Chapter VIII: Fuzzy Logics with Enriched Language

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1 Introduction

The basic language of t-norm based fuzzy logics is composed of the strong conjunction $\&$, the lattice (or additive) conjunction $\wedge$, the implication $\rightarrow$ and the truth-constant $\overline{0}$ denoting *falsum*. From these primitive connectives,[1] other usual connectives are definable: the truth-constant $\overline{1}$ is defined as $\overline{0} \rightarrow \overline{0}$; the residual negation $\neg$, where $\neg\varphi$ is defined as $\varphi \rightarrow \overline{0}$; the equivalence $\leftrightarrow$, where $\varphi \leftrightarrow \psi$ is defined as $(\varphi \rightarrow \psi) \,\&\, (\psi \rightarrow \varphi)$; or the lattice disjunction $\vee$, where $\varphi \vee \psi$ is defined as $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$. The properties of these connectives may heavily vary depending on the particular semantics of the different t-norm based logics we consider. As a matter of example, consider the negation $\neg$. It turns out that e.g. in Łukasiewicz logic Ł, this negation is involutive, so $\neg\neg\varphi$ is equivalent to φ, while in Gödel logic G, the negation behaves very differently: it is, in fact, a pseudo-complementation and satisfies the axiom $\neg(\varphi \wedge \neg\varphi)$. Therefore, in some cases, we might need to use an involutive negation in the framework of Gödel logic, or, vice versa, we might need to use a Gödel negation in the framework of Łukasiewicz logic. Thus, in order to increase the expressive power of a given logic, it might be interesting to study expansions of a logic with different additional connectives. Indeed, developments in the field of fuzzy logic in a broad sense (like the study of De Morgan triples, the use of linguistic hedges and evaluated formulas in fuzzy logic applications, etc.) have led to the study of a number of expansions of fuzzy logics with additional connectives with varying arity. In this chapter we have selected some of the most relevant systems among such expansions.

In Section 2, we consider expansions of a logic L_* of a continuous t-norm $*$ with a set of truth-constants $\overline{r}$ for each r belonging to a countable subalgebra $\boldsymbol{C}$ of the standard L_*-algebra $[0,1]_*$. In Section 3, we deal with expansions of core fuzzy logics with truth-stressing and truth-depressing hedges, modelled as unary connectives. Their intended interpretations on standard algebras are non-decreasing mappings $h\colon [0,1] \rightarrow [0,1]$ such that $h(0) = 0$ and $h(1) = 1$: so, they respect the Boolean truth-values; moreover h is required to be subdiagonal ($h(x) \leq x$ for all $x \in [0,1]$) in case of a truth-stresser, and superdiagonal ($h(x) \geq x$ for all $x \in [0,1]$) in case of a truth-depresser. In Section, 4 we consider expansions of core fuzzy logics with an involutive negation $\sim$, that is, a negation such that $\sim\sim\varphi$ is equivalent to φ, which is not usually the case with the negation $\neg$

[1] In logics above BL, the lattice conjunction $\wedge$ is also definable, namely $\varphi \wedge \psi$ is $\varphi \,\&\, (\varphi \rightarrow \psi)$.

definable from the implication and the truth-constant $\overline{0}$, where $\neg\varphi$ is $\varphi \to \overline{0}$. Finally, in Section 5 we consider specially relevant expansions for Łukasiewicz logic, namely, the so-called Rational Łukasiewicz logic and different expansions including the product conjunction, eventually leading to the logics ŁΠ and $Ł\Pi\frac{1}{2}$.

2 Expansions with truth-constants

T-norm based fuzzy logics are basically logics of *comparative truth*. In fact, the residuum $\Rightarrow$ of a (left-continuous) t-norm $*$ satisfies the condition $x \Rightarrow y = 1$ if, and only if , $x \leq y$ for all $x, y \in [0,1]$. This means that a formula $\varphi \to \psi$ is a logical consequence of a theory if the truth degree of φ is at most as high as the truth degree of ψ in any interpretation which is a model of the theory. In fact, the logic of continuous t-norms presented in Hájek's seminal book [46] only deals with valid formulas and deductions taking 1 as the only truth value to be preserved by inference (in the sense of yielding true consequences from true premises for each interpretation). This line has been followed by the majority of papers written since then in the setting of many-valued systems of mathematical fuzzy logic, including this handbook. However, in general, these truth-preserving logics do not exploit in depth neither the idea of comparative truth nor the potentiality of dealing with explicit partial truth that a many-valued logic setting offers.

The idea of comparative truth is pushed forward in the so-called *logics preserving truth-degrees*, studied in [5, 37], where a deduction is valid if, and only if, the degree of truth of the premises is less than or equal to the degree of truth of the conclusion: in fact they preserve lower bounds of truth values. Actually, since Gödel logic is the only t-norm based logic enjoying the classical deduction-detachment theorem, it is the only case where both notions of logic coincide.

On the other hand, in some situations one might be also interested in explicitly representing and reasoning with intermediate degrees of truth. A way to do so, while keeping the truth preserving framework, is to introduce truth-constants into the language. This approach actually goes back to Pavelka [75], who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz logic obtained by adding to the language a truth-constant $\overline{r}$ for each real $r \in [0,1]$, together with some additional axioms. Pavelka proved that his logic is strongly complete in a non-finitary sense (known as Pavelka-style completeness), heavily relying on the continuity of Łukasiewicz truth-functions.

Similar expansions with truth-constants for other propositional t-norm based fuzzy logics can be analogously defined. However, Pavelka-style completeness cannot be obtained in those cases, since Łukasiewicz logic is the only t-norm based logic whose truth-functions are continuous. A more general approach was developed in a series of papers [12, 26, 30–33, 78] where, rather than Pavelka-style completeness, the authors focused on the usual notion of completeness of a logic. It is interesting to note that in this approach: (1) the logic to be expanded with truth-constants has to be the logic of a given left-continuous t-norm; (2) the expanded logic is still a truth-preserving logic, but its richer language admits formulas of type $\overline{r} \to \varphi$, implying that, when their evaluation equals 1, the truth degree of φ is greater or equal than r; and (3) the expanded logic is still algebraizable in the sense of Blok and Pigozzi.

