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Relating adjoint negations with strong adjoint negations

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Abstract. Adjoint negations, whose definition is based on the implications of an adjoint triple, arise as a generalization of residuated negations. Recently, interesting properties of these negation operators have been introduced [5]. In this paper, a comparative survey with weak negations studied by Trillas, Esteva and Domingo [10, 13] is presented. Moreover, the relationship between weak and strong negations, introduced by these authors, is extended to adjoint negations. These technical developments lead us to increase the number of applications of adjoint negations.

Key words: residuated negations; weak and strong negations; adjoint triples.

1 Introduction

Negation operators play an important role in several frameworks and they have widely been studied in [8, 10, 20]. From residuated implications of a t-norm [4, 12, 19], it is defined the residuated negation defined from the residuated implication as $\neg x = x \to 0$. In addition, weak negations are one of the most general negation operators, which have heavily been studied by Trillas, Esteva and Domingo [10, 11, 13, 20]. In this paper, we will work with adjoint triples in order to consider more general negation operators.

Adjoint triples were firstly considered in [15, 18] taking into account the adjoint conjunctor and only one implication. They have been used as basic operators in Logic Programming [17], general substructural logics [3], Fuzzy Formal Concept Analysis [16], Fuzzy Relation Equations [9] and Rough Set Theory [7], providing more flexibility and increasing the range of applications.

From the implications of an adjoint triple, we define the generalization of the residuated negation which are called adjoint negations. Since they are associated with an adjoint triple with respect to three different posets, these negation operators are defined on two different posets. Dealing with this general structure is helpful in the applications as it has been highlighted in [1, 2, 9].

In this paper, we will compare adjoint negations with weak negations and we will show that adjoint negations are more general. Besides, a bijection between adjoint negations and strong adjoint negations will be presented, following the idea introduced by Trillas, Esteva and Domingo in [10, 13], in order to establish the relationship between adjoint negations and strong adjoint negations.

2 Adjoint negations and weak negations

Adjoint triples, which generalize triangular norms and their residuated implications [14], are considered to decrease the mathematical requirements of the basic operators used in several frameworks. In this paper, adjoint triples will be used in order to define adjoint negations. For that reason, we will start introducing the notion of adjoint triple.

Definition 1. *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *be posets and* $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ *be mappings, then* $(\&, \swarrow, \nwarrow)$ *is an* adjoint triple *with respect to* P_1, P_2, P_3 *if:*

$$x \leq_1 z \swarrow y \quad \textit{iff} \quad x \,\&\, y \leq_3 z \quad \textit{iff} \quad y \leq_2 z \nwarrow x \tag{1}$$

where $x \in P_1$, $y \in P_2$ *and* $z \in P_3$. *The condition* (1) *is called* adjoint property.

If adjoint triples are used in environments that require finiteness such as Fuzzy Formal Concept Analysis to obtain a finite concept lattice [6, 16] and Fuzzy Relation Equations to guarantee the existence of minimal solutions [9], then it is important that adjoint triples are defined on regular partitions of the unit interval $[0, 1]$.

Example 1. Given $m \in \mathbb{N}$, the set $[0, 1]_m$ is a regular partition of $[0, 1]$ in m pieces, for example $[0, 1]_2 = \{0, 0.5, 1\}$ divides the unit interval into two pieces.

A discretization of the Łukasiewicz t-norm is the operator $\&_{\mathrm{L}}^{*}\colon [0, 1]_{20} \times [0, 1]_8 \to [0, 1]_{100}$ defined, for each $x \in [0, 1]_{20}$ and $y \in [0, 1]_8$ as:

$$x \,\&_{\mathrm{L}}^{*}\, y = \frac{\lceil 100 \cdot \max(0, x + y - 1) \rceil}{100}$$

whose residuated implications $\swarrow_{\mathrm{L}}^{*}\colon [0, 1]_{100} \times [0, 1]_8 \to [0, 1]_{20}$, $\nwarrow_{\mathrm{L}}^{*}\colon [0, 1]_{100} \times [0, 1]_{20} \to [0, 1]_8$ are defined as:

$$z \swarrow_{\mathrm{L}}^{*} y = \frac{\lfloor 20 \cdot \min\{1, 1 - y + z\} \rfloor}{20} \qquad z \nwarrow_{\mathrm{L}}^{*} x = \frac{\lfloor 8 \cdot \min\{1, 1 - x + z\} \rfloor}{8}$$

where $\lceil \_ \rceil$ and $\lfloor \_ \rfloor$ are the ceiling and the floor functions, respectively. Hence, the triple $(\&_{\mathrm{L}}^{*}, \swarrow_{\mathrm{L}}^{*}, \nwarrow_{\mathrm{L}}^{*})$ is an adjoint triple. □

Now, we recall the definition of adjoint negations which is given from the implications of an adjoint triple and generalize the notion of residuated negation [4, 12, 19]. Adjoint negations are defined on two different posets since they are associated with an adjoint triple with respect to three different posets.

