Ricardo Oscar
Rodriguez ⓘ
Olim Frits Tuyt
Francesc Esteva ⓘ
Lluís Godo ⓘ

Simplified Kripke Semantics for K45-Like Gödel Modal Logics and Its Axiomatic Extensions

Abstract. In this paper we provide a simplified, possibilistic semantics for the logics K45($\mathbf{G}$), i.e. a many-valued counterpart of the classical modal logic K45 over the $[0,1]$-valued Gödel fuzzy logic $\mathbf{G}$. More precisely, we characterize K45($\mathbf{G}$) as the set of valid formulae of the class of possibilistic Gödel frames $\langle W, \pi \rangle$, where W is a non-empty set of worlds and $\pi : W \to [0,1]$ is a possibility distribution on W. We provide decidability results as well. Moreover, we show that all the results also apply to the extension of K45($\mathbf{G}$) with the axiom (D), provided that we restrict ourselves to normalised Gödel Kripke frames, i.e. frames $\langle W, \pi \rangle$ where π satisfies the normalisation condition $\sup_{w \in W} \pi(w) = 1$.

Keywords: Possibilistic semantics, Epistemic logics, Many-valued modal logics.

1. Introduction

Possibilistic logic [10,11,13] is a well-known uncertainty logic to reason with graded (epistemic) beliefs on classical propositions by means of necessity and possiblity measures. In this setting, (graded) epistemic states of an agent are represented by possibility distributions. If W is a set of classical interpretations, or possible worlds, for a given propositional language $\mathcal{L}$, i.e. W is the set of mappings $w : \mathcal{L} \mapsto \{0,1\}$ respecting the classical truth-functions, a *possibility distribution* on W is a mapping $\pi : W \mapsto [0,1]$. If $\sup_{w \in W} \pi(w) = 1$, π is called *normalized*. Such a map π ranks interpretations according to its plausibility level: $\pi(w) = 0$ means that w is rejected, $\pi(w) = 1$ means that w is fully plausible, while $\pi(w) < \pi(w')$ means that w' is more plausible than w. A possibility distribution π induces a pair of dual possibility and necessity measures on propositions, defined respectively as:

$$\Pi(\varphi) = \sup\{\pi(w) \mid w \in W, w(\varphi) = 1\}$$
$$N(\varphi) = \inf\{1 - \pi(w) \mid w \in W, w(\varphi) = 0\}.$$

Given a graded epistemic state π, $\Pi(\varphi)$ measures the degree in which φ is plausible or possible, in the sense that at least a world satisfying φ is plausible (to some degree), while $N(\varphi)$ measures to what degree φ can be considered certain, in the sense that all worlds falsifying φ are implausibe (to some degree). Both measures are dual, in fact, $\Pi(\varphi)$ $= 1 - N(\neg\varphi)$, so that the degree of possibility of a proposition φ equates the degree in which $\neg\varphi$ is not certain. If the normalized condition over possibility distribution is dropped, then we gain the ability to deal with inconsistency. In [12], a possibility distribution which satisfies $\sup_{w \in W} \pi(w) < 1$ is called sub-normal. In this case, given a set W of classical interpretations, a degree of inconsistency can be defined in the following way:

$$\mathrm{inc}(W) = 1 - \sup_{w \in W} \pi(w)$$

When the normalised possibility distribution π is $\{0, 1\}$-valued, i.e. when π is the characteristic function of a subset $\emptyset \neq E \subseteq W$, then the structure $\langle W, \pi \rangle$, or better $\langle W, E \rangle$, can be seen in fact as a K45 frame. Indeed, it is known (see e.g. [19]) that modal logic K45, which is sound and complete w.r.t. the class of Kripke frames $\langle W, R \rangle$ where R is a euclidean and transitive binary relation, also has a simplified semantics given by the subclass of frames $\langle W, E \rangle$, where E is a subset of W (understanding E as its corresponding binary relation R_E defined as $R_E(w, w')$ iff $w' \in E$).

When we go beyond the classical framework of two valued logic to many-valued frameworks, one has to come up with appropriate extensions of the notion of necessity and possibility measures for multiple truth values. Several generalisations have been proposed in the literature, most of them (if not all) take the following form:

$$\Pi(\varphi) = \sup_{w \in W} \{\pi(w) * w(\varphi)\}$$
$$N(\varphi) = \inf_{w \in W} \{\pi(w) \Rightarrow w(\varphi)\},$$

where W is the set of many-valued interpretations $w : \mathcal{L} \mapsto S$ on a suitable linearly ordered scale S (usually the real unit interval $[0, 1]$), $\pi : W \to S$ is a possibility distribution on W, and where $*$ and $\Rightarrow$ are a pair of, respectively, a many-valued conjunction and a many-valued implication operators on S. Different formalisations make different choices for the these operators. For instance, in [18] the authors define a modal logic, called MVKD45, to reason about possibility and necessity degrees of propositions in the frame of the finite-valued Łukasiewicz logics $Ł_k$, with truth-values in the sets $S_k = \{0, 1/k, \ldots, (k-1)/k, 1\}$, where $x * y = \min(x, y)$ and $x \Rightarrow y = \max(1 - x, y)$. This approach was later generalised in [2] to define

similar modal logics over any finite MTL-chain, using truth-constants, and where $*$ and $\Rightarrow$ are taken resp. as the monoidal operation and its residuum operation of the given MTL-chain. On the other hand, in his book [17], for a given propositional axiomatic extension $\mathcal{C}$ of his *basic fuzzy* logic BL, Hájek defines a fuzzy extension of KD45, that he calls *FMBEL($\mathcal{C}$)*, as the set of valid formulas in the class of possibilistic $\mathcal{C}$-Kripke models $\langle W, \pi, e \rangle$, with values over arbitrary $\mathcal{C}$-algebras, and with similar semantics for the modal possibility and necessity operators.[1] Hájek writes "we shall present no definitive results, only some observations", and he leaves as an open problem to find an elegant axiomatization for *FMBEL($\mathcal{C}$)* with models over all $\mathcal{C}$-algebras [17, Chapter 8, pp. 226-228]. Some attempts to solve this open problem have been done, considering that the most natural semantics for a fuzzy counterpart of $KD45$ is the one of Hájek's possibilistic Kripke models or frames with values on BL-algebras. However, it has not been an easy problem to deal with. In fact, reviewing the literature on logical formalizations of fuzzy possibilistic reasoning, one can find results which have a limited scope since they only apply to particular cases of fuzzy modal logics (see e.g. the overview in [2]). In particular, some authors consider cases where either the $\mathcal{C}$-algebras are finite, sometimes expanded with truth constants and the Monteiro-Baaz's Δ operator (as in [2]), and/or where the modal language is restricted. The latter is the case of the so-called two-tiered logics, see e.g. [9,16], where the language is defined in a two-level manner: non-modal formulas are formulas from a given propositional logic L_1 (e.g. classical or a fuzzy propositional logic) and then modal formulas are propositional combinations (according to a second fuzzy logic L_2) of atomic modal formulas $\Box \varphi$ and $\Diamond \varphi$, where $\varphi \in L_1$. This kind of languages allow neither formulas with nested modalities nor formulas mixing both non-modal and modal subformulas.

In this paper we consider the particular context of Gödel fuzzy logic [17] extended with two modal operators $\Box$ and $\Diamond$ with a full general language. In this setting, epistemic states will be modelled as well by possibility distributions $\pi \colon W \to [0,1]$, but now worlds from W are considered to be Gödel propositional $[0,1]$-valued interpretations, that induce the following generalized possibility and necessity measures over Gödel logic propositions:[2]

$$\Pi(\varphi) = \sup_{w \in W} \{\min(\pi(w), w(\varphi))\}$$

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[1] In fact he considers countably-many modalities $\Box_n$, $\Diamond_n$ for each natural n.

[2] Strictly speaking, the possibility measure is indeed a generalization of the classical one, but the necessity measure is not, since $x \Rightarrow 0 \neq 1 - x$ and hence $\Pi(\varphi) \neq 1 - N(\neg\varphi)$.

$$N(\varphi) = \inf_{w \in W} \{\pi(w) \Rightarrow_G w(\varphi)\},$$

where $\Rightarrow_G$ is Gödel implication, that is, for each $x, y \in [0, 1]$,

$$x \Rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$$

These expressions agree with the ones commonly used in many-valued modal Kripke frames $\langle W, R \rangle$ to respectively evaluate modal formulas $\Diamond\varphi$ and $\Box\varphi$ (see for example [1] and references therein) when the $[0, 1]$-valued accessibility relation $R : W \times W \to [0, 1]$ is of the particular form $R(w, w') = \pi(w')$, for any $w, w' \in W$, for of a given possibility distribution $\pi : W \to [0, 1]$.

Actually, modal extensions of Gödel fuzzy logic have been studied by Caicedo and Rodriguez [6], providing sound and complete axiomatizations for different classes of $[0, 1]$-valued Kripke models. These structures are triples $\mathfrak{M} = \langle W, R, e \rangle$, where W is a set of worlds, $R : W \times W \to [0, 1]$ is a $[0, 1]$-valued accessibility relation, as above, and $e : W \times Var \to [0, 1]$ is such that, for every $w \in W$, $e(w, \cdot)$ is a $[0, 1]$-valued Gödel evaluation of propositional variables (more details in next section) that extends to modal formulas as follows:

$$e(w, \Diamond\varphi) = \sup_{w' \in W} \{\min(R(w, w'), e(w', \varphi))\},$$

$$e(w, \Box\varphi) = \inf_{w' \in W} \{R(w, w') \Rightarrow_G e(w', \varphi)\}.$$

We will denote by $\mathcal{K}45(\mathbf{G})$ the class of $[0, 1]$-models $\mathfrak{M} = \langle W, R, e \rangle$ where R satisfies the following many-valued counterpart of the classical properties:

- *Transitivity*: $\forall w, w', w'' \in W$, $\min(R(w, w'), R(w', w'')) \leq R(w, w'')$
- *Euclidean*: $\forall w, w', w'' \in W$, $\min(R(w, w'), R(w, w'')) \leq R(w', w'')$.

In this setting, the class $\Pi\mathcal{G}$ of *possibilistic Kripke models* $\langle W, \pi, e \rangle$, where $\pi : W \to [0, 1]$ is a possibility distribution (not necessarily normalized) on the set of worlds W, can be considered as the subclass of $\mathcal{K}45(\mathbf{G})$ models $\langle W, R, e \rangle$ where R is independent of the world in its first argument, in the sense that $R(w, w') = \pi(w')$ for all $w, w' \in W$. Since $\Pi\mathcal{G} \subsetneq \mathcal{K}45(\mathbf{G})$, it follows that the set $Val(\mathcal{K}45(\mathbf{G}))$ of valid formulas in the class of $\mathcal{K}45(\mathbf{G})$ is a subset of the set $Val(\Pi\mathcal{G})$ of valid formulas in the class $\Pi\mathcal{G}$, i.e. $Val(\mathcal{K}45(\mathbf{G})) \subseteq Val(\Pi\mathcal{G})$. The interesting question is therefore whether the converse inclusion $Val(\Pi\mathcal{G}) \subseteq Val(\mathcal{K}45(\mathbf{G}))$ holds, and thus whether the class $\Pi\mathcal{G}$ provides a simplified possibilistic semantics for the modal logic $K45(\mathbf{G})$.

In [3] the authors claim to provide a positive answer, not for the class $\mathcal{K}45(\mathbf{G})$ models but for the subclass of $\mathcal{K}D45(\mathbf{G})$ models, i.e. those $\mathcal{K}45(\mathbf{G})$ models $\mathfrak{M} = \langle W, R, e \rangle$ further satisfying the many-valued counterpart of seriality:

- *Seriality*: $\forall w \in W$, $\sup_{w' \in W} R(w, w') = 1$.

Indeed, they prove that the logic $KD45(\mathbf{G})$, complete w.r.t. the class of $\mathcal{K}D45(\mathbf{G})$ models, is also complete w.r.t. the class $\Pi^*\mathcal{G}$ of possibilistic models $\langle W, \pi, e \rangle$, where $\pi : W \to [0, 1]$ is a *normalized* possibility distribution on W. Unfortunately, it has to be noted that the completeness proof in [3] has some flaws. In this paper we provide a correct proof, not only for the completeness of $KD45(\mathbf{G})$ with respect to its corresponding class of possibilistic frames, but also for the weaker logic $K45(\mathbf{G})$ accounting for partially inconsistent possibilistic Kripke frames.

This paper is organized as follows: Sect. 2 introduces the main notions about minimum propositional modal Gödel logic and its relational semantics; Sect. 3 then formally introduces a calculus for the logic $K45(G)$, some of its extensions and several of its main theorems; Sect. 4 presents our simplified possibilistic semantics and the results of completeness; Sect. 5 is devoted to analyze the finite model property for the new semantics; finally, Sect. 6 provides some conclusions. We gather in an appendix some technical proofs.

2. Preliminaries on Propositional and Modal Gödel Logic

This section is devoted to preliminaries on the Gödel fuzzy logic $\mathbf{G}$. We present their syntax and semantics, their main logical properties and the notation we use throughout the article.

The language of Gödel propositional logic $\mathcal{L}(Var)$ is built as usual from a countable set of propositional variables Var, the constant $\bot$ and the binary connectives $\wedge$ (conjunction) and $\to$ (implication).

$$\top := \bot \to \bot$$
$$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$
$$\neg\varphi := \varphi \to \bot$$
$$\varphi \equiv \psi := (\varphi \to \psi) \wedge (\psi \to \varphi).$$

The following are the *axioms* of G:

$$(A1) \ \ (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$(A2) \ \ (\varphi \wedge \psi) \to \varphi$$

(A3) $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$

(A4a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \wedge \psi) \rightarrow \chi)$

(A4b) $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$

(A6) $\varphi \rightarrow (\varphi \wedge \varphi)$

(A7) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$

(A8) $\bot \rightarrow \varphi$

The *deduction rule* of G is modus ponens.

