Base Belief Revision for finitary monotonic logics.*

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Abstract

We slightly improve on characterization results already in the literature for base revision. We show that in order to axiomatically characterize revision operators in a logic the only conditions this logic is required to satisfy are: finitarity and monotonicity. A characterization of limiting cases of revision operators, full meet and maxichoice, is also offered. We also distinguish two types of bases, naturally arising in the context of fuzzy logics.

1 Preliminaries

We introduce in this section the concepts and results needed for results. This section contains a brief exposition of partial meet belief change, and abstract and t-norm based fuzzy logics[1].

1.1 Partial meet Base Belief Change

Belief change is the study of how some theory T (non-necessarily closed, as we use the term) in a given language L can adapt to new incoming information $\varphi \in L$ (inconsistent with T, in the interesting case). The main operations are: *revision*, where the new input must follow from the revised theory, which is to be consistent, and *contraction* where the input must not follow from the contracted theory. In the classical paper [1], by Alchourrón, Gärdenfors and Makinson, partial meet revision and contraction operations were characterized for closed theories in, essentially, monotonic compact logics with the deduction property. Their work put in solid grounds this newly established area of research, opening the way for other formal studies involving new objects of change, operations (see [10] for a comprehensive list) or logics. Change operators can be defined by the following method, adapted from [1]. *Partial meet* consists in (i) generating all logically maximal ways to adapt T to the new sentence (those subtheories of T making further information loss logically unnecessary), (ii) selecting some of these possibilities, (iii) forming their meet, and, optionally, (iv) performing additional steps (if required by the operation). Then a set

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[1] We will use throughout the paper relational $\vdash_{\mathcal{S}}$ and functional $\mathrm{Cn}_{\mathcal{S}}$ notation indistinctively, where $\vdash_{\mathcal{S}}$ is a consequence relation and $\mathrm{Cn}_{\mathcal{S}}$ its associated closure operator.

of axioms is provided to capture these partial meet operators, by showing equivalence between satisfaction of these axioms and being a partial meet operator[2]. In addition, new axioms may be introduced to characterize the limiting cases of selection in step (ii), full meet and maxichoice selection types. Finally, results showing the different operation types can be defined each other are usually provided too. A *base* is an arbitrary set of formulas, the original requirement of logical closure being dropped. Base belief change for the same logical framework than AGM was characterized by Hansson (see [5], [6]). The results for contraction and revision were improved in [7] (by Hansson and Wassermann): for contraction (Theorem 3.8) it is shown that finitarity and monotony suffice, while for revision (Theorem 3.17) their proof depends on a further condition, *Non-contravention*: for all sentences φ, if $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T \cup \{\varphi\})$, then $\neg\varphi \in \mathrm{Cn}_{\mathcal{S}}(T)$. Observe this condition holds in logics having (i) the deduction property and (ii) the structural axiom of Contraction[3]. We show *Non-contravention* can be dropped in the characterization of revision if we replace unprovability (remainders) by consistency in the definition of partial meet.

1.1.1 Abstract fuzzy logics and t-norm based fuzzy logics.

Given a logic $\mathcal{S}$ with language $\mathbf{Fm}$, one can consider a fuzzy extension of it from some lattice of degrees W by considering the next elements: (i) a *fuzzy base* u as a function $u : \mathbf{Fm} \to W$ mapping each $\mathbf{Fm}$-formula φ some degree $u(\varphi) \in W$, obtaining a *signed* language $\mathcal{F}_{\mathbf{Fm}}(W)$; (ii) some fuzzy deduction operator $D_{\mathcal{S}} : \mathcal{F}_{\mathbf{Fm}}(W) \to \mathcal{F}_{\mathbf{Fm}}(W)$ mapping bases to deductively closed bases; (iii) additionally, a fuzzy semantics for the fuzzy deduction operator may be supplied. In [2], the authors consider fuzzy bases to be generated by associating values from some *complete distributive* lattice W, i.e. such that for any $U \subseteq W$, $\sup(U), \inf(U) \in W$ exist. The resulting revision operation is quite elegant, but in this setting one cannot define contraction to be sound w.r.t. the axiom of (Success), $(\varphi/r) \notin T \ominus (\varphi/r)$, due to base functionality.