In this section we describe the expansions with truth-constants of logics of continuous t-norms in a general setting.[2] Actually, we provide a full description of completeness results for the expansions of logics of continuous t-norms with a set of truth-constants $\{\overline{r} \mid r \in C\}$, for a suitable countable $C \subseteq [0,1]$, when (i) the t-norm is a finite ordinal sum of Łukasiewicz, Gödel and Product components and (ii) the set of truth-constants *covers* the whole unit interval in the sense that each component contains at least one value of C in its interior.

This section is structured as follows. After this introduction, for historical reasons we first introduce Rational Pavelka logic, a simplified version of Pavelka logic defined by Hájek [46]. In Section 2.2 we introduce a general notion of expanded logics with truth-constants and their algebraic semantics. In Sections 2.3 and 2.4, we describe the structure and relevant algebraic properties of the expanded linearly ordered algebras, which are needed to obtain the completeness results reported in Sections 2.5 and 2.6. Section 2.7 deals with completeness results when restricting the language to evaluated formulas. In Section 2.8, we also consider expansions with the Monteiro–Baaz $\triangle$ connective as well as an alternative approach to the use of the $\triangle$ connective. Finally, after mentioning some open questions in Section 2.9, we overview in Section 2.10 the expansions of first-order logics and their main results.

2.1 Rational Pavelka logic

Hájek [46] showed that Pavelka's logic could be significantly simplified while keeping the completeness results. Indeed, Rational Pavelka logic (RPL), see [45, 46], is the expansion of Łukasiewicz logic Ł by adding a truth-constant $\overline{r}$ for each rational $r \in [0,1]$ together with the following book-keeping axioms for truth-constants:

$$\text{(RPL1)} \quad \overline{r} \,\&\, \overline{s} \;\leftrightarrow\; \overline{r *_{Ł} s}$$
$$\text{(RPL2)} \quad \overline{r} \to \overline{s} \;\leftrightarrow\; \overline{r \Rightarrow_{Ł} s}$$

where $*_{Ł}$ and $\Rightarrow_{Ł}$ are Łukasiewicz t-norm and implication respectively. An evaluation e of propositional variables into the real unit interval $[0,1]$ is extended to an RPL-evaluation of arbitrary formulas as in Łukasiewicz logic with the additional requirement that $e(\overline{r}) = r$ for each rational r.

Notice that a formula of the form $\overline{r} \to \varphi$ gets value 1 under an evaluation e whenever φ gets a value by e greater or equal than r. Therefore, the RPL-formula $\overline{r} \to \varphi$ expresses that the truth-value of φ is at least r. Similarly, $\varphi \to \overline{r}$ expresses that the truth-value of φ is at most r.

As usual, a *theory* T over RPL is just a set of formulas. The notion of proof, denoted $\vdash_{\mathrm{RPL}}$, is defined as usual from the axioms of RPL and *modus ponens*. A theory T is *consistent* if $T \nvdash \overline{0}$. Furthermore, a theory T is *linear* if $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$ for each pair of RPL-formulas φ, ψ.

Given a theory T, the *truth degree* of a formula φ in T is defined as

$$||\varphi||_T = \inf\{e(\varphi) \mid e \text{ is a model of } T\},$$

[2] For the case of expansions of some left-continuous t-norms the reader is referred to [30, 33].

and the *provability degree* of φ over T as

$$|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{RPL}} \overline{r} \to \varphi\}.$$

REMARK 2.1.1. The provability degree is a supremum, which is not necessarily a maximum; for an infinite T, $|\varphi|_T = 1$ does not always imply $T \vdash \varphi$. (Still, this works for a finite T, see [53] and [46, Theorem 3.3.14].)

The (Pavelka-style) form of strong completeness for RPL says that the provability degree of φ in T equals the truth degree of φ over T, that is, $||\varphi||_T = |\varphi|_T$. To prove this we need some preliminary lemmas (that the reader can consult in [46]).

LEMMA 2.1.2.

(1) *$T \vdash \overline{0}$ iff $T \vdash \overline{r}$ for some $r < 1$.*

(2) *Each consistent theory T can be extended to a consistent and complete theory T'.*

(3) *If T does not prove $(\overline{r} \to \varphi)$ then $T \cup \{\varphi \to \overline{r}\}$ is consistent.*

(4) *If T is consistent and complete, then* $\sup\{r \mid T \vdash \overline{r} \to \varphi\} = \inf\{r \mid T \vdash \varphi \to \overline{r}\}$.

Proof. (1) It easily follows from the fact that if $r < 1$, then there exists n such that $r *_{Ł} \overset{n}{\ldots} *_{Ł} r = 0$.

(2) This is a well-known fact for Łukasiewicz logic, which is not invalidated by the presence of the truth-constants.

(3) Assume $T \cup \{\varphi \to \overline{r}\}$ is inconsistent, hence $T \cup \{\varphi \to \overline{r}\} \vdash \overline{0}$. By the local deduction theorem of Łukasiewicz logic, there is n such that $T \vdash (\varphi \to \overline{r})^n \to \overline{0}$. But $(\varphi \to \psi)^n \vee (\psi \to \varphi)^n$ is a theorem of Łukasiewicz logic, therefore RPL proves $(\varphi \to \overline{r})^n \vee (\overline{r} \to \varphi)^n$ as well. Thus obviously $T \vdash (\overline{0})^n \vee (\varphi \to \overline{r})^n$. Hence $T \vdash (\varphi \to \overline{r})^n$. Therefore, we obtain a contradiction.

(4) Since for each r, either $T \vdash \varphi \to \overline{r}$ or $T \vdash \overline{r} \to \varphi$, it suffices to show that $T \vdash \overline{r} \to \varphi$ and $T \vdash \varphi \to \overline{s}$ implies $r \leq s$. Assume $r > s$. Then we would get $T \vdash \overline{r} \to \overline{s}$, i.e. $T \vdash \overline{r \Rightarrow_{Ł} s}$, but $r \Rightarrow_{Ł} s < 1$, and thus T would be inconsistent. □

LEMMA 2.1.3. *If T is consistent and complete, the provability degree commutes with the connectives, i.e.*

$$|\overline{r}|_T = r, \qquad |\neg\varphi|_T = 1 - |\varphi|_T, \qquad |\varphi \to \psi|_T = |\varphi|_T \Rightarrow_{Ł} |\psi|_T.$$

Proof. The case of truth-constants is easy. For the case of the negation, we have $|\neg\varphi|_T = \sup\{r \mid T \vdash \overline{r} \to \neg\varphi\} = \sup\{r \mid T \vdash \varphi \to \overline{1-r}\} = \sup\{1-r \mid T \vdash \varphi \to \overline{r}\} = 1 - \inf\{r \mid T \vdash \varphi \to \overline{r}\} = 1 - |\varphi|_T$.

