ABOUT STRONG STANDARD COMPLETENESS OF PRODUCT LOGIC

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Abstract

Propositional Product Logic is known to be standard finite-strong complete but not a strong complete, that is, it is complete for deductions only from finite sets of premises with respect to evaluations on the standard product chain over the real unit interval. On the other hand, Montagna has defined a logical system, an axiomatic extension of the Hájek's Basic Fuzzy Logic BL with an storage operator and an infinitary rule, which was proved to be standard strong complete (i.e. for deductions from possibly infinite theories) with respect to the standard BL chains. In particular, the expansion of Product Logic with the infinitary rule and Monteiro-Baaz Delta operator is standard strong complete. In this paper we generalize this result to the case of having rational truth constants in the language, and provide alternative infinitary rules better adapted to the final goal of our ongoing research, that is to study modal extensions over product fuzzy logic.

Keywords: BL logic, Product logic, strong completeness, infinitary rules, archimedean product chains, storage and Δ-operators.

1 INTRODUCTION

The standard product algebra is the algebra defined on the real unit interval by the product t-norm and its residuum which will be denoted as $[0,1]_\Pi$ and which operations are $\min, \max, \odot, \rightarrow_\odot, \neg_\odot$ and the constants $0, 1$, i.e., $[0,1]_\Pi = \langle [0,1], \min, \max, \odot, \rightarrow_\odot, \neg_\odot, 0, 1\rangle$.

In [3] the authors defined Product Logic as the propositional logic which language is obtained from an enumerable set of propositional variables by means of the binary connectives $\odot, \rightarrow$ and the the constant $\bot$, defined by the following set of axiom and Modus Ponens as the only inference rule:

$(A1)$ $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$;
$(A2)$ $\varphi \odot \psi \rightarrow \varphi$;
$(A3)$ $\varphi \odot \psi \rightarrow \psi \odot \varphi$;
$(A4)$ $\varphi \odot (\varphi \rightarrow \psi) \rightarrow (\psi \odot (\psi \rightarrow \varphi))$;
$(A5)$ $(\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\varphi \odot \psi) \rightarrow \chi)$;
$(A6)$ $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$;
$(A7)$ $\bot \rightarrow \varphi$;
$(A_\Pi 1)$ $\neg\neg\chi \rightarrow ((\varphi \odot \chi \rightarrow \psi \odot \chi) \rightarrow (\varphi \rightarrow \psi))$;
$(A_\Pi 2)$ $\neg(\varphi \odot \varphi) \rightarrow \neg\varphi$;

In that paper the authors proved completeness for theorems with respect to the standard product algebra and this result was generalized to finite strong standard completeness in [2], i.e., for every finite set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_\Pi \varphi \text{ if and only if } \Gamma \models_{[0,1]_\Pi} \varphi$$

But the completeness result is not valid for infinite theories as the following example shows.

Example 1.1. *Let p, q be propositional variables and let $\Gamma = \{p \rightarrow q^n \mid n \in N\}$ where q^n is the abbreviation of $p\&p...\&p$. For any evaluation v on $[0,1]_\Pi$, it is obvious that if $v(p \rightarrow q^n) = 1$ for all $n \in N$, then $v(p) = 0$ or $v(q) = 1$ which is equivalent to $v(\neg p \vee q) = 1$, i.e.,*

$$\Gamma \models_{[0,1]_\Pi} \neg p \vee q$$

However, any syntactic proof can only use a finite number of formulas of Γ. Suppose that from a finite subset Γ_0 of Γ we can prove $\neg p \vee q$. By finite strong standard completeness this means that for each evaluation v that evaluate formulas of Γ_0 to 1 (suppose that k is the greatest exponent appearing in formulas of Γ_0), then $v(\neg p \vee q) = 1$. Let $v(q) = 0.5$ and $v(p) \leq 0.5^k$. Then $v(p \rightarrow q^n) = 1$ for any $n \leq k$ ($v(\psi) = 1$ for any $\psi \in \Gamma_0$) but $v(\neg p \vee q) = 0.5 \neq 1$.