In contrast, the original framework of fuzzy logics presenting them as axiomatic extensions of Hájek's Basic Logic BL (the latter capturing logical tautologies common to each t-norm based logic) constitute the most well-known fuzzy logics (see [4] for a reference); an additional advantage of this approach is direct definability of the corresponding graded logics; these are obtained by adding truth-constants $\overline{r}$ to the language, restricting their evaluation to $e(\overline{r}) = r$ and adding the so-called book-keeping axioms $\overline{r}\&\overline{s} \equiv \overline{r * s}$, $\overline{r} \to \overline{s} \equiv \overline{r \Rightarrow s}$, where $*$ is some t-norm and $\Rightarrow$ its residuum. Even if the real interval $[0, 1]$, taken as the set of truth-degrees, does not capture all complete distributive lattices W it is considered to be sufficiently general for most purposes.

We prove a characterization theorem for base revision in any finitary monotonic logic, so instead of having to prove compactness for fuzzy logics in a case-by-case fashion within Gerla's framework, as it is done in examples from [2], we can take any (finitary) logical calculus abounding in the literature and directly obtain a charac-

[2]Other known formal mechanisms defining change operators can be classified into two broad classes: *selection*-based mechanisms include selection functions on remainder sets and incision functions on kernels; *ranking*-based mechanisms include entrenchments and systems of spheres.

[3]If $T \cup \{\varphi\} \vdash_{\mathcal{S}} \varphi \to \overline{0}$, then by the deduction property $T \vdash_{\mathcal{S}} \varphi \to (\varphi \to \overline{0})$; i.e. $T \vdash_{\mathcal{S}} (\varphi\&\varphi) \to \overline{0}$. Finally, by modus ponens from the axiom of contraction, we obtain $T \vdash_{\mathcal{S}} \varphi \to \overline{0}$.

terization of the revision operation therein. This is the case, for instance, of logical calculi associated to fuzzy logics of some fundamental t-norm, as studied in Hájek's [4]. Also, we can directly deal with graded logics, since the fuzzy version (in the sense of [2]) of some propositional language $\mathbf{Fm}_{\mathrm{Prop}}$ gives us only signed languages (where truth-constants cannot appear within φ). Another advantage is the possibility of directly considering graded logics with added truth-constants from a *countable* (hence possibly non-complete) lattice $\mathcal{C}$, as e.g. $\mathcal{C} = [0,1] \cap \mathbb{Q}$ in the case of Rational Pavelka Logic Ł$([0,1]_{\mathbb{Q}})$.

2 Multiple base revision for finitary monotonic logics.

Without loss of generality, we assume the language to contain a constant $\overline{0}$ for *falsity*.

Definition 1. ([11], [2]) Given some monotonic logic $\mathcal{S}$, let T_0, T_1 be theories. We say T_0 is *consistent* if $T_0 \nvdash_{\mathcal{S}} \overline{0}$, and define the set of *subsets* of T_0 *maximally consistent with* T_1 as follows: $X \in Con(T_0, T_1)$ iff:

(i) $X \subseteq T_0$,
(ii) $X \cup T_1$ is consistent, and
(iii) For any X' such that $X \subsetneq X' \subseteq T_0$, we have $X' \cup T_1$ is inconsistent

Now we prove some properties of Con which will be helpful for the characterization theorems of base belief change operators for arbitrary finitary monotonic logics.

