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Nilpotent Minimum logic preserving non-falsity and consistency operators

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***Abstract*—Nilpotent Minimum logic (NML) is one of the fuzzy logics of the family of t-norm based logics that is particularly interesting because it enjoys two important properties: its residual negation is involutive and it satisfies a form of deduction theorem. In this paper we study a paraconsistent companion of NML that captures a weak notion of logical consequence that preserves non-zero truth-values from the premises to the conclusions. Moreover, we also consider its expansion with a consistency operator in the sense of the logics of formal inconsistency (LFI).**

***Index Terms*—Fuzzy logic; Nilpotent minimum logic; paraconsistent logic; non-falsity preserving companion; consistency operators.**

I. Introduction

Nilpotent Minimum logic (NML for short) is a distinguished member of the family of formal systems of mathematical fuzzy logic (MFL) [11], introduced by two of the authors of this paper in [6] as a particular extension of the so-called Monoidal t-norm based fuzzy logic (MTL), a very general logic whose equivalent algebraic semantis is the variety of prelinear (commutative, bounded, integral) residuated lattices, also known as MTL-algebras, that is generated by the subclass of algebras with domain the real unit interval $[0, 1]$ and defined by left-continuous t-norms[1], see [13]. In fact, the logic NM was originally defined in [6] as the axiomatic extension of MTL by the involutive negation axiom

$$\text{(INV)} \quad \neg\neg\varphi \to \varphi$$

and the (weak) nilpotent minimum axiom

$$\text{(WNM)} \quad (\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi).$$

NML is an algebraizable logic, as all the axiomatic extensions of MTL, and the corresponding variety of NM-algebras is generated by a single algebra on the real unit interval $[0, 1]$, called *standard NM-algebra*, defined by the Nilpotent minimum t-norm and its residuum, see Section 2.

The NM logic together with all its axiomatic and finitary extensions has been exhaustively studied by Gispert in [8], [9]. They are all explosive with respect to its residual negation $\neg\varphi = \varphi \to 0$. This means that any theory T containing or deriving both a formula φ and its negation $\neg\varphi$ is contradictory, and hence it can derive any formula. In other words, the explosion rule with respect to $\neg$:

$$\text{(Exp)} \quad \frac{\varphi \quad \neg\varphi}{\bot}$$

is valid in NML. From a semantical point of view, this is so, because the only designated value for NML is truth-value 1: a formula φ is a logical consequence of a theory T in NML if, under any evaluation, φ gets value 1 whenever all the premises in T get value 1 as well. It is clear then that no evaluation can assign the truth-value 1 to both φ and $\neg\varphi$.

When the (Exp) rule with respect to a negation $\neg$ is not valid for a logic, the logic is called $\neg$-*paraconsistent*, that is, when deduction from theory having a contradiction does not immediately trivialise. Paraconsistent variants of fuzzy logics have been already studied under the paradigm of the so-called *truth-preserving* logics. In these logics, φ is a consequence of T if, under any evaluation, φ gets a value at least as high as all the values got by the premises in T. In the case of NML, this variant, denoted NML$^{\leq}$ is paraconsistent since $\varphi \wedge \neg\varphi$ can get a value greater than 0. In this paper our main aim is to study and characterise another paraconsistent variant of NML obtained by taking the semi-open interval $(0, 1]$ as set of designated values, i.e. when we only exclude the *falsum* truth-value. In this logic, that will be denoted nf-NML (nf for non-falsity), φ follows from T if, under any evaluation, φ does not get value 0 whenever all the premises in T do not get value 0. The logic nf-NML is paraconsistent as well, and it lies between NML and NML$^{\leq}$.

II. Preliminaries: the NM logic

The *nilpotent minimum logic*, NML for short, was firstly introduced by two of the authors in [6] in order to formalize the logic of the nilpotent minimum t-norm, that was defined by Fodor in [7] as an example of an involutive left continuous t-norm which is not continuous.[2]

The language of NML consists of countably many propositional variables $p_1, p_2, \ldots$, binary connectives $\wedge$ (weak or lattice conjunction), $*$ (strong conjunction), $\to$ (implication), and the truth constant $\overline{0}$. Formulas, which will be denoted by

[1]A t-norm $*$ is a binary operation in $[0, 1]$ which is commutative, associative, non-decreasing and having 1 as neutral element and 0 as absorbent elements.