Montagna presented in [4] an axiomatic system with storage operator and an infinitary inference rule such that propositional logic BL is standard complete. In that paper he also studied the case of Product Logic. In such a case storage operator coincides with the Monteiro-Baaz Delta operator Δ (whose interpretation in a product chain is $\Delta(1) = 1$, $\Delta(x) = 0$ for $x \neq 1$.). Therefore the result of

Montagna is an axiomatic system of Product Logic with Δ obtained by adding to the axiomatic system of Product Logic the axioms of Δ and its generalization rule, a new infinitary rule (R^M):

$(A_\Delta 1)$ $\Delta\varphi \vee \neg\Delta\varphi$; $(A_\Delta 4)$ $\Delta\varphi \to \Delta\Delta\varphi$;
$(A_\Delta 2)$ $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$; $(A_\Delta 5)$ $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$;
$(A_\Delta 3)$ $\Delta\varphi \to \varphi$; $(\mathbf{G}_\Delta)$ $\frac{\varphi}{\Delta\varphi}$;

$$(\mathbf{R^M}): \frac{C \vee (A \to B^n),\ for\ all\ n}{C \vee \neg A \vee B}$$

whose intuitive meaning is that of fixing that the algebras associated with this logic are archimedean (if $0 > x, y > 1$, then there is an n such that $x^n < y$).

On the other hand, in [5], the expansion of product logic with rational truth constants was studied, and it was proven that the extension of product logic with the Δ axioms from before and the following axiomatization for the constants is finitely strong standard complete with respect to the canonical standard product algebra with δ, denoted by $[0,1]_{\mathfrak{c}\Pi}$ (where the rational designated symbols are interpreted by its name).

$$(A_{\mathscr{C}}1)\ \overline{r} \odot \overline{s} \leftrightarrow \overline{r \cdot s}; \qquad (A_{\mathscr{C}}2)\ (\overline{r} \to \overline{s}) \leftrightarrow \overline{r \to_\Pi s}\ ^1;$$
$$(A_{\mathscr{C}}3)\ \Delta\overline{r} \leftrightarrow \overline{\delta(r)};$$

where $\delta(1) = 1$ and $\delta(r) = 0$ if $r < 1$. We will call this logic (Product logic with Δ operator and rational constants) Π^+. Note than in the following sections, when we talk about truth constants we will implicitly assume they are *rational* constants. More specifically we will say a product chain $\mathbf{A}$ with constants has *proper (rational) constants* if $\overline{c}^{\mathbf{A}} \neq \overline{d}^{\mathbf{A}}$ for any $c \neq d$. From [5] we know that either $\mathbf{A}$ has proper constants or $\overline{c}^{\mathbf{A}} = 1$ for any constant $c \neq 0$.

This paper focuses on proving the generalization of Montagna's result about (infinite) strong completeness to product logic plus constants. We will provide an axiomatic extension of Π^+ that is strongly complete with respect to the canonical standard product algebra. Also, we will modify Montagna's rule to obtain a simpler characterization of archimedeanity in the particular case we have rational constants, with the idea of using these rules in an ongoing work concerning a modal expansion of this logic.

2 SOME ALGEBRAIC RESULTS

In this section we will provide some technical results that will be useful for proving the completeness of our logic. First, from [1] we have that there exist two functors $\mathfrak{G} : PL_c \leftrightarrow LG : \mathfrak{B}$ that induce a natural equivalence between the category of product chains and that of the lattice-ordered abelian groups, so given a product chain $\mathbf{A}$,

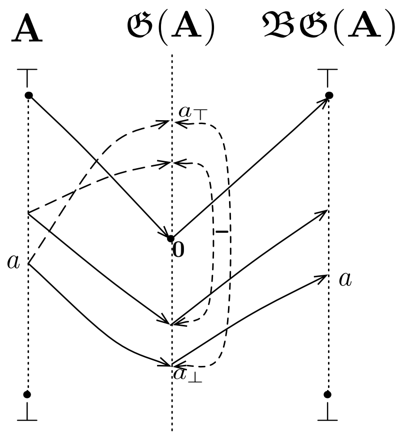

Figure 1: Intuitive meaning of $\mathfrak{G}$ and $\mathfrak{B}$

$\mathfrak{B}\mathfrak{G}(\mathbf{A}) \cong \mathbf{A}$. In particular, in the case of product chains, we can let $\mathfrak{G}(\mathbf{A}) = \{a_\bot :\ a \in A \setminus \{\bot, \top\}\} \cup \{a_\top :\ a \in A \setminus \{\bot, \top\}\} \cup \{\mathbf{0}\}$ where $\mathbf{0}$ is the neutral element and

