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Some Categorical Equivalences for Nelson Algebras with Consistency Operators

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Abstract

The aim of this paper is to present categorical equivalences involving Nelson algebras with a consistency operator. These algebraic structures are the algebraic semantics of a paraconsistent logic, actually a logic of formal inconsistency, based on Nelson logic, also known as constructive logic with strong negation. In particular, we will extend a well-known relationship between Nelson algebras/lattices and Heyting algebras with a boolean filter to these expanded structures in terms of categorical equivalences.

Keywords: Nelson algebras, Heyting algebras, Consistency operators, Categorical equivalence.

1 Introduction

In a recent paper [5], the authors have studied logics of formal inconsistency (LFIs) that can be defined as degree-preserving companions of logics of (bounded, integral, commutative) distributive involutive residuated lattices (dIRLs) with a consistency operator. Special attention is paid to the class of Nelson lattices, the subvariety of dIRLs satisfying the following equation

$$(((x*x) \to y) \wedge ((\sim y * \sim y) \to \sim x)) \to (x \to y) = 1,$$

called Nelson equation. Nelson lattices are term equivalent to the so-called Nelson algebras, that are the algebraic structures arising when an involutive negation $\sim$ is added to Heyting algebras, related to Nelson logic, also known as constructive logic with strong negation. Note that, in the prelinear case, Nelson lattices become Nilpotent Minimum algebras while Heyting algebras become Gödel algebras, two well-known varieties of algebras falling within the hierarchy of algebraic structures related to mathematical fuzzy logic systems.

LFIs is a family of paraconsistent logics that internalize in the object language a notion of consistency by means of a specific connective $\circ$ (primitive or definable) with the following intended meaning: although LFIs are non-explosive logics in general, the connective $\circ$ allows to recover the explosion property from a formula φ and its negation $\neg\varphi$ whenever they are deemed to be consistent, in the sense of φ falling under the scope of $\circ$. In this paper, the algebraic counterpart of the consistency connective in the class of dIRLs, where the equation $x \wedge \sim x = 0$ is not valid in general, will be played by unary operators in dIRL-algebras, that we will call *consistency operators*, satisfying the following minimal properties:

($\circ$1) $x \wedge \sim x \wedge \circ(x) = 0$

($\circ$2) if $x \wedge \sim x \wedge y = 0$ then $y \leq \circ(x)$

Condition ($\circ$1) stands for the requirement that x and $\sim x$ are explosive when put together with $\circ(x)$. Finally, condition ($\circ$2) guarantees that $\circ(x)$ is the maximum value satisfying ($\circ$1). Also we will consider *boolean* consistency operators, i.e. those operators that also satisfy the Booleanity condition:

($\circ$3) $\circ(x) \vee \sim \circ(x) = 1$.

In [5], the extensions of dIRLs with consistency operators are studied from a general algebraic point of view. In this paper we focus on the class of Nelson lattices (or algebras) with consistency operators and we extend to these expanded structures a well-known relationship between Nelson algebras/lattices and Heyting algebras with a boolean filter in terms of categorical equivalences.

The paper is structured as follows. After this introduction, some needed algebraic preliminaries are gathered in Section 2. In Section 3 we formally define Nelson algebras with consistency operators and prove some basic results. Then, in Section 4, we present some first algebraic relationships between Nelson algebras with consistency operators and Heyting algebras with dual pseudocomplement. These algebraic results are lift to a categorical equivalence in Section 5, where we prove that the algebraic category of Nelson algebras with a consistency operator is equivalent to a category whose objects are pairs $(\mathbf{H}, F)$ where $\mathbf{H}$ is a Heyting algebra and F is a boolean filter of $\mathbf{H}$ such that every element of F admits a dual pseudocomplement. In Section 6 we study some particular cases and in particular we focus on categorical equivalences for prelinear algebras. Finally, in Section 7 we conclude and present our future work on this subject.

2 Algebraic preliminaries

The algebraic structures that will be central to this paper are Nelson algebras defined as follows

Definition 2.1. A *Nelson algebra* is a system $\mathbf{A} = (A, \vee, \wedge, \Rightarrow, \sim, 0, 1)$ of type $(2,2,2,1,0,0)$ such that its reduct $(A, \vee, \wedge, \sim, 0, 1)$ is a Kleene algebra and $\Rightarrow$ satisfies the following conditions for all $x, y, z \in A$:

(N1) $x \Rightarrow x = 1$;

(N2) $x \wedge (x \Rightarrow y) = x \wedge (\sim x \vee y)$;

(N3) $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$;

(N4) $x \Rightarrow (y \Rightarrow z) = (x \wedge y) \Rightarrow z$.

Notice that, for every Nelson algebra $\mathbf{A} = (A, \vee, \wedge, \Rightarrow, \sim, 0, 1)$ the unary operator $\sim$ is an involutive negation. Moreover, one can define, in $\mathbf{A}$ an additional negation operator as $\neg x = x \Rightarrow 0$. In what follows it will be convenient to adopt the redundant signature $(\wedge, \vee, \Rightarrow, \neg, \sim, 0, 1)$ for Nelson algebras.