Gödel logic can be obtained in fact as the axiomatic extension of Hájek's Basic Fuzzy Logic BL [17] (which is the logic of continuous t-norms and their residua) by means of the contraction axiom (A6) $\varphi \rightarrow (\varphi \wedge \varphi)$. Since the unique idempotent continuous t-norm is the minimum, this yields that Gödel logic is strongly complete with respect to its standard fuzzy semantics that interprets formulas over the structure $[0,1]_G := \langle [0,1], \min, \Rightarrow_G, 0, 1 \rangle$,[3] i.e. semantics defined by truth-evaluations e such that $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$, $e(\varphi \rightarrow \psi) = e(\varphi) \Rightarrow_G e(\psi)$ and $e(\bot) = 0$.

Gödel logic can also be seen as the axiomatic extension of intuitionistic propositional logic by the pre-linearity axiom $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$. In fact, it is sound and complete in the following stronger sense, see [5].

PROPOSITION 2.1. *(i) If $T \cup \{\varphi\} \subseteq \mathcal{L}(Var)$, then $T \vdash_G \varphi$ implies $\inf\{v(\theta) \mid \theta \in T\} \leq v(\varphi)$ for any valuation $v : Var \rightarrow [0,1]$.*

(ii) If T is countable, and $T \nvdash_G \varphi_{i_1} \vee .. \vee \varphi_{i_n}$ for each finite subset of a countable family $\{\varphi_i\}_{i \in I}$ there is an evaluation $v : \mathcal{L}(Var) \rightarrow [0,1]$ such that $v(\theta) = 1$ for all $\theta \in T$ and $v(\varphi_i) < 1$ for all $i \in I$.

We mention in passing that the algebraic semantics of Gödel logic is given by the variety of prelinear Heyting algebras, also known as Gödel algebras. A Gödel algebra is a structure $\mathbf{A} = \langle A, *, \Rightarrow, 0, 1 \rangle$ which is a bounded integral commutative residuated lattice satisfying the contraction equation

$$x * x = x,$$

and the pre-linearity equation

$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1,$$

where $x \vee y = ((x \Rightarrow y) \Rightarrow y) * ((y \Rightarrow x) \Rightarrow x))$.

Let us consider a modal expansion of Gödel logic with two operators $\Box$ and $\Diamond$. The set of formulas $\mathcal{L}_{\Box\Diamond}(Var)$ is built as $\mathcal{L}(Var)$ (always assuming

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[3]Called *standard* Gödel algebra.

countability of the set of propositional variables Var) but extending the set of operations with two unary symbols $\Box$ and $\Diamond$. Whenever Var is clear from the context we will simply write $\mathcal{L}_{\Box\Diamond}$.

In the style introduced by Fitting [14,15] and studied in the works mentioned in the introduction, we define the Gödel Modal Logic as arising from its semantic definition. This is given by enriching usual Kripke models with evaluations over the previous standard algebra, as in [5,6] and others. Formally:

DEFINITION 2.1. A *Gödel-Kripke model* is a structure $\langle W, R, e \rangle$ where W is a non-empty set of so-called *worlds*, and $R \colon W \times W \to [0,1]$ and $e \colon W \times Var \to [0,1]$ are arbitrary mappings.

The evaluation e can be uniquely extended to a map with domain $W \times \mathcal{L}_{\Box\Diamond}$ in such a way that it is a propositional Gödel homomorphism (for the propositional connectives) and where the modal operators are interpreted as infima and suprema, namely: for all $w \in W$,

- $e(w, \bot) := 0$,
- $e(w, \varphi \wedge \psi) := \min(e(w, \varphi), e(w, \psi))$,
- $e(w, \varphi \vee \psi) := \max(e(w, \varphi), e(w, \psi))$,
- $e(w, \varphi \to \psi) := e(w, \varphi) \Rightarrow_G e(w, \psi)$,
- $e(w, \Box\varphi) := \bigwedge_{w' \in W}(R(w, w') \Rightarrow_G e(w', \varphi))$,
- $e(w, \Diamond\varphi) := \bigvee_{w' \in W} \min(R(w, w'), e(w', \varphi))$.

A formula φ is *valid* in a Gödel-Kripke model $\langle W, R, e \rangle$ if $e(w, \varphi) = 1$ for all $w \in W$. We will denote by $\mathcal{G}$ the class of Gödel Kripke models and will say that a formula φ is $\mathcal{G}$-*valid*, written $\models_{\mathcal{G}} \varphi$, if φ is valid in all Gödel Kripke models.

In their paper [6] Caicedo and Rodriguez define the logic $K(\mathbf{G})$ as the smallest set of formulas containing some axiomatic version of Gödel-Dummet propositional calculus, that is, Heyting calculus plus the prelinearity law, and the following additional axioms and rule:

$$(K_\Box) \; \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \qquad (K_\Diamond) \; \Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$$
$$(P) \; \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi) \qquad (F_\Box) \; \Box\top$$
$$(FS2) \; (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi) \qquad (Nec) \; \text{from } \varphi \text{ infer } \Box\,\varphi$$

They denote deduction in this system as $\vdash_{K(\mathbf{G})}$ and they show the following result:

THEOREM 2.1. ([6, Theorem 3.1]) *Let $\varphi \in \mathcal{L}_{\Box\Diamond}$. Then*

$$\vdash_{K(\mathbf{G})} \varphi \text{ if and only if } \models_{\mathcal{G}} \varphi.$$

Proofs with assumptions are allowed with the restriction that (Nec) can be applied only when the premise is a theorem. This restriction allows for the following convenient reduction (see [5]).

LEMMA 2.1. *Let $ThK(\mathbf{G})$ be the set of theorems of $K(\mathbf{G})$ with no assumptions, then for any theory T and formula φ in $\mathcal{L}_{\Box\Diamond} : T \vdash_{K(\mathbf{G})} \varphi$ if and only if $T \cup ThK(G) \vdash_{\mathbf{G}} \varphi$ where $\vdash_{\mathbf{G}}$ denotes deduction in Gödel fuzzy logic G.*

The following are some theorems of $K(\mathbf{G})$, see [6]:

$$T1.\ \neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$$
$$T2.\ \neg\neg\Box\varphi \to \Box\neg\neg\varphi$$
$$T3.\ \Diamond\neg\neg\varphi \to \neg\neg\Diamond\varphi$$
$$T4.\ (\Box\varphi \to \Diamond\psi) \vee \Box((\varphi \to \psi) \to \psi)$$
$$T5.\ \Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$$

The first one is an axiom in Fitting's systems in [14], the next two were introduced in [6], the fourth one will be useful in our completeness proof and is the only one depending on prelinearity. The last is known as the first connecting axiom given by Fischer Servi. In addition, it is interesting to notice that the following rule is derivable:

$$(Nec_{\Diamond}) \text{ from } \varphi \to \psi \text{ infer } \Diamond\varphi \to \Diamond\psi$$

Indeed, if $\vdash_{K(\mathbf{G})} \varphi \to \psi$, then $\vdash_{K(\mathbf{G})} \Box(\varphi \to \psi)$ by Nec, and by using axiom (P) and modus ponens we get $\vdash_{K(\mathbf{G})} \Diamond\varphi \to \Diamond\psi$.

In the next section, we focus on an extension of $K(\mathbf{G})$ for which we are able to simplify the Gödel Kripke semantics introduced in this section.

3. The Logic $K45(\mathbf{G})$

The logic $K45(\mathbf{G})$ is defined by adding to $K(\mathbf{G})$ the following axioms:

$$(4_{\Box})\quad \Box\varphi \to \Box\Box\varphi \qquad\qquad (4_{\Diamond})\quad \Diamond\Diamond\varphi \to \Diamond\varphi$$
$$(5_{\Box})\quad \Diamond\Box\varphi \to \Box\varphi \qquad\qquad (5_{\Diamond})\quad \Diamond\varphi \to \Box\Diamond\varphi$$

And we define the logic $KD45(\mathbf{G})$ by adding to $K45(\mathbf{G})$ the following axiom:

$$(D)\quad \Diamond\top$$

and $KT45(\mathbf{G})$ by adding the following axioms to $K45(\mathbf{G})$:

$$(T_\Box)\quad \Box\varphi \to \varphi \qquad\qquad\qquad (T_\Diamond)\quad \varphi \to \Diamond\varphi$$

A first observation is that the following is a theorem of $KD45(\mathbf{G})$:

$$(D')\ \Box\varphi \to \Diamond\varphi$$

Indeed, we can replace $\top$ by $\varphi \to \varphi$ in Axiom (D) and then use T5. In fact, (D) and (D') are interderivable in $K(\mathbf{G})$, that is, we have both $\Diamond\top \vdash_{K(\mathbf{G})}$ $\Box\varphi \to \Diamond\varphi$ and $\Box\varphi \to \Diamond\varphi \vdash_{K(\mathbf{G})} \Diamond\top$, the latter holding by first instantiating (D') with $\varphi = \top$ and getting $\Box\top \to \Diamond\top$, and then using that $\Box\top$ is an axiom of $K(\mathbf{G})$.

Next we show that in $K45(\mathbf{G})$ some iterated modalities can be simplified. This is in accordance with our intended simplified semantics for $K45(\mathbf{G})$ that will be formally introduced in the next section.

PROPOSITION 3.1. *The logic $K45(\mathbf{G})$ proves the following schemes:*

$(F_{\Diamond\Box})$ $\Diamond\Box\top \leftrightarrow \Diamond\top$ $\qquad$ $(G45)$ $(\Box\varphi \to \Diamond\psi) \to \Box(\Box\varphi \to \Diamond\psi)$

$(U_\Diamond)$ $\Diamond\Diamond\varphi \leftrightarrow \Diamond\varphi$ $\qquad\qquad$ $(U_\Box)$ $\Box\Box\varphi \leftrightarrow \Box\varphi$

$(T4_\Box)$ $(\Box\varphi \to \Diamond\Box\varphi) \vee \Box\varphi$ $\quad$ $(T4_\Diamond)$ $(\Box\Diamond\varphi \to \Diamond\varphi) \vee \Box\Diamond\varphi$

$(Sk_\Diamond)$ $(\Diamond\top \to \Diamond\varphi) \leftrightarrow \Box\Diamond\varphi$ $\quad$ $(T4'_\Diamond)$ $(\Box\Diamond\varphi \to \Diamond\varphi) \vee (\Diamond\top \to \Diamond\varphi)$

PROOF. $(F_{\Diamond\Box})$ is an immediate consequence of $F_\Box$ and $Nec_\Diamond$. As for schemes $U_\Diamond$ and $U_\Box$, axioms $4_\Box$ and $4_\Diamond$ give one direction of them. The opposite directions, together with the rest of schemes, are obtained as follows:
Proof $(U_\Diamond)$:

$$\Diamond\varphi \to \Box\Diamond\varphi \text{ axiom } (5_\Diamond)$$
$$\Box(\varphi \to \Diamond\varphi) \text{ by } MP \text{ and } (FS2)$$
$$\Diamond\varphi \to \Diamond\Diamond\varphi \text{ by } MP \text{ and } (P)$$

Proof $(U_\Box)$:

$$\Diamond\Box\varphi \to \Box\varphi \quad \text{axiom } (5_\Box)$$
$$\Box(\Box\varphi \to \varphi) \text{ by } MP \text{ and } (FS2)$$
$$\Box\Box\varphi \to \Box\varphi \text{ by } MP \text{ and } (K_\Box)$$

Proof $(T4_\Box)$:

$$(\Box\Box\varphi \to \Diamond\Box\varphi) \vee \Box\Box\varphi \text{ by (T4), replacing } \varphi \text{ and } \psi \text{ by } \Box\varphi$$
$$(\Box\varphi \to \Diamond\Box\varphi) \vee \Box\varphi \quad \text{ by } (U_\Box)$$

Proof $(T4_\Diamond)$:

$$(\Box\Diamond\varphi \to \Diamond\Diamond\varphi) \vee \Box\Diamond\varphi \text{ by (T4), replacing } \varphi \text{ and } \psi \text{ by } \Diamond\varphi$$
$$(\Box\Diamond\varphi \to \Diamond\varphi) \vee \Box\Diamond\varphi \quad \text{ by } (U_\Diamond)$$

Proof ($G45$) :

$$(\Box\varphi \to \Diamond\psi) \to (\Box\varphi \to \Box\Diamond\psi) \qquad \text{by applying } (5_\Diamond)$$
$$(\Box\varphi \to \Box\Diamond\psi) \to (\Diamond\Box\varphi \to \Box\Diamond\psi) \quad \text{by applying } (5_\Box)$$
$$(\Diamond\Box\varphi \to \Box\Diamond\psi) \to \Box(\Box\varphi \to \Diamond\psi) \text{ by } (FS2)$$

Proof ($Sk_\Diamond$) :

$$\Box(\top \to \Diamond\varphi) \to (\Diamond\top \to \Diamond\Diamond\varphi) \text{ by (P)}$$
$$\Box\Diamond\varphi \to (\Diamond\top \to \Diamond\varphi) \qquad\quad \text{by } (U_\Diamond) \text{ and equivalences}$$
$$(\Diamond\top \to \Box\Diamond\varphi) \to \Box(\top \to \Diamond\varphi) \text{ by } (FS2)$$
$$(\Diamond\top \to \Diamond\varphi) \to \Box\Diamond\varphi \qquad\quad \text{by } (5_\Diamond)$$

Proof ($T4'_\Diamond$): using ($T4_\Diamond$) and ($Sk_\Diamond$)

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Moreover, if we restrict ourselves to formulas starting with $\Box$ or $\Diamond$ we can prove the following property.

LEMMA 3.1. *Let $X = \{\Box\theta, \Diamond\theta : \theta \in \mathcal{L}_{\Box,\Diamond}\}$. If $\varphi \in X$ then the schemas*

$$\varphi \to \Box\varphi \quad \text{and} \quad \Diamond\varphi \to \varphi$$

are theorems of $K45(\mathbf{G})$.

PROOF. We check that $K45(\mathbf{G})$ derives $\varphi \to \Box\varphi$ if $\varphi \in X$, the other schema is similar. In fact, we have two cases: if $\varphi = \Box\psi$, then $\varphi \to \Box\varphi$ is in fact one direction of ($U_\Box$); if $\varphi = \Diamond\psi$, then this is axiom ($5_\Diamond$). ∎

From now on we will use $ThK45(\mathbf{G})$ to denote the set of theorems of $K45(\mathbf{G})$. We close this section with the following observation.