- $a_\bot +_{\mathfrak{G}(\mathbf{A})} b_\bot = (a \odot_{\mathbf{A}} b)_\bot$;
- $a_\top +_{\mathfrak{G}(\mathbf{A})} b_\top = (a \odot_{\mathbf{A}} b)_\top$;
- $a_\top +_{\mathfrak{G}(\mathbf{A})} b_\bot = \begin{cases} (b \to_{\mathbf{A}} a)_\bot & \text{if } a \leq_{\mathbf{A}} b \\ (a \to_{\mathbf{A}} b)_\top & \text{if } b <_{\mathbf{A}} a \end{cases}$
- $-a_\top = a_\bot,\ -a_\bot = a_\top$;

and the l-order is given by

- $a_\bot < b_\top$ for any $a, b \in \mathbf{A}$;
- $a_\bot \leq b_\bot$ iff $a \leq b$;
- $a_\top \leq b_\top$ iff $a \geq b$;

And given a group $\langle G, +, \leq, 0, \mathfrak{B}(G) = \{g \in G :\ g \leq 0\} \cup \{\bot\}$, with

- $a \odot b = \begin{cases} a + b & \text{if } a, b \in G \\ \bot & \text{otherwise} \end{cases}$
- $\neg a = \bot$ if $a \in G$, and $\neg\bot = 0_G$

and the order is inherited from G, letting $\bot \leq g$ for any $g \in G$.

For simplicity, we will denote by $\mathbb{R}_+$ the additive group of the real numbers, $\langle \mathbb{R}, +, -, 0 \rangle$.

We can now formulate and prove the following technical result.

Lemma 2.1. *Let* $\mathbf{A}$ *be an archimedean product chain with proper constants. Then, for each* $\overline{c}$ *(*$c \in (0,1)_{\mathbb{Q}}$*) there*

[1] $r \to_\Pi s = s/r$ if $r > s$ and 1 otherwise

is a complete embedding ϕ_c from A to $\mathfrak{B}(\mathbb{R}_+)$ such that $\phi_c(\overline{d}^{\mathbf{A}}) = -\log_c d$.

Proof. We can define

$$\phi_c(x) := \begin{cases} 0 & \text{if } x = \top \\ -\infty & \text{if } x = \bot \\ -\sup\{\frac{n}{m} : x^m \leq_{\mathbf{A}} (\overline{c}^{\mathbf{A}})^n\} & \text{otherwise} \end{cases}$$

It is simple to see (using that the rationals are dense in the reals, that $c < 1$ and that the logarithm is a continuous function) that $\log_c d = sup\{r \in \mathbb{Q} : \ d \leq c^r\}$, and so, by axiom $(A_{\mathscr{C}}1)$, $\phi_c(\overline{d}^{\mathbf{A}}) = -\log_c d$.

On the other hand, by the definitions of operations and order in $\mathfrak{G}(\mathbf{A})$ we can construct the following chain of equivalences, for $x, \overline{c} \in \mathbf{A}$:

$$\begin{aligned} m \cdot x_\top \geq n \cdot (\overline{c}^{\mathbf{A}})_\top \quad &\text{iff} \quad (x^m)_\top \geq ((\overline{c}^{\mathbf{A}})^n)_\top \\ &\text{iff} \quad x^m \leq_{\mathbf{A}} (\overline{c}^{\mathbf{A}})^n \end{aligned}$$

Then, we have that $\sup\{\frac{n}{m} : m \cdot x_\top \geq (\overline{c}^{\mathbf{A}})_\top\} = \sup\{\frac{n}{m} : z^m \leq_{\mathbf{A}} (\overline{c}^n)^{\mathbf{A}}\}$.

We end up having that ϕ_c can be equally given by $\mathfrak{B}(\phi_c')$, where $\phi_c' \colon \mathfrak{G}(A) \to \mathbb{R}_+$ is the complete embedding given by

$$\phi_c'(x) = \begin{cases} 0 & \text{if } x = \mathbf{0} \\ \sup\{\frac{n}{m} : m \cdot x \geq (\overline{c}^{\mathbf{A}})_\top\} & \text{if } x > \mathbf{0} \\ -\sup\{\frac{n}{m} : m \cdot (-x) \geq (\overline{c}^{\mathbf{A}})_\top\} & \text{if } x < \mathbf{0} \end{cases}$$

To prove ϕ_c' is indeed a complete embedding, we refer to the Proof of [4, Prop. 3]. ϕ_c' is a particular case of the family of embeddings ϕ constructed there. Note in that proof, the *unit* is an arbitrary element of the algebra, while here we fix it to an element that coincides with the interpretation of constant $\overline{c}$, i.e. $\overline{c}^{\mathbf{A}}$, which is in particular an element of $\mathbf{A}$ too. Then, from this general case we have that ϕ_c' is a complete embedding, and then, so is $\phi_c = \mathfrak{B}(\phi_c')$. □

To be able to prove completeness with respect to the canonical standard product algebra, what is needed is a complete embedding into that algebra. This will be now easy to prove using the previous intermediate embedding and composing it with the adequate one from $\mathfrak{B}(\mathbb{R}_+)$ into $[0,1]_\Pi$.