Nelson algebras form a variety which is isomorphic to that of Nelson *lattices* (cf. [3, Theorem 3.11]), that is, bounded distributive commutative integral residuated lattices $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ satisfying the so called Nelson equation:

$$(((x * x) \rightarrow y) \wedge ((\sim y * \sim y) \rightarrow \sim x)) \rightarrow (x \rightarrow y) = 1$$

where $\sim$ stands for the residual negation, that is, $\sim x$ stands for $x \rightarrow 0$.

The second class of algebras that will play a main role in this paper is that of Heyting algebras, that constitute the algebraic semantics of intuitionistic logic [9].

Definition 2.2. A *Heyting algebra* $\mathbf{H} = (H, \vee, \wedge, *, \rightarrow, \perp, \top)$ is a bounded distributive commutative integral residuated lattice such that $x * y = x \wedge y$ for all $x, y \in H$.

By the very definition of a Heyting algebra $\mathbf{H}$, the $*$ operator is redundant in the signature. Furthermore, one can always define the residual negation operator $\neg x = x \rightarrow \perp$. For convenience, we will henceforth consider Heyting algebras in the signature $(\vee, \wedge, \rightarrow, \neg, \perp, \top)$.

For every Nelson algebra $\mathbf{A} = (A, \vee, \wedge, \Rightarrow, \neg, \sim, 0, 1)$ and for every $a \in A$, let $a^* = \sim\neg a$. Define $A^* = \{a^* \mid a \in A\}$ with operations $a \star^* b = (a \star b)^*$ for all binary operations $\star$, and $\neg^* a = (\neg a)^*$. Then the algebra $\mathbf{A}^* = (A^*, \vee^*, \wedge^*, \Rightarrow^*, \neg^*, 0, 1)$ is a Heyting algebra.

For every Heyting algebra $\mathbf{H}$, a filter F of $\mathbf{H}$ is said to be *Boolean* provided that the quotient structure $\mathbf{H}/F$ is a boolean algebra. Let hence $\mathbf{H}$ be a Heyting algebra and F a Boolean filter of $\mathbf{H}$. Define:

$$N_F(\mathbf{H}) = \{(a, b) \in H \times H \mid a \wedge b = \perp \text{ and } (a \vee b) \in F\}.$$

Consider operations on $N_F(\mathbf{H})$ as follows:

- $(a, b) \vee (c, d) = (a \vee c, b \wedge d)$
- $(a, b) \wedge (c, d) = (a \wedge c, b \vee d)$
- $(a, b) \Rightarrow (c, d) = (a \rightarrow c, a \wedge d)$
- $\neg(a, b) = (\neg a, a)$
- $\sim(a, b) = (b, a)$

Then we have the following relationships between Nelson and Heyting algebras.

Theorem 2.3 ([11])**.** *(1) For each Heyting algebra* $\mathbf{H}$ *and boolean filter* F *of* $\mathbf{H}$*, the structure* $\mathbf{N}_F(\mathbf{H}) = (N_F(\mathbf{H}), \vee, \wedge, \Rightarrow, \neg, \sim, (\perp, \top), (\top, \perp))$ *is a Nelson algebra such that* $\mathbf{N}_F(\mathbf{H})^*$ *is isomorphic to* $\mathbf{H}$.

(2) For every Nelson algebra $\mathbf{A}$ *there is a boolean filter* F *on* $\mathbf{A}^*$ *such that* $\mathbf{A}$ *and* $\mathbf{N}_F(\mathbf{A}^*)$ *are isomorphic.*

Furthermore the following holds.

Lemma 2.4 ([4, Theorem 5.2])**.** *For every Nelson algebra* $\mathbf{A}$*, the lattice* $Fil(\mathbf{A})$ *of its filters is isomorphic to the lattice* $Fil(\mathbf{A}^*)$ *of filters of the Heyting algebra* $\mathbf{A}^*$*. As a consequence the lattice* $Con(\mathbf{A})$ *of congruences of* $\mathbf{A}$ *and* $Con(\mathbf{A}^*)$ *of congruences of* $\mathbf{A}^*$ *are isomorphic as well.*

Therefore we can prove the following properties relating Nelson algebras $\mathbf{A}$ and their corresponding Heyting algebras $\mathbf{A}^*$.