LEMMA 3.2. *If T is a finite set of formulas, $T \vdash_{K45(\mathbf{G})} \varphi$ iff $\vdash_{K45(\mathbf{G})} T^\wedge \to \varphi$, where $T^\wedge = \bigwedge\{\psi \mid \psi \in T\}$.*

PROOF. By Lemma 2.1, we have $T \vdash_{K45(\mathbf{G})} \varphi$ iff $T \cup ThK45(\mathbf{G}) \vdash_G \varphi$. By the deduction theorem of Gödel logic, the latter is equivalent to $ThK45(\mathbf{G}) \vdash_G T^\wedge \to \varphi$, and by (i) again, this is equivalent to $\vdash_{K45(\mathbf{G})} T^\wedge \to \varphi$.

REMARK 3.1. It is worth noting that for any valuation v such that $v(ThK45(\mathbf{G})) = 1$ there is no formula φ such that $v(\Diamond\top) < v(\nabla\varphi) < 1$ with $\nabla \in \{\Box, \Diamond\}$ because both formulae $(\Box\varphi \to \Diamond\varphi) \vee \Box\varphi$ and $\Diamond\varphi \to \Diamond\top$ are in $ThK45(\mathbf{G})$.

4. Simplified Kripke Semantics and Completeness

In this section we will show that $K45(\mathbf{G})$ is complete with respect to a class of simplified Kripke Gödel frames.

DEFINITION 4.1. A *possibilistic Kripke frame*, or Π-*frame*, is a structure $\langle W, \pi \rangle$ where W is a non-empty set of *worlds*, and $\pi : W \to [0, 1]$ is a *possibility distribution* over W.

A *possibilistic Gödel Kripke model*, ΠG-*model* for short, is a triple $\langle W, \pi, e \rangle$ where $\langle W, \pi \rangle$ is a Π-frame and $e : W \times Var \to [0, 1]$ provides a Gödel evaluation of variables in each world. For each $w \in W$, $e(w, -)$ extends to arbitrary formulas in the usual way for the propositional connectives and for modal operators in the following way:

$$e(w, \Box\varphi) := \inf_{w' \in W} \{\pi(w') \Rightarrow_G e(w', \varphi)\}$$
$$e(w, \Diamond\varphi) := \sup_{w' \in W} \{\min(\pi(w'), e(w', \varphi))\}.$$

If π is *normalised*, i.e. if $\sup_{w \in W} \pi(w) = 1$, then $\langle W, \pi, e \rangle$ will be called a *normalised* possibilistic Gödel Kripke model, or $\Pi^* G$-*model*. A formula φ is *valid* in a ΠG-model $\langle W, \pi, e \rangle$ if $e(w, \varphi) = 1$ for all $w \in W$.

We will denote by $\Pi\mathcal{G}$ the class of possibilistic Gödel Kripke models, and by $\Pi^*\mathcal{G}$ the subclass of normalised models. We say that a formula φ is $\Pi\mathcal{G}$-*valid*, written $\models_{\Pi\mathcal{G}} \varphi$, if φ is valid in all possibilistic Gödel Kripke models, and $\Pi^*\mathcal{G}$-*valid*, written $\models_{\Pi^*\mathcal{G}} \varphi$, if φ is valid in all normalised possibilistic Gödel Kripke model.

Observe that the evaluation of formulas beginning with a modal operator is in fact independent from the current world. Also note that the $e(-, \Box\varphi)$ and $e(-, \Diamond\varphi)$ are in fact generalisations for Gödel logic propositions of the necessity and possibility degrees of φ introduced in Sect. 1 for classical propositions, although now they are not dual (with respect to Gödel negation) any longer.

In the rest of this section we are going to show in detail a weak completeness proof of the logic $K45(\mathbf{G})$ (resp. $KD45(\mathbf{G})$) with respect to the class $\Pi\mathcal{G}$ (resp. the subclass $\Pi^*\mathcal{G}$) of possibilistic Gödel Kripke models. In fact one can prove a slightly stronger result, namely completeness for deductions from finite theories.

We start with the case of $K45(\mathbf{G})$. In what follows, for any formula φ we denote by $Sub(\varphi) \subseteq \mathcal{L}_{\Box\Diamond}$ the set of subformulas of φ and containing the formula $\bot$. Moreover, let $X := \{\Box\theta, \Diamond\theta : \theta \in \mathcal{L}_{\Box\Diamond}\}$ be the set of formulas in $\mathcal{L}_{\Box\Diamond}$ beginning with a modal operator; then $\mathcal{L}_{\Box\Diamond}(Var) = \mathcal{L}(Var \cup X)$. That is, any formula in $\mathcal{L}_{\Box\Diamond}(Var)$ may be seen as a propositional Gödel formula built from the extended set of propositional variables $Var \cup X$. In addition, for a given formula φ, let $\sim_\varphi$ be the equivalence relation on $[0, 1]^{Var \cup X} \times [0, 1]^{Var \cup X}$ defined as follows:

$$u \sim_\varphi w \text{ iff } \forall \psi \in Sub(\varphi) : u(\Box\psi) = w(\Box\psi) \text{ and } u(\Diamond\psi) = w(\Diamond\psi).$$

Now, assume that a formula φ is not a theorem of $K45(\mathbf{G})$. Hence by completeness of Gödel calculus and Lemma 2.1, there exists a Gödel valuation v such that $v(ThK45(\mathbf{G})) = 1$ and $v(\varphi) < 1$. With the valuation v now fixed, we follow the usual canonical model construction, defining a canonical ΠG-model $\mathfrak{M}_\varphi^v$ in which we will show φ is not valid.

The *canonical model* $\mathfrak{M}_\varphi^v = \langle W_\varphi^v, \pi_\varphi^v, e_\varphi^v \rangle$ is defined as follows:

- W_φ^v is the set $\{u \in [0,1]^{Var \cup X} \mid u \sim_\varphi v \text{ and } u(ThK45(\mathbf{G})) = 1\}$.
- $\pi_\varphi^v(u) = \inf_{\psi \in Sub(\varphi)}\{\min(v(\Box\psi) \Rightarrow_G u(\psi), u(\psi) \Rightarrow_G v(\Diamond\psi))\}$.
- $e_\varphi^v(u, p) = u(p)$ for any $p \in Var$.

In this context, we call the elements of $\Delta_\varphi := \{\Box\theta, \Diamond\theta : \theta \in Sub(\varphi)\}$ the *fixed points* of the Canonical Model. Note that having $\nu(ThK45(\mathbf{G})) = 1$ does not guarantee that ν belongs to the canonical model because it may not take the appropriated values for the fixed points, i.e. it may be that $u \not\sim_\varphi \nu$. However, the next lemma shows how to, under certain conditions, transform such an evaluation into another belonging to the canonical model.

LEMMA 4.1. *Let $u \in W_\varphi^v$ and let $\nu : Var \cup X \to [0,1]$ be a Gödel valuation. Define $\delta = \max\{u(\lambda) : \nu(\lambda) < 1 \text{ and } \lambda \in \Delta_\varphi\}$ and $\beta = \min\{u(\lambda) : \nu(\lambda) = 1 \text{ and } \lambda \in \Delta_\varphi\}$. If ν satisfies the following conditions:*

a. $\nu(ThK45(\mathbf{G})) = 1$;

b. *for all $\lambda \in X$, we have $u(\lambda) > \delta$ implies $\nu(\lambda) = 1$;*

c. *for any $\psi, \phi \in \{\lambda : u(\lambda) \leq \delta \text{ and } \lambda \in X\}$: $u(\psi) < u(\phi)$ implies $\nu(\psi) < \nu(\phi)$;*

d. *for any $\psi, \phi \in \Delta_\varphi : u(\psi) \leq u(\phi)$ implies $\nu(\psi) \leq \nu(\phi)$,*

then there exists a Gödel valuation $w \in W_\varphi^v$ such that for any $\varepsilon > 0$ with $\delta + \varepsilon < \beta$, and for any formulae ψ and ϕ, the following conditions hold:

1. $\nu(\psi) = 1$1. implies $w(\psi) \geq \delta + \varepsilon$.

2. $\nu(\psi) < 1$ implies $w(\psi) < \delta + \varepsilon$.

3. $1 \neq \nu(\psi) \leq \nu(\phi)$ implies $w(\psi) \leq w(\phi)$.

4. $\nu(\psi) < \nu(\phi)$ implies $w(\psi) < w(\phi)$.

PROOF. First of all, notice that if ν satisfies the conditions **c** and **d**, then necessarily $\delta < \beta$. Indeed, suppose $\delta \geq \beta$. Then there are at least two formulas θ_1 and θ_2 in Δ_φ such that $\nu(\theta_1) < 1$, $\nu(\theta_2) = 1$ and $\delta \geq u(\theta_1) \geq u(\theta_2) \geq \beta$. Note that the case $u(\theta_1) > u(\theta_2)$ is not possible because it would

violate condition **c**, and the case $u(\theta_1) = u(\theta_2)$ is also impossible because it would then violate condition **d**.

Let $B = \{\nu(\lambda) : \lambda \in \Delta_\varphi, \nu(\lambda) < 1\} \cup \{0\} = \{b_0 = 0 < b_1 < \ldots < b_N\}$. Obviously, $b_N < 1$. For each $0 \leq i \leq N$, pick $\lambda_i \in \Delta_\varphi$ such that $\nu(\lambda_i) = b_i$. Define now a continuous strictly function $g : [0,1] \mapsto [0, \delta + \varepsilon) \cup \{1\}$ such that

$$g(1) = 1$$
$$g(b_i) = u(\lambda_i) \text{ for every } 0 \leq i \leq N$$
$$g[(b_N, 1)] = (\delta, \delta + \varepsilon)$$

Notice that $\delta = g(b_N)$. In addition, define another continuous strictly increasing function $h : [0,1] \mapsto [\delta + \varepsilon, 1]$ such that

$$h(0) = \delta + \varepsilon$$
$$h[(0, \beta)] = (\delta + \varepsilon, \beta)$$
$$h(x) = x, \text{ for } x \in [\beta, 1]$$

Then we define the valuation $w : Var \cup X \to [0,1]$ as follows:

$$w(p) = \begin{cases} g(\nu(p)) & \text{if } \nu(p) < 1 \\ h(u(p)) & \text{if } \nu(p) = 1. \end{cases}$$

Next step is to prove that w satisfies the required Properties 1–4. We prove it by induction on the maximum of the complexity of both formulae ψ, ϕ.

First, we consider the base case when both ψ and ϕ belong to $Var \cup X$. Then

1. By definition of w, $\nu(\psi) = 1$ implies $w(\psi) = h(u(\psi)) \geq \delta + \varepsilon$. The same happens for ϕ.

2. By definition of w, $\nu(\psi) < 1$ implies $w(\psi) = g(\nu(\psi)) < \delta + \varepsilon$. The same happens for ϕ.

3. Suppose $1 \neq \nu(\psi) \leq \nu(\phi)$. Since $\nu(\psi) < 1$, by definition of w, we have $w(\psi) = g(\nu(\psi)) < \delta + \varepsilon$. Now, we analyse different cases. If $\nu(\phi) < 1$ then $w(\phi) = g(\nu(\phi)) \geq g(\nu(\psi)) = w(\psi)$ because g is strictly increasing. Otherwise, If $\nu(\phi) = 1$ then $w(\phi) = h(u(\phi)) \geq \delta + \varepsilon > w(\psi) = g(\nu(\psi))$.

4. Suppose $\nu(\psi) < \nu(\phi)$. In this case, the proof is similar to the previous one. If $\nu(\psi) < 1$ and $\nu(\phi) = 1$ then $w(\phi) = h(u(\phi)) \geq \delta + \varepsilon > g(\nu(\psi)) = w(\psi)$. On the contrary, if both $\nu(\psi) < 1$ and $\nu(\phi) < 1$ then $w(\phi) = g(\nu(\phi)) > g(\nu(\psi)) = w(\psi)$ because g is strictly increasing.

This case will become important when we try to prove that w satisfies the axioms of $K45(\mathbf{G})$.

In fact, in case both formulae ψ and ϕ belong to $Var \cup X$ we can prove that two further conditions hold:

5. If $\nu(\psi) = \nu(\phi) = 1$ and $u(\psi) \leq u(\phi)$ then $w(\psi) \leq w(\phi)$.
6. If $\nu(\psi) = \nu(\phi) = 1$ and $u(\psi) < u(\phi)$ then $w(\psi) < w(\phi)$.

Indeed, by definition of w, we have $w(\psi) = h(u(\psi))$ and $w(\phi) = h(u(\phi))$, and since h is strictly increasing, $u(\psi) \leq u(\phi)$ (resp. $u(\psi) < u(\phi)$) implies $h(u(\psi)) \leq h(u(\phi))$ (resp. $h(u(\psi)) < h(u(\phi)))$, as desired. These properties will become important when proving that w satisfies the axioms of $K45(\mathbf{G})$.

Now, we consider the inductive step.

CLAIM. *Induction hypothesis (IH): if Properties 1–4 are satisfied by formulas with complexity at most n, then these properties also hold for formulae with complexity at most $n + 1$.*

The proof of this claim is quite technical and it is moved to the appendix.

Finally, we prove that $w \in W_\varphi^v$, i.e. we prove that both $w \sim_\varphi v$ and $w(ThK45(\mathcal{G})) = 1$.

(i) By definition, $w \sim_\varphi v$ iff $w(\Box\phi) = v(\Box\phi)$ and $w(\Diamond\phi) = v(\Diamond\phi)$ for all $\phi \in Sub(\varphi)$. Let $A \in \{\Box\phi, \Diamond\phi\} \subseteq \Delta_\varphi$. By definition of w, $w(A) = g(\nu(A))$ if $\nu(A) < 1$, and $w(A) = h(u(A))$, otherwise. Since $A \in \Delta_\varphi$, if $\nu(A) < 1$ then, by construction $(u(A) \leq \delta)$, we have $g(\nu(A)) = u(A) \leq \delta$, and if $\nu(A) = 1$, again by construction $(u(A) \geq \beta)$, we have $h(u(A)) = u(A)$. So, we have proved that $w \sim_\varphi u$, but by hypothesis $u \sim_\varphi v$ as well, thus $w \sim_\varphi v$ as well.