Theorem 2.2. *(c.f. [4, Prop. 4]) A product-chain with Δ operator and (proper) constants is archimedean iff it can be embedded in the canonical standard algebra by a complete embedding.*

Proof. Let $P_c \colon \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ be defined by

$$P_c(x) := \begin{cases} c^{-x} & \text{if } x \in \mathbb{R}^- \\ 0 & \text{if } x = \bot \end{cases}$$

Observe that the function P_c is clearly a complete embedding from $\mathfrak{B}(\mathbb{R})$ to $[0,1]_\Pi$ (it is a continuous monotone increasing function, since $0 < c < 1$ and its domain is $\mathbb{R}^- \cup \{\bot\}$), and $P_c(-1) = c$.

We can now proceed to prove that $P_c \circ \phi_c$ is a complete embedding from $\mathbf{A}$ to $[0,1]_\Pi$. Since both are complete embeddings, the only thing that is necessary to prove is that for any constant $\overline{d}^{\mathbf{A}} \in (0,1)_\mathbb{Q}$ we have that $P_c \circ \phi_c(\overline{d}^{\mathbf{A}}) = d$. But by definition, $\phi_c(\overline{d}) = \phi_c(\overline{d}^{\mathbf{A}}) = -log_c d$, and thus $P_c(\phi_c(\overline{d})) = c^{-(-log_c d)} = d$. □

The last result we will need is an alternative characterization of archimedean algebras, which is independent of the existence of constants symbols.

Lemma 2.3. *([4, Lemma. 10]) Let $\mathbf{A}$ be a product chain. Then $\mathbf{A}$ is archimedean if and only if it satisfies the following condition:*
(+) If there are $x, y \in \mathbf{A}$ such that $x \leq y^n$ for all n, then $x \leq \Delta y$.

3 AXIOMATIC SYSTEM AND STRONG COMPLETENESS

We let Π^* to be the logic defined from Π^+ plus the following rules:

$$(\mathbf{R_1})\ \frac{\overline{c} \to \varphi, \text{ for all } c \in (0,1)_\mathbb{Q}}{\varphi} \qquad (\mathbf{R_2})\ \frac{\varphi \to \overline{c}, \text{ for all } c \in (0,1)_\mathbb{Q}}{\neg\varphi}$$

And we will let $\vdash_{\Pi^*}$ be defined from the clauses of Π^+ and the following two clauses:

1. If $\{\gamma_c\}_{c \in (0,1)_\mathbb{Q}}$ are derivations of $\overline{c} \to \varphi$ from a set Γ of assumptions, then $\{\gamma_c\}_{c \in (0,1)_\mathbb{Q}}$ is a derivation of φ.
2. If $\{\gamma_c\}_{c \in (0,1)_\mathbb{Q}}$ are derivations of $\varphi \to \overline{c}$ from a set Γ of assumptions, then $\{\gamma_c\}_{c \in (0,1)_\mathbb{Q}}$ is is a derivation of $\neg\varphi$.

These two rules are valid in the canonical standard product algebra, which is archimedean. This provides the soundness of the logic Π^*.

To prove that Π^* enjoys strong completeness with respect to $[0,1]_{\mathfrak{c}\Pi}$, we will follow an usual precourse: First, seeing that it is complete with respect to the product chains with constants (and, in fact, it is so with respect to *archimedean* product chains with *proper* constants). And then, using that any archimedean product chain with proper constants can be embedded into $[0,1]_\Pi$ with canonical constants by a complete embedding, we can conclude the proof.

To achieve this, we will provide some useful results regarding the logic Π^*.

Lemma 3.1. *Π^* is an extension of Π^+, i.e. for any $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\Pi^+} \varphi$ implies $\Gamma \vdash_{\Pi^*} \varphi$.*

From the previous observation, we can use finite strong completeness of Π^+ with respect to $[0,1]_{\mathscr{C}\Pi}$ to obtain useful results. It is easy to check the validity of the following statements in that algebra, and so, prove they are valid in our logic.