For the case of the implication, we show the two inequalities:

(a) $|\varphi|_T \Rightarrow_{Ł} |\psi|_T = \inf\{s \mid T \vdash \varphi \to \overline{s}\} \Rightarrow_{Ł} \sup\{r \mid T \vdash \overline{r} \to \psi\} = \sup\{s \Rightarrow_{Ł} r \mid T \vdash \varphi \to \overline{s}, T \vdash \overline{r} \to \psi\} \leq \sup\{t \mid T \vdash \overline{t} \to (\varphi \to \psi)\} = |\varphi \to \psi|_T$. Notice that here the continuity of $\Rightarrow_{Ł}$ plays a crucial role.

(b) Assume there exist rationals $t, t' \in [0,1]$ such that $|\varphi|_T \Rightarrow_{*_Ł} |\psi|_T < t < t' < |\varphi \to \psi|_T$. Express t as $r \Rightarrow_Ł s$ for some $r < |\varphi|_T$ and some $s > |\psi|_T$. Then $T \vdash \overline{r} \to \varphi$ and $T \vdash \psi \to \overline{s}$, and hence $T \vdash (\varphi \to \psi) \to (\overline{r} \to \overline{s})$, $T \vdash (\varphi \to \psi) \to \overline{t}$, $T \vdash \overline{t'} \to (\varphi \to \psi)$, and thus $T \vdash \overline{t'} \to \overline{t}$, i.e. $T \vdash \overline{t' \Rightarrow_Ł t}$. But since $t' > t$, we have $t' \Rightarrow_Ł t < 1$ and thus T is inconsistent. Therefore $|\varphi|_T \Rightarrow_Ł |\psi|_T \geq |\varphi \to \psi|_T$. □

From these lemmas, one can finally prove the following Pavelka's style completeness for RPL.

THEOREM 2.1.4. *In* RPL *we have* $||\varphi||_T = |\varphi|_T$, *for any theory* T *and any formula* φ.

Proof. The inequality $|\varphi|_T \leq ||\varphi||_T$ is derivable from the soundness of RPL. To prove the other inequality it is enough to show that for each rational $r < \|\varphi\|_T$, $T \vdash \overline{r} \to \varphi$, or equivalently, if $T \not\vdash \overline{r} \to \varphi$ then $r \geq \|\varphi\|_T$. But if $T \not\vdash \overline{r} \to \varphi$, then $T \cup \{\varphi \to \overline{r}\}$ is consistent. In that case, $T \cup \{\varphi \to \overline{r}\}$ has a consistent complete extension T', and by Lemma 2.1.3, the evaluation defined as $e(p_i) = |p_i|_{T'}$ is a model of T' and $e(\varphi \to \overline{r}) = 1$, and thus $e(\varphi) \leq r$ and hence $\|\varphi\|_T \leq \|\varphi\|_{T'} \leq r$. □

Actually in his papers [75], Pavelka proved a more general completeness result. In fact what he proves is that one can expand the logic with an arbitrary set of additional connectives whose real semantics are defined by "fitting" (finitary) operations on the real unit interval $[0,1]$.[3] In the framework of RPL, a very similar result is obtained in [49] for the expansion of RPL with product conjunction. Namely, the logic RPL^+ is defined as the expansion of RPL with a new connective $\odot$ and having as axioms those of RPL plus:

(RPL^+1) $\quad (\varphi \to \psi) \to ((\varphi \odot \chi) \to (\psi \odot \chi))$

(RPL^+2) $\quad (\varphi \to \psi) \to ((\chi \odot \varphi) \to (\chi \odot \psi))$

(RPL^+3) $\quad \overline{r} \odot \overline{s} \leftrightarrow \overline{r \cdot s}$

where $\cdot$ denotes product of reals. The first two axioms clearly stand for the monotonicity conditions of $\odot$ and the third is the book-keeping axiom on truth-constants with the product operation. Evaluations e of RPL^+ formulas are defined as in RPL together with the additional requirement that $e(\varphi \odot \psi) = e(\varphi) \cdot e(\psi)$. Moreover, the notions of truth and provability degrees of RPL^+-formulas in a theory are defined in the same way as in RPL. Then the following completeness theorem holds.

THEOREM 2.1.5. *In* RPL^+ *we have* $||\varphi||_T = |\varphi|_T$, *for any theory* T *and any formula* φ.

Proof. The proof mimics the one for RPL and basically one has to extend Lemma 2.1.3 to $\odot$, that is, one has to prove that $|\varphi \odot \psi|_T = |\varphi|_T \cdot |\psi|_T$. Remark that due to axioms (RPL^+1) and (RPL^+2), in RPL^+ we have that if $T \vdash \varphi_1 \to \psi_1$ and $T \vdash \varphi_2 \to \psi_2$ then we also have $T \vdash \varphi_1 \odot \varphi_2 \to \psi_1 \odot \psi_2$.

[3] An operation $O\colon [0,1]^n \to [0,1]$ *fits* the standard MV-chain $[0,1]_Ł$ if there is exists natural numbers $k_1, \ldots, k_n$ such that for any $x_1, \ldots, x_n, y_1, \ldots, y_n \in [0,1]$, holds: $(x_1 \Leftrightarrow_Ł y_1)^{k_1} \otimes \cdots \otimes (x_n \Leftrightarrow_Ł y_n)^{k_n} \leq O(x_1, \ldots, x_n) \Leftrightarrow_Ł O(y_1, \ldots, y_n)$, where $\Leftrightarrow_Ł$ is defined as $x \Leftrightarrow_Ł y = \min\{x \Rightarrow_Ł y, y \Rightarrow_Ł x\}$.