(ii) We first prove that all axioms of $K45(\mathbf{G})$ are evaluated to 1 by w. The axioms of $\mathcal{G}$ are evaluated to 1 by any Gödel valuation. As for the specific axioms of $K45(\mathbf{G})$ (i.e. axioms $(4_\Box), (4_\Diamond), (5_\Box), (5_\Diamond)$), observe that all these axioms are of the form $\phi \rightarrow \psi$ for some $\phi, \psi \in X$. Then it is enough to prove that for any $\varphi, \psi \in X$, if $\nu(\phi \rightarrow \psi) = 1$ then $w(\phi \rightarrow \psi) = 1$. Indeed, if $\nu(\phi \rightarrow \psi) = 1$, then $\nu(\phi) \leq \nu(\psi)$ and we have two possibilities:

- If $1 \neq \nu(\phi) \leq \nu(\psi)$, then by Property 3, we have $w(\phi) \leq w(\psi)$.
- If $\nu(\phi) = \nu(\psi) = 1$, then by definition of w, $w(\phi) = h(u(\phi))$ and $w(\psi) = h(u(\psi))$. But since $u \in W_\varphi^v$, we have $u(\phi \rightarrow \psi) = 1$, because we assume that $\phi \rightarrow \psi \in K45(\mathbf{G})$ with $\phi, \psi \in X$. Thus $u(\phi) \leq u(\psi)$ and, by Property 5, we know $w(\phi) = h(u(\phi)) \leq h(u(\psi)) = w(\psi)$.

Finally, let us consider the axioms of $K(\mathbf{G})$ (Sect. 2). We have to prove that $w(\chi) = 1$ for each of such axioms χ, knowing by assumption that $\nu(\chi) = 1$. The case of $(F_\Box)$ is easy since $(F_\Box) \in X$, and then, by definition, $w(F_\Box) = h(u(F_\Box)) = 1$. As for axiom $(K_\Box)$, note that, using propositional reasoning, it can be equivalently expressed first as $(\Box(\phi \to \psi) \wedge \Box\phi) \to \Box\psi$, and then as $(\Box(\phi \to \psi) \to \Box\psi) \vee (\Box\phi \to \Box\psi)$. Therefore, if $\nu(K_\Box) = 1$, it means either $\nu(\Box(\phi \to \psi) \to \Box\psi) = 1$ or $\nu(\Box\phi \to \Box\psi) = 1$. But these two cases concern implications of formulas from X, and hence we are in the same situation above as with the specific axioms of $K45(\mathbf{G})$, and hence using the same reasoning, we can conclude that either $w(\Box(\phi \to \psi) \to \Box\psi) = 1$ or $w(\Box\phi \to \Box\psi) = 1$.

The case of axioms $(K_\Diamond)$ and (P) can be with in an analogous way, as they can be written as a disjunction of implications of formulas from X. So the case left is that of axiom $(FS2)$, $(\Diamond\phi \to \Box\psi) \to \Box(\phi \to \psi)$.

By hypothesis, we know $\nu(FS2) = 1$, that is $\nu(\Diamond\phi) \Rightarrow_G \nu(\Box\psi) \leq \nu(\Box(\phi \to \psi))$. According to that, we have to prove $w(FS2) = 1$ as well. Notice that, since $\psi \to (\phi \to \psi)$ is a tautology, $u(\Box(\psi \to (\phi \to \psi))) = \nu(\Box(\psi \to (\phi \to \psi))) = 1$. Then, by definition, $w(\Box(\psi \to (\phi \to \psi))) = 1$ as well and because axiom (K) is valid for w, we have $w(\Box\psi \to \Box(\phi \to \psi)) = 1$, i.e. $w(\Box\psi) \leq w(\Box(\phi \to \psi))$.

Now, we consider the following cases:

- Case $u(\Diamond\phi) \leq u(\Box\psi)$. Then $u(\Box(\phi \to \psi)) = 1$ which implies $\nu(\Box(\phi \to \psi)) = 1$. Hence, by construction, $w(\Box(\phi \to \psi)) = h(u(\Box(\phi \to \psi))) = 1$.
- Case $u(\Diamond\phi) > u(\Box\psi)$. Here we distinguish three subcases:
    - $u(\Box\psi) \leq \delta < u(\Diamond\phi)$: by Conditions **b** and **c**, $\nu(\Diamond\phi) = 1$ and $\nu(\Box\psi) < 1$, respectively. Therefore, by Property 4, $w(\Box\psi) < w(\Diamond\phi)$ and thus $w(\Diamond\varphi \to \Box\psi) = w(\Box\psi) \leq w(\Box(\phi \to \psi))$ and hence $w(FS2) = 1$.
    - $\delta < u(\Box\psi) < u(\Diamond\phi)$: by Condition **b**, $1 = \nu(\Diamond\phi) = \nu(\Box\psi) \leq \nu(\Box(\phi \to \psi))$. Thus, by construction, $w(FS2) = (h(u(\Diamond\varphi)) \Rightarrow_G h(u(\Box\psi))) \Rightarrow_G h(u(\Box(\phi \to \psi)) = 1$.
    - $u(\Box\psi) < u(\Diamond\phi) \leq \delta$: by Condition **c**, $\nu(\Box\psi) < \nu(\Diamond\phi)$, and by Property 4, $w(\Box\psi) < w(\Diamond\phi)$, and hence $w(\Diamond\varphi \to \Box\psi) = w(\Box\psi) \leq w(\Box(\phi \to \psi))$, and hence $w(FS2) = 1$.

So far, we have proved that $w(\phi) = 1$ if ϕ is a $K45(\mathbf{G})$ axiom. To conclude the proof, we need to extend this result to the rest of the formulas in $ThK45(\mathbf{G})$. Recall that a formula ϕ in the set $ThK45(\mathbf{G})$ is either an

axiom of $K45(\mathbf{G})$ or ϕ can be proved from the axioms and rules of inference $K45(\mathbf{G})$. In the latter case, there is non-empty finite sequence of formulae $\phi_1, \phi_2, \ldots, \phi_n$, with $\phi_n = \psi$, and each ϕ_i is either an axiom or it has been obtained by application of an inference rule on some of the preceding formulae $\phi_1, \ldots, \phi_{i-1}$. We proceed by induction on the length n of the sequence. The unique interesting case is when ϕ_n is obtained by applying the (Nec) rule to some ϕ_j with $1 \leq j < n$, i.e. $\phi_n = \Box\phi_j$. It is clear that $\phi_j \in ThK45(\mathbf{G})$, and by the inductive hypothesis, $w(\phi_j) = 1$. Of course, $\Box\phi_j \in ThK45(\mathbf{G})$ as well, and thus $\nu(\Box\xi) = 1$ too. But since $\Box\phi_j \in X$, by definition of w, we hace $w(\psi) = w(\Box\phi_j) = h(u(\Box\phi_j)) = 1$. This finishes the proof of the lemma. ∎

Completeness will follow from the next truth-lemma.

LEMMA 4.2. (Truth-lemma) *For any $\psi \in Sub(\varphi)$ and any $u \in W_\varphi^v$, $e_\varphi^v(u, \psi) = u(\psi)$.*

PROOF. For simplicity, we will write W^c, π^c and e^c for $W_\varphi^v, \pi_\varphi^v$ and e_φ^v, respectively. We prove the identity by induction on the complexity of the formulas in $Sub(\varphi)$, considered now as elements of $\mathcal{L}_{\Box\Diamond}(Var)$. For $\bot$ and the propositional variables in $Sub(\varphi)$ the equation holds by definition. The only non trivial inductive steps are: $e^c(u, \Box\psi) = u(\Box\psi)$ and $e^c(u, \Diamond\psi) = u(\Diamond\psi)$ for $\Box\psi, \Diamond\psi \in Sub(\varphi)$. By the inductive hypothesis we may assume that $e^c(u', \psi) = u'(\psi)$ for every $u' \in W^c$; thus we must prove:

$$\inf_{u' \in W^c} \{\pi^c(u') \Rightarrow_G u'(\psi)\} = u(\Box\psi) \tag{1}$$

$$\sup_{u' \in W^c} \{\min(\pi^c(u'), u'(\psi))\} = u(\Diamond\psi) \tag{2}$$

By definition, $\pi^c(u') \leq (v(\Box\psi) \Rightarrow_G u'(\psi))$ and $\pi^c(u') \leq (u'(\psi) \Rightarrow_G v(\Diamond\psi))$ for any $\psi \in Sub(\varphi)$ and $u' \in W$; therefore, $u(\Box\psi) = v(\Box\psi) \leq (\pi^c(u') \Rightarrow_G u'(\psi))$ and $\min(\pi^c(u'), u'(\psi)) \leq v(\Diamond\psi) = u(\Diamond\psi)$. Taking infimum over u' in the first inequality and the supremum in the second we get

$$u(\Box\psi) \leq \inf_{u' \in W^c} \{\pi^c(u') \Rightarrow_G u'(\psi)\} \quad \text{and} \quad \sup_{u' \in W^c} \{\min(\pi^c(u'), u'(\psi))\} \leq u(\Diamond\psi).$$

Hence, if $u(\Box\psi) = 1$ and $u(\Diamond\psi) = 0$ we directly obtain (1) and (2), respectively. Therefore, it only remains to prove the next two claims for $\Box\psi, \Diamond\psi \in Sub(\varphi)$.

CLAIM 1. *If $u(\Box\psi) = \alpha < 1$ then, for every $\varepsilon > 0$, there exists a valuation $w \in W^c$ such that $\pi^c(w) > w(\psi)$ and $w(\psi) < \alpha + \varepsilon$, and thus $(\pi^c(w) \Rightarrow_G w(\psi)) < \alpha + \varepsilon$.*

CLAIM 2. *If $u(\Diamond\psi) = \alpha > 0$ then, for any small enough $\varepsilon > 0$, there exists a valuation $w' \in W^c$ such that $\min(w'(\psi), \pi^c(w')) \geq \alpha - \varepsilon$.*

The proofs of these two claims are rather involved and can be found in the appendix. ∎

THEOREM 4.1. (Weak completeness $K45(\mathbf{G})$) *For any formula φ in $\mathcal{L}_{\Box\Diamond}$, $\models_{\Pi\mathcal{G}} \varphi$ iff $\vdash_{K45(\mathbf{G})} \varphi$.*

PROOF. One direction is soundness, and it is easy to check that the axioms are valid in the class $\Pi\mathcal{G}$ of models. As for the other direction, assume $\nvdash_{K45(\mathbf{G})} \varphi$. Then $ThK45(\mathbf{G}) \nvdash_G \varphi$ by Lemma 2.1, and thus there is, by Proposition 2.1, a Gödel valuation $v : Var \cup X \to [0,1]$ such that $v(\varphi) < v(ThK45(\mathbf{G})) = 1$. Then v is a world of the canonical model $\mathfrak{M}_\varphi^v$ and by Lemma 4.2, $e_\varphi^v(v, \varphi) = v(\varphi) < 1$. Thus $\not\models_{\Pi\mathcal{G}} \varphi$. ∎

In addition, it is easy to generalize the last proof of completeness for deductions from finite theories.

THEOREM 4.2. (Finite strong completeness $K45(\mathbf{G})$) *For any finite theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$, we have: $T \models_{\Pi\mathcal{G}} \varphi$ iff $T \vdash_{K45(\mathbf{G})} \varphi$.*

PROOF. The proof is an easy adaptation of the one of weak completeness. We will only mention the main differences. If $T \nvdash_{K45(\mathbf{G})} \varphi$, by completeness of Gödel logic and Lemma 3.2, there exists a Gödel valuation v such that $v(ThK45(\mathbf{G})) = 1$, $v(T) = 1$ and $v(\varphi) < 1$. Now, in order to build a canonical model, we need to take into account not only v and φ but also T. To do that we have to follow the very same steps as before but replacing everywhere the set $Sub(\varphi)$ of subformulas of φ by the larger set $Sub(T, \varphi)$ of subformulas of $T \cup \{\varphi\}$, i.e. $Sub(T, \varphi) = \bigcup_{\psi \in T \cup \{\varphi\}} Sub(\psi)$. Let us denote by $\mathfrak{M}_{T,\varphi}^v = \langle W^c, \pi^c, e^c \rangle$ the canonical model built accordingly, where $v \in W^c$. Note that there is no need of any modification in neither Lemma 4.1 nor the Truth Lemma 4.2 (except for replacing $Sub(\varphi)$ by $Sub(T, \varphi)$ in its statement). Then the theorem follows by observing that Lemma 4.2 guarantees that $e^c(v, \psi) = v(\psi) = 1$ for all $\psi \in T$ while $e^c(v, \varphi) = v(\varphi) < 1$. Therefore, $T \not\models_{\Pi\mathcal{G}} \varphi$. ∎

Actually, the proofs for weak and finite strong completeness of $K45(\mathbf{G})$ with respect to the class of simplified possibilistic models $\Pi\mathcal{G}$ easily generalize to completeness of $KD45(\mathbf{G})$, the axiomatic extension $K45(\mathbf{G})$ with axiom D, with respect to the class of *normalized* possibilistic models $\Pi^*\mathcal{G}$.