[2]Actually, Pei showed later in [15] that NML and NM-algebras are equivalent to Wang's $\mathcal{L}^*$ logic and R_0-algebras, respectively [16], [17].

lower case greek letters $\varphi, \psi, \chi, \ldots$, are recursively defined from propositional variables, connectives and truth-constant as usual. Further definable connectives and constants are as follows: $\neg\varphi$ stands for $\varphi \to \overline{0}$ and $\overline{1}$ stands for $\neg\overline{0}$.

As already mentioned in the previous section, NML is defined as the axiomatic extension of the monoidal t-norm logic MTL, also introduced in [6], by the axioms

$$\text{(INV)} \quad \neg\neg\varphi \to \varphi$$
$$\text{(WNM)} \quad (\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi).$$

Axiom (INV) forces the negation to be involutive, and axiom (WNM) forces the strong conjunction $*$ to coincide with the lattice or weak conjunction $\wedge$ wherever it does not vanish. It is worth observing that NML enjoys the following form of deduction theorem:

$$\Gamma \cup \{\varphi\} \vdash_{NM} \psi \text{ iff } \Gamma \vdash_{NM} \varphi^2 \to \psi,$$

where φ^2 is a shorthand for $\varphi * \varphi$. It is well known that NML is algebraizable and the class $\mathbb{NM}$ of all nilpotent minimum algebras is its equivalent algebraic quasivariety semantics [6].

A *nilpotent minimum algebra* (*NM-algebra*) $\mathbf{A} = \langle A, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ is an involutive MTL-algebra (i.e. a bounded, commutative, integral, involutive, prelinear residuated lattice) that satisfies the following equation

$$\text{(WNM)} \quad (x * y \to \mathbf{0}) \vee (x \wedge y \to x * y) \approx \mathbf{1}.$$

We say that an NM-algebra is an NM-chain provided that its underlying lattice order (defined as $x \le y$ if $x \to y = \mathbf{1}$) is total. Since the class $\mathbb{NM}$ of all NM-algebras is a proper subvariety of MTL-algebras, it inherits the subdirect representation of MTL-algebras, and thus each NM-algebra is representable as a subdirect product of NM-chains (see [6, Proposition 3]).

NM-chains can be easily characterised. Namely, given a bounded totally ordered set $(A, \le)$, with upper bound $\mathbf{1}$ and lower bound $\mathbf{0}$, equipped with an involutive negation $\neg$, then defining for every $a, b \in A$,

$$a*b = \begin{cases} \mathbf{0}, & \text{if } b \le \neg a \\ a \wedge b, & \text{otherwise} \end{cases} \quad a \to b = \begin{cases} \mathbf{1}, & \text{if } a \le b \\ \neg a \vee b, & \text{otherwise,} \end{cases}$$

where $\wedge$ and $\vee$ denote meet and join in $(A, \le)$, it follows that $\mathbf{A} = \langle A, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ is an NM-chain. And moreover, every NM-chain is of this form. In particular, when $A = [0,1], \wedge = \min, \vee = \max$, and $\neg x = 1 - x$, then $\mathbf{A} = [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ is called the *standard NM-algebra*. It turns out that the variety $\mathbb{NM}$ is generated by the standard algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$, and this means that the logic NML is sound and complete w.r.t. the semantics given by evaluations of formulas on $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$.

Theorem 2.1: [6] For any set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{NM} \varphi$ iff, for every $[0,1]_{NM}$-evaluation e, if $e(\psi) = 1$ for all $\psi \in \Gamma$, then $e(\varphi) = 1$.

III. Non-falsity preserving companion of NML

We start by generalising the usual notion of 1-preserving logical consequence by considering the consequence relations $\models_{(a}$ and $\models_a$ that respectively preserve values above $a \in [0,1]$ both in a strict (for $a < 1$) and non-strict sense (for $0 > a$).

Definition 3.1: For any finite set of formulas $\Gamma \cup \{\varphi\}$, we define:

- $\Gamma \models_{(a} \varphi$ if, for any NM-evaluation e, if $e(\psi) > a$ for all $\psi \in \Gamma$, then $e(\varphi) > a$.
- $\Gamma \models_a \varphi$ if, for any NM-evaluation e, if $e(\psi) \ge a$ for all $\psi \in \Gamma$, then $e(\varphi) \ge a$.