Theorem 2.5. *For every Nelson algebra* $\mathbf{A}$ *the following properties hold:*

1. $\mathbf{A}$ *is subdirectly irreducible iff* $\mathbf{A}^*$ *is subdirectly irreducible;*

2. **A** *is directly indecomposable iff* $\mathbf{A}^*$ *is directly indecomposable.*

Proof. (1) is [4, Corollary 5.3]. Let us hence prove (2). Recall that an algebra is directly indecomposable (d.i. for short) iff its unique pair of factor congruences is (Δ,∇). Assume that **A** is not d.i. and let (Θ,Θ') be a non trivial pair of factor congruences. Let λ be the isomorphism between $Con(\mathbf{A})$ and $Con(\mathbf{A}^*)$ of Lemma 2.4. Thus, since both **A** and $\mathbf{A}^*$ are congruence permutable, $(\lambda(\Theta),\lambda(\Theta'))$ is a non trivial pair of factor congruences of $\mathbf{A}^*$, whence $\mathbf{A}^*$ is not d.i. as well. The other direction can be proven in an analogous way. □

3 Nelson algebras with consistency operators

Definition 3.1. A *Nelson algebra with consistency operator* is a pair $(\mathbf{A},\circ)$ where **A** is a Nelson algebra and $\circ : A \to A$ satisfies the following conditions:

($\circ$1) $\circ(x) = \max\{z \in A \mid x \wedge {\sim}x \wedge z = 0\}$

($\circ$2) $x \wedge {\sim}x \wedge \circ(x) = 0$

The consistency operator $\circ$ is said to be *boolean* provided that is also satisfies the following condition:

($\circ$3) $\circ(x) \vee {\sim}\circ(x) = 1$.

For convenience, let us represent **A** as $\mathbf{N}_F(\mathbf{H})$ for **H** being a Heyting algebra and F a boolean filter of **H** as in Theorem 2.3. Thus, every $x \in A$ is of the form (a,b) for $a,b \in H$ and $a \wedge b = \bot$ and $(a \vee b) \in F$ and the above condition ($\circ$1) can be reformulated in the following way: for all $(a,b) \in \mathbf{A}$,

$$\begin{aligned}
&\circ(a,b)\\
&= \max\{(z,z') \in A \mid (a,b) \wedge {\sim}(a,b) \wedge (z,z') = (\bot,\top)\}\\
&= \max\{(z,z') \in A \mid (a,b) \wedge (b,a) \wedge (z,z') = (\bot,\top)\}\\
&= \max\{(z,z') \in A \mid (a \wedge b, b \vee a) \wedge (z,z') = (\bot,\top)\}\\
&= \max\{(z,z') \in A \mid (\bot, a \vee b \vee z') = (\bot,\top)\},
\end{aligned}$$

where the last equality holds because $a \wedge b = \bot$ (in **H**) for all $(a,b) \in A$. Therefore, since the order in **A** is defined, with respect to the order of **H**, as $(a,b) \le (c,d)$ iff $a \le c$ and $b \ge d$, one has that

$$\begin{aligned}
&\circ(a,b) = (c,d) \text{ iff}\\
&\quad d = \min\{z' \in H \mid a \vee b \vee z' = \top\} \text{ and}\\
&\quad c = \max\{z \mid z \wedge d = \bot, (z \vee d) \in F\}.
\end{aligned} \tag{1}$$

Proposition 3.2. *Let* **H** *be a Heyting algebra,* F *a boolean filter and* **A** *the Nelson algebra* $\mathbf{N}_F(\mathbf{H})$*. Then, the operator* $\circ : (a,b) \mapsto (c,d)$ *where* c *and* d *are as in Equation (1) satisfies* $\circ(a,b) = \max\{(z,z') \in A \mid (a,b) \wedge {\sim}(a,b) \wedge (z,z') = (\bot,\top)\}$.

Proof. Let us first notice that $\{(z,z') \in A \mid (a,b) \wedge {\sim}(a,b) \wedge (z,z') = (\bot,\top)\} = \{(z,z') \in H \times H \mid z \wedge z' = 0, (z \vee z') \in F, a \vee b \vee z' = \top\}$. Assume, by way of contradiction, that $\circ(a,b) = (c,d) \ne \max\{(z,z') \in H \times H \mid z \wedge z' = 0, (z \vee z') \in F, a \vee b \vee z' = \top\}$. Thus there exists $(e,f) \in H \times H$ with $(e,f) > (c,d)$ and such that

(a) $a \vee b \vee f = \top$;

(b) $e \wedge f = \bot$;

(c) $(e \vee f) \in F$.

By the absurdum hypothesis that $(e,f) > (c,d)$ it follows that $f \le d$. If $f < d$, together with (a), we reach a contradiction because $d = \min\{z' \in H \mid a \vee b \vee z' = \top\}$. Thus assume that $f = d$ and hence $(e,f) = (e,d) > (c,d)$ gives that $e > c$. By (b) and (c), $e \wedge d = \bot$ and $(e \vee d) \in F$. Therefore $e > c$ and both belong to $\{z \mid z \wedge d = \bot, (z \vee d) \in F\}$ (with the same d). Thus $c \ne \max\{z \mid z \wedge d = \bot, (z \vee d) \in F\}$ which contradicts our hypothesis. □

The following example should clarify the above claim.