Lemma 3.2. *The following formulae are theorems of* Π^**:*
$(Th.1) : (\Delta\varphi \to \psi \odot \theta) \leftrightarrow ((\Delta\varphi \to \psi) \odot (\Delta\varphi \to \theta))$;
$(Th.2) : \Delta(\varphi \to \psi) \vee \Delta(\psi \to \varphi)$;
$(Th.3) : \neg\Delta(\overline{r} \to \overline{r} \odot \overline{c})$ *for any* $0 < c < 1$;
$(Th.4) : ((\Delta\varphi \to \alpha) \wedge (\Delta\psi \to \alpha)) \to (\Delta(\varphi \vee \psi) \to \alpha)$;

Lemma 3.3. *The following deductions hold in* Π^**:*
$(D.1) : \varphi \vee (\overline{c} \to \psi)) \vdash \overline{c} \to (\varphi \vee \psi)$ *for* $0 < r < 1$;
$(D.2) : (\varphi \to \overline{c}) \vee \Delta\alpha \vdash (\neg\Delta\alpha \wedge \varphi) \to \overline{c}$;
$(D.3) : \overline{c}^n \to \varphi^n \vdash \overline{c} \to \varphi$;

Now, we can easily prove that the Deduction Theorem keeps working in the Π^* logic.

Lemma 3.4 (Deduction Theorem)**.** *For any set of formulas over the product logic language with* Δ *and rational constants* $\Gamma \cup \{\alpha, \varphi\}$*, it holds that*

$$\Gamma \cup \{\alpha\} \vdash_{\Pi^*} \varphi \text{ iff } \Gamma \vdash_{\Pi^*} \Delta\alpha \to \varphi.$$

Proof. The right to left direction is simple from $(\mathbf{G}_\Delta)$ and *MP*. For the other sense, we can proceed by induction on the derivation. The induction steps corresponding to Modus Ponens or Generalization of Δ are easy.
We now consider the case in which the last rule applied in the proof is (R_1). Thus, assume φ was obtained from $\{\overline{c} \to \varphi\}_{c \in (0,1)_\mathbb{Q}}$ using rule (R_1) (which are previous steps of the proof). Then, by induction hypothesis, $\Gamma \vdash_{\Pi^*} \Delta\alpha \to (\overline{c} \to \varphi)$ for all c. By $(A.5)$ we have that $\Gamma \vdash_{\Pi^*} \overline{c} \to (\Delta\alpha \to \varphi)$ for all c. Now, by rule (R_1), $\Gamma \vdash_{\Pi^*} \Delta\alpha \to \varphi$.
We now consider the case where the last rule applied was (R_2). Suppose $\varphi \equiv \neg\psi$ and that it was obtained from $\{\psi \to \overline{c}\}_{c \in (0,1)_\mathbb{Q}}$ through rule (R_2). Then, by induction hypothesis, $\Gamma \vdash_{\Pi^*} \Delta\alpha \to (\psi \to \overline{c})$ for each c. Then, $\Gamma \vdash_{\Pi^*} (\psi \wedge \Delta\alpha) \to \overline{c}$ for each c, and by rule (R_2), this leads to $\Gamma \vdash_{\Pi^*} \neg(\psi \wedge \Delta\alpha)$. Now, since we know the DeMorgan Laws keep holding, $\Gamma \vdash_{\Pi^*} \neg\Delta\alpha \vee \neg\psi$. Since $\Delta\alpha \vee \neg\Delta\alpha$ is a theorem, this is equivalent to $\Gamma \vdash_{\Pi^*} \Delta\alpha \to \neg\psi$, which was the desired result. □

We can start now with the completeness proof. First, we can see that we can extend any Π^*- theory to a complete theory over Π^+ closed under R_1 and R_2.

To do that we will make use of the following technical lemma to prove how we can close our extension under (R_1).

Lemma 3.5. *(c.f. [4, Lemma 6])* Π^* *is closed under*

$$(\mathbf{R_1}') : \frac{\chi \vee (\overline{c} \to \varphi), \text{ for all } c}{\chi \vee \varphi}$$

Proof. Suppose $T \vdash_{\Pi^*} \chi \vee (\overline{c} \to \varphi)$ for all c. By $(D.1)$, $T \vdash_{\Pi^*} \overline{c} \to (\varphi \vee \chi)$ for all c. Then, by rule R_1, $T \vdash_{\Pi^*} \varphi \vee \chi$. □

We can express the more concrete result we will be using in the following way, which is a direct application from the previous lemma and the axioms and rules of Δ.

Corollary 3.6. *(c.f. [4, Lemma 7]) If for all c,* $\Gamma \vdash_{\Pi^*} \Delta(\varphi \to \overline{c}) \to \alpha$ *and* $\Gamma \vdash_{\Pi^*} \Delta\varphi \to \alpha$ *then* $\Gamma \vdash_{\Pi^*} \alpha$.