Note that $\models_1$, by the above completeness result, coincides with $\vdash_{NM}$. Also, it is easy to check that $\models_{(a}$ is paraconsistent iff $a < 1/2$, while $\models_a$ is paraconsistent iff $a \le 1/2$. Moreover, one can show that many of these logics colapse.

Proposition 3.1: The following properties hold:

1) $\models_a$, $\models_{(a}$ and $\models_1$ are the same logic for all $a \in (1/2, 1)$.
2) $\models_a$, $\models_{(a}$ and $\models_{(0}$ are the same logic for all $a \in (0, 1/2)$.

In the rest of the section we axiomatise the logic $\models_{(0}$, that we will refer to as the *non-falsity preserving* companion of NML. We borrow the terminology of 'non-falsity preserving logic' from Avron [1], where the author considers a similar companion for Łukasiewicz logic, although in fact the logic defined there is the non-falsity preserving companion of only the $\{\wedge, \vee, \neg\}$-fragment of Łukasiewicz logic, and the idea of the proof is totally different.

Next lemma is a key observation that tightly relates both logics $\models_1$ and $\models_{(0}$ through the negation connective.

Lemma 3.1: For every formula φ,

$$\psi \models_{(0} \varphi \text{ iff } \neg\varphi \models_1 \neg\psi \quad (\text{iff } \models_1 (\neg\varphi)^2 \to \neg\psi)$$

In particular, $\models_{(0} \varphi$ if, and only if, $\models_1 \neg(\neg\varphi)^2$.

Proof: By definition, $\psi \models_1 \varphi$ iff for every NM-evaluation e, if $e(\psi) = 1$ then $e(\varphi) = 1$; that is, if $e(\varphi) < 1$ then $e(\psi) < 1$, for all e; that is, $e(\neg\varphi) > 0$ then $e(\neg\psi) > 0$, for all e; iff $\neg\varphi \models_{(0} \neg\psi$. ■

Now we define an axiomatic system aimed to syntactically capture the logical consequence $\models_0$ that preserves the non-falsity.

Definition 3.2: The *non-falsity preserving companion* of NML, denoted nf-NML, is the logic defined by the following axioms and rules:

- Axioms: those of NML
- The rule of Adjunction (Adj): from φ and ψ derive $\varphi \wedge \psi$
- The rule (r-MP^2): from φ and $\varphi \to \neg(\neg\psi)^2$ derive ψ, whenever $\vdash_{NM} \varphi \to \neg(\neg\psi)^2$

Finally, using the above lemma and some further adjustments, we can prove that nf-NML is sound and complete w.r.t. the intended semantics. Details can be found in the paper [10] currently under submission.

Theorem 3.1: For any set of formulas $\Gamma \cup \{\varphi\}$, it holds that $\Gamma \vdash_{\text{nf-NM}} \varphi$ iff $\Gamma \models_{(0} \varphi$.

Since NM-algebras are *locally finite* (i.e. the NM-subalgebra generated by a finite set of elements of a given NM-algebra is finite), the logic enjoys the finite model property, and thus it is decidable. Furthermore, due to the direct relation between $\models_{(0}$ and $\models_1$ shown in Lemma 3.1 above, the computational complexity of deciding whether a deduction $\Gamma \vdash_{\text{nf-NM}} \varphi$ holds, with Γ finite, is the same as in the case of the NML logic, which is known to be **coNP**-complete, see e.g. [12], [14].

IV. The nf-NM logic and consistency operators

Among the plethora of paraconsistent logics proposed in the literature, the Logics of Formal Inconsistency (LFIs), proposed in [4] (see also [2]), play an important role, since they internalize in the object language the very notion of consistency by means of a specific unary connective (primitive or not), usually denoted as $\circ$.

Definition 4.1: Let L be a logic defined in a language L containing a negation $\neg$ and a unary operator $\circ$ whose deduction system is denoted by $\vdash$. L is an LFI (with respect to $\neg$ and $\circ$) if the following conditions hold:

(i) $\varphi, \neg\varphi \not\vdash \psi$, for some formulas φ, ψ, i.e. L is not explosive w.r.t. $\neg$,
(ii) $\circ(\varphi), \varphi \not\vdash \psi$, for some formulas φ and ψ,
(iii) $\circ(\varphi), \neg\varphi \not\vdash \psi$, for some formulas φ and ψ, and
(iv) $\varphi, \neg\varphi, \circ(\varphi) \vdash \bot$, for every formula φ.