(a) $|\varphi|_T \cdot |\psi|_T = \sup\{s \mid T \vdash \overline{s} \to \varphi\} \cdot \sup\{r \mid T \vdash \overline{r} \to \psi\} = \sup\{s \cdot r \mid T \vdash \overline{s} \to \varphi, T \vdash \overline{r} \to \psi\} \leq \sup\{t \mid T \vdash \overline{t} \to (\varphi \odot \psi)\} = |\varphi \odot \psi|_T$.

(b) Assume there exist rationals $t, t' \in [0,1]$ such that $|\varphi|_T \cdot |\psi|_T < t < t' < |\varphi \odot \psi|_T$. Clearly we can express t as $r \cdot s$ for some $r > |\varphi|_T$ and some $s > |\psi|_T$. Then $T \vdash \varphi \to \overline{r}$ and $T \vdash \psi \to \overline{s}$, and hence $T \vdash (\varphi \odot \psi) \to (\overline{r} \odot \overline{s})$, $T \vdash (\varphi \odot \psi) \to \overline{t}$, $T \vdash \overline{t'} \to (\varphi \odot \psi)$, and thus $T \vdash \overline{t'} \to \overline{t}$, i.e. $T \vdash \overline{t' \Rightarrow_{Ł} t}$. But since $t' > t$, we have $t' \Rightarrow_{Ł} t < 1$ and thus T is inconsistent.

Therefore $|\varphi|_T \cdot |\psi|_T = |\varphi \odot \psi|_T$. □

Looking at the above proof, one realizes that the same proof would apply if the conjunction $\odot$ is semantically interpreted by another continuous t-norm $*$ closed over the rationals, that is, if we replace axiom $(\mathrm{RPL}^{+}3)$ by

$(\mathrm{RPL}^{+}_{*}3)$ $\quad \overline{r} \odot \overline{s} \leftrightarrow \overline{r * s}$

then the resulting logic would enjoy the same Pavelka's style completeness. Therefore, this shows that the monotonicity axioms $(\mathrm{RPL}^{+}1)$ and $(\mathrm{RPL}^{+}2)$ plus the book-keeping axiom $(\mathrm{RPL}^{+}_{*}3)$ suffice to axiomatize (à la Pavelka) any continuous t-norm closed over the rationals.

2.2 Expansions of the logic of a continuous t-norm with truth-constants

A complete analogy between RPL and other logics L_* of continuous t-norms $*$ is not possible since Łukasiewicz logic is the only logic L_* with continuous real truth-functions. Still, one can consider similar expansions with analogous book-keeping axioms and investigate classical completeness properties. This is the main goal of the rest of this section.

Let L_* be the logic of a continuous t-norm $*$, i.e., the extension of BL such that for any finite set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathrm{L}_*} \varphi \qquad \text{iff} \qquad \Gamma \models_{[0,1]_*} \varphi.$$

As proved in Chapter V, whenever $*$ is a continuous t-norm, the logic L_* is finitely axiomatizable. Moreover, Chapter V gives a finite axiomatization of L_* as an axiomatic extension of BL.

The goal of this section is to define and study the expansion of any L_* by adding a countable set of truth-constants.

DEFINITION 2.2.1. *Let $[0,1]_*$ be the real L_*-chain and $\boldsymbol{C}$ its countable subalgebra. We define the expanded language $\mathcal{L}_C = \mathcal{L} \cup \{\overline{r} \mid r \in C \setminus \{0,1\}\}$, where $\mathcal{L}$ is the language of L_*. The logic $\mathrm{L}_*(\boldsymbol{C})$ is the expansion of L_* in the language of $\mathcal{L}_C$ obtained by adding the so-called book-keeping axioms:*

$$\overline{r} \,\&\, \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow_* s}$$

where $\Rightarrow_*$ is the residuum of $*$.

Since these logics are core fuzzy logics, sharing *modus ponens* as the only inference rule, they have the same local deduction-detachment theorem as BL. In fact, the proof for BL also applies here.

THEOREM 2.2.2. *For every $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}_C}$, $\Gamma, \varphi \vdash_{\mathrm{L}_*(\boldsymbol{C})} \psi$ if, and only if, there is a natural $k \geq 1$ such that $\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi^k \to \psi$.*

The algebraic counterpart of the $\mathrm{L}_*(\boldsymbol{C})$ logics is defined in the natural way.

DEFINITION 2.2.3. *Let $*$ be a continuous t-norm and $\boldsymbol{C}$ a countable subalgebra of $[0,1]_*$. A structure $\boldsymbol{A} = \langle A, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \{\overline{r}^{\boldsymbol{A}} \mid r \in C\}\rangle$ is an $\mathrm{L}_*(\boldsymbol{C})$-algebra if:*

(1) $\langle A, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ *is an L_*-algebra, and*

(2) *for every $r, s \in C$ the following identities hold:*

$$\overline{r}^{\boldsymbol{A}} \,\&^{\boldsymbol{A}}\, \overline{s}^{\boldsymbol{A}} = \overline{r * s}^{\boldsymbol{A}}$$
$$\overline{r}^{\boldsymbol{A}} \to^{\boldsymbol{A}} \overline{s}^{\boldsymbol{A}} = \overline{r \Rightarrow_* s}^{\boldsymbol{A}}.$$

Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_C}$, we define $\Gamma \models_{\boldsymbol{A}} \varphi$ iff for all evaluations e on $\boldsymbol{A}$ (i.e. such that $e(\overline{r}) = \overline{r}^{\boldsymbol{A}}$), we have $e(\varphi) = \overline{1}^{\boldsymbol{A}}$ whenever $e(\psi) = \overline{1}^{\boldsymbol{A}}$ for all $\psi \in \Gamma$.

Real chains *are $\mathrm{L}_*(\boldsymbol{C})$-chains whose underlying domain is the real unit interval $[0,1]$. In particular the* canonical $\mathrm{L}_*(\boldsymbol{C})$*-chain is* $[0,1]_{\mathrm{L}_*(\boldsymbol{C})} = \langle [0,1], *, \Rightarrow_*, \min, \max,$ $\{r \mid r \in C\}\rangle$*, i.e. the $\mathcal{L}_{\boldsymbol{C}}$-expansion of $[0,1]_*$ where the truth-constants are interpreted as themselves.*

We denote the set of interpretations of truth-constants over $\boldsymbol{A}$ by $\boldsymbol{C}^{\boldsymbol{A}}$.