Lemma 2. *Let $\mathcal{S}$ be some finitary logic and T_0 a theory. For any $X \subseteq T_0$, if $X \cup T_1$ is consistent, then X can be extended to some Y with $Y \in \mathrm{Con}(T_0, T_1)$.*

Proof. Let $X \subseteq T_0$ with $X \cup T_1 \nvdash_{\mathcal{S}} \overline{0}$. Consider the poset $(T^*, \subseteq)$, where $T^* = \{Y \subseteq T_0 : X \subseteq Y \text{ and } Y \cup T_1 \nvdash_{\mathcal{S}} \overline{0}\}$. Let $\{Y_i\}_{i \in I}$ be a chain in $(T^*, \subseteq)$; that is, each Y_i is a subset of T_0 and consistent with T_1. Hence, $\bigcup_{i \in I} Y_i \subseteq T_0$; since $\mathcal{S}$ is finitary, $\bigcup_{i \in I} Y_i$ is also consistent with T_1 and hence is an upper bound for the chain. Applying Zorn's Lemma, we obtain an element Z in the poset with the next properties: $X \subseteq Z \subseteq T$ and Z maximal w.r.t. $Z \cup \{\varphi\} \nvdash_{\mathcal{S}} \overline{0}$. Thus $X \subseteq Z \in \mathrm{Con}(T, \varphi)$. □

Remark 3. Considering $X = \emptyset$ in the preceding lemma, we infer: if T_1 is consistent, then $\mathrm{Con}(T_0, T_1) \neq \emptyset$.

For simplicity, we assume that input base T_1 (to revise T_0 by) is consistent[4].

Definition 4. Let T_0 be a theory. A *selection function* for T_0 is a function

$$\gamma : \mathcal{P}(\mathcal{P}(\mathbf{Fm})) - \{\emptyset\} \longrightarrow \mathcal{P}(\mathcal{P}(\mathbf{Fm})) - \{\emptyset\}$$

such that for all $T_1 \subseteq \mathbf{Fm}$, $\gamma(\mathrm{Con}(T_0, T_1)) \subseteq \mathrm{Con}(T_0, T_1)$ and $\gamma(\mathrm{Con}(T_0, T_1))$ is non-empty.

[4] Observe one could define for T_1 inconsistent: $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, \{\overline{1}\})$, so in case T_0 was consistent this definition would make $\mathrm{Con}(T_0, T_1) = \{T_0\}$, and otherwise it would select consistent subtheories of T_0.

2.1 Base belief revision.

The axioms to characterize (multiple) base revision operators for finitary monotonic logics $\mathcal{S}$ are the following:

(F1)	$T_1 \subseteq T_0 \circledast T_1$	(Success)
(F2)	If T_1 is consistent, then $T_0 \circledast T_1$ is also consistent.	(Consistency)
(F3)	$T_0 \circledast T_1 \subseteq T_0 \cup T_1$	(Inclusion)
(F4)	For all $\psi \in \mathbf{Fm}$, if $\psi \in T_0 - T_0 \circledast T_1$ then, there exists T' with $T_0 \circledast T_1 \subseteq T' \subseteq T_0 \cup T_1$ and such that $T' \nvdash_{\mathcal{S}} \overline{0}$ but $T' \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0})$	(Relevance)
(F5)	If for all $T' \subseteq T_0 (T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0} \Leftrightarrow T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0})$ then $T_0 \cap (T_0 \circledast T_1) = T_0 \cap (T_0 \circledast T_2)$	(Uniformity)

Given some theory $T_0 \subseteq \mathbf{Fm}$ and selection function γ for T, we define partial meet revision operator $\circledast_\gamma$ for T_0 as follows:

$$T_0 \circledast_\gamma T_1 \;=\; \bigcap \gamma(\mathrm{Con}(T_0, T_1)) \cup T_1$$

Definition 5. Let $\mathcal{S}$ be some finitary logic, and T_0 a theory. Then $\circledast : \mathcal{P}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$ is a *revision operator* for T_0 iff $\circledast = \circledast_\gamma$ for some selection function γ for T_0.