Example 3.3. Let **H** be the 5 elements directly indecomposable Heyting algebra (see the left hand-side of Fig. 1) and let F be the boolean filter $\uparrow a = \{a,b,\top\}$. Then,

$$N_F(\mathbf{H}) = \{(\bot,\top),(\bot,b),\\ (\bot,a),(a,\neg a),(\neg a,a),(a,\bot),(b,\bot),(\top,\bot)\}.$$

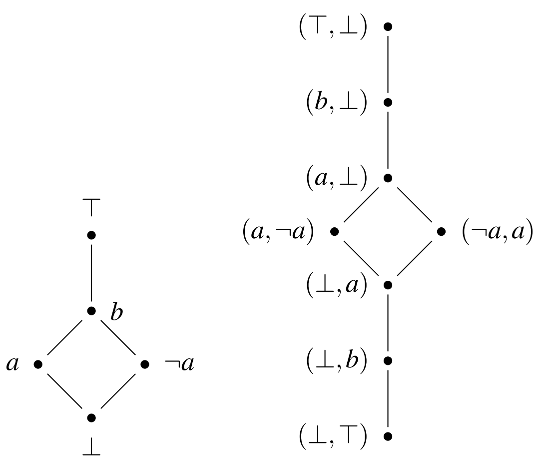

Figure 1: The 5 elements d.i. Heyting algebra (on the left) and the d.i. Nelson algebra $\mathbf{N}_F(\mathbf{H})$ (on the right).

Operations on $N_F(\mathbf{H})$ are defined as above and they make it a (d.i.) Nelson algebra. Notice that, in particular, the order between the elements of $\mathbf{N}_F(\mathbf{H})$ elements is as in Fig. 1 (righ-hand-side).

Now, let us define $\circ(a,\neg a) = \max\{(z,z') \in H \times H \mid (a,\neg a) \wedge (\neg a, a) \wedge (z,z') = (\bot,\top)\}$. To this end let us first compute $d = \min\{z' \in H \mid a \vee \neg a \vee z' = \top\}$, thus $d = \top$. Then compute $c = \max\{z \in H \mid z \wedge \top = \bot$ and $z \vee \top \geq a\}$ which gives $c = \bot$. Therefore $\circ(a,\neg a) = (\bot,\top)$. Notice that such $\circ(a,\neg a)$ also satisfies ($\circ$2).

It is also easy to compute, by the same method, that $\circ(\bot,a) = \circ(a,\bot) = \circ(\bot,b) = \circ(b,\bot) = (\bot,\top)$, while $\circ(\bot,\top) = \circ(\top,\bot) = (\top,\bot)$. Thus, $(\mathbf{N}_F(\mathbf{H}),\circ)$ is in particular a simple algebra.

4 Nelson algebras with a consistency operator and Heyting algebras with dual pseudocomplement

In order to extend the results of [3] to Nelson algebras with a consistency operator, we need to restrict our attention to those Heyting algebras that define, for some of its elements, the unary operation of *dual pseudo-complement* $\neg a = \min\{z \in H \mid a \vee z = \top\}$.

Notice that, if $\mathbf{H}^+ = (H,\wedge,\vee,\to,\leftarrow,\bot,\top)$ is a *double Heyting algebra* as defined in [10] the operator $\neg$ is definable define for all $a \in H$ and in fact $\neg a = a \leftarrow \top$. In particular, for all $a \in H$, we have

$$\neg a = a \to \bot = \max\{z \in H \mid a \wedge z = \bot\}$$

and

$$\neg a = \top \leftarrow a = \min\{z \in H \mid a \vee z = \top\}.$$

Therefore, if $\mathbf{H}^+$ is a double Heyting algebra, F is a boolean congruence of its Heyting reduct and $\mathbf{N}_F(\mathbf{H})$ is the Nelson algebra defined as in Section 2 one can always define $\circ$ on $\mathbf{N}_F(\mathbf{H})$ in the following way:

$$\circ(a,b) = (\neg(\neg(a \vee b)), \neg(a \vee b)). \quad (2)$$

Before proving that the above defined operator satisfies the condition of $\circ$, let us show the following easy fact.

Lemma 4.1. *For every Heyting algebra* $\mathbf{H}$*, for every boolean filter* F *of* $\mathbf{H}$ *and for every* $a \in H$*,* $a \vee \neg a \in F$*.*

Proof. Since F is boolean $\mathbf{H}/F$ is a boolean algebra. Thus for all $[a]_F \in \mathbf{H}/F$, $[a]_F \vee \neg[a]_F = [\top]_F$, that is, $[a \vee \neg a]_F = [\top]_F$ and hence $a \vee \neg a \in F$. □

Then $\neg(a \vee b)$ is by definition the $\min\{z \mid (a \vee b) \vee z = \top\}$ and $\neg(\neg(a \vee b)) = \max\{z \mid \neg(a \vee b) \wedge z = 0\}$. Set $d = \neg(a \vee b)$ and $c = \neg(\neg(a \vee b))$. By Lemma 4.1, $(d \vee c) = (d \vee \neg d)\Theta\top$. Therefore the following easily holds.