COROLLARY 4.1. (Finite strong completeness $KD45(\mathbf{G})$) *For any finite theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$, we have: $T \models_{\Pi^*\mathcal{G}} \varphi$ iff $T \vdash_{KD45(\mathbf{G})} \varphi$.*

PROOF. We only consider the proof of weak completeness, its extension to a proof of finite strong completeness can be then devised as in Theorem 4.2. Indeed, the proof only needs small adaptations to the one for the case of $K45(\mathbf{G})$. To start with, for technical reasons that will become clear later, we will actually prove that $\nvdash_{KD45(\mathbf{G})} \varphi'$ implies $\nvDash_{\Pi^*\mathcal{G}} \varphi'$, where $\varphi' = \Diamond\top \to \varphi$. Note that since $\Diamond\top$ is an axiom of $KD45(\mathbf{G})$ and is valid in the class of frames $\Pi^*\mathcal{G}$, the former condition is indeed equivalent to prove that $\nvdash_{KD45(\mathbf{G})} \varphi$ implies $\nvDash_{\Pi^*\mathcal{G}} \varphi$. Next, in the definition of the set of worlds $W^v_{\varphi'}$ of the canonical model $\mathfrak{M}^v_{\varphi'}$, we need to replace the condition $u(ThK45(\mathbf{G})) = 1$ by $u(ThKD45(\mathbf{G})) = 1$, i.e. we define

$$W^v_{\varphi'} = \{u \in [0,1]^{Var \cup X} \mid u \sim_{\varphi'} v \text{ and } u(ThKD45(\mathbf{G})) = 1\},$$

and analogously in the condition **a.** of Lemma 4.1. Then in the proof of this lemma (item (ii) after the **Claim**), one has to further check that $w(\Diamond\top) = 1$. Observe that $\Diamond\top$ is provably equivalent to $\Box\top \to \Diamond\top$, but this formula is of the form $\phi \to \psi$ for some $\phi, \psi \in X$, and hence it falls under the cases already considered in the proof. Thus Lemma 4.1 holds, and the same happens with Lemma 4.2, that holds as well without any modification. Moreover, Lemma 4.2 allows us to prove that the canonical model belongs in fact to the class $\Pi^*\mathcal{G}$ of normalized possibilistic models. Indeed, by (2) it follows that

$$\sup_{u' \in W^v_{\varphi'}} \{\min(\pi^c(u'), u'(\psi))\} = u(\Diamond\psi)$$

for every $\Diamond\psi \in Sub(\varphi')$, and since $\Diamond\top \in Sub(\varphi')$ and $u \in W^v_{\varphi'}$ (and hence $u(\Diamond\top) = 1$), we finally have

$$1 = u(\Diamond\top) = \sup_{u' \in W^v_{\varphi'}} \{\min(\pi^c(u'), u'(\top))\} = \sup_{u' \in W^v_{\varphi'}} \pi^c(u'),$$

in other words, π^c is normalized and thus $\mathfrak{M}^v_{\varphi'} \in \Pi^*\mathcal{G}$. In summary, we have found a model $\mathfrak{M}^v_{\varphi'} = \langle W^v_{\varphi'}, \pi^c, e^v_{\varphi'} \rangle \in \Pi^*\mathcal{G}$ and a world $v \in W^v_{\varphi'}$ such that $e^v_{\varphi'}(v, \varphi') = e^v_{\varphi'}(v, \varphi) < 1$, and therefore $\nvDash_{\Pi^*\mathcal{G}} \varphi$. ∎

We would like to finish this section by noting that the same kind of proof for weak and finite strong completeness for $K45$ can also be easily adapted to the logic $KT45(\mathbf{G})$, which is in fact equivalent to $KT5(\mathbf{G})$, since Axiom (4) is derivable in $KT5(\mathbf{G})$. For that, we need to adapt some details of our original proof. First, we can take the same definition of the canonical model but considering $ThKT5(\mathbf{G})$ instead of $ThK45(\mathbf{G})$. Next, we follow with Lemma 4.1 where we change condition **a.** by $\nu(ThKT5(\mathbf{G})) = 1$. It is easy to check that the same proof goes through except that we need to prove now

that $w(\Box\varphi \to \varphi) = w(\varphi \to \Diamond\varphi) = 1$. But again this is easy to be verified, and the rest of the proof is working well. Then, we are are able to prove the following:

THEOREM 4.3. (Weak completeness $KT45(\mathbf{G})$) *For any formula φ in $\mathcal{L}_{\Box\Diamond}$:*
$$\models_{\Pi^5\mathcal{G}} \varphi \text{ iff } \vdash_{KT5(\mathbf{G})} \varphi,$$
where $\Pi^5\mathcal{G}$ is the class of possibilistic frames $\langle W, \pi \rangle$ that validate axioms $(T_\Box)$ and $(T_\Diamond)$.

This result is not new however, since it turns out that $\langle W, \pi \rangle$ validates $(T_\Box)$ and $(T_\Diamond)$ iff π is such that $\pi(w) = 1$ for every $w \in W$. In other words, $\Pi^5\mathcal{G}$ is in fact the class of universal models, the simplified semantics for $S5(\mathbf{G})$, a result that is well-known in the literature [8].

5. Decidability

So far, we have shown that $Val(\Pi\mathcal{G}) = ThK45(\mathbf{G})$, i.e. the set of valid formulas in $\Pi\mathcal{G}$, the class of all $\Pi\mathcal{G}$-frames, coincides with the set of theorems of the logic $K45(\mathbf{G})$, or in other words, the logic $K45(\mathbf{G})$ is sound and complete with respect to $\Pi\mathcal{G}$-frames. It is natural to ask whether the logic $K45(\mathbf{G})$ is decidable. Unfortunately, it is easy to check that under the possibilistic semantics the logic $K45(\mathbf{G})$ does not satisfy the finite model property. Indeed, consider the formula

$$\Box\neg\neg p \to \neg\neg\Box p$$

where p is a propositional variable. This formula is not valid in the model $\mathfrak{M} = \langle \mathbb{N}, \pi, e \rangle$, where for all $n \in \mathbb{N}$,

$$\pi(n) = 1 \text{ and } e(n, p) = \tfrac{1}{n+1}.$$

Then, for all $n \in \mathbb{N}$ we have:

$$e(n, \neg\neg p) = (\frac{1}{n+1} \Rightarrow_G 0) \Rightarrow_G 0 = 1,$$

$$e(n, \Box p) = \inf_{n\in\mathbb{N}}\{\pi(n) \Rightarrow_G e(n, p)\} = \inf_{n\in\mathbb{N}}\{1 \Rightarrow_G \frac{1}{n+1}\} = 0,$$

$$e(n, \Box\neg\neg p) = \inf_{n\in\mathbb{N}}\{\pi(n) \Rightarrow_G e(n, \neg\neg p)\} = \inf_{n\in\mathbb{N}}\{1 \Rightarrow_G 1\} = 1,$$

and hence $e(n, \Box\neg\neg p \to \neg\neg\Box p) = 0$. However, $\Box\neg\neg p \to \neg\neg\Box p$ is valid in any $\Pi\mathcal{G}$-model $\langle W, \pi, e \rangle$ where W is finite.

Nevertheless, in [7] an alternative semantics was provided for $K(\mathbf{G})$ which admits the finite model property. In this section, we adapt that semantics

in order to obtain an equivalent class of ΠG-models. For that, we are going to use the same strategy used in [7], i.e. by limiting the truth-values of modal formulae to a finite number of possibilities. According to this idea, we propose the following adaptation of the original semantics given in [7].

DEFINITION 5.1. A ΠGF-*model* is a quadruple $\mathfrak{M} = \langle W, \pi, T, e \rangle$, where $\langle W, \pi, e \rangle$ is a ΠG-model and $T \in \mathcal{P}_{<\omega}([0,1])$ is a finite set of truth values satisfying $\{0,1\} \subseteq T \subseteq [0,1]$. The valuation e is extended to formulas using the same clauses in ΠG-models for non-modal connectives, and using the following revised clauses for modal connectives:

$$e(w, \Box\varphi) = \max\left\{ r \in T : r \leq \inf_{w' \in W} \{\pi(w') \Rightarrow_G e(w', \varphi)\} \right\},$$

$$e(w, \Diamond\varphi) = \min\left\{ r \in T : r \geq \sup_{w' \in W} \{\min(\pi(w'), e(w', \varphi))\} \right\}.$$

A formula φ is said to be *valid* in $\mathfrak{M}$ if $e(w, \varphi) = 1$ for all $w \in W$. We will denote by $\Pi\mathcal{GF}$ the class of all ΠGF-models.

Notice that now the formula $\varphi = \Box\neg\neg p \rightarrow \neg\neg\Box p$ has a finite ΠGF-counter-model. Indeed, consider the ΠGF-model $\mathfrak{M}_0 = \langle W, \pi, T, e \rangle$ with $W = \{a\}$, $\pi(a) = 1$, $T = \{0,1\}$, and such that $e(a,p) = \frac{1}{2}$. Then we have:

- $e(a, \neg p) = 0$, $\pi(a) \Rightarrow_G e(a, \neg\neg p) = 1$, and so $e(a, \Box\neg\neg p) = 1$.
- Further, $e(a, \Box p) = 0$ (since $\pi(a) \Rightarrow_G e(a,p) = \frac{1}{2}$, and 0 is the next smallest element of T);
- Hence, $e(a, \neg\Box p) = 1$ and $e(a, \neg\neg\Box p) = 0$.
- Therefore, $1 = e(a, \Box\neg\neg p) > e(a, \neg\neg\Box p) = 0$ and $\Box\neg\neg p \rightarrow \neg\neg\Box p$ is not valid in $\mathfrak{M}_0$.

Next, we are going to prove that both semantics characterize the same logic. First we need the following lemmas to prove the main result.

LEMMA 5.1. *Let $\mathfrak{M} = \langle W, \pi, T, e \rangle$ be an ΠGF-model. Given an order-embedding $h \colon [0,1] \rightarrow [0,1]$ satisfying $h(0) = 0$, $h(1) = 1$, and for any $t \in T$, $h(t) = t$, consider $\widehat{\mathfrak{M}} = \langle W, \widehat{\pi}, T, \widehat{e} \rangle$, with $\widehat{\pi}(w) = h(\pi(w))$ and $\widehat{e}(w, p) = h(e(w, p))$ for all $w \in W$ and $p \in \mathrm{Var}$. Then, for all $\varphi \in \mathcal{L}_{\Box\Diamond}(Var)$ and $w \in W$:*

$$\widehat{e}(w, \varphi) = h(e(w, \varphi))$$

PROOF. It is a special case of part (c) of Lemma 1 in [7]. ∎

Now, we provide the key construction of a ΠG-model taking the same truth values for formulae as a given ΠGF-model.

LEMMA 5.2. *For any ΠGF-model $\mathfrak{M} = \langle W, \pi, T, e \rangle$, there is a ΠG-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{\pi}, \widehat{e} \rangle$ with $W \subseteq \widehat{W}$, such that $\widehat{e}(w, \varphi) = e(w, \varphi)$ for all $\varphi \in \mathcal{L}_{\square\lozenge}(Var)$ and $w \in W_{\mathfrak{M}}$.*

PROOF. We proceed similarly to the proof of Lemma 4 in [7]. Given a ΠGF-model $\mathfrak{M} = \langle W, \pi, T, e \rangle$, we construct an associated ΠG-model $\widehat{\mathfrak{M}}$ directly by taking infinitely many copies of $\mathfrak{M}$. Assume $T = \{\alpha_1, \ldots, \alpha_n\}$ with $0 = \alpha_1 < \ldots < \alpha_n = 1$ and, using Lemma 5.1, define a family of order-embeddings $\{h_k\}_{k \in \mathbb{Z}^+}$ from $[0, 1]$ into $[0, 1]$ satisfying $h_k(0) = 0$ and $h_k(1) = 1$, and such that for all $i \leq n - 1$:

$$
\begin{array}{rcll}
h_k(\alpha_i) & = & \alpha_i & k \in \mathbb{Z}^+ \\
h_k[(\alpha_i, \alpha_{i+1})] & = & (\alpha_i, \min(\alpha_i + \frac{1}{k}, \alpha_{i+1})) & \text{even } k \in \mathbb{Z}^+ \\
h_k[(\alpha_i, \alpha_{i+1})] & = & (\max(\alpha_i, \alpha_{i+1} - \frac{1}{k}), \alpha_{i+1}) & \text{odd } k \in \mathbb{Z}^+.
\end{array}
$$

For all $k \in \mathbb{Z}^+$, we define a ΠG-model $\widehat{\mathfrak{M}}_k = \langle \widehat{W}_k, \widehat{\pi}_k, \widehat{e}_k \rangle$ such that each $\widehat{W}_k$ is a copy of W with distinct worlds, $\widehat{\pi}_k = h_k(\pi)$ and $\widehat{e}_k(w^k, \varphi) = h_k(e(w, \varphi))$ for each copy w^k of $w \in W$ and $\varphi \in \mathcal{L}_{\square\lozenge}(Var)$. We also let $\widehat{W}_0 = W$, $\widehat{\pi}_0 = \pi$ and $\widehat{e}_0 = e$. Then we define $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{\pi}, \widehat{e} \rangle$, where

$$
\widehat{W} = \bigcup_{k \in \mathbb{N}} \widehat{W}_k \quad \text{and for } \widehat{w} \in \widehat{W}_k : \quad \widehat{\pi}(\widehat{w}) = \widehat{\pi}_k(\widehat{w}) \quad ; \quad \widehat{e}(\widehat{w}, p) = \widehat{e}_k(\widehat{w}, p).
$$

It then suffices to prove that $\widehat{e}(w, \varphi) = e(w, \varphi)$ for all $\varphi \in \mathcal{L}_{\square\lozenge}(Var)$ and $w \in W$, proceeding by an induction on the complexity of φ. The base case follows directly from the definition of $\widehat{e}$. For the inductive step, the cases for the non-modal connectives follow easily using the induction hypothesis. Let us just consider the case $\varphi = \square\psi$, the case $\varphi = \lozenge\psi$ being very similar. There are two possibilities. Suppose first that

$$
e(w, \square\psi) = \max\{r \in T : r \leq \inf\{\pi(w') \Rightarrow_G e(w', \psi) : w' \in W\}\} = 1.
$$

This means that, for all $w' \in W$, $\pi(w') \leq e(w', \psi)$ and hence, for all $k \in \mathbb{Z}^+$ and $\widehat{w'} \in \widehat{W}$: $h_k(\pi(w')) = \widehat{\pi}_k(w'^k) = \widehat{\pi}(\widehat{w'}) \leq h_k(e(w', \psi)) = \widehat{e}_k(w'^k, \psi) = \widehat{e}(\widehat{w'}, \psi)$. It follows that

$$
\widehat{e}(\widehat{w}, \square\psi) = \inf\{\widehat{\pi}(\widehat{w'}) \Rightarrow_G \widehat{e}(\widehat{w'}, \psi) : \widehat{w'} \in \widehat{W}\} = 1 = e(w, \square\psi).
$$

Now suppose that $e(w, \Box\psi) = \alpha_i < 1$, for some $i \leq m - 1$. Then, $\pi(v) \Rightarrow_G e(v, \psi) \geq \alpha_i$ for all $v \in W$, and thus $(\star)$: $\widehat{\pi}(\widehat{v}) \Rightarrow_G \widehat{e}(\widehat{v}, \psi) \geq \alpha_i$ for all $\widehat{v} \in \widehat{W}$, by construction using the order-embeddings $\{h_k\}_{k \in \mathbb{Z}^+}$.

