p_1), \dots, v(p_n) \rangle \in E_\delta(e(p_1), \dots, e(p_n))$ then $v(\psi) = 0$.
2. If $e(\psi) \neq 0$ then for every $\varepsilon > 0$ there is $E_\delta(e(p_1), \dots, e(p_n))$ such that for every evaluation v on $[0,1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}$, if $\langle v(p_1), \dots, v(p_n) \rangle \in E_\delta(e(p_1), \dots, e(p_n))$ then $|v(\psi) - e(\psi)| < \varepsilon$.

Therefore, there is an evaluation v on $[0,1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}$ that maps to 1 the premises while maps the con-

clusion to some lower value and hence $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \not\models_{[0,1]_{\Pi(\mathcal{C})}^{\mathbb{Q}}} \bar{s} \rightarrow \psi$. \square

For the two remaining cases the result is obtained following a reasoning analogous to that of [6] for the standard semantics.

Lemma 4.2. *Let $a \in (0,1]_{\mathbb{Q}}$ and $b \in (\frac{1}{2},1]_{\mathbb{Q}}$ and define a pair of mappings $f_a, f^b : [0,1]_{\mathbb{Q}} \rightarrow [0,1]_{\mathbb{Q}}$ as follows:*

$$f_a(x) = \begin{cases} 1, & \text{if } x \geq a \\ x, & \text{otherwise} \end{cases}$$

$$f^b(x) = \begin{cases} 1, & \text{if } x \geq b \\ 0, & \text{if } x \leq 1 - b \\ x, & \text{otherwise} \end{cases}$$

Then f_a is a homomorphism with respect to the Gödel truth functions, and f^b is a homomorphism with respect to the Nilpotent Minimum truth functions. Therefore, if e is an evaluation over $[0,1]_{G(\mathcal{C})}^{\mathbb{Q}}$ (resp. over $[0,1]_{NM(\mathcal{C})}^{\mathbb{Q}}$) then $e_a = f_a \circ e$ (resp. $e^b = f^b \circ e$) is also an evaluation over the same algebra.

Theorem 4.3. *For every countable $\mathcal{C} \subseteq [0,1]_{\Pi}^{\mathbb{Q}}$, the logics $G(\mathcal{C})$ and $NM(\mathcal{C})$ enjoy the CanFSQC restricted to evaluated formulae.*

Proof. Consider first the case of Gödel logic. Assume again that $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \cup \{\bar{s} \rightarrow \psi\}$ is a finite set of evaluated formulae such that $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\} \not\vdash_{G(\mathcal{C})} \bar{s} \rightarrow \psi$. By the Deduction-detachment Theorem we have $\not\vdash_{G(\mathcal{C})} \bigwedge_{i=1}^n (\bar{r}_i \rightarrow \varphi_i) \rightarrow (\bar{s} \rightarrow \psi)$, and hence by the CanQC, there is an evaluation e on the rational canonical chain which is not a model of $(\bigwedge_{i=1}^n (\bar{r}_i \rightarrow \varphi_i) \rightarrow (\bar{s} \rightarrow \psi))$. We must find another evaluation e' which is model of $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\}$ and not of $\bar{s} \rightarrow \psi$.

If e is a model of every $\bar{r}_i \rightarrow \varphi_i$, then we can take $e' = e$ and the problem is solved. Otherwise, there exists some $1 \leq j \leq n$ for which $r_j > e(\varphi_j)$ and thus $e(\bar{r}_j \rightarrow \varphi_j) = e(\varphi_j) < 1$. Let $J = \{j \mid r_j > e(\varphi_j)\}$ and let $a = e(\bigwedge_{i=1}^n \bar{r}_i \rightarrow \varphi_i) = \min\{e(\varphi_j) \mid j \in J\}$. Then the evaluation e' such that $e' = e_a$ over the propositional variables does the job. Namely, by the previous lemma, over Gödel formulae we have $e' = e_a \geq e$, so e' is still a model

of $\bar{r}_i \rightarrow \varphi_i$ for every $i \in \{1, \dots, n\} \setminus J$. But now, $e'(\varphi_j) = 1$ for every $j \in J$, so e' is also a model of $\{\bar{r}_i \rightarrow \varphi_i \mid i = 1, \dots, n\}$. On the other hand, since $e(\bigwedge_{i=1}^n (\bar{r}_i \rightarrow \varphi) \rightarrow (\bar{s} \rightarrow \psi)) < 1$, it must be $s > e(\psi)$ and $a = e(\bigwedge_{i=1}^n (\bar{r}_i \rightarrow \varphi_i)) > e(\psi)$. Now, by the previous lemma, $e'(\psi) = e_a(\psi) = e(\psi)$, hence $e'(\bar{s} \rightarrow \psi) = e(\bar{s} \rightarrow \psi) < 1$.

For the case of Nilpotent Minimum logic the proof runs analogously by using the evaluations e^b defined in the previous lemma. \square

The obtained results are summarized in Table 5.

	L(C)	II(C)	G(C)	NM(C)
QC	Yes	Yes	Yes	Yes
FSQC	Yes	Yes	Yes	Yes
SQC	Yes	Yes	Yes	Yes
CanQC	Yes	Yes	Yes	Yes
CanFSQC	Yes	Yes	Yes	Yes
CanSQC	No	No	No	No

Table 5: Rational completeness properties for propositional fuzzy logics with truth-constants restricted to (positively) evaluated formulae.

5 CONCLUSIONS

In this paper we have proposed the rational semantics for fuzzy logics expanded with truth-constants as a new topic for research. The collection of first results we have presented show the interest of the approach, as the rational semantics has demonstrated to provide better completeness properties for the main propositional fuzzy logics. We plan to extend these results in forthcoming papers by addressing the following problems:

- (i) extend the investigation on rational completeness properties to wider classes of logics based on continuous and weak nilpotent t-norms;
- (ii) study rational completeness properties for first-order predicate fuzzy logics, as it has been done for the standard semantics in [9]; and

- (iii) investigate rational completeness results for expansions of logics with the projection connective Δ , both in the propositional and the first-order case.

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