Lemma 6. *Condition* $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, T_2)$ *is equivalent to the antecedent of Axiom* (F5)

$$\forall T' \subseteq T_0 \; (T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0} \Leftrightarrow T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0})$$

Proof. (If-then) Assume $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, T_2)$ and let $T' \subseteq T_0$ with $T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0}$. By Lemma 2, T' can be extended to $X \in \mathrm{Con}(T_0, T_1)$. Hence, by assumption we get $T' \subseteq X \in \mathrm{Con}(T_0, T_2)$ so that $T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0}$ follows. The other direction is similar. (Only if) This direction follows from the definition of $\mathrm{Con}(T_0, \cdot)$. □

Theorem 7. *Let $\mathcal{S}$ be a finitary monotonic logic. For any $T_0 \subseteq \mathbf{Fm}$, $T_1 \subseteq \mathbf{Fm}$ and function $\circledast : \mathcal{P}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$:*

$$\circledast \textit{ satisfies } (\mathrm{F1}) - (\mathrm{F5}) \text{ iff } T_0 \circledast T_1 = T_0 \circledast_\gamma T_1, \textit{ for some } \gamma$$

Proof. (Soundness) Given some partial meet revision operator $\circledast_\gamma$ for T_0, we prove $\circledast_\gamma$ satisfies (F1) – (F5). (F1) – (F3) hold by definition of $\circledast_\gamma$. (F4) Let $\psi \in T_0 - T_0 \circledast_\gamma T_1$. Hence, $\psi \notin T_1$ and for some $X \in \gamma(\mathrm{Con}(T_0, T_1))$, $\psi \notin X$. Simply put $T' = X \cup T_1$: by definitions of $\circledast_\gamma$ and Con we have (i) $T_0 \circledast_\gamma T_1 \subseteq T' \subseteq T_0 \cup T_1$ and (ii) T' is consistent (since T_1 is). We also have (iii) $T' \cup \{\psi\}$ is inconsistent (otherwise $\psi \in X$ would follow from maximality of X and $\psi \in T_0$, hence contradicting our previous step $\psi \notin X$). (F5) We have to show, assuming the antecedent of(F5), that $T_0 \cap (T_0 \circledast_\gamma T_1) = T_0 \cap (T_0 \circledast_\gamma T_2)$. We prove the $\subseteq$ direction only since the other is similar. Assume, then, for all $T' \subseteq T_0$,

$$T' \cup T_1 \nvdash_{\mathcal{S}} \overline{0} \;\Leftrightarrow\; T' \cup T_2 \nvdash_{\mathcal{S}} \overline{0}$$

and let $\psi \in T_0 \cap (T_0 \circledast_\gamma T_1)$. This set is just $T_0 \cap (\bigcap \gamma(Con(T_0, T_1)) \cup T_1)$ which can be transformed into $(T_0 \cap \bigcap \gamma(Con(T_0, T_1)) \cup (T_0 \cup T_1)$, i.e. $\bigcap \gamma(Con(T_0, T_1)) \cup (T_0 \cup T_1)$ (since $\bigcap \gamma(Con(T_0, T_1)) \subseteq T_0$). Case $\psi \in \bigcap \gamma(Con(T_0, T_1))$. Then we use Lemma 6 upon the assumption to obtain $\bigcap \gamma(Con(T_0, T_1)) = \bigcap \gamma(Con(T_0, T_2))$, since γ is a function. Case $\psi \in T_0 \cap T_1$. Then $\psi \in X$ for all $X \in \gamma(Con(T_0, T_1))$, by maximality of X. Hence, $\psi \in \bigcap \gamma(Con(T_0, T_1))$. Using the same argument than in the former case, $\psi \in \bigcap \gamma(Con(T_0, T_2))$. Since we also assumed $\psi \in T_0$, we obtain $\psi \in T_0 \cap (T_0 \circledast_\gamma T_2)$. (Completeness) Let $\circledast$ satisfy (F1) – (F5). We have to show that for some selection function γ and any T_1, $T_0 \circledast T_1 = T \circledast_\gamma T_1$. We define first

$$\gamma(\mathrm{Con}(T_0, T_1)) = \{X \in \mathrm{Con}(T_0, T_1) : X \supseteq T_0 \cap T_0 \circledast T_1\}$$

We prove that (1) γ is well-defined, (2) γ is a selection function and (3) $T_0 \circledast T_1 = T \circledast_\gamma T_1$.