Notice that $\boldsymbol{C}^{\boldsymbol{A}}$ is closed under the operations of the algebra $\boldsymbol{A}$, i.e. $\langle \boldsymbol{C}^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}},$ $\wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \bar{0}^{\boldsymbol{A}}, \bar{1}^{\boldsymbol{A}}\rangle$ is a subalgebra of $\boldsymbol{A}$.

It is worth noticing that it is not always possible to equip any L_*-algebra with an arbitrary set of constants from a subalgebra of $[0,1]_*$. For instance, it is not possible to equip a finite MV-chain with truth-constants from the whole subalgebra of rationals of $[0,1]_{\text{Ł}}$.

Since $\mathrm{L}_*(\boldsymbol{C})$ is an expansion of L_* without new rules of inference, by [21], $\mathrm{L}_*(\boldsymbol{C})$ is a semilinear logic. As a consequence, each $\mathrm{L}_*(\boldsymbol{C})$-algebra is a subdirect product of chains and thus the logic $\mathrm{L}_*(\boldsymbol{C})$ is complete not only with respect to the full variety but also with respect to the chains of the variety.

To describe real completeness results requires a deeper insight into $\mathrm{L}_*(\boldsymbol{C})$-chains. This is done in the next subsection. Actually, for technical reasons (see the remarks at the end of this section), we will restrict ourselves to logics $\mathrm{L}_*(\boldsymbol{C})$ satisfying the following two conditions:

(**C1**) $*$ is a continuous t-norm that is a *finite* ordinal sum of the basic components (we will denote by **CONT-fin** the set of such continuous t-norms).

(**C2**) each component of the t-norm contains at least one value of C different from the bounds of the component.

2.3 About the structure of real $\mathrm{L}_*(\boldsymbol{C})$-chains

Suppose that $*$ is a continuous t-norm in **CONT-fin** whose decomposition as ordinal sum of isomorphic copies of the three basic components is $\bigoplus_{i\in I}[a_i, b_i]_{*_i}$.

DEFINITION 2.3.1. *Let $\boldsymbol{A}$ be an $\mathrm{L}_*(\boldsymbol{C})$-chain. $\boldsymbol{C}^{\boldsymbol{A}}$ will denote the subalgebra of $\boldsymbol{A}$ defined over $\{\overline{r}^{\boldsymbol{A}} \mid r \in C\}$ and $F_{\boldsymbol{C}}(\boldsymbol{A})$ will denote the set of the truth-constants interpreted as 1 in $\boldsymbol{A}$, i.e. $F_{\boldsymbol{C}}(\boldsymbol{A}) = \{r \in C \mid \overline{r}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}\}$.*

LEMMA 2.3.2. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be non-trivial $\mathrm{L}_*(\boldsymbol{C})$-chains with the same $\mathcal{L}$-reduct (i.e. possibly differing only on the interpretation of truth-constants). Then:*

- (i) *$F_{\boldsymbol{C}}(\boldsymbol{A})$ is a proper filter of $\boldsymbol{C}$.*
- (ii) *$\boldsymbol{C}/F_{\boldsymbol{C}}(\boldsymbol{A}) \cong \boldsymbol{C}^{\boldsymbol{A}}$.*
- (iii) *If $\boldsymbol{A} \cong \boldsymbol{B}$, then $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B})$.*
- (iv) *If $r, s \in C \setminus F_{\boldsymbol{C}}(\boldsymbol{A})$ and $r < s$, then $\overline{r}^{\boldsymbol{A}} < \overline{s}^{\boldsymbol{A}}$.*

Proof. (i) Clearly $1 \in F_{\boldsymbol{C}}(\boldsymbol{A})$. If $r \in F_{\boldsymbol{C}}(\boldsymbol{A})$ and $s \in C$, $s > r$, then $s \in F_{\boldsymbol{C}}(\boldsymbol{A})$ because by the book-keeping axioms and the definability of min and max we have $\overline{s}^{\boldsymbol{A}} = \max\{\overline{r}^{\boldsymbol{A}}, \overline{s}^{\boldsymbol{A}}\} = \overline{1}^{\boldsymbol{A}}$. Moreover if $r, s \in F_{\boldsymbol{C}}(\boldsymbol{A})$ then $r * s \in F_{\boldsymbol{C}}(\boldsymbol{A})$, since $\overline{r * s}^{\boldsymbol{A}} = \overline{r}^{\boldsymbol{A}} \,\&\, \overline{s}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}$.

(ii) Consider the function $f\colon \boldsymbol{C} \to \boldsymbol{C}^{\boldsymbol{A}}$ defined by $f(r) = \overline{r}^{\boldsymbol{A}}$. It is clear that f is a surjective homomorphism and $Ker f = F_{\boldsymbol{C}}(\boldsymbol{A})$, so $\boldsymbol{C}/F_{\boldsymbol{C}}(\boldsymbol{A}) \cong \boldsymbol{C}^{\boldsymbol{A}}$.

(iii) If $\boldsymbol{A} \cong \boldsymbol{B}$, then it is clear that $\boldsymbol{C}^{\boldsymbol{A}} \cong \boldsymbol{C}^{\boldsymbol{B}}$, so $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B})$.

(iv) If $r < s \notin F_{\boldsymbol{C}}(\boldsymbol{A})$, then $\overline{r}^{\boldsymbol{A}} \leq \overline{s}^{\boldsymbol{A}}$ since the book-keeping axioms imply that the order must be preserved. On the other hand, if $\overline{r}^{\boldsymbol{A}} = \overline{s}^{\boldsymbol{A}}$, then $[r]_{F_{\boldsymbol{C}}(\boldsymbol{A})} = [s]_{F_{\boldsymbol{C}}(\boldsymbol{A})}$, which implies $s \to r \in F_{\boldsymbol{C}}(\boldsymbol{A})$. So, we obtain a contradiction. In fact:

- (a) If $r, s \in (a_i, b_i)$ and $[a_i, b_i]$ is a Łukasiewicz component, then $s \to r$ belongs to $F_{\boldsymbol{C}}(\boldsymbol{A})$, which implies that the minimum of the component also belongs to $F_{\boldsymbol{C}}(\boldsymbol{A})$. Therefore $[a_i, b_i] \subseteq F_{\boldsymbol{C}}(\boldsymbol{A})$, i.e. a contradiction.
- (b) If $r, s \in (a_i, b_i)$ and $[a_i, b_i]$ is a Product component, then $s \to r \in F_{\boldsymbol{C}}(\boldsymbol{A})$, which implies: if $r = 0$ then $0 \in F_{\boldsymbol{C}}(\boldsymbol{A})$, which is a contradiction; and if $r \neq 0$ then there exists n such that $r > (s \to r)^n$, and, thus, $r, s \in F_{\boldsymbol{C}}(\boldsymbol{A})$, i.e., again, a contradiction.
- (c) Finally, if $r * s = \min\{r, s\}$ then $s \to r = r \in F_{\boldsymbol{C}}(\boldsymbol{A})$, a contradiction. □

Notice that the first three properties in the previous lemma also hold for the general case of $*$ being a left-continuous t-norm. However, we do make use of the continuity of $*$ in the proof of the last one.[4] Actually this lemma describes all possible interpretations of the truth-constants over $\mathrm{L}_*(\boldsymbol{C})$-chains. For instance, for every filter F we can define

[4] In [30] it is proved that (iv) is also valid when $*$ is a Weak Nilpotent Minimum t-norm but it could fail for a general left-continuous t-norm.

an $\mathrm{L}_*(\boldsymbol{C})$-algebra over $[0,1]_*$ interpreting $\overline{r}$ as 1 if $r \in F$ and as r otherwise. We will denote this algebra by $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$. An easy computation shows that it is indeed an $\mathrm{L}_*(\boldsymbol{C})$-chain. Notice that the canonical real algebra corresponds to the case $F = \{1\}$. Moreover, if the t-norm has only Łukasiewicz or Product components, there are as many $\mathrm{L}_*(\boldsymbol{C})$-algebras over $[0,1]_*$ (up to isomorphism) as proper filters of $\boldsymbol{C}$.

PROPOSITION 2.3.3. *Let $*$ be a continuous t-norm that is a finite ordinal sum of Łukasiewicz and Product components. Let $X = \{[\boldsymbol{A}] \mid \boldsymbol{A}$ is a real $\mathrm{L}_*(\boldsymbol{C})$-algebra over $[0,1]_*\}$ be the set of isomorphism classes of $\mathrm{L}_*(\boldsymbol{C})$-algebras over $[0,1]_*$ and let $Fi(\boldsymbol{C})$ be the set of proper filters of $\boldsymbol{C}$. Then, the function $\Phi\colon X \to Fi(\boldsymbol{C})$ such that for every $\boldsymbol{A} \in X$, $\Phi([\boldsymbol{A}]) = F_{\boldsymbol{C}}(\boldsymbol{A})$, is a bijection.*

Proof. Φ is well-defined because of (iii) of Lemma 2.3.2. For an easier notation we will simply write $\Phi(\boldsymbol{A})$ instead of $\Phi([\boldsymbol{A}])$. Φ is clearly onto because $\Phi([0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}) = F$. Thus, we have to prove that Φ is also injective. Suppose that $\Phi(\boldsymbol{A}) = \Phi(\boldsymbol{B})$, i.e. $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B})$. Then, we have $\boldsymbol{C}^{\boldsymbol{A}} \cong \boldsymbol{C}/F_{\boldsymbol{C}}(\boldsymbol{A}) = \boldsymbol{C}/F_{\boldsymbol{C}}(\boldsymbol{B}) \cong \boldsymbol{C}^{\boldsymbol{B}}$. In the following, denoting by h the isomorphism between $\boldsymbol{C}^{\boldsymbol{A}}$ and $\boldsymbol{C}^{\boldsymbol{B}}$, we show how to extend it as a function $h\colon [0,1] \to [0,1]$ making $\boldsymbol{A}$ and $\boldsymbol{B}$ isomorphic as well.

(1) If $* = *_{\mathrm{Ł}}$ (the Łukasiewicz t-norm), the only proper filter of $\boldsymbol{C}$ is $\{1\}$, and thus $\boldsymbol{C}^{\boldsymbol{A}} \cong \boldsymbol{C}^{\boldsymbol{B}} \cong \boldsymbol{C}$. But taking into account that any two isomorphic subalgebras of $[0,1]_{*_{\mathrm{Ł}}}$ coincide (see e.g. [10, Corollary 7.2.6]), $\boldsymbol{C}^{\boldsymbol{A}} = \boldsymbol{C}^{\boldsymbol{B}} = \boldsymbol{C}$, and thus necessarily $\boldsymbol{A} = \boldsymbol{B}$.

(2) If $* = *_{\Pi}$ (the product t-norm), there are only two proper filters, $\{1\}$ and $C \setminus \{0\}$ and thus we have two types of $\Pi(\boldsymbol{C})$-chains over $[0,1]_{\Pi}$ corresponding to the cases that $F = \{1\}$ ($\Pi(\boldsymbol{C})$-chains such that for each pair $r < s$ in C, $\overline{r}^{\mathcal{A}} < \overline{s}^{\boldsymbol{A}}$) and the case $C \setminus \{0\}$ ($\Pi(\boldsymbol{C})$-chains such that $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$ for all $r \neq 0$). If $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B}) = \{1\}$, then $\boldsymbol{C}^{\boldsymbol{A}} \cong \boldsymbol{C}^{\boldsymbol{B}} \cong \boldsymbol{C}$, and by [78, Theorem 2] we obtain $\boldsymbol{A} \cong \boldsymbol{B}$. If $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B}) = C \setminus \{0\}$, the result is trivial.

(3) If $*$ is any continuous t-norm that is a finite ordinal sum of Łukasiewicz or Product components, then all possible proper filters are either of the form $[a,1]$ where a is the minimum of a Łukasiewicz or product component, or of the form $(a,1]$ where a is the minimum of a product component. The result is proved by applying the previous cases to each component of its decomposition not included in the filter. □

Notice that the last result is not valid when $*$ is the minimum t-norm, as the following counterexample shows. Take $C = \mathsf{Q} \cap [0,1]$, $F = \{1\}$ and the following chains over $[0,1]_{\mathrm{G}}$:

(1) The canonical $\mathrm{G}(\boldsymbol{C})$-chain, i.e. the chain $\boldsymbol{A}$ obtained by interpreting each $\overline{r}$ as its intended value r.