Lemma 4.2. *Let* $\mathbf{H}$ *be a Heyting algebra,* F *a boolean filter of* $\mathbf{H}$*. The dual pseudo-complement of* $a \vee b$*,* $\neg(a \vee b)$*, exists for all those* a,b *in* H *such that* $a \wedge b = 0$ *and* $a \vee b \in F$ *iff* $\circ(a,b)$ *exists in the Nelson algebra* $\mathbf{N}(\mathbf{H},F)$ *and* $\circ(a,b) = (\neg d, d)$*, where* $d = \neg(a \vee b)$*. Furthermore,* $\circ(a,b)$ *satisfies* ($\circ$3)*, that is* $\circ$ *is boolean, if and only if the dual pseudocomplement of every element of the filter* F *exists and is boolean.*

In the light of the above Lemma, it is hence clear that, in order for a Heyting algebra $\mathbf{H}$ and a boolean filter F to allow $\mathbf{N}_F(\mathbf{H})$ to admit a consistency operator, it is sufficient that $\neg$ exists for all the elements of F. In the next section we will consider such a case.

5 Categories and equivalences

In this section we establish equivalences between categories that involve Heyting algebras (with extra structure) and Nelson algebras with consistency operators. Besides the results which can reasonably be regarded as natural extensions of those provided in [4] and [11] between Nelson algebras and Heyting algebras with a boolean filter (or a boolean congruence), this section is intended also to highlight which are the necessary and sufficient properties that are needed to add to Heyting algebras with boolean filters, to fully capture consistency operators in Nelson algebras.

For every Heyting algebra $\mathbf{H}$, let $BPF(\mathbf{H})$ be the set of the boolean filters of $\mathbf{H}$ further satisfying the following property:

(DP) For every $x \in F$, $\neg x$ exists in $\mathbf{H}$.

Thus, $BPF(\mathbf{H})$ is the set of boolean filters of $\mathbf{H}$ for which each element x of any F in this set has a dual pseudocomplement in $\mathbf{H}$.

Definition 5.1. Consider the set **HB** made of pairs $(\mathbf{H},F)$ such that $\mathbf{H}$ is a Heyting algebra and $F \in BPF(\mathbf{H})$, and consider morphims defined as follows: given two pairs $(\mathbf{H},F)$ and $(\mathbf{H}',F')$ a morphism h between them is a map such that:

(m1) h is a Heyting homomorphism between $\mathbf{H}$ and $\mathbf{H}'$,

(m2) $h(F) \subseteq F'$,

(m3) for all $x \in F$, $h(\neg x) = \neg' h(x)$.

Morphisms are closed under composition. Indeed, if $h : (\mathbf{H},F) \to (\mathbf{H}',F')$, $f : (\mathbf{H}',F') \to (\mathbf{H}'',F'')$, then fh is a homomorphism of $\mathbf{H}$ to $\mathbf{H}''$, $fh(F) \subseteq F''$ because set-theoretical inclusion is compositional, and if $x \in F$, then $fh(\neg x) = f(\neg' h(x))$. Since $h(F) \subseteq F'$, $h(x) \in F'$. Hence, by (m3), $f(\neg' h(x)) = \neg'' fh(x)$.

Obviously the identity map of each object $id : (\mathbf{H},F) \to (\mathbf{H},F)$ is a morphism and moreover, the composition of morphisms is associative: if $h : (\mathbf{H},F) \to (\mathbf{H}',F')$, $f : (\mathbf{H}',F') \to (\mathbf{H}'',F'')$ and $g : (\mathbf{H}'',F'') \to (\mathbf{H}''',F''')$ are morphisms, $g(fh) = (gf)h$ because so are the composition of homomorphisms and set-theoretical inclusion. Therefore, **HB** is a category.

Now we prove that **HB** is equivalent to the (algebraic) category **NC** of Nelson algebra with a consistency operator, whose objects and morphisms are defined in the obvious way. To this end, let us consider the map $\mathscr{F} : \mathbf{HB} \to \mathbf{NC}$:

- For each object $(\mathbf{H},F) \in \mathbf{HB}$, $\mathscr{F}(\mathbf{H},F) = (\mathbf{N}_F(\mathbf{H}),\circ)$, where $(\mathbf{N}_F(\mathbf{H})$ is the Nelson algebra of pairs $(a,b) \in H \times H$ such that $a \wedge b = \bot$ and $a \vee b \in F$ as in the above section, and $\circ(a,b) = (\neg\urcorner(a\vee b), \urcorner(a \vee b))$ as in (2).
- For each morphism $h : (\mathbf{H},F) \to (\mathbf{H}',F')$ of **HB**, $\mathscr{F}(h) : \mathscr{F}(\mathbf{H},F) \to \mathscr{F}(\mathbf{H}',F')$ is so defined: for all $(a,b) \in \mathscr{F}(\mathbf{H},F)$,