A consistency operator in an LFI logic can be primitive or it can be defined from other connectives of the language.

However, the nf-NM logic, although it is paraconsistent, it is not an LFI. Obviously, the consistency operator $\circ$ is not a primitive connective, but as we will show below it is not definable either. Anyway, similarly to what was done in the case of fuzzy logics preserving degrees of truth [5], we can expand nf-NML with a consistency operator $\circ$, that is, a unary operator such that the expanded logic satisfies the above properties (i)-(iv) and hence it becomes an LFI. This will be done in the rest of this section.

In the following, we will denote by $\mathcal{L}_\circ$ the expansion of the language of NML with $\circ$. And given a unary operator $\circ : [0,1] \to [0,1]$ we will denote by $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}^\circ}$ the the expansion of the standard algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ with $\circ$[3] and by $\models^\circ_{(0}$ the consequence relation that preserves non-falsity (defined as in Def. 3.1 for $a = 0$) but over the expanded language $\mathcal{L}_\circ$ and where evaluations interpret formulas on the expanded algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}^\circ}$.

We start by considering the most general semantical conditions on $\circ$ such that the logic $\models^\circ_{(0}$ is an LFI, that is, such that the following conditions are satisfied:

- $\circ\varphi, \varphi, \neg\varphi \models^\circ_{(0} \bot$
- $\varphi, \circ\varphi \not\models^\circ_{(0} \bot$
- $\neg\varphi, \circ\varphi \not\models^\circ_{(0} \bot$

It immediately follows that these conditions are satisfied if, and only if, in the algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}^\circ}$ the following conditions are in turn satisfied:

- for all $x \in [0,1]$, $x \wedge \neg x \wedge \circ x = 0$,
- there exists $x \in [0,1]$, such that $x \wedge \circ x \neq 0$,
- there exists $x \in [0,1]$, such that $\circ x \wedge \neg x \neq 0$,

which can be equivalently replaced by the next three conditions on $\circ$:

(C0) $\circ x = 0$ for all $x \in (0,1)$,
(1-NZ) $\circ 1 > 0$,
(0-NZ) $\circ 0 > 0$.

Definition 4.2: A unary operator $\circ : [0,1] \to [0,1]$ that satisfies conditions (C0), (1-NZ) and (0-NZ) will be called a *basic* consistency operator.

As we anticipated, such basic consistency operators are not definable in $[0,1]_{NM}$, and more generally in any NM-algebra. An argument for this claimn is the following. Since the 2-element Boolean $\mathbf{2}$ algebra over $\{0,1\}$ is a subalgebra of any NM-chain, if $\circ$ were definable (by a unary term), the only consistency operator that could be definable would be the one where $\circ(0) = \circ(1) = 1$, since this is the only compatible possibility when restricting $\circ$ to $\mathbf{2}$. Now, consider the NM-homomorphism $h : [0,1]_{NM} \to NM_3$, where NM_3 is the NM-subalgebra of $[0,1]_{NM}$ on the set $\{0,1/2,1\}$, defined as $h(x) = 1$ if $x > 1/2$, $h(1/2) = 1/2$ and $h(x) = 0$ if $x < 1/2$. Then it should be $h(\circ(x)) = \circ(h(x))$, but if $1/2 < x < 1$ or $0 < x < 1/2$, we have $h(\circ(x)) = 0$ while $\circ(h(x)) = 1$, contradiction.

In the following, we will call an element $x \in [0,1]$ *stricly positive* (SP) if $1/2 < x < 1$ and *strictly negative* (SN) if $0 < x < 1/2$.

It turns out that one cannot distinguish in $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ the case $\circ(0) = a$ from the case $\circ(0) = b$ if both a and b are SP or SN, because there exists an isomorphism f of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ such that $f(a) = b$. Therefore, from conditions (1-NZ) and (0-NZ) above, we are left only four significant cases to consider for the values $\circ(0)$ and $\circ(1)$, that can be characterized by equations and inequations in $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$. The proof is easy and thus omitted.