Moreover, the following are some quite simple but useful remarks that come from axiom $(\mathscr{A}_\Pi^\mathscr{C} 1)$ and $(Th.3)$.

Lemma 3.7. *The following conditions hold for any consistent theory T and* $c \in (0,1)_\mathbb{Q}$*:*

1. *If* $T \vdash_{\Pi^*} \varphi \to \overline{c}$*, for any consistent theory* T^+ *extending T,* $T^+ \nvdash_{\Pi^*} \overline{c'} \to \varphi$ *for some* c'.
2. *If* $T \vdash_{\Pi^*} \overline{c} \to \varphi$*, for any consistent theory* T^+ *extending T,* $T^+ \nvdash_{\Pi^*} \varphi \to \overline{c \cdot c}$.

Now we are ready to prove it is possible to extend a theory to another one complete and closed under (R_1) and (R_2). For $\Gamma \subset Fm$ we will define $C_{\Pi^*}(\Gamma) := \{\varphi : \Gamma \vdash_{\Pi^*} \varphi\}$, the closure of Γ under Π^*.

Theorem 3.8 (Prime extension)**.** *Given a theory T such that* $T \nvdash_{\Pi^*} \alpha$*, then there is a complete theory* T^+ *that extends T such that* $T^+ \nvdash_{\Pi^*} \alpha$ *and such that it is closed under* (R_1) *and* (R_2).

Proof. As Montagna noticed, it is interesting to observe that by the presence of infinitary rules, many standard constructions do not work. For instance, one might be tempted to use Zorn's lemma to obtain a maximal theory closer under R_1 and R_2 Γ^+ extending Γ such that $\varphi \notin \Gamma^+$. But in this case Zorn's lemma does not apply, as the union of a chain of (R_1, R_2)-theories may fail to be a theory closer under R_1 and R_2: for instance, let $\Gamma_n = K_{\Pi^*}(\{\overline{\frac{1}{n}} \to \psi\})$. Then, $\Gamma_1 \subseteq \ldots \subseteq \Gamma_n \ldots$, and every Γ_n is a theory closer under R_1, but their union Γ^+ is not, as $\psi \notin \Gamma^+$. Thus we will proceed in another way.

Let $\langle \varphi_n, \psi_n \rangle$ be an enumeration of all the possible couples of formulae from $\mathscr{L}_{\mathscr{C},\Delta}$. We can define a series of theories T_n such that:

1. $T_0 = C_{\Pi^*}(T) \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq \ldots$;
2. For each n, $\alpha \notin T_n$.
3. For each n, either $\varphi_n \to \psi_n \in T_{2n+1}$ or $\psi_n \to \varphi_n \in T_{2n+1}$. (linearity)
4. For each n, if φ_n or ψ_n is a constant symbol $\overline{r_n}$ (and in that case, we name χ_n the other formula):
    - If $\overline{r_n} \to \chi_n \in T_{2n+1}$, then either $\chi_n \to \overline{c} \in T_{2n+2}$ for some $1 > c > 0$ or $\chi_n \in T_{2n+2}$. (closure under (R_1)).

- If $\overline{r_n} \to \chi_n \notin T_{2n+1}$, then either $\overline{c} \to \chi_n \in T_{2n+2}$ for some $c \in (0, r_n]_{\mathbb{Q}}$ or $\neg\chi_n \in T_{2n+2}$. (closure under (R_2)).

Indeed, we define them as follows:

Step 0: $T_0 := C_{\Pi^*}(T)$.

Step $2n+1$: If $\alpha \notin C_{\Pi^*}(T_{2n} \cup \{\varphi_n \to \psi_n\})$, put $T_{2n+1} := C_{\Pi^*}(T_{2n} \cup \{\varphi_n \to \psi_n\})$. Otherwise, put $T_{2n+1} := C_{\Pi^*}(T_{2n} \cup \{\psi_n \to \varphi_n\})$. Notice that in any case, $\alpha \notin T_{2n+1}$. Otherwise, $T_{2n} \vdash_{\Pi^{c,\Delta}_{R_1,R_2}} \Delta(\varphi_n \to \psi_n) \to \alpha$ and $T_{2n} \vdash_{\Pi^*} \Delta(\psi_n \to \varphi_n) \to \alpha$ (by the deduction theorem). Then, by $(Th.4)$, $T_{2n} \vdash_{\Pi^*} \alpha$, which is a contradiction.