$$\mathscr{F}(h)(a,b) = (h(a),h(b)).$$

For every object $(\mathbf{H},F)$, $\mathscr{F}(\mathbf{H},F)$ is an object in **NC**. $\mathscr{F}(h)$ actually maps $\mathscr{F}(\mathbf{H},F)$ to $\mathscr{F}(\mathbf{H}',F')$. Indeed, let $h : (\mathbf{H},F) \to (\mathbf{H}',F')$ and let us denote $\mathscr{F}(\mathbf{H},F)$ by $(\mathbf{N},\circ)$ and $\mathscr{F}(\mathbf{H}',F')$ by $(\mathbf{N}',\circ')$. Then, for all $(a,b) \in N$, $(h(a),h(b)) \in N' \times N'$. Moreover $h(a) \wedge h(b) = h(a \wedge b) = h(\bot) = \bot$ because h is a Heyting homomorphism and $a \wedge b = \bot$. Furthermore, $h(a) \vee h(b) = h(a \vee b) \in F'$ because $a \vee b \in F$ and $(m2)$.

Proposition 5.2. *The map $\mathscr{F} : \mathbf{HB} \to \mathbf{NC}$ is a functor.*

Proof. In the light of the observations above, it is left to prove that for every morphism $h : (\mathbf{H},F) \to (\mathbf{H}',F')$, $\mathscr{F}(h)$ is a morphism of **NC**. In particular, we have to show that, denoting $\mathscr{F}(\mathbf{H},F)$ by $(\mathbf{N},\circ)$ and $\mathscr{F}(\mathbf{H}',F')$ by $(\mathbf{N}',\circ')$, for all $x \in N$, $\mathscr{F}(h)(\circ x) = \circ'\mathscr{F}(h)(x)$. Recall that $x \in \mathbf{N}$ iff $x = (a,b)$ for $a,b \in H$, $a \wedge b = \bot$ and $a \vee b \in F$ and $\circ(a,b) = (\neg\urcorner(a\vee b), \urcorner(a\vee b))$. Thus,

$$\begin{aligned}
\mathscr{F}(h)(\circ x) &= \mathscr{F}(h)(\neg\urcorner(a\vee b), \urcorner(a\vee b)) \\
&= (h(\neg\urcorner(a\vee b)), h(\urcorner(a\vee b))) \\
&= (\neg'\urcorner'(h(a)\vee h(b)), \urcorner'(h(a)\vee h(b))) \\
&= \circ'(h(a),h(b)) \\
&= \circ'\mathscr{F}(h)(x)
\end{aligned}$$

where the last equality follows, in particular, by the property (m3) of Definition 5.1. □

Recall from [8] that a functor between two categories yields a categorical equivalence iff it is full, faithful and essentially surjective.

Theorem 5.3. *The functor $\mathscr{F}$ establishes a categorical equivalence between* **HB** *and* **NC***.*

Proof. Let us start showing that $\mathscr{F}$ is essentially surjective. Let $(\mathbf{N},\circ)$ any object of **NC**. Recall from [3] that there exists an Heyting algebra $\mathbf{H}$ and a boolean filter F of $\mathbf{H}$ such that $\mathbf{N} \cong \mathbf{N}_F(\mathbf{H})$. Since $\circ$ exists in $\mathbf{N}$, by Lemma 4.2, the dual pseudocomplement $\urcorner$ exists in $\mathbf{H}$ for all elements of F. Therefore, $F \in BPF(\mathbf{H})$ and $(\mathbf{H},F) \in \mathbf{HB}$. Let us define $\circ^*$ on $\mathbf{N}_F(\mathbf{H})$ as usual, $\circ^*(a,b) = (\neg\urcorner(a\vee b), \urcorner(a,b))$. Then it is clear that $(\mathbf{N},\circ) \cong (\mathbf{N}_F(\mathbf{H}),\circ^*) = \mathscr{F}(\mathbf{H},F)$.

Now we prove that $\mathscr{F}$ is full and faithful, i.e., for each pair of objects $(\mathbf{H},F)$, $(\mathbf{H}',F')$, $\mathscr{F}$ establishes a bijection between the set of morphisms of between $(\mathbf{H},F)$ and $(\mathbf{H}',F')$ and the set of morphisms of $\mathscr{F}(\mathbf{H},F)$ and $\mathscr{F}(\mathbf{H}',F')$. In particular we need to prove that the map λ that maps each morphism $h : (\mathbf{H},F) \to (\mathbf{H}',F')$ in the morphism $\mathscr{F}(h) : \mathscr{F}(\mathbf{H},F) \to \mathscr{F}(\mathbf{H}',F')$ is a bijection.