Proposition 4.1: For $\mathsf{x} \in \{0,1\}$, the following conditions hold:

[x-1] $\circ(\mathsf{x}) = 1$ is equivalently characterized by the equation $\neg(\circ(\mathsf{x})) = 0$,
[x-SP] $\circ(\mathsf{x}) \in (1/2, 1)$ is characterized by the inequation $(\circ(\mathsf{x}))^2 \wedge \neg(\circ(\mathsf{x})) > 0$,
[x-fix] $\circ(\mathsf{x}) = 1/2$ is characterized by the inequation $(\circ(\mathsf{x}) \leftrightarrow \neg(\circ(\mathsf{x})))^2 > 0$,
[x-SN] $\circ(\mathsf{x}) \in (0, 1/2)$ is characterized by the inequation $\circ(\mathsf{x}) \wedge (\neg(\circ(\mathsf{x})))^2 > 0$.

Combining these four conditions for $\mathsf{x} = 1$ and $\mathsf{x} = 0$, we obtain sixteen types of basic consistency operators $\circ$. In particular, the operator satisfying the conditions [1-1] and [0-1] is the maximal consistency operator $\circ_{max}$, i.e. the one such that $\circ_{max}(0) = \circ_{max}(0) = 1$.

Proposition 4.2: Two interesting properties of consistency operators are the following:

[3] Without risk of confusion, we will use the same symbol $\circ$ to denote the connective and a generic unary operation on the unit real interval $[0,1]$.

(i) The operator $\circ_{max}$ and Baaz-Monteiro's projection operator[4] Δ are interdefinable.
(ii) The maximal consistency operator $\circ_{max}$ (and the Δ operator) is definable from any of the sixteen types of consistency operators except from the one defined by the pair of conditions [0-SN] and [1-SN].

Proof: (i) To prove the first item it is enough to check that $\Delta(x) = \circ_{max}(x) \wedge x$ and also that $\circ_{max}(x) = \Delta(x \vee \neg x)$.

(ii) The second item can be proved by checking the following cases:

- if both $\circ(0), \circ(1) \geq 1/2$, then
  $\circ_{max}(x) = \neg((\neg(\circ(x)))^2)$ and $\Delta(x) = \circ_{max}(x) \wedge x$.
- if $\circ(1) \geq 1/2$ and $\circ(0) \in (0, 1/2)$, then
  $\Delta(x) = \neg((\neg(\circ(x))^2)^2)$ and $\circ_{max}(x) = \Delta(x \vee \neg x)$.
- finally, if $\circ(1) \in (0, 1/2)$ and $\circ(0) \geq 1/2$, then
  $\Delta(x) = \neg((\neg(\circ(\neg x))^2)^2)$ and $\circ_{max}(x) = \Delta(x \vee \neg x)$.

■

Note that the converse of the previous results does not hold in the sense that if we add $\circ_{max}$ to the algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$, it is not possible to recover the previous consistency operators, of course with the exception of $\circ_{max}$ itself, because Δ and $\circ_{max}$ are crisp operators (i.e. they only take values 0 or 1) and the operations of the algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ are classical when restricted to $\{0, 1\}$.

In the next section we focus our attention to the case of adding the maximal consistency operator $\circ_{max}$ to the logic nf-NML. The expansions of nf-NML with the remaining fifteen cases of consistency operators can be dealt in a similar way, except for the case of operators where both $\circ(0)$ and $\circ(1)$ are SN.

V. Expanding nf-NML with the maximal consistency operator $\circ_{max}$

In this final part of the paper we formally define and axiomatise the expansion of the logic nf-NM with the maximal consistency operator $\circ_{max}$, i.e. the basic consistency operator $\circ$ further satisfying:

[1-1] $\circ(1) = 1$
[0-1] $\circ(0) = 1$

As already noted before, the crucial observation is that, in this case, $\circ_{max}$ and the Baaz-Monteiro operator Δ are interdefinable: $\Delta(x) = \circ_{max}(x) \wedge x$, and $\circ_{max}(x) = \Delta(x \vee \neg x)$.

We start by axiomatising first the 1-preserving logic $\models_1^{\circ_{max}}$ and then, based on that, we will axiomatise the non-falsity preserving logic $\models_{(0}^{\circ_{max}}$. In the following we introduce the following abbreviation: $\Delta\varphi := (\circ\varphi) \wedge \varphi$.