There are two subcases. First, suppose there is at least one $w' \in W$ such that $\pi(w') \Rightarrow_G e(w', \psi) = \alpha_i$; call it w'_0. This means that $\pi(w'_0) > e(w'_0, \psi) = \alpha_i$ and for all $k \in \mathbb{Z}^+$, $\widehat{e}(\widehat{w'}_0, \psi) = \widehat{e}_k(w'^k_0, \psi) = h_k(e(w'_0, \psi)) = h_k(\alpha_i) = \alpha_i$. Since $\pi(w'_0) > \alpha_i$, also for all $k \in \mathbb{Z}^+$, $\widehat{\pi}(\widehat{w'}_0) = \widehat{\pi}_k(w'^k_0) = h_k(\pi(w'_0)) > \alpha_i = \widehat{e}(\widehat{w'}_0, \psi)$, and hence, using $(\star)$, we have:

$$\widehat{e}(\widehat{w}, \Box\psi) = \inf\{\widehat{\pi}(\widehat{v}) \Rightarrow_G \widehat{e}(\widehat{v}, \psi) : \widehat{v} \in \widehat{W}\} = \alpha_i = e(w, \Box\psi).$$

Now suppose $\pi(w') \Rightarrow_G e(w', \psi) > \alpha_i$ for all $w' \in W$. Since $e(w, \Box\psi) = \max\{r \in T : r \leq \inf\{\pi(w') \Rightarrow_G e(w', \psi) : w' \in W\}\} = \alpha_i$, there is at least one $w' \in W$ such that $\pi(w') \Rightarrow_G e(w', \psi) \in (\alpha_i, \alpha_{i+1})$; call it w'_0. Then, by construction, for any $\epsilon > 0$ there is a $k \in \mathbb{Z}^+$ such that $\widehat{\pi}(w'^k_0) \Rightarrow_G \widehat{e}(w'^k_0, \psi) \in (\alpha_i, \alpha_i + \epsilon)$. Using $(\star)$, this ensures that

$$\widehat{e}(\widehat{w}, \Box\psi) = \inf\{\widehat{\pi}(v) \Rightarrow_G \widehat{e}(v, \psi) : v \in \widehat{W}\} = \alpha_i = e(w, \Box\psi).$$

This ends the proof. ∎

As an immediate consequence of Lemma 5.2, we have the next corollary.

COROLLARY 5.1. $Val(\Pi\mathcal{G}) \subseteq Val(\Pi\mathcal{G}\mathcal{F})$.

The next lemma paves the way to prove in Theorem 5.1 that the logic $K45(\mathbf{G})$ has the finite model property and offers a bound on the complexity. We call a set $\Sigma \subseteq \mathcal{L}_{\Box\Diamond}(Var)$ a *fragment* if it is closed under subformulas.

LEMMA 5.3. *Let $\Sigma \subseteq \mathcal{L}_{\Box\Diamond}(Var)$ be a finite fragment. Then, for any ΠG-model $M = \langle W, \pi, e \rangle$, there is a finite ΠGF-model $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{\pi}, \widehat{T}, \widehat{e} \rangle$ with $\widehat{W} \subseteq W$, such that $\widehat{e}(w, \varphi) = e(w, \varphi)$ for all $\varphi \in \Sigma$ and $w \in \widehat{W}$, and moreover, $|\widehat{W}| + |\widehat{T}| \leq 2|\Sigma|$.*

PROOF. Let $\Sigma \subseteq \mathcal{L}_{\Box\Diamond}(Var)$ be a finite fragment, $M = \langle W, \pi, e \rangle$ a ΠG-model. First, define $\Sigma_\Box$ as the set of all box-formulas in Σ, $\Sigma_\Diamond$ as the set of all diamond-formulas in Σ, and Σ_{Var} as the set of all variables in Σ. Let us also define $e_w[\Delta] = \{e(w, \varphi) : \varphi \in \Delta\}$ for any $w \in W$ and $\Delta \subseteq \mathcal{L}_{\Box\Diamond}(Var)$. In addition, let $e_w[\Sigma_\Box \cup \Sigma_\Diamond] \cup \{0, 1\} = \{\alpha_1, \ldots, \alpha_n\}$ with $0 = \alpha_1 < \ldots < \alpha_n = 1$.

Next, we choose a finite number of $w' \in W$. For each $\Box\psi \in \Sigma_\Box$ such that $e(w, \Box\psi) = \alpha_i < 1$, choose a $w' = w'_{\Box\psi} \in W$ such that $e(w'_{\Box\psi}, \psi) < \alpha_{i+1}$, and for each $\Diamond\psi \in \Sigma_\Diamond$, such that $e(w, \Diamond\psi) = \alpha_i > 0$, choose a $w' = w'_{\Diamond\psi} \in W$ such that $e(w'_{\Diamond\psi}, \psi) > \alpha_{i-1}$. Then let $\widehat{W} = \{w\} \cup \{w'_{\Box\psi} \in W : \Box\psi \in \Sigma_\Box\} \cup$

$\{w'_{\Diamond\psi} \in W : \Diamond\psi \in \Sigma_\Diamond\}$. Clearly $\widehat{W} \subseteq W$ is finite. We define $\widehat{\mathfrak{M}} = \langle \widehat{W}, \widehat{\pi}, \widehat{T}, \widehat{e}\rangle$ where $\widehat{T} = e_w[\Sigma_\Box \cup \Sigma_\Diamond] \cup \{0,1\}$, and both $\widehat{\pi}$ and $\widehat{e}$ are equal to π and e restricted to $\widehat{W}$, respectively. It then follows by induction on complexity of φ that $\widehat{e}(w,\varphi) = e(w,\varphi)$ for all $w \in \widehat{W}$ and $\varphi \in \Sigma$. The base case follows from the definition of $\widehat{e}$. For the inductive step, let $\varphi \in \Sigma$ be of the form $\varphi = \Box\psi$ (the non-modal cases follow directly, using the induction hypothesis). We need to consider the same two cases it was considered in previous Lemma 5.2. First, note that since $\widehat{W} \subseteq W$, it is always true that

$$\inf\{\pi(w') \Rightarrow_G e(w',\psi) : w' \in W\} \leq \inf\{\widehat{\pi}(\widehat{w'}) \Rightarrow_G \widehat{e}(\widehat{w'},\psi) : \widehat{w'} \in \widehat{W}\}.$$

Now suppose that $1 = e(w,\Box\psi)$. Then, because $1 \in \widehat{T}$,

$$\begin{aligned}
1 = e(w,\Box\psi) &= \inf\{\pi(w') \Rightarrow_G e(w',\psi) : w' \in W\}\\
&\leq \inf\{\widehat{\pi}(\widehat{w'}) \Rightarrow_G \widehat{e}(\widehat{w'},\psi) : \widehat{w'} \in \widehat{W}\}\\
&\leq \max\{r \in \widehat{T} : r \leq \inf\{\widehat{\pi}(\widehat{w'}) \Rightarrow_G \widehat{e}(\widehat{w'},\psi) : \widehat{w'} \in \widehat{W}\}\}\\
&= \widehat{e}(w,\Box\psi).
\end{aligned}$$

For the second case, $e(w,\Box\psi) = \inf\{\pi(w') \Rightarrow_G e(w',\psi) : w' \in W\} = \alpha_i < 1$ for some $i \in \{1,\ldots,n-1\}$. According to our choise, there exists a $w'_{\Box\psi} \in \widehat{W}$ such that $\alpha_i \leq \widehat{\pi}(w'_{\Box\psi}) \Rightarrow_G \widehat{e}(w'_{\Box\psi},\psi) < \alpha_{i+1}$. Hence,

$$\alpha_i \leq \inf\{\widehat{\pi}(\widehat{w'}) \Rightarrow_G \widehat{e}(\widehat{w'},\psi) : \widehat{w'} \in \widehat{W}\}\} \leq \widehat{\pi}(w'_{\Box\psi}) \Rightarrow_G \widehat{e}(w'_{\Box\psi},\psi) < \alpha_{i+1}.$$

Thus

$$\widehat{e}(w,\Box\psi) = \max\{r \in \widehat{T} : r \leq \inf\{\widehat{\pi}(\widehat{w'}) \Rightarrow_G \widehat{e}(\widehat{w'},\psi) : \widehat{w'} \in \widehat{W}\}\} = \alpha_i.$$

The diamond-case case follows similarly to the box-case and is therefore omitted.

Finally, we note that by the construction of $\widehat{\mathfrak{M}}$, $|\widehat{W}| \leq |\Sigma_\Box \cup \Sigma_\Diamond| + 1 \leq |\Sigma|$ and $|\widehat{T}| \leq |\Sigma_\Box \cup \Sigma_\Diamond| + 2 \leq |\Sigma|$, and therefore $|\widehat{W}| + |\widehat{T}| \leq 2|\Sigma|$. ∎

As a direct consequence of the above lemma we have the converse inclusion of Corollary 5.1.

COROLLARY 5.2. $Val(\Pi\mathcal{GF}) \subseteq Val(\Pi\mathcal{G})$.

Finally, we can state the main result of this section, taking into account that, if the length of φ is n, the smallest set $\Sigma(\varphi)$ closed by subformulas containing φ is such that $|\Sigma(\varphi)| \leq n + 2$.
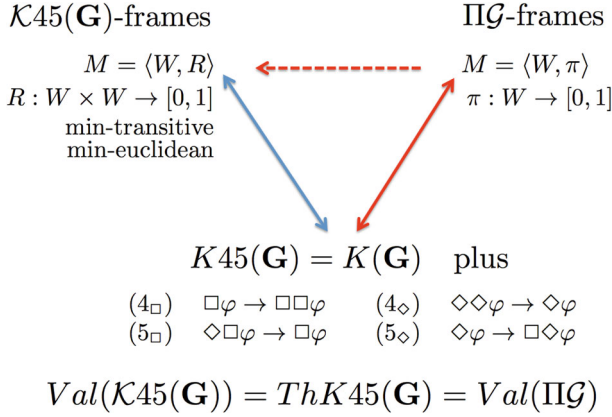
Figure 1. $K45(\mathbf{G})$ and its relational and possibilistic semantics

THEOREM 5.1. *For each $\varphi \in \mathcal{L}_{\Box\Diamond}(Var)$ of length n: $\models_{\Pi\mathcal{G}} \varphi$ iff φ is valid in all (finite) $\Pi\mathcal{G}F$-models $\mathfrak{M} = \langle W, \pi, T, e \rangle$ satisfying $|W| + |T| \leq 2(n + 2)$.*

As a direct consequence we get the decidability of $K45(\mathbf{G})$.

COROLLARY 5.3. *The logic $K45(\mathbf{G})$ is decidable.*

Similarly, one can prove that the logic $KD45(\mathbf{G})$ is decidable as well.

COROLLARY 5.4. *The logic $KD45(\mathbf{G})$ is decidable.*

6. Conclusions

In this paper we have studied the logic over Gödel fuzzy logic arising from many-valued Gödel Kripke models with possibilistic semantics, and have shown that it actually corresponds to a simplified semantics for the logic $K45(\mathbf{G})$, the extension of Caicedo and Rodriguez's bi-modal Gödel logic with many-valued versions of the well-known modal axioms 4 and 5. See Fig. 1 for a diagram of the results. We have also considered the extension with the axiom D, the logic $KD45(\mathbf{G})$, and have shown to be captured by normalised possibilistic Gödel Kripke models. In this way, we have obtained many-valued Gödel generalizations of the results reported by Pietruszczak in [19] about simplified semantics for several classical modal logics. We have also shown the decidability of those logics.

It is worth noting that the truth-value of a formula $\Diamond\varphi$ in a possibilistic Kripke model is indeed a proper generalization of the possibility measure of φ when φ is a classical proposition, however the semantics of $\Box\varphi$ is not.

This is due to the fact that the negation in Gödel logic is not involutive. Therefore, a first open problem we leave for further research is to consider to extension of the logic $K45(\mathbf{G})$ with an involutive negation and investigate its possibilistic semantics. Finally, we would like to study the connection between our simplified possiblistic semantics and the pseudomonadic algebras proposed in [4].

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Appendix

Claim from Lemma 4.1. Induction hypothesis (IH): if Properties 1–4 are satisfied by formulas with complexity at most n then these properties keep holding for formulae with complexity at most $n + 1$.

PROOF. This case will become important when we try to prove that w satisfies the axioms of $K45(\mathbf{G})$.

We consider different cases according to the main operator being either $\wedge$ or $\rightarrow$. First, we prove Properties 1 and 2, that involve only one formula:

Case $\psi = \psi_1 \wedge \psi_2$.

1. Let $\nu(\psi_1 \wedge \psi_2) = 1$. Then $\nu(\psi_1) = \nu(\psi_2) = 1$. In such a case, from the inductive hypothesis, we conclude that $w(\psi_1) \geq \delta + \varepsilon$ and $w(\psi_2) \geq \delta + \varepsilon$. Therefore, $w(\psi_1 \wedge \psi_2) \geq \delta + \varepsilon$.

2. Let $\nu(\psi_1 \wedge \psi_2) < 1$. Then, without loss of generality, we can assume that $\nu(\psi_1 \wedge \psi_2) = \nu(\psi_1) \leq \nu(\psi_2)$. In the case that $\nu(\psi_1) < 1$, we can apply the inductive hypothesis and conclude that $w(\psi_1) < \delta + \varepsilon$ and $w(\psi_1) \leq w(\psi_2)$, according to Properties 2 and 3, respectively. Therefore, we obtain $w(\psi_1 \wedge \psi_2) = w(\psi_1) < \delta + \varepsilon$.

Case $\psi = \psi_1 \rightarrow \psi_2$.