(Inj) Suppose h_1,h_2 are two morphisms $(\mathbf{H},F) \to (\mathbf{H}',F')$ and $h_1 \neq h_2$. In particular let $x \in H$ be such that $h_1(x) \neq h_2(x)$. Thus, $x \wedge \neg x = \bot$ and $x \vee \neg x \in F$ by Lemma 4.1. Therefore $(x,\neg x) \in \mathbf{N}_F(\mathbf{H})$ and $\lambda(h_1)(x,\neg x) = (h_1(x),h_1(\neg x)) \neq (h_2(x),h_2(\neg x)) = \lambda(h_2)(x,\neg x)$. Whence λ is injective.

(Sur) Let $k : (\mathbf{N},\circ) \to (\mathbf{N}',\circ')$ be a morphism of **NC**. By previous results, $\mathbf{N}$ and $\mathbf{N}'$ can be uniquely represented as $\mathbf{N}_F(\mathbf{H})$ and $\mathbf{N}_{F'}(\mathbf{H}')$ for ($\mathbf{H}$ and $\mathbf{H}'$ Heyting algebras and F,F' boolean filters. Moreover, for each homomorphism $k : \mathbf{N} \to \mathbf{N}'$ there exists $h : \mathbf{H} \to \mathbf{H}'$ such that for all $(a,b) \in N$, $k(a,b) = (h(a),h(b))$. Notice that hence $h(F) \subseteq h(F')$ for otherwise, if there exists $x \in F$ and $h(x) \notin F'$, one would have that $(x,\bot) \in N$ (since $x \wedge \bot = \bot$ and $x \vee \bot = x \in F$ by hypothesis), while $k(x,\bot) = (h(x),h(\bot)) \notin N'$ because, in particular, $h(x) \vee h(\bot) = h(x) \notin F'$. Therefore (m2) of Definition 5.1 is satisfied. Our further hypothesis that k preserves $\circ$, gives us in particular that, for every $x \in F$,

$$\begin{aligned}
k(\circ(x,\bot)) &= k(\neg\urcorner(x), \urcorner(x)) \\
&= (\neg'\urcorner'h(x), \urcorner'h(x)) \\
&= \circ'k(x,\bot).
\end{aligned} \tag{3}$$

Let us prove that h satisfies (m3) of Definition 5.1, i.e., for all $x \in F$, $h(\urcorner x) = \urcorner'h(x)$. Again assume that for some $x \in F$, $h(\urcorner x) \neq \urcorner'h(x)$. Then $k(\circ(x,\bot)) = k(\neg\urcorner x, \urcorner x) = (h(\neg\urcorner x), h(\urcorner x))$, while $\circ'(k(x,\bot)) = \circ'(h(x),h(\bot)) = \circ'(h(x),\bot) = (\neg'\urcorner'h(x), \urcorner'h(x))$ and hence $k(\circ(x,\bot)) \neq \circ'k(x,\bot)$ and a contradiction has been reached.

Therefore $\mathscr{F}$ is full, faithful and essentially surjective and therefore the claim is settled. □

6 Categorical equivalences for some relevant particular cases

Let us now consider the full subcategory **HBB** of **HB** obtained by restricting it to those objects $(\mathbf{H}, F)$ in which $F \in BPF(\mathbf{H})$ and, for all $x \in F$ its dual pseudo-complement is a *boolean element* of **H**, that is to say for all $x \in F$, $\neg x$ exists and $\neg x \vee \neg\neg x = \top$.

On the other side let **NBC** be the full subcategory of **NC** whose objects are Nelson algebras with a *boolean* consistency operator $(\mathbf{A}, \circ)$ as in Definition 3.1.

The following result can be easily proved adapting the proof of Theorem 5.3 taking into account Lemma 4.2.

Corollary 6.1. *The restriction of $\mathscr{F}$ to* **HBB** *establishes a categorical equivalence between* **HBB** *and* **NCB**.

As we recall in Section 2, the variety of Nelson algebras and the variety of Nelson lattices are isomorphic, whence they are isomorphic as algebraic category. As for Nelson lattices, if **H** is a Heyting algebra and F is a boolean filter of **H**, the construction that defines the Nelson algebra $\mathbf{N}_F(\mathbf{H})$ of Section 2, can be slightly modified so as to determine the Nelson lattice (isomorphically) associated to $\mathbf{N}_F(\mathbf{H})$. Indeed, define $N_F(\mathbf{H})$, and the operations $\vee$, $\wedge$ as in the case of $\mathbf{N}_F(\mathbf{H})$. Furthermore, for all $(a,b),(c,d) \in N_F(\mathbf{H})$, put

- $(a,b) \to (c,d) = ((a \to_H c) \wedge (b \to_H d), a \wedge d)$;
- $(a,b) * (c,d) = (a \wedge c, (a \to_H d) \wedge (c \to_H b))$.[1]

Then, the stricture $\mathbf{NL}_F(\mathbf{H}) = (N_F(\mathbf{H}), \vee, \wedge, *, \to, (\bot, \top), (\top, \bot))$ is a the Nelson lattice isomorphically corresponding to $\mathbf{N}_F(\mathbf{H})$.