Step $2n+2$: If both φ_n, ψ_n are different from constant symbols, let $T_{2n+2} := T_{2n+1}$; If at least one of $\{\varphi_n, \psi_n\}$ is a constant symbol, let $\overline{r_n}$ be it and χ_n be the other formula. Then, we have two cases:

1. If $\overline{r_n} \to \chi_n \in T_{2n+1}$ then we have two subcases:
   - If $\alpha \notin C_{\Pi^*}(T_{2n+1} \cup \{\chi_n \to \overline{c}\})$ for some c. Then let $T_{2n+2} := C_{\Pi^*}(T_{2n+1} \cup \{\chi_n \to \overline{c}\})$.
   - If $\alpha \in C_{\Pi^*}(T_{2n+1} \cup \{\chi_n \to \overline{c}\})$ for any $c \in (0,1)_{\mathbb{Q}}$, by the Deduction theorem we have that $T_{2n+1} \vdash_{\Pi^*} \Delta(\chi_n \to \overline{c}) \to \alpha$ for each c. By Corollary 3.6 ,and since $T_{2n+1} \nvdash_{\Pi^*} \alpha$, we know that $T_{2n+1} \nvdash_{\Pi^*} \Delta\chi_n \to \alpha$. Then, put $T_{2n+2} := K_R(T_{2n+1,R_2} \cup \{\chi_n\})$.
2. If $\overline{r_n} \to \chi_n \notin T_{2n+1}$ (i.e., $\chi_n \to \overline{r_n} \in T_{2n+1}$) then we have two subcases again:
   - If $\alpha \notin C_{\Pi^*}(T_{2n+1} \cup \{\overline{c} \to \chi_n\})$ for some $c \in (0, r_n]_{\mathbb{Q}}$, let $T_{2n+2} := C_{\Pi^*}(T_{2n+1} \cup \{\overline{c} \to \chi_n\})$.
   - If $\alpha \in C_{\Pi^*}(T_{2n+1} \cup \{\overline{c} \to \chi_n\})$ for any $c \in (0,1)_{\mathbb{Q}}$, then (by the Deduction Theorem) we have that $T_{2n+1} \vdash_{\Pi^*} \Delta(\overline{c} \to \chi_n) \to \alpha$ for each c. Using $(Th.4)$, we have that $T_{2n+1} \vdash_{\Pi^*} \Delta(\chi_n \to \overline{c}) \vee \alpha$ for each c. And so, by G_Δ, $T_{2n+1} \vdash_{\Pi^*} \Delta(\chi_n \to \overline{c}) \vee \Delta\alpha$ for each c. By $(Th.7)$, this implies that $T_{2n+1} \vdash_{\Pi^*} (\neg\Delta\alpha \wedge \chi_n) \to \overline{c})$ for each c, and so, by rule (R_2), we know that $T_{2n+1} \vdash_{\Pi^*} \neg(\neg\Delta\alpha \wedge \chi_n)$. By the DeMorgan Laws, and using G_Δ and axiom $(\mathscr{A}^{\Delta}_{\Pi}3)$ we have that $T_{2n+1} \vdash_{\Pi^*} \alpha \vee \Delta\neg\chi_n$. Then, $T_{2n+1} \nvdash_{\Pi^*} \Delta\neg\chi_n \to \alpha$ (otherwise, $T_{2n+1} \vdash_{\Pi^*} \alpha$, which is a contradiction).
     Then, let $T_{2n+2} := C_{\Pi^*}(T_{2n+1} \cup \{\neg\chi_n\})$.

It is clear that the sequence $\{T_n\}_{n\in\omega}$ satisfies conditions 1.,2.,3. and 4. . Then, let $T^+ := \bigcup_{n\in\omega} T_n$. Clearly, T^+ is a complete theory extending T and $\alpha \notin T^+$. We just need to see it is closed under (R_1) and (R_2).

On the one hand suppose that $\overline{r} \to \varphi \in T^+$ for some $0 < r < 1$. Then, by our construction, there must be an n such that $\overline{r} \equiv \overline{r_n}$ and $\varphi \equiv \chi_n$ (this implication has been added at step $2n+1$). Then, by construction, we have two options: either $\varphi \to \overline{c} \in T_{2n+2}$ for some c or $\varphi \in T_{2n+2}$. In the first case, by Lemma 3.7(1) it does not hold that T^+ contains $\overline{d} \to \varphi$ for any constant d (so (R_1) will not be applied over this formula). On the second case, we have that $\varphi \in T^+$, so it is closed under R_1 for this formula.