(2) The real $\mathrm{G}(\boldsymbol{C})$-chain $\boldsymbol{B}$ obtained by interpreting the truth-constants as:

$$\overline{r}^{\boldsymbol{B}} = r \quad \text{if} \quad r > \frac{1}{2} \qquad \text{and} \qquad \overline{r}^{\boldsymbol{B}} = \frac{r}{2} \quad \text{if} \quad r \leq \frac{1}{2}.$$

It is clear that $\boldsymbol{C} = \boldsymbol{C}^{\boldsymbol{A}} \equiv \boldsymbol{C}^{\boldsymbol{B}}$, but it is impossible to extend the isomorphism between $\boldsymbol{C}^{\boldsymbol{A}}$ and $\boldsymbol{C}^{\boldsymbol{B}}$ to an isomorphism of the full interval $[0,1]_*$.

From now on, for every filter F of $\boldsymbol{C}$ we will say that an $\mathrm{L}_*(\boldsymbol{C})$-chain $\boldsymbol{A}$ is of *type* F if $F = F_{\boldsymbol{C}}(\boldsymbol{A})$.

To finish this section, we point out that, as we mentioned in the proof of Proposition 2.3.3, any subalgebra $\boldsymbol{C}$ of $[0,1]_\Pi$ has only two filters: $F = \{1\}$ and $F = C \setminus \{0\}$, and hence we have only two types of $\Pi(\boldsymbol{C})$-algebras, which will be referred to (as in [78]) as type I and type II, respectively. If we restrict ourselves to the real chains, there is only one $\Pi(\boldsymbol{C})$-chain of type I, which is the canonical $\Pi(\boldsymbol{C})$-chain $[0,1]_{\Pi(\boldsymbol{C})}$, and also only one chain of type II, denoted $[0,1]^*_{\Pi(\boldsymbol{C})}$. The following result, that will be used in Section 2.6, relates these two real $\Pi(\boldsymbol{C})$-chains. The interested reader may find the proof in [78].

PROPOSITION 2.3.4. *The real* $\Pi(\boldsymbol{C})$*-algebra of type II,* $[0,1]^*_{\Pi(\boldsymbol{C})}$*, belongs to the variety generated by the (canonical)* $\Pi(\boldsymbol{C})$*-algebra of type I,* $[0,1]_{\Pi(\boldsymbol{C})}$*, and hence the variety generated by the class of real* $\Pi(\boldsymbol{C})$*-chains is* $\mathbb{V}([0,1]_{\Pi(\boldsymbol{C})})$.

2.4 Partial embeddability property

In order to study completeness of $\mathrm{L}_*(\boldsymbol{C})$ logics, we need results about the partial embeddability $\mathrm{L}_*(\boldsymbol{C})$-chains into real ones. In this section we will show that most of these logics enjoy this partial embeddability property with respect to their classes $\mathrm{L}_*(\boldsymbol{C})$-chains over $[0,1]$.

DEFINITION 2.4.1. *We say that a logic* $\mathrm{L}_*(\boldsymbol{C})$ *has the partial embeddability property if, and only if, for every filter* F *of* $\boldsymbol{C}$ *and every subdirectly irreducible* $\mathrm{L}_*(\boldsymbol{C})$*-chain* $\boldsymbol{A}$ *of type* F*,* $\boldsymbol{A}$ *is partially embeddable into* $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$.

PROPOSITION 2.4.2. $\mathrm{G}(\boldsymbol{C})$ *has the partial embeddability property.*

Proof. Let $\boldsymbol{A}$ be a linearly ordered $\mathrm{G}(\boldsymbol{C})$-algebra of type F, and let X be a finite subset of A. Let g be an order-preserving injection of X into $[0,1]$ satisfying

$$g(\overline{r}^{\boldsymbol{A}}) = \begin{cases} 1 & \text{if } r \in F, \\ r & \text{otherwise.} \end{cases}$$

So defined, g clearly gives a partial embedding of $\boldsymbol{A}$ into the real $\mathrm{G}(\boldsymbol{C})$-chain of type F, $[0,1]^F_{\mathrm{G}(\boldsymbol{C})}$. □

PROPOSITION 2.4.3. $\Pi(\boldsymbol{C})$ *has the partial embeddability property.*

Proof. For linearly ordered $\Pi(\boldsymbol{C})$-algebras of type II, the problem reduces to the well-known partial embeddability property of product chains into the standard product chain $[0,1]_{*\Pi}$.

Therefore, let $\boldsymbol{A}$ be a linearly ordered $\Pi(\boldsymbol{C})$-algebra of type I, and let E be a finite subset of A. Denote by C_E the set $\{r \in C \mid \overline{r}^{\boldsymbol{A}} \in E\}$. We have to show that there exists a one-to-one mapping $h\colon E \to [0,1]$ satisfying the following conditions:

(i) h preserves the order,

(ii) $h(\overline{r}^{\boldsymbol{A}}) = r$ for all $r \in C_E$,

(iii) if $x, y, z \in E$ and $z = x * y$ then $h(x) \cdot h(y) = h(z)$,

(iv) If $x, y, z \in E$ and $z = x \Rightarrow y$ then $h(x) \Rightarrow_\Pi h(y) = h(z)$.

Let $\widetilde{\boldsymbol{C}_E}$ be the Π-algebra generated by C_E. Note that the Π-algebra generated by E is naturally a $\Pi(\widetilde{\boldsymbol{C}_E})$-algebra, which will be denoted by $\boldsymbol{A}_E$.

CLAIM 2.4.4. *$\boldsymbol{A}_E$ is isomorphic to a $\Pi(\widetilde{\boldsymbol{C}_E})$-algebra $\boldsymbol{D}$ such that the following conditions are satisfied:*

(1) $\boldsymbol{D} = \mathbf{P}(\mathcal{G})$ with $\mathcal{G}$ being a subgroup of $(\mathsf{R}^+)^k_{lex}$, where k is a natural number,

(2) there is an integer l and a real number $\alpha > 0$, such that, for every positive $r \in \widetilde{C_E}$, we have $\overline{r}^{\boldsymbol{D}} = \omega_{k,l}(r^\alpha)$,

where, for any $x \in (0, 1]$ and natural $1 \leq l \leq k$, $\omega_{k,l}(x) = \langle 1, \ldots, 1, x, 1, \ldots, 1\rangle \in (\mathsf{R}^+)^k$, with x being at coordinate with index l.