Definition 5.1: $\text{NML}_\circ^{\max}$ is the logic defined by the following axioms and rules:

- Axioms of NML
- Consistency Axioms:
  (C0) $\neg(\circ\varphi \wedge \varphi \wedge \neg\varphi)$
  ($\top$-1) $\circ\top$
  ($\bot$-1) $\circ\bot$
- Modus ponens: (MP) $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$
- Congruence rule:
  (Cong) $\dfrac{(\varphi \leftrightarrow \psi) \vee \chi}{(\circ\varphi \leftrightarrow \circ\psi) \vee \chi}$

[4]Recall that the Baaz-Monteiro unary operator Δ on the unit interval $[0, 1]$ is defined as $\Delta(1) = 1$ and $\Delta(x) = 0$ for $x < 1$.

Observe that it is easy to check that the following three inference rules

$$\frac{\varphi}{\circ\varphi}, \quad \frac{\neg\varphi}{\circ\varphi}, \quad \frac{\varphi}{\Delta\varphi}$$

are derivable in $\text{NML}_\circ^{\max}$ from the axioms ($\top$-1) and ($\bot$-1) and the rule (Cong). Moreover, one can also check that the formula $\circ\varphi \vee \neg\circ\varphi$, stating that $\circ$ is a Boolean operator, can be proved to be a theorem of the logic as well: by applying the (Cong) rule to the axiom (C0), equivalently expressed as $\varphi \vee \neg\varphi \vee \neg\circ\varphi$, one gets $\circ\varphi \vee \neg\varphi \vee \neg\circ\varphi$, and applying (Cong) again, one gets $\circ\varphi \vee \circ\varphi \vee \neg\circ\varphi$, which is equivalent to $\circ\varphi \vee \neg\circ\varphi$.

Theorem 5.1: $\text{NML}_\circ^{\max}$ is a sound and complete axiomatisation of $\models_1^{\circ_{max}}$.

Proof: (Sketch) First of all, note that $\text{NML}_\circ^{\max}$ is an expansion of NM with axioms plus the (Cong) inference rule, so the logics keeps being algebraizable, and hence it is strongly complete with respect to the class (quasivariety) of $\text{NML}_\circ^{\max}$-algebras. Moreover, the (Cong) inference rule is closed by disjunctions (thanks to the addition of the clause '$\vee\chi$' in the premise and in the conclusion of the rule). Then, by results in [3], the quasi-variety of $\text{NM}_\circ^{\max}$-algebras is semilinear, that is, it is generated by its linearly ordered members. Hence, if an equation fails in an $\text{NML}_\circ^{\max}$-algebra, it also fails in a $\text{NML}_\circ^{\max}$-chain. The final observation is the fact that every embedding from a countable NM-chain into $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ (which always exists) easily extends to an embedding from a $\text{NM}_\circ^{\max}$-chain into the standard algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}_\circ^{\max}}$, hence if an equation fails in a $\text{NM}_\circ^{\max}$-chain it will aso fail in the standard chain $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}_\circ^{\max}}$. Therefore, if $\Gamma \not\models \varphi$ there will always exist an evaluation over an evaluation e on $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ such that $e(\Gamma) = 1$ and $e(\varphi) < 1$. ■

It is worth noticing that, from this completeness result, it follows that the set of axioms for the Δ operator (defined above as $\Delta\varphi := (\circ\varphi) \wedge \varphi$), as proposed e.g. in [11] to syntactically characterizing it, are provable in $\text{NML}_\circ^{\max}$, since they are obviously valid formulas for $\models_1^{\circ_{max}}$.

Now we move from the logic $\models_1^{\circ_{max}}$ to the paraconsistent logic $\models_{(0}^{\circ_{max}}$. Note that $\models_{(0}^{\circ_{max}}$ can be described in terms of $\models_1^{\circ_{max}}$ by using the Δ connective. Namely, it holds that

$$\begin{aligned} &\{\varphi_1, \ldots, \varphi_n\} \models_{(0}^{\circ_{max}} \psi \\ &\quad \text{iff } \{\nabla\varphi_1, \ldots, \nabla\varphi_n\} \models_1^{\circ_{max}} \nabla\psi, \\ &\quad \text{iff } \models_1^{\circ_{max}} (\nabla\varphi_1 \wedge \ldots \wedge \nabla\varphi_n) \to \nabla\psi, \\ &\quad \text{iff } \models_1^{\circ_{max}} \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \nabla\psi, \end{aligned}$$

where $\nabla = \neg\Delta\neg$. Indeed, for any evaluation e, it holds that $e(\varphi) > 0$ iff $e(\neg\Delta\neg\varphi) = 1$.