Definition 2. *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two posets, $(P_3, \leq_3, \bot_3)$ be a lower bounded poset and $(\&, \swarrow, \nwarrow)$ an adjoint triple with respect to P_1, P_2 and P_3. The mappings $n_n \colon P_1 \to P_2$ and $n_s \colon P_2 \to P_1$ defined, for all $x \in P_1$, $y \in P_2$ as*

$$n_n(x) = \bot_3 \nwarrow x \qquad n_s(y) = \bot_3 \swarrow y$$

are called adjoint negations *with respect to P_1 and P_2.*

The operators n_s and n_n satisfying that $x = n_s(n_n(x))$ and $y = n_n(n_s(y))$, for all $x \in P_1$ and $y \in P_2$, are called strong adjoint negations.

Considering the adjoint triple $(\&^*_{\mathrm{L}}, \swarrow^*_{\mathrm{L}}, \nwarrow^*_{\mathrm{L}})$ presented in Example 1, we introduce the next example of adjoint negations.

Example 2. The adjoint negations $n_s \colon [0,1]_8 \to [0,1]_{20}$ and $n_n \colon [0,1]_{20} \to [0,1]_8$ obtained from the adjoint triple $(\&^*_{\mathrm{L}}, \swarrow^*_{\mathrm{L}}, \nwarrow^*_{\mathrm{L}})$ are defined as:

$$n_s(y) = \frac{\lfloor 20 \cdot (1-y) \rfloor}{20} \qquad n_n(x) = \frac{\lfloor 8 \cdot (1-x) \rfloor}{8}$$

Observe that the choice of the posets is fundamental. If the adjoint conjunctor is defined as $\&^*_{\mathrm{L}} \colon [0,1]_k \times [0,1]_t \to [0,1]_p$, the corresponding adjoint negations will be $n_s \colon [0,1]_t \to [0,1]_k$ and $n_n \colon [0,1]_k \to [0,1]_t$. Therefore,

(i) If $t = k$, then it is easy to verify that n_s and n_n are strong adjoint negations.
(ii) If $t \neq k$, the obtained adjoint negations are not strong adjoint negations, in general. □

One of the most general negation operators are weak negations, which have widely been studied by Trillas and Esteva et al [10, 11, 13, 20]. In order to compare adjoint negations with weak negations, we will remind the next definition.

Definition 3 ([20]). *Given a mapping $n \colon [0,1] \to [0,1]$ is said to be a* weak negation *if the following conditions hold, for all $x, y \in [0,1]$.*

1. *$n(1) = 0$;*
2. *if $x \leq y$ then $n(y) \leq n(x)$;*
3. *$x \leq n(n(x))$.*

We will say that n is a strong negation *if the equality $x = n(n(x))$ holds, for all $x \in [0,1]$.*

The next theorem shows that adjoint negations are a generalization of weak negations.

Theorem 1. *If the mapping $n \colon [0,1] \to [0,1]$ is a weak negation, then there exists an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to the poset $([0,1], \leq)$ satisfying $n = n_s = n_n$.*

Once we have presented this result, we will study if the relation between weak and strong negations defined on a complete lattice studied in [10] can be extended to adjoint negations. This relationship ensures that weak negations can be defined uniquely from strong negations.

3 Relation between adjoint negations and strong adjoint negations

In this section, we will introduce the main result of this paper which proves that there exists an one to one correspondence between adjoint negations defined on two posets and strong adjoint negations defined on two complete meet-semilattices.

For that purpose, we will consider two posets $(P, \leq_P)$, $(Q, \leq_Q)$ and two complete meet-semilattices $(P', \preceq_{P'})$, $(Q', \preceq_{Q'})$ with maximum elements $\top_{P'}$ and $\top_{Q'}$, respectively, such that $P' \subseteq P$ and $Q' \subseteq Q$. From now on, the set of pair of adjoint negations (n_s, n_n) with respect to P and Q satisfying that $n_s(P) = Q'$ and $n_n(Q) = P'$, will be denoted as $N_{(P',Q')}(P, Q)$ and the set of pairs of strong adjoint negations (n'_s, n'_n) with respect to P' and Q' will be denoted as $SN(P', Q')$.

A bijection between $N_{(P',Q')}(P, Q)$ and $SN(P', Q')$ is obtained, as the following theorem shows.