On the other hand, suppose that $\varphi \to \overline{r} \in T^+$, $0 < r < 1$. Then, by our construction, there must be an n such that $\overline{r} \equiv \overline{r_n}$ and $\varphi \equiv \chi_n$ (this implication has been added at step $2n+1$). Then, by construction, we have two options: either $\overline{c_0} \to \varphi \in T_{2n+2}$ for some $c_0 \in (0, r]_{\mathbb{Q}}$ or $\neg\varphi \in T_{2n+2}$. In the first case, by Lemma 3.7(2) it does not hold that T^+ contains $\varphi \to \overline{c}$ for any $c \in (0,1)_{\mathbb{Q}}$ (so (R_2) will not be applied over this pair of formula-constant). On the second case, we have that $\neg\varphi \in T^+$ so it is closed under R_2 for this formula. □

Once we have this prime extension of a theory, closed under R_1 and R_2, we know that its Lindenbaum sentence algebra is a product chain with constants $[\overline{c}]_{\Gamma^+}$. Now, to prove we can embed it into $[0,1]$, we will prove a similar result to [4, Lemma. 4], but using the new infinitary rules.

Lemma 3.9. *Let Γ be any complete theory closed under R_1 and R_2 over Π^+, and let $\mathscr{L}_\Gamma$ denote the Lindenbaum sentence algebra of Γ. Then $\mathscr{L}_\Gamma$ can be embedded into the canonical standard product algebra $[0,1]_{\mathfrak{c}\Pi}$ by a complete embedding.*

Proof. We can prove $(+)$ from 2.3 by cases. Suppose there is $c \in (0,1)_{\mathbb{Q}}$ such that $\overline{c} \to x \in \Gamma$.

> **Claim:** $\Gamma \vdash_{\Pi^*} \overline{c} \to \phi^k$ *for all k implies that* $\Gamma \vdash_{\Pi^*} \overline{d} \to \phi$ *for all* $d \in (0,1)_{\mathbb{Q}}$.
>
> Proof: Suppose that $\Gamma \vdash_{\Pi^*} c \to \varphi^k$ for all k. We know $(0,1)_{\mathbb{Q}}$ is archimedean, so for each d there is an n_d such that $d^{n_d} < c$. Then in particular $\Gamma \vdash_{\Pi^*} \overline{d}^{n_d} \to \varphi^{n_d}$. By $(D.3)$, $\Gamma \vdash_{\Pi^*} \overline{d} \to \varphi$.

Then, $\overline{c} \to y \in \Gamma$ for all c, so, by rule (R_1), $y \in \Gamma$, so $\Delta y \in \Gamma$ (by G_Δ) and so, for any element, and in particular for x, $x \to \Delta y \in \Gamma$.

If there is no $c \in (0,1)_{\mathbb{Q}}$ such that $\overline{c} \to x \in \Gamma$, since Γ is complete we have that $x \to \overline{c} \in \Gamma$ for all $c \in (0,1)_{\mathbb{Q}}$. Then, by rule (R_2), $\neg x \in \Gamma$, and then again, trivially, $x \to \Delta y \in \Gamma$ (for any y).

Finally, we have that $\mathscr{L}_\Gamma$ is achimedean, and so, by Theorem 3.9, there is a complete embedding from it into $[0,1]_{\mathfrak{c}\Pi}$. □

As a corollary of the previously presented results, we can prove the main result of this paper: the strong standard completeness of our logic.

Theorem 3.10 (Strong Standard Completeness)**.** *For any set of formulas* $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\Pi^*} \varphi$ *if and only if* $\Gamma \models_{[0,1]_{c\Pi}} \varphi$.

4 CONCLUSIONS AND FUTURE WORK

In this paper presented an axiomatization of Product logic that is strong standard complete with respect to the standard product chain (with Delta operator and rational truth constants). This axiomatization is inspired in the one given by Montagna in [4] of Basic Logic (strongly) complete with respect to standard chains (with storage operator). As in Montagna paper we need to make use of Delta operator (the storage operator on archimedean product chains) and of infinitary inference rules.

As future work we are interested in study modal product logic, the logic obtained as expansion of Product logic with the modal operators of necessity and possibility defined as in the setting of many-valued modal logics. A first step would be the axiomatization of modal Product logic that is complete with respect to Kripke structures defined by crisp accessibility relation defined over the standard product chain $[0,1]_\Pi$. The results of the present paper seem to provide a good background to work towards the completeness proof for the above mentioned modal product logic. Moreover, knowing similar results for Gödel and Łukasiewicz modal logics it seems reasonable to think about completeness results for a modal BL logic.

Acknowlegdements

The authors acknowledge support of the Spanish projects EdeTRI (TIN2012-39348-C02-01) and AT (CONSOLIDER CSD 2007- 0022). Amanda Vidal is supported by a CSIC JAE Predoc grant.

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