Now we introduce an axiomatic system for the logic $\models_{(0}^{\circ_{max}}$.

Definition 5.2: $\mathsf{nf}\text{-NML}_\circ^{\max}$ is the logic defined by the following axioms and rules:

- Axioms of $\text{NML}_\circ^{\max}$
- Rule of Adjunction: (Adj) $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- Restricted Modus Ponens:
  (r-MP) $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$, if $\vdash_{\text{NML}_\circ} \varphi \to \psi$
- Restricted Congruence:
  (r-Cong) $\dfrac{(\varphi \leftrightarrow \psi) \vee \chi}{(\circ\varphi \leftrightarrow \circ\psi) \vee \chi}$, if $\vdash_{\text{NML}_\circ} (\varphi \leftrightarrow \psi) \vee \chi$
- Reversed Necessitation for ∇:
  (r-∇Nec) $\dfrac{\nabla\varphi}{\varphi}$

Observe that the rule of necessitation for ∇:

$$(\nabla\text{Nec}) \quad \frac{\varphi}{\nabla\varphi},$$

which is the reverse of (r-∇Nec), is derivable. In fact, it is a direct consequence of the fact that, by definition, $\neg\Delta\neg\varphi = (\neg\circ\neg\varphi)\vee\varphi$. On the other hand, from (r-∇Nec) it easily follows that the rule

$$\frac{\neg\circ\neg\varphi}{\varphi},$$

is also derivable since clearly $\neg\circ\neg\varphi \to (\neg\circ\neg\varphi) \vee \varphi$ is a theorem of $\text{NML}_\circ^{\max}$.

Theorem 5.2: $\mathsf{nf}\text{-NML}_\circ^{\max}$ is a sound and complete axiomatisation of $\models_{(0}^{\circ_{max}}$.

Proof: Suppose $\{\varphi_1, \ldots, \varphi_n\} \models_{(0}^{\circ_{max}} \psi$. Then, as observed above, this holds iff $\models_1^{\circ_{max}} \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \nabla\psi$, and by completeness of $\text{NML}_\circ^{\max}$, iff $\vdash_{NM_\circ^{\max}} \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \nabla\psi$. Therefore, in $\text{NML}_\circ^{\max}$ there is a proof

$$\Pi_1, \ldots, \Pi_r = \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \nabla\psi,$$

where each Π_i (with $1 \leq i < r$) is either an axiom of $\text{NML}_\circ^{\max}$, it has been obtained from a previous Π_k by the (Cong) rule, or has been obtained from previous Π_k, Π_j $(k, j < r)$ by the application of Modus ponens rule. Then, in order to get a proof of φ from $\psi_1, \ldots, \psi_n$ in $\mathsf{nf}\text{-NML}_\circ^{\max}$ we only need do the following:

(i) add two previous steps Π_0^1 and Π_0^2, where
  - $\Pi_0^1 = \varphi_1 \wedge \ldots \wedge \varphi_n$, obtained from the premises by the (Adj) rule,
  - $\Pi_0^2 = \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n)$, obtained from Π_0^1 by the (∇Nec) rule

(ii) add two final steps Π_{r+1} and Π_{r+2}, where
  - $\Pi_{r+1} = \neg\Delta\neg\psi$, obtained by the application of the (r-MP) rule to Π_0 and Π_r, and
  - $\Pi_{r+2} = \psi$, obtained by applying the rule (r-ΔNec) to Π_{r+1}.

Therefore, the sequence $\Pi_0^1, \Pi_0^2, \Pi_1, \ldots \Pi_r, \Pi_{r+1}, \Pi_{r+2}$ is a proof of ψ from $\{\varphi_1, \ldots, \varphi_n\}$ in the logic $\mathsf{nf}\text{-NML}_\circ^{\max}$, with the proviso that the applications of the modus ponens and the (Cong) rules in the original proof $\Pi_1, \ldots \Pi_r$ in $\text{NML}_\circ^{\max}$ have to be replaced now by applications of the corresponding restricted rules (r-MP) and (r-Cong). ■