1. Let $\nu(\psi_1 \rightarrow \psi_2) = 1$. Then $\nu(\psi_1) \leq \nu(\psi_2)$. First, we consider the case $\nu(\psi_1) < 1$ where we can apply Property 3 by the inductive hypothesis, obtaining that $w(\psi_1) \leq w(\psi_2)$ and, hence, $w(\psi_1 \rightarrow \psi_2) = 1 \geq \delta + \varepsilon$. Now, we turn to the case $\nu(\psi_1) = 1 = \nu(\psi_2)$. In this situation, by applying Property 1 on ψ_2, we conclude that $w(\psi_1 \rightarrow \psi_2) \geq w(\psi_2) \geq \delta + \varepsilon$.

2. Let $\nu(\psi_1 \rightarrow \psi_2) < 1$. Then $\nu(\psi_1) > \nu(\psi_2)$. By inductive hypothesis, by applying Property 4, we have that $w(\psi_1) > w(\psi_2)$, and hence $w(\psi_1 \rightarrow \psi_2) = w(\psi_2)$, but since $\nu(\psi_2) < 1$, then by Property 2, $w(\psi_2) < \delta + \varepsilon$.

Next we prove Properties 3 and 4, that involve two formulas. We proceed by considering the four possible combinations.

Case $\psi = \psi_1 \wedge \psi_2$ and $\phi = \phi_1 \wedge \phi_2$. In this case the proof is easy because, without loss of generality, we can assume that $\nu(\psi_1 \wedge \psi_2) = \nu(\psi_1)$ and $\nu(\phi_1 \wedge \phi_2) = \nu(\phi_1)$.

3. Let $1 \neq \nu(\psi_1 \wedge \psi_2) \leq \nu(\phi_1 \wedge \phi_2)$. According to our assumption, this means $1 \neq \nu(\psi_1) \leq \nu(\phi_1)$, and by applying induction hypothesis we obtain $w(\psi_1) \leq w(\phi_1)$ and $w(\psi_1) < \delta + \varepsilon$. Now, we need to consider two additional cases:

   a. If $\nu(\psi_2) = 1$ then, by IH, $w(\psi_2) \geq \delta + \varepsilon$. Thus, we obtain $w(\psi_1 \wedge \psi_2) = w(\psi_1) < \delta + \varepsilon$. In addition, if $\nu(\phi_1) < 1$ then, by IH (Property 3), $w(\phi_1) \leq w(\phi_2)$ and $w(\psi_1 \wedge \psi_2) \leq w(\phi_1 \wedge \phi_2)$. On the contrary, if $\nu(\phi_1) = 1$ then, $\nu(\phi_2) = 1$ and $w(\phi_1 \wedge \phi_2) \geq \delta + \varepsilon > w(\psi_1 \wedge \psi_2)$.

b. If $\nu(\psi_2) < 1$ then, by IH on Properties 2 and 3, $w(\psi_1) \leq w(\psi_2) < \delta + \varepsilon$ which implies $w(\psi_1) = w(\psi_1 \wedge \psi_2) \leq w(\phi_1)$. The proof of this case is completed by reproducing the both alternatives when $\nu(\phi_1) < 1$ and $\nu(\phi_1) = 1$, which were analyzed in the previous item.

4. Let $\nu(\psi_1 \wedge \psi_2) < \nu(\phi_1 \wedge \phi_2)$. Again, this means $\nu(\psi_1) < \nu(\phi_1)$. By IH, we obtain $w(\psi_1) < w(\phi_1)$ and $w(\psi_1) < \delta + \varepsilon$ (Properties 4 and 2, respectively). The rest of the proof runs as before.

Case $\psi = \psi_1 \rightarrow \psi_2$ and $\phi = \phi_1 \wedge \phi_2$. Again, without loss of generality, we can assume that $\nu(\phi_1 \wedge \phi_2) = \nu(\phi_1)$.

3. Let $1 \neq \nu(\psi_1 \rightarrow \psi_2) \leq \nu(\phi_1 \wedge \phi_2)$. Since $\nu(\psi_1 \rightarrow \psi_2) < 1$, we conclude that $\nu(\psi_1) > \nu(\psi_2)$ and $1 \neq \nu(\psi_2) \leq \nu(\phi_1)$. By IH, we know that $w(\psi_2) \leq w(\phi_1)$ and $w(\psi_1) > w(\psi_2)$. In addition, if $1 \neq \nu(\phi_1) \leq \nu(\phi_2)$ then, by IH, we have $w(\psi_1 \rightarrow \psi_2) = w(\psi_2) \leq w(\phi_1) = w(\phi_1 \wedge \phi_2)$. On the contrary, if $\nu(\phi_1) = 1 = \nu(\phi_2)$ then $w(\phi_1 \wedge \phi_2) \geq \delta + \varepsilon > w(\psi_2) = w(\psi_1 \rightarrow \psi_2)$.

4. Let $\nu(\psi_1 \rightarrow \psi_2) < \nu(\phi_1 \wedge \phi_2)$. As in the previous proof, we can conclude that $\nu(\psi_1) > \nu(\psi_2)$ and so by Property 4, $w(\psi_1) > w(\psi_2)$, i.e. $w(\psi_1 \rightarrow \psi_2) = w(\psi_2)$. Moreover, as $\nu(\psi_2) < \nu(\phi_1)$, $w(\psi_2) < w(\phi_1)$ again by Property 4. If $1 \neq \nu(\phi_1) \leq \nu(\phi_2)$ then, by IH, $w(\phi_1) = w(\phi_1 \wedge \phi_2)$. On the contrary, if $\nu(\phi_1) = 1 = \nu(\phi_2)$ then, by IH, $w(\phi_1 \wedge \phi_2) \geq \delta + \varepsilon > w(\psi_2) = w(\psi_1 \rightarrow \psi_2)$.

Case $\psi = \psi_1 \wedge \psi_2$ and $\phi = \phi_1 \rightarrow \phi_2$. Again, without loss of generality, we can assume that $\nu(\psi_1 \wedge \psi_2) = \nu(\psi_1)$.

3. Let $1 \neq \nu(\psi_1 \wedge \psi_2) \leq \nu(\phi_1 \rightarrow \phi_2)$. According to our assumption, we know $\nu(\psi_1 \wedge \psi_2) = \nu(\psi_1) < 1$. Then, by IH, we obtain $w(\psi_1 \wedge \psi_2) = w(\psi_1) < \delta + \varepsilon$. If $\nu(\phi_1 \rightarrow \phi_2) < 1$ then $\nu(\phi_1) > \nu(\phi_2) \geq \nu(\psi_1)$ and, by IH, we know $w(\phi_1 \rightarrow \phi_2) = w(\phi_2) \geq w(\psi_1) = w(\psi_1 \wedge \psi_2)$. On the contrary, $\nu(\phi_1 \rightarrow \phi_2) = 1$ implies $\nu(\phi_1) \leq \nu(\phi_2)$. In this case, if $\nu(\phi_1) < 1$ then, by IH $w(\phi_1) \leq w(\phi_2)$ which implies $w(\phi_1 \rightarrow \phi_2) = 1$. If $\nu(\phi_1) = 1 = \nu(\phi_2)$ then, by IH, $w(\phi_1 \rightarrow \phi_2) \geq w(\phi_2) \geq \delta + \varepsilon > w(\psi_1) = w(\psi_1 \wedge \psi_2)$.

4. Let $\nu(\psi_1 \wedge \psi_2) < \nu(\phi_1 \rightarrow \phi_2)$. In this case, we can proceed analogously to the previous proofs. For this reason, we leave to the reader to complete the details.

Case $\psi = \psi_1 \to \psi_2$ and $\phi = \phi_1 \to \phi_2$.

3. Let $1 \neq \nu(\psi_1 \to \psi_2) \leq \nu(\phi_1 \to \phi_2)$. As $\nu(\psi_1 \to \psi_2) \neq 1$, we conclude $\nu(\psi_1 \to \psi_2) = \nu(\psi_2) < 1$. In addition, if $\nu(\phi_1 \to \phi_2) < 1$ then $\nu(\phi_1 \to \phi_2) = \nu(\phi_2)$. Thus, if the truth values of both implications are less than one, we have $1 \neq \nu(\psi_2) \leq \nu(\phi_2)$. By IH, we know $w(\psi_2) \leq w(\phi_2)$, $w(\psi_1) > w(\psi_2)$ and $w(\phi_1) > w(\phi_2)$. Therefore, $w(\psi_1 \to \psi_2) = w(\psi_2) \leq w(\phi_2) = w(\phi_1 \to \phi_2)$. On the contrary, if $\nu(\phi_1 \to \phi_2) = 1$ then $\nu(\phi_1 \leq \nu(\phi_2)$. If $\nu(\phi_1) < 1$ then, by IH, $w(\phi_1 \to \phi_2) = 1$. In the opposite case, if $\nu(\phi_1) = 1 = \nu(\phi_2)$ then, by IH, $w(\phi_1 \to \phi_2) \geq w(\phi_2) \geq \delta + \varepsilon > w(\psi_2) = w(\psi_1 \to \psi_2)$.

4. Let $\nu(\psi_1 \to \psi_2) < \nu(\phi_1 \to \phi_2)$. If both $\nu(\psi_1 \to \psi_2) < 1$ and $\nu(\phi_1 \to \phi_2) < 1$ then we can proceed as in the previous proof and we can apply IH concluding that $w(\psi_1 \to \psi_2) = w(\psi_2) < w(\phi_2) = w(\phi_1 \to \phi_2)$. On the contrary, if $\nu(\phi_1 \to \phi_2) = 1$ then we need to consider two alternatives. First, $1 \neq \nu(\phi_1) \leq \nu(\phi_2)$ implies by IH that $w(\psi_1 \to \psi_2) = w(\psi_2) < \delta + \varepsilon < 1 = w(\phi_1 \to \phi_2)$. Second, if $\nu(\phi_1) = 1 = \nu(\phi_2)$ then, by IH, $w(\phi_1 \to \phi_2) \geq w(\phi_2) \geq \delta + \varepsilon > w(\psi_2) = w(\psi_1 \to \psi_2)$.

This finishes the proof of the claim. ∎

Claim 1 from Lemma 4.2. If $u(\Box\psi) = \alpha < 1$, for every $\varepsilon > 0$, there exists a valuation $w \in W^c$ such that $\pi^c(w) > w(\psi)$ and $w(\psi) < \alpha + \varepsilon$, and thus $\pi^c(w) \Rightarrow_G w(\psi) = w(\psi) < \alpha + \varepsilon$.

PROOF. The proof is achieved in two stages:

- First producing a valuation $\nu \in W$ satisfying $\nu(\psi) < 1$ and preserving the relative ordering conditions the values $w(\theta)$ must satisfy, conditions which may be coded by a theory $\Gamma_{\psi,u}$;

- And then moving the values $\nu(\theta)$, for $\theta \in Sub(\varphi)$, to the correct valuation w by composing ν with an increasing function of $[0,1]$, using Lemma 4.1.

Assume $u(\Box\psi) = \alpha < 1$, and define the following set of formulas (all formulas involved ranging in $\mathcal{L}_{\Box\Diamond}(Var)$):

$$\begin{aligned}
\Gamma_{\psi,u} = \{&\chi : \chi \in X \text{ and } u(\chi) > \alpha\} \\
\cup \{&\lambda \to \theta : \lambda \in \Delta_\varphi \text{ and } u(\lambda) \leq u(\Box\theta)\} \\
\cup \{&(\theta \to \lambda) \to \lambda : \lambda \in \Delta_\varphi \text{ and } u(\lambda) < u(\Box\theta) < 1\} \\
\cup \{&\theta \to \lambda : \lambda \in \Delta_\varphi \text{ and } u(\Diamond\theta) \leq u(\lambda)\} \\
\cup \{&(\lambda \to \theta) \to \theta : \lambda \in \Delta_\varphi \text{ and } u(\Diamond\theta) < u(\lambda) < 1\} \\
\cup \{&(\chi_1 \to \chi_2) \to \chi_2 : \chi_1, \chi_2 \in X \text{ and } u(\chi_2) < u(\chi_1) \leq \alpha\}
\end{aligned}$$

Then we can check that $u(\Box\xi) > \alpha$ for each $\xi \in \Gamma_{\psi,u}$. Indeed, first recall that, by $U_\Box$ and $U_\Diamond$ of Proposition 3.1, for any $\lambda \in X$ (in particular Δ_φ) we have $u(\Diamond\lambda) \leq u(\lambda) \leq u(\Box\lambda)$ in $\mathsf{K45}(\mathbf{G})$. We analyse case by case. For the first set of formulas, it is clear by construction. For the second set, we have $u(\Box(\lambda \to \theta)) \geq u(\Diamond\lambda \to \Box\theta) = u(\Diamond\lambda) \Rightarrow_G u(\Box\theta) \geq u(\lambda) \Rightarrow_G u(\Box\theta) = 1$, by $FS2$. For the third, by FS2 and P, we have

$$u(\Box((\theta \to \lambda) \to \lambda)) \geq u(\Diamond(\theta \to \lambda) \to \Box\lambda) \geq u((\Box\theta \to \Diamond\lambda) \to \Box\lambda)$$

$$\geq u((\Box\theta \to \lambda) \to \Box\lambda)$$

$$= u(\lambda) \Rightarrow_G u(\Box\lambda) = 1,$$

since $u(\Diamond\lambda) \leq u(\lambda) \leq u(\Box\lambda)$ and $u(\lambda) < u(\Box\theta)$. The fourth and fifth cases are very similar to the second and third ones. As for the sixth case, by (T4), either $u(\Box\chi_1 \to \Diamond\chi_2) = 1$ or $u(\Box((\chi_1 \to \chi_2) \to \chi_2)) = 1$. According to $(U_\Box)$, $(U_\Diamond)$, $(5_\Box)$ and $(5_\Diamond)$, we have that, for any φ, $\Diamond\Box\varphi \to \Box\varphi$ and $\Diamond\varphi \to \Box\Diamond\varphi$ are theorems of $K45(G)$, and thus $u(\Box\chi_1) \geq u(\chi_1)$ and $u(\Diamond\chi_2) \leq u(\chi_2)$, from where it follows that

$$u(\Box\chi_1 \to \Diamond\chi_2) = u(\Box\chi_1) \Rightarrow_G u(\Diamond\chi_2) \leq u(\chi_1) \Rightarrow_G u(\chi_2) = u(\chi_2) < 1.$$

Therefore, it holds that $u(\Box((\chi_1 \to \chi_2) \to \chi_2)) = 1$.