(1) Assume (i) $\mathrm{Con}(T_0, T_1) = \mathrm{Con}(T_0, T_2)$; we have to prove that $\gamma(\mathrm{Con}(T_0, T_1)) = \gamma(\mathrm{Con}(T_0, T_2))$. Applying Lemma 6 to (i) we obtain the antecedent of (F5). Since $\circledast$ satisfies this axiom, we have (ii) $T_0 \cap T_0 \circledast T_1 = T_0 \cap T \circledast T_2$. By the above definition of γ, $\gamma(\mathrm{Con}(T_0, T_1)) = \gamma(\mathrm{Con}(T_0, T_2))$ follows from (i) and (ii).

(2) Since T_1 is consistent, by Remark 3 we obtain $\mathrm{Con}(T_0, T_1)$ is not empty; we have to show that $\gamma(\mathrm{Con}(T_0, T_1))$ is not empty either (since the other condition $\gamma(\mathrm{Con}(T_0, T_1)) \subseteq \mathrm{Con}(T_0, T_1)$ is met by the above definition of γ). We have $T_0 \cap T_0 \circledast T_1 \subseteq T_0 \circledast T_1$; the latter is consistent and contains T_1, by (F2) and (F1), respectively; thus, $(T_0 \cap T_0 \circledast T_1) \cup T_1$ is consistent; from this and $T_0 \cap T_0 \circledast T_1 \subseteq T_0$, we deduce by Lemma 2 that $T_0 \cap T_0 \circledast T_1$ is extensible to some $X \in \mathrm{Con}(T_0, T_1)$. Thus, exists some $X \in \mathrm{Con}(T_0, T_1)$ such that $X \supseteq T_0 \cap T_0 \circledast T_1$. In consequence, $X \in \gamma(\mathrm{Con}(T_0, T_1)) \neq \emptyset$.

For (3), we prove first $T_0 \circledast T_1 \subseteq T_0 \circledast_\gamma T_1$. Let $\psi \in T_0 \circledast T_1$. By (F3), $\psi \in T_0 \cup T_1$. Case $\psi \in T_1$: then trivially $\psi \in T_0 \circledast_\gamma T_1$ Case $\psi \in T_0$. Then $\psi \in T_0 \cap T_0 \circledast T_1$. In consequence, for any $X \in \mathrm{Con}(T_0, T_1)$, if $X \supseteq T_0 \cap T_0 \circledast T_1$ then $\psi \in X$. This implies, by definition of γ above, that for all $X \in \gamma(\mathrm{Con}(T_0, T_1))$ we have $\psi \in X$, so that $\psi \in \bigcap \gamma(\mathrm{Con}(T_0, T_1)) \subseteq T_0 \circledast_\gamma T_1$. In both cases, we obtain $\psi \in T_0 \circledast_\gamma T_1$.

Now, we prove the other direction: $T_0 \circledast_\gamma T_1 \subseteq T_0 \circledast T_1$. Let $\psi \in \bigcap \gamma(\mathrm{Con}(T_0, T_1)) \cup T_1$. By (F1), we have $T_1 \in T_0 \circledast T_1$; then, in case $\psi \in T_1$ we are done. So we may assume $\psi \in \bigcap \gamma(\mathrm{Con}(T_0, T_1))$. Now, in order to apply (F4), let X be arbitrary with $T \circledast T_1 \subseteq X \subseteq T_0 \cup T_1$ and X consistent. Consider $X \cap T_0$: since $T_1 \subseteq T_0 \circledast T_1 \subseteq X$ implies $X = X \cup T_1$ is consistent, so is $(X \cap T_0) \cup T_1$. Together with $X \cap T_0 \subseteq T_0$, by Lemma 2 there is $Y \in \mathrm{Con}(T_0, T_1)$ with $X \cap T_0 \subseteq Y$. In addition, since $T_0 \circledast T_1 \subseteq X$ implies $T_0 \circledast T_1 \cap T_0 \subseteq X \cap T_0 \subseteq Y$ we obtain $Y \in \gamma(\mathrm{Con}(T_0, T_1))$, by the definition of γ above. Condition $X \cap T_0 \subseteq Y$ also implies $(X \cap T_0) \cup T_1 \subseteq Y \cup T_1$. Observe that from $X \subseteq X \cup T_1$ and $X \subseteq T_0 \cup T_1$ we infer that $X \subseteq (X \cup T_1) \cap (T_0 \cup T_1)$. From the latter being identical to $(X \cap T_0) \cup T_1$ and the fact that $(X \cap T_0) \cup T_1 \subseteq Y \cup T_1$, we obtain that $X \subseteq Y \cup T_1$. Since $\psi \in Y \in \mathrm{Con}(T_0, T_1)$, we have $Y \cup T_1$ is consistent with ψ, so its subset X is also consistent with ψ. Finally, we may apply *modus tollens* on Axiom (F4) to obtain that $\psi \notin T_0 - T_0 \circledast T_1$, i.e. $\psi \notin T_0$ or $\psi \in T_0 \circledast T_1$. But since the former is false, the latter must be the case. $\square$