Nelson lattices with consistency operators are defined as in the case of Nelson algebras (recall Definition 3.1) and clearly the algebraic category **NLC** whose objects are Nelson lattices with a consistency operator is equivalent to **NC** and **HB**.

It is well known that Gödel algebras [7] form the proper subvariety of Heyting algebras satisfying the prelinearity equation

$$\text{(Pre)} \quad (x \to y) \vee (y \to x) = \top.$$

Moreover, it is proved in [3] that the construction defining the Nelson lattice $\mathbf{NL}_F(\mathbf{H})$ from a Heyting algebra **H** and a boolean filter F preserves prelinearity. Moreover, prelinear Nelson lattices (i.e., Nelson lattices satisfying (Pre)) coincides with *Nilpotent Minimum* algebras introduced in [6].

From the categorical perspective, the full subcategory **GB** of **HB**, whose objects are pairs $(\mathbf{G}, F)$ for **G** being a Gödel algebra, and the category **NMC** of Nilpotent Minimum algebras with a consistency operator, are equivalent. A similar result holds if we consider the category **GBB**, the full subcategory of **HBB** restricted to Gödel algebras, and the category **NMCB** with objects being Nilpotent Minimum algebras with a boolean consistency operator. Thus, we conclude with the following corollary.

Corollary 6.2. *The restriction of $\mathscr{F}$ to* **GB** *establishes a categorical equivalence between* **GB** *and* **NMC** *and the restriction of $\mathscr{F}$ to* **GBB** *establishes a categorical equivalence between* **GBB** *and* **NMCB**.

7 Conclusions

In the present paper we presented some categorical equivalences involving from one side Nelson algebras with a consistency operator and, from the other, Heyting algebras with a boolean filter and the dual pseudo-complement operator.

As we remarked in [5], Nelson algebras with consistency operators provide an algebraic semantics for a logic of formal inconsistency (LFI) based on Nelson logic. In that paper, indeed, we introduced algebraic semantics for more general LFIs based on distributive involutive residuated lattices (dIRLs) of which Nelson algebras are a particular case.

In our future work, then, we will be concerned with extending the categorical equivalences presented here to the more general setting of dIRLs or some of its relevant subvarieties.

As for the prelinear case, an almost direct byproduct of our construction shows that Nilpotent Minimum algebras (a.k.a. prelinear Nelson algebras) with consistency operator form an algebraic category that turns out to be equivalent to that of Gödel algebras (a.k.a. prelinear Heyting algebras) with a boolean filter and dual pseudocomplement. In turn, (finite) Gödel algebras and Nilpotent Minimum algebras with (or without) a negation fix-point are dually equivalent, as categories, to the category of finite forests (see [1] and [2]). Thus it will be interesting to investigate up to which extent the duality with the category of finite forests extends once we also consider consistency operators to Nilpotent Minimum algebras, or equivalently, a dual pseudocomplement to Gödel algebras.

[1] Notice that in the above expressions we denote by $\to$ the residuum in Nelson lattices, while $\to_H$ denotes the residuum of the Heyting algebra **H**.

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References

[1] S. Aguzzoli, S. Bova, B. Gerla. Free algebras and functional representation for fuzzy logics, in: Cintula, P., Hájek, P., Noguera, C. (eds.), Handbook of Mathematical Fuzzy Logic - Volume 2, Chap. IX. Studies in Logic, vol. 38, pp. 713–791. College Publications, London, 2011.

[2] S. Aguzzoli, T. Flaminio, S. Ugolini. Equivalences between subcategories of MTL-algebras via Boolean algebras and prelinear semihoops, J. Log. Comput. 27(8), 2525–2549, 2017.

[3] M. Busaniche, R. Cignoli. Constructive Logic with Strong Negation as a Substructural Logic, J. Log. Comput. 20(4): 761–793, 2010.

[4] R. Cignoli. The class of Kleene algebras satisfying an interpolation property and Nelson algebras, Algebra Universalis 23: 262–292, 1986.

[5] F. Esteva, A. Figallo-Orellano, T. Flaminio and L. Godo. Logics of Formal Inconsistency Based on Distributive Involutive Residuated Lattices. J. Log. Comput., https://doi.org/10.1093/logcom/exab029, 2021.

[6] F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets and Systems, vol. 124, pp. 271–288, 2001.

[7] P. Hájek. Metamathematics of fuzzy logic, Trends in Logic–Studia Logica Library, vol. 4, Kluwer Academic Publishers, 1998.

[8] S. Mac Lane. Categories for the working mathematician, New York: Springer, 1998.

[9] D.E. Rutherford. Introduction to Lattice Theory, Oliver and Boyd, 1965.

[10] H. P. Sankappanavar. Heyting Algebras with Dual Pseudocomplementation, Pacific Journal of Mathematics 117(2): 405–415, 1985.

[11] A. Sendlewski. Nelson Algebras Through Heyting Ones: I, Studia Logica 49(1): 105–126, 1990.