VI. Conclusions

The Nilpotent Minimum logic NML is an axiomatic extension of the Monoidal t-norm based fuzzy logic MTL that enjoys nice properties. In this paper we have explored the definition and axiomatisation of the logic nf-NML, the non-falsity preserving companion of the Nilpotent Minimum logic. nf-NML is a $\neg$-paraconsistent logic, but it does not belong to the family of well-behaved paraconsistent logics known as Logics of Formal Inconsistency. These logics are characterised by having in its language a unary connective $\circ$ (primitive or definable) by which one can internalize in the object language the notion of consistency. To remedy this problem we have considered expanding nf-NML with a consistency operator, and have presented a complete axiomatic system for this expansion in the particular case of the maximum consistency operator $\circ_{\max}$. Nevertheless, let us notice that the same kind of approach could be used to define the logics corresponding to each of the remaining fourteen basic consistency operators described in Proposition 4.1 for which the Δ operator is definable, see (ii) of Proposition 4.2. To do this, in Definition 5.1 one has to:

(1) Replace axioms ($\top$-1) and ($\bot$-1) respectively by suitable axioms corresponding to any pair of conditions [x-SP], [x-fix], [x-SN].

(2) Change the working definition of Δ in terms of $\circ$ in Definition 5.1 (i.e. $\Delta\varphi := (\circ\varphi) \wedge \varphi$) in each case according to the three expressions shown in the proof of Proposition 4.2.

As future work we plan to generalise the approach to other axiomatic extensions of MTL with a (primitive or definable) involutive negation.

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References

[1] A. Avron. Paraconsistent fuzzy logic preserving non-falsity. *Fuzzy Sets and Systems* 292, 75-84, 2016.

[2] W. Carnielli, M.E. Coniglio. *Paraconsistent Logic: Consistency, Contradiction and Negation.* Vol. 40 of Logic, Epistemology, and the Unity of Science, Springer, 2016.

[3] P. Cintula, C. Noguera C. A general framework for mathematical fuzzy logic. In: Cintula P, Hájek P, Noguera C (eds) *Handbook of Mathematical Fuzzy Logic - Volume 1.* Studies in logic, Mathematical Logic and Foundations, vol 37. College Publications, London, pp. 103-207.

[4] W. Carnielli, M.E. Coniglio, and J. Marcos. Logics of Formal Inconsistency. In Dov Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic* (2nd. edition), volume 14, pages 1–93. Springer, 2007.
[5] M.E. Coniglio, F. Esteva, L. Godo. Logics of formal inconsistency arising from systems of fuzzy logic, *Logic Journal of the IGPL* 22, n. 6: 880–904, 2014.
[6] F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124, 271–288, 2001.
[7] J. Fodor, *Nilpotent minimum and related connectives for fuzzy logic*, Proc. FUZZ–IEEE '95, 1995, pp. 2077–2082.
[8] J. Gispert. Axiomatic extensions of the nilpotent minimum logic. *Reports on Mathematical Logic* 37: 113-123, 2003
[9] J. Gispert. Finitary Extensions of the Nilpotent Minimum Logic and (Almost) Structural Completeness. *Studia Logica* 106(4): 789-808 (2018)
[10] J. Gispert, F. Esteva, L. Godo. On Nilpotent Minimum logics defined by lattice filters. Submitted
[11] P. Hájek. *Metamathematics of Fuzzy Logic*. Vol. 4 of Trends in Logic - Studia Logica Library. Kluwer, 1998.
[12] Z. Haniková. Computational complexity of propositional fuzzy logics. Chapter X of *Handbook of Mathematical Fuzzy Logic*, Volume 2, College Publications, pp. 793–851, 2011.
[13] S. Jenei and F. Montagna. A completeness proof of Esteva and Godo's MTL logic. *Studia Logica* 70: 183-192, 2002.
[14] E. Marchioni. On computational complexity of semilinear varieties. *Journal of Logic and Computation*, 18(6): 941-958, 2008.
[15] D. Pei. On equivalent forms of fuzzy logic systems NM and IMTL. *Fuzzy Sets and Systems* 138, 187-195, 2003.
[16] G.J. Wang. A formal deductive system for fuzzy propositional calculus, *Chinese Sci. Bull.* 42, 1521-1526, 1997.
[17] G.J. Wang. On the logic foundation of fuzzy reasoning, *Inform. Sci.* 117, 47-88, 1997.