2.1.1 Full meet and maxichoice base revision operators.

The previous result can be extended to limiting cases of selection functions formally defined as follows:

Definition 8. A revision operator for T is *full meet* if it is generated by the selection function $\gamma_{\text{fm}} = Id$: $\gamma(\text{Con}(T_0, T_1)) = \text{Con}(T_0, T_1)$; that is,

$$T_0 \circledast_{\text{fm}} T_1 = (\bigcap \text{Con}(T_0, T_1)) \cup T_1$$

A revision operator for T_0 is *maxichoice* if it is generated by a selection function of type $\gamma_{\text{mc}}(\text{Con}(T_0, T_1)) = \{X\}$, for some $X \in \text{Con}(T_0, T_1)$, in which case $T_0 \circledast_{\gamma_{\text{mc}}} T_1 = X \cup T_1$.

To characterize *full meet* and *maxichoice* revision operators for some theory T_0 in any finitary logic, we define the next additional axioms:

$$\begin{array}{ll} \text{(FM)} & \text{For any } X \subseteq \mathbf{Fm} \text{ with } T_1 \subseteq X \subseteq T_0 \cup T_1 \\ & \quad X \nvdash_{\mathcal{S}} \overline{0} \text{ implies } X \cup (T_0 \circledast T_1) \nvdash_{\mathcal{S}} \overline{0} \\ \text{(MC)} & \text{For all } \psi \in \mathbf{Fm} \text{ with } \psi \in T_0 - T_0 \circledast T_1 \text{ we have} \\ & \quad T_0 \circledast T_1 \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0} \end{array}$$

Theorem 9. *Let $T_0 \subseteq \mathbf{Fm}$ and $\circledast$ be a function $\circledast : \mathcal{P}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$. Then the following hold:*

$$\begin{array}{ll} \text{(fm)} & \circledast \textit{ satisfies } \text{(F1)} - \text{(F5)} \textit{ and } \text{(FM) iff } \circledast = \circledast_{\gamma_{\text{fm}}} \\ \text{(mc)} & \circledast \textit{ satisfies } \text{(F1)} - \text{(F5)} \textit{ and } \text{(MC) iff } \circledast = \circledast_{\gamma_{\text{mc}}} \end{array}$$

Proof. We prove (fm) first. (Soundness): We know $\circledast_{\gamma_{\text{fm}}}$ satisfies (F1) – (F5) so it remains to be proved that (FM) holds. Let X be such that $T_1 \subseteq X \subseteq T_0 \cup T_1$ and $X \nvdash_{\mathcal{S}} \overline{0}$. From the latter and $X - T_1 \subseteq (T_0 \cup T_1) - T_1 \subseteq T_0$ we infer by Lemma 2 that $X - T_1 \subseteq Y \in \text{Con}(T_0, T_1)$, for some Y. Notice $X = X' \cup T_1$ and that for any $X'' \in \text{Con}(T_0, T_1) X'' \cup T_1$ is consistent and

$$T_0 \circledast_{\gamma_{\text{fm}}} T_1 = (\bigcap \text{Con}(T_0, T_1)) \cup T_1 \subseteq X' \subseteq X''$$