In this claim, $\mathbf{P}(\mathcal{G})$ denotes the Π-algebra defined from the negative cone of the linearly ordered Abelian group $\mathcal{G}$,[5] and $(\mathsf{R}^+)^k_{lex}$ denotes the ordered Abelian group obtained as the lexicographic product of k copies of the multiplicative group of positive reals. The proof of this claim is rather technical and can be found in [78, Proposition 12].

CLAIM 2.4.5. *For every* finite *subset E' of D, there is a mapping $\delta\colon E' \to [0, 1]$ satisfying the following conditions:*

(i) *δ preserves the order,*

(ii) *$\delta(\overline{r}^{\boldsymbol{D}}) = r$ for all $r \in C_E$,*

(iii) *if $x, y, x * y \in E$ then $\delta(x) \cdot \delta(y) = \delta(x * y)$,*

(iv) *if $x, y, x \Rightarrow y \in E$ then $\delta(x) \Rightarrow_\Pi \delta(y) = \delta(x \Rightarrow y)$.*

Proof. The candidates for δ are restrictions to E of functions $g\colon \mathcal{G} \to \mathsf{R}^+$ of the form

$$g((x_1, x_2, \ldots, x_k)) = (x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \cdots \cdot x_k^{\varepsilon_k})^\beta,$$

where $\varepsilon_i, \beta > 0$. Each of these functions is a homomorphism w.r.t. the product of $\mathcal{G}$. Hence, for every choice of ε_i and β, the restriction of g to E satisfies (iii). By the assumption, for every $r \in C^*$, $\overline{r}^{\boldsymbol{D}} = \omega_{k,l}(r^\alpha)$. Therefore, for every choice of ε_i and β, we have $g(\overline{r}^{\boldsymbol{D}}) = r^{\alpha \cdot \varepsilon_l \cdot \beta}$, where $\alpha \cdot \varepsilon_l \cdot \beta > 0$. By choosing $\beta = 1/(\alpha \cdot \varepsilon_l)$, we obtain that the restriction of g to E' satisfies (ii).

[5] Product algebras are closely related to ordered Abelian groups [46], in fact a linearly ordered Π-algebra without the bottom element can be identified with the negative cone of a linearly ordered Abelian group.

Let us prove that it is possible to choose the ε_i in such a way that the restriction of g to E' satisfies (i). We classify the pairs of distinct values in E' according to the first index i_0, where the values are different. Pairs which satisfy $i_0 = k$ are ordered correctly for any positive value of ε_k. Pairs satisfying $i_0 = k-1$ may be put into the right order by choosing $\varepsilon_{k-1} = 1$ and ε_k small enough to guarantee that the difference (measured as a ratio) in the $(k-1)$-th coordinate is always larger than the difference in the k-th coordinate. In fact, if the exponents $\varepsilon_{k-1} = 1, \varepsilon_k$ guarantee the right order of the pairs with $i_0 = k-1$, then the exponents $\varepsilon_{k-1} = t, t \cdot \varepsilon_k$, for any positive t, guarantee the order as well. Hence, when it is necessary to put the pairs with $i_0 = k-2$ into the right order, we choose $\varepsilon_{k-2} = 1$ and t small enough so that the difference in the $(k-2)$-th coordinate is always larger than the differences contributed by $(k-1)$-th and k-th coordinates. Since we preserve the ratio between ε_{k-1} and ε_k, we do not destroy the already correct order of pairs with $i_0 = k-1$. We proceed in a similar way for pairs with smaller and smaller i_0.

The condition (iv), the preservation of existing implications in E', is a consequence of h being order preserving (i), and the preservation of existing products (iii). This ends the proof of the claim. □

Now, take E' to be the image of E under the isomorphism between $\boldsymbol{A}_E$ and $\boldsymbol{D}$. Applying Claim 2.4.5 to $\boldsymbol{D}$ and E' with $C = \widetilde{C_E}$, we obtain an embedding δ, whose composition with the above isomorphism has the required properties of h. □

PROPOSITION 2.4.6. Ł$(\boldsymbol{C})$ *has the partial embeddability property into the canonical* Ł$(\boldsymbol{C})$*-chain.*

Proof. Let X be a finite subset of an Ł$(\boldsymbol{C})$-chain $\boldsymbol{A} = \langle A, \wedge, \vee, \otimes, \to, \{\overline{r}^{\boldsymbol{A}} \mid r \in C\}\rangle$. We have to prove that there is a mapping $f\colon X \to [0,1]$ such that:

- if $x, y, x \circ y \in X$, then $f(x \circ y) = f(x) \circ' f(y)$
  for $\circ = \otimes$ and $\circ' = *_{Ł}$, or for $\circ' = \to$ and $\circ' = \Rightarrow_{Ł}$,
- for any $r \in C$ such that $\overline{r}^{\boldsymbol{A}} \in X$, $f(\overline{r}^{\boldsymbol{A}}) = r$.

It is well known that if an MV-algebra $\boldsymbol{A}$ is isomorphic to $\Gamma(\mathcal{G}, u)$ for some ℓ-group $\mathcal{G}$ with strong unit u, and if $\boldsymbol{S}$ is a subalgebra of $\boldsymbol{A}$, then there is a (unique) sub-ℓ-group $\mathcal{E}$ of $\mathcal{G}$ such that $u \in \mathcal{E}$ and $\mathcal{S} \cong \Gamma(\mathcal{E}, u)$ (see [10]).

Since $\boldsymbol{C}$ is a countable subalgebra of the standard MV-algebra $\Gamma(\mathsf{R}, 1) = [0,1]_{Ł}$, it is isomorphic to $\Gamma(\mathcal{H}, 1)$ for a *unique* sub-ℓ-group $\mathcal{H}$ of R such that $1 \in H$. Moreover, the product chain $\mathbf{P}(\mathcal{H})$