The fact that $u(\Box\xi) > \alpha$ for each $\xi \in \Gamma_{\psi,u}$ implies

$$\Gamma_{\psi,u} \nvdash_{K45(\mathbf{G})} \psi,$$

otherwise there would exist $\xi_1, \ldots, \xi_k \in \Gamma_{\psi,u}$ such that $\xi_1, \ldots, \xi_k \vdash_{K45(\mathbf{G})} \psi$. In such a case, we would have $\Box\xi_1, \ldots, \Box\xi_k \vdash_{K45(\mathbf{G})} \Box\psi$ by Nec and $K_\Box$. Then $\Box\xi_1, \ldots, \Box\xi_k, ThK45(\mathbf{G}) \vdash_G \Box\psi$ by Lemma 2.1, and thus by Proposition 2.1 (i), and recalling that $u(ThK45(\mathbf{G})) = 1$, we would have

$$\alpha < \inf u(\{\Box\xi_1, \ldots, \Box\xi_k\} \cup ThK45(\mathbf{G})) \leq u(\Box\psi) = \alpha,$$

a contradiction. Therefore, by Proposition 2.1 (ii) there exists a valuation $\nu : Var \cup X \mapsto [0,1]$ such that $\nu(\Gamma_{\psi,u} \cup ThK45(\mathbf{G})) = 1$ and $\nu(\psi) < 1$. This implies the following relations between u and ν, that we list for further use. Given $\lambda, \lambda_1, \lambda_2 \in \Delta_\varphi$, $\theta \in \mathcal{L}_{\Box\Diamond}(Var)$ and $\chi, \chi_1, \chi_2 \in X$, we have:

#**1**. If $u(\chi) > \alpha$ then $\chi \in \Gamma_{\psi,u}$, and hence $\nu(\chi) = 1$.

#**2**. If $u(\lambda) \leq u(\Box\theta)$ then $\nu(\lambda) \leq \nu(\theta)$ (since then $\lambda \to \theta \in \Gamma_{\psi,u}$). In particular, if $\lambda_1, \lambda_2 \in \Delta_\varphi$ and $u(\lambda_1) \leq u(\Box\lambda_2)$ then $\nu(\lambda_1) \leq \nu(\lambda_2)$. Furthermore, if $\Box\theta \in \Delta_\varphi$ then $\nu(\Box\theta) \leq \nu(\theta)$. That means, taking $\theta = \psi$, $\nu(\Box\psi) \leq \nu(\psi) < 1$.

#**3**. If $u(\lambda) < u(\Box\theta) < 1$ then $\nu(\lambda) < \nu(\theta)$ or $\nu(\lambda) = 1$ (since then $(\theta \to \lambda) \to \lambda) \in \Gamma_{\psi,u}$). In particular, if $\lambda_1, \lambda_2 \in \Delta_\varphi$, $u(\lambda_1) < u(\lambda_2)$ and $u(\lambda_2) \leq u(\Box\psi) = \alpha$ then $\nu(\lambda_1) < \nu(\psi) < 1$ and thus $\nu(\lambda_1) < \nu(\lambda_2)$. This means that ν preserves in a strict sense the order values by u of the formulas $\lambda \in \Delta_\varphi$ such that $u(\lambda) \leq \alpha$.

#**4**. If $u(\Diamond\theta) \leq u(\lambda)$ then $\nu(\theta) \leq \nu(\lambda)$ (because $\theta \to \lambda \in \Gamma_{\psi,u}$). In particular, if $\Diamond\theta \in \Delta_\varphi$ then $\nu(\theta) \leq \nu(\Diamond\theta)$.

#**5**. If $u(\Diamond\theta) < u(\lambda) < 1$ then $\nu(\theta) < \nu(\lambda)$ or $\nu(\theta) = 1$. In particular, if $\lambda_1, \lambda_2 \in \Delta_\varphi$ and $u(\Diamond\lambda_1) \leq u(\lambda_1) < u(\lambda_2) \leq \alpha = u(\Box\psi)$ then $\nu(\lambda_1) < \nu(\lambda_2)$. Furthermore, if $u(\lambda_2) > 0$ then $\nu(\lambda_2) > 0$ (taking $\lambda_1 := \Diamond\bot$ since $u(\bot) = u(\Diamond\bot) = 0$).

#**6**. If $u(\chi_2) < u(\chi_1) \leq \alpha$ then $\nu(\chi_2) < \nu(\chi_1) < 1$. Indeed, note that, by construction, $\nu(\psi) < 1$, that implies, by #**2**, $\nu(\Box\psi) < 1$. In addition, we know $\Box\psi \in \Delta_\phi$ and $u(\Box\psi) = \alpha$. Therefore, according to #**4**, if $\chi_1 \in X$ and $u(\Diamond\chi_1) \leq u(\chi_1) \leq u(\Box\psi) = u(\Box\Box\psi)$, then $\nu(\chi_1) \leq \nu(\Box\Box\psi) = \nu(\Box\psi) < 1$. Finally, since $(\chi_1 \to \chi_2) \to \chi_2 \in \Gamma_{\psi,u}$ we have $\nu(\chi_2) < \nu(\chi_1) < 1$.

#**7**. If $u(\lambda_1) \leq u(\lambda_2)$ then $\nu(\lambda_1) \leq \nu(\lambda_2)$. Note that this is a particular case of #**2**. Certainly, if $\lambda_1, \lambda_2 \in \Delta_\varphi$, then by Lemma 3.1 we have $u(\lambda_2) \leq u(\Box\lambda_2)$, and, hence, by #**2**, $\nu(\lambda_1) \leq \nu(\lambda_2)$.

According to the properties #1–#7, it is clear that ν satisfies the conditions **a-d** of Lemma 4.1. Indeed, from #1 and #2 it follows that

$$\alpha = \max\{u(\lambda) : \nu(\lambda) < 1 \text{ and } \lambda \in \Delta_\varphi\},$$

because, by construction of $\Gamma_{\psi,u}$ and by #1, $\forall \lambda \in \Delta_\varphi$, if $u(\lambda) > \alpha$ then $\nu(\lambda) = 1$ and by #2, $\nu(\Box\psi) < 1$. Therefore α coincides with the value δ defined in Lemma 4.1.

Consequently, for all $\varepsilon > 0$ such that $\alpha + \varepsilon < \beta$, where β is defined as in Lemma 4.1, there exists a valuation $w \in W^v$ such that $w(\psi) < \alpha + \varepsilon$. Then in order to finish our proof, it remains to show that:

$$\pi^c(w) = \inf_{\lambda \in sub(\varphi)} \min(v(\Box\lambda) \Rightarrow_G w(\lambda), w(\lambda) \Rightarrow_G v(\Diamond\lambda)) > w(\psi) \quad (3)$$

To do so, we will prove that, for any $\lambda \in sub(\varphi)$, both implications in (3) are greater than or equal to $\alpha + \varepsilon$.[4] First we prove it for the first implication by cases:

– If $v(\Box\lambda) \leq \alpha < 1$ then $v(\Box\lambda) \Rightarrow_G w(\lambda) = 1$. Indeed, first of all, by #2, from $u(\Box\lambda) = v(\Box\lambda) \leq \alpha = u(\Box\psi)$ it follows $\nu(\Box\lambda) \leq \nu(\psi) < 1$. Now, since $u(\Box\lambda) \leq u(\Box\lambda)$, by #2, we have $1 \neq \nu(\Box\lambda) \leq \nu(\lambda)$, and by 3 of Lemma 4.1 we have $v(\Box\lambda) = w(\Box\lambda) \leq w(\lambda)$. Then $v(\Box\lambda) \Rightarrow_G w(\lambda) = 1$.

– If $v(\Box\lambda) > \alpha$ then by #1 and #2, $1 = \nu(\Box\lambda) \leq \nu(\lambda)$. Therefore, by 1 of Lemma 4.1, $w(\lambda) \geq \alpha + \varepsilon$ which implies $v(\Box\lambda) \Rightarrow_G w(\lambda) > \alpha$.

For the second implication we also consider two cases:

– If $v(\Diamond\lambda) = u(\Diamond\lambda) > \alpha$ then $\nu(\Diamond\lambda) = 1$ by #1, so $w(\lambda) \geq w(\Diamond\lambda) \geq \alpha + \varepsilon$. it is obvious that $w(\lambda) \Rightarrow_G v(\Diamond\lambda) > \alpha$ because by Lemma 4.1 we know $w(\lambda) \geq \alpha + \varepsilon$.

– If $u(\Diamond\lambda) \leq \alpha$, then by #4 we obtain $\nu(\lambda) \leq \nu(\Diamond\lambda) < 1$. Then by Lemma 4.1 we have $w(\lambda) \leq w(\Diamond\lambda) = v(\Diamond\lambda)$ and thus $w(\lambda) \Rightarrow_G v(\Diamond\lambda) = 1$.

∎

Claim 2 from Lemma 4.2. If $u(\Diamond\psi) = \alpha > 0$ then, for any small enough $\varepsilon > 0$, there exists a valuation $w' \in W$ such that $\min(w'(\psi), \pi^c(w')) \geq \alpha - \varepsilon$.

PROOF. Assume $u(\Diamond\psi) = \alpha > 0$[5]. Then, we take $\delta = \max\{u(\lambda) < \alpha : \lambda \in \Delta_\varphi\}$ (note that the last set is not empty because $\Diamond\bot$ belongs to it) and define $\Gamma'_{\psi,u}$ in a similar way that it was defined in the proof of Claim 1:

$$\Gamma'_{\psi,u} = \{\chi : \chi \in X \text{ and } u(\chi) > \delta\}$$
$$\cup \{\lambda \to \theta : \lambda \in \Delta_\varphi \text{ and } u(\lambda) \leq u(\Box\theta)\}$$
$$\cup \{(\theta \to \lambda) \to \lambda : \lambda \in \Delta_\varphi \text{ and } u(\lambda) < u(\Box\theta) < 1\}$$
$$\cup \{\theta \to \lambda : \lambda \in \Delta_\varphi \text{ and } u(\Diamond\theta) \leq u(\lambda)\}$$
$$\cup \{(\lambda \to \theta) \to \theta : \lambda \in \Delta_\varphi \text{ and } u(\Diamond\theta) < u(\lambda) < 1\}$$
$$\cup \{(\chi_1 \to \chi_2) \to \chi_2 : \chi_1, \chi_2 \in X \text{ and } u(\chi_2) < u(\chi_1) \leq \delta\}$$

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[4]Remember that $u \sim_\varphi v \sim_\varphi w$.

[5]In this context, α plays the role of β in Lemma 4.1.

If $u(\Diamond\psi) = \alpha > 0$, then we let $U_{\psi,u} = \psi \to \theta$ where $\theta \in \Delta_\varphi$ and $u(\theta) = \delta$. We claim that

$$\Gamma'_{\psi,u} \nvdash_{K45(\mathbf{G})} U_{\psi,u} \ ,$$

otherwise there would exist $\theta_1, \ldots, \theta_n \in \Gamma'_{\psi,u}$ such that $\vdash_{K45(\mathbf{G})} (\theta_1 \wedge \ldots \wedge \theta_n) \to U_{\psi,u}$, and then we would have $\vdash_{K45(\mathbf{G})} \Box(\theta_1 \wedge \ldots \wedge \theta_n) \to \Box(\psi \to \theta)$, which would imply $\vdash_{K45(\mathbf{G})} (\Box\theta_1 \wedge \ldots \wedge \Box\theta_n) \to (\Diamond\psi \to \Box\theta)$. In that case, evaluating with u it would yield $1 = u(\Box\theta_1 \wedge \ldots \wedge \Box\theta_n) \Rightarrow_G u(\Diamond\psi \to \theta)$, contradiction, since $u(\Box\theta_1 \wedge \ldots \wedge \Box\theta_n) > \delta$ and $u(\Diamond\psi \to \theta) = \delta$ (because by definition $u(\theta) < u(\Diamond\psi)$).

Therefore, there is an evaluation ν' such that $\nu'(ThK45(\mathbf{G})) = \nu'(\Gamma'_{\psi,u})$ $= 1$, $\nu'(\Diamond\psi) = 1$ and $\nu'(\psi \to \theta) < 1$. Hence, Condition **a**, in Lemma 4.1 is satisfied. Condition **b** is also satisfied because by definition of $\Gamma'_{\psi,u}$ we know that $\nu'(\chi) = 1$ for every $\chi \in X$ such that $u(\chi) > \delta$. In addition, we can verify Condition **c** by considering Property #6 of Claim 1 with δ instead of α.

Therefore, for any $\varepsilon > 0$ with $\delta + \varepsilon < \alpha$ there exists a valuation $w' \in W$ such that $w'(\psi) \geq \delta + \varepsilon$. Note that in this case $\delta < \alpha = 1$. That means we are always able to find an appropriate valuation w' such that $w'(\psi)$ is arbitrarily close to α.

It remains to show that $\pi^c(w') \geq \alpha$. Indeed, by construction, it holds that $\forall \theta \in \Delta_\varphi$ and $u(\Box\theta) \leq \delta : u(\Box\theta) \leq w'(\theta) \leq u(\Diamond\theta)$, and hence $\min(u(\Box\theta) \Rightarrow_G w'(\theta), w'(\theta) \Rightarrow_G u(\Diamond\theta)) \geq \alpha$. This finishes the proof. ∎

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R. O. Rodriguez
UBA. FCEyN. Departamento de Computación
UBA-CONICET. Inst. de Invest. en Cs. de la Computación
Buenos Aires
Argentina
`ricardo@dc.uba.ar`

O. F. Tuyt
Mathematical Institute
University of Bern
Bern
Switzerland
`olim.tuyt@math.unibe.ch`

F. Esteva, L. Godo
Artificial Intelligence Research Institute (IIIA)
Spanish National Research Council (CSIC)
Bellaterra
Spain
`godo@iiia.csic.es`

F .Esteva
`esteva@iiia.csic.es`