Hence $X \subseteq X''$, so that $T_0 \circledast_{\gamma_{\text{fm}}} T_1 \cup X \subseteq X''$. Since the latter is consistent, $T_0 \circledast_{\text{fm}} T_1 \cup X \nvdash_{\mathcal{S}} \overline{0}$. (Completeness) Let $\circledast$ satisfy (F1) – (F5) and (FM). It suffices to prove that $X \in \gamma(\text{Con}(T_0, T_1)) \Leftrightarrow X \in \text{Con}(T_0, T_1)$; but we already know that $\circledast = \circledast_{\gamma}$, for selection function γ (for T_0) defined by: $X \in \gamma(\text{Con}(T_0, T_1)) \Leftrightarrow T_0 \cap T_0 \circledast T_1 \subseteq X$. It is enough to prove, then, that $X \in \text{Con}(T_0, T_1)$ implies $X \supseteq T_0 \cap T_0 \circledast T_1$. Let $X \in \text{Con}(T_0, T_1)$ and let $\psi \in T_0 \cap T_0 \circledast T_1$. Since $\psi \in T_0$ and $X \in \text{Con}(T_0, T_1)$, we have by maximality of X that either $X \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$ or $\psi \in X$. We prove the former case to be impossible: assuming it we would have $T_1 \subseteq X \cup T_1 \subseteq T_0 \cup T_1$. By (FM), $X \cup T_1 \cup (T_0 \circledast T_1) \nvdash_{\mathcal{S}} \overline{0}$. Since $\psi \in T_0 \circledast T_1$, we would obtain $X \cup \{\psi\} \nvdash_{\mathcal{S}} \overline{0}$, hence contradicting the case assumption; since the former case is not possible, we have $\psi \in X$. Since X was arbitrary, $X \in \text{Con}(T_0, T_1)$ implies $X \subseteq T_0 \cap T_0 \circledast T_1$ and we are done.

For (mc): (Soundness) We prove (MC), since (F1) – (F5) follow from $\circledast_{\gamma_{\text{mc}}}$ being a partial meet revision operator. Let $X \in \text{Con}(T_0, T_1)$ be such that $T_0 \circledast_{\gamma_{\text{mc}}} \varphi = X \cup T_1$

and let $\psi \in T_0 - T_0 \circledast_{\gamma_{\rm mc}} T_1$. We have $\psi \notin X \cup T_1 = T_0 \circledast T_1$. Since $\psi \in T_0$ and $X \in \mathrm{Con}(T_0, T_1)$, $X \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$. Finally $T_0 \circledast T_1 \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$. (Completeness) Let $\circledast$ satisfy (F1) – (F5) and (MC). We must prove $\circledast = \circledast_{\gamma_{\rm mc}}$, for some maxichoice selection function $\gamma_{\rm mc}$. Let $X, Y \in \mathrm{Con}(T_0, T_1)$; we have to prove $X = Y$. In search of a contradiction, assume the contrary, i.e. $\psi \in X - Y$. We have $\psi \notin \bigcap \gamma(\mathrm{Con}(T_0, T_1))$ and $\psi \in X \subseteq T_0$. By MC, $T_0 \circledast T_1 \cup \{\psi\} \vdash_{\mathcal{S}} \overline{0}$. Since $T_0 \circledast T_1 \subseteq X$, we obtain $X \cup \{\psi\}$ is also inconsistent, contradicting previous $\psi \in X \nvdash_{\mathcal{S}} \overline{0}$. Thus $X = Y$ which makes $\circledast = \circledast_{\gamma_{\rm mc}}$, for some maxichoice selection function $\gamma_{\rm mc}$. □

2.2 Two types of bases.

The original definition of base is simply a set of formulas. In the context of graded (or signed) logics, although, an alternative notion of basehood naturally arises: $\mathcal{C}$-closed bases. We adapt the following definition from [5].