Theorem 2. *There exists an one to one correspondence between $N_{(P',Q')}(P, Q)$ and $SN(P', Q')$.*

As a consequence, the next corollary is straighforwardly obtained.

Corollary 1. *Given a pair of strong adjoint negations (n'_s, n'_n) with respect to P' and Q', there exists a pair of adjoint negations (n_s, n_n) with respect to P and Q defined as:*

$$n_s(p) = n'_s(z_p) \quad \textit{with} \quad z_p = \bigwedge\nolimits_{P'}\{y \in P' \mid p \leq y\}$$
$$n_n(q) = n'_n(z_q) \quad \textit{with} \quad z_q = \bigwedge\nolimits_{Q'}\{x \in Q' \mid q \leq x\}$$

such that $n_{s_{|P'}} = n'_s$, $n_{n_{|Q'}} = n'_n$, and $n_s(P) = Q'$, $n_n(Q) = P'$.

There exist cases in which we can define only one pair of strong adjoint negations with respect to P' and Q'. Then, applying the previous theorem and corollary, only one pair of adjoint negations can be defined with respect to P and Q, as the following examples shows:

Example 3. Given $P' = \{p', \top_{P'}\}$ and $Q' = \{q', \top_{Q'}\}$. The unique pair of strong adjoint negations (n'_s, n'_n) with respect to $(P', \preceq_{P'})$ and $(Q', \preceq_{Q'})$, is defined as $n'_s(p') = \top_{Q'}$, $n'_s(\top_{P'}) = q'$ and $n'_n(q') = \top_{P'}$, $n'_n(\top_{Q'}) = p'$. Then, there exists only one pair of adjoint negations (n_s, n_n) with respect to P and Q, being $(P, \leq_P)$ and $(Q, \leq_Q)$ two posets with maximum elements $\top_P \in P$ and $\top_Q \in Q$, such that $n_s(P) = Q'$ and $n_n(Q) = P'$. By Corollary 1, n_s and n_n are defined as follows:

$$n_s(p) = \begin{cases} \top_{Q'} & \text{if} \quad p \leq_P p' \\ q' & \text{otherwise} \end{cases} \qquad n_n(q) = \begin{cases} \top_{P'} & \text{if} \quad q \leq_Q q' \\ p' & \text{otherwise} \end{cases}$$

for all $p \in P$ and $q \in Q$.

Example 4. Let $(P' = \{a, b, c\}, \preceq_{P'})$ and $(Q' = \{x, y, z\}, \preceq_{Q'})$ two complete meet-semilattices such that $a \preceq_{P'} b \preceq_{P'} c$ and $x \preceq_{Q'} y \preceq_{Q'} z$. The pair (n'_s, n'_n), defined as $n'_s(a) = z$, $n'_s(b) = y$, $n'_s(c) = x$ and $n'_n(x) = c$, $n'_n(y) = b$, $n'_n(z) = a$, is the unique pair of strong adjoint negations (n'_s, n'_n) with respect to P' and Q'. Therefore, applying Theorem 2, there exists only one pair of adjoint negations (n_s, n_n) with respect to P and Q, the posets given in Figure 1, such that $n_s(P) = Q'$ and $n_n(Q) = P'$. By Corollary 1, n_s and n_n are defined as follows:

$$n_s(p) = \begin{cases} z & \text{if} \quad p = a \\ y & \text{if} \quad p \in \{b, d\} \\ x & \text{if} \quad p = c \end{cases} \qquad n_n(q) = \begin{cases} c & \text{if} \quad q = x \\ b & \text{if} \quad q = y \\ a & \text{if} \quad q = z \end{cases}$$

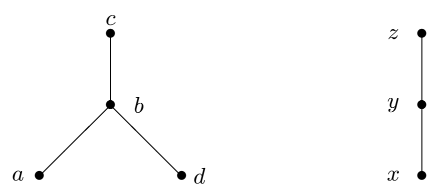

Fig. 1. The posets $(P, \leq_P)$ (left side) and $(Q, \leq_Q)$ (right side) of Example 4

4 Conclusions and further work

We have shown that adjoint negations are more general than weak negations studied by Trillas, Esteva and Domingo [10, 11, 13, 20]. Specifically, we have proven that every weak negation can be obtained from the implications of an adjoint triple. Moreover, an interesting generalization of the relation between weak and strong negations defined on a complete lattice studied in [10] has been presented. In this paper, a bijection between adjoint negations defined on two posets and strong adjoint negations defined on two complete meet-semilattices is shown.

As a further work, we will continue studying more properties of adjoint negations and possible applications of these operators. In addition, we will study the existence of an algorithm capable of computing the number of strong adjoint negations which can be defined on two complete meet-semilattices.

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