Definition 10. Given some monotonic logic $\mathcal{S}$ with language $\mathbf{Fm}$, let A and B be two subsets of $\mathbf{Fm}$. Then A is *B-closed* iff $\mathrm{Cn}_{\mathcal{S}}(A) \cap B \subseteq A$. We define $\mathrm{Cn}_{\mathcal{C}}(T) = \{(\varphi, r') : (\varphi, r) \in A, \text{ for } r, r' \in \mathcal{C} \text{ with } r \geq r'\}$, where $\mathcal{C}$ is some set of truth-constants. If $\mathcal{C} \subseteq [0,1]$ then a base $T \subseteq L(\mathcal{C})$ is $\mathcal{C}$-closed if T is $\mathrm{Cn}_{\mathcal{C}}(T)$ closed.

Observe [2]'s proposal forces us to work with Cn_W-closed bases, whenever $\mathbf{Fm}_{\rm Prop}$ is taken as the language to define some fuzzy deduction system $(\mathbf{Fm}_{\rm Prop}, W, D)$ in the sense of Gerla. The following results prove $\circledast_\gamma$ operators preserve $\mathcal{C}$-closure, hence Theorem 7 also applies to $\mathcal{C}$-closed bases.

Proposition 11. *If T_0, T_1 are $\mathcal{C}$-closed graded (or signed) bases, for any partial meet revision operator $\circledast_\gamma$, $T_0 \circledast_\gamma T_1$ is also a $\mathcal{C}$-closed graded (or signed) base.*

Proof. We prove the claim for graded bases, since the proof for signed bases is similar. Since T_0 is $\mathcal{C}$-closed, by maximality of $X \in \gamma(\mathrm{Con}(T_0, T_1))$ we have X is also $\mathcal{C}$-closed, for any such X. Let $(\psi, s) \in \bigcap \gamma(\mathrm{Con}(T_0, T_1))$ and $s' <_{\mathcal{C}} s$ for some $s' \in \mathcal{C}$. Then $(\psi, s) \in X$ for any $X \in \gamma(\mathrm{Con}(T_0, T_1))$ implies $(\psi, s') \in X$ for any such X. Hence $\bigcap \gamma(\mathrm{Con}(T_0, T_1))$ is $\mathcal{C}$-closed. Finally, since T_1 is $\mathcal{C}$-closed, we deduce $\bigcap \gamma(\mathrm{Con}(T_0, T_1)) \cup T_1$ is also $\mathcal{C}$-closed. □

Corollary 12. *Assume $\mathcal{S}$ and $\mathcal{C}$ are as before and let $\circledast$ be an operator $\circledast : \mathcal{P}(\mathbf{Fm}) \to \mathcal{P}(\mathbf{Fm})$ for some $\mathcal{C}$-closed graded bases T_0, T_1. Then,*

$$\circledast \textit{ satisfies } (\mathrm{F1}) - (\mathrm{F5}) \quad \textit{iff} \quad \textit{there is some selection function } \gamma \textit{ s.t. } T_0 \circledast T_1 = T_0 \circledast_\gamma T_1$$

At least for some logics related to Łukasiewicz t-norm, both approaches differ in the revision output. Hence, this distinction in basehood has important consequences.

Example 13. (In **RPL**) Let $\mathcal{C} = \mathbb{Q} \cap [0,1]$ and define $T_0 = \{(\varphi, 0.5), (\varphi, 0.7)\}$ and $T_1 = \mathrm{Cn}_{\mathcal{C}}(T_0)$. The only possible selection functions γ_0 and γ_1 result in:

$$\begin{array}{rcl} T_0 \circledast_{\gamma_0} (\neg\varphi, 0.4) & = & \{(\varphi, 0.5), (\neg\varphi, 0.4)\}, \text{ while} \\ T_1 \circledast_{\gamma_1} \mathrm{Cn}_{\mathcal{C}}(\{(\neg\varphi, 0.4)\}) & = & \mathrm{Cn}_{\mathcal{C}}(\{(\varphi, 0.6), (\neg\varphi, 0.4)\}) \end{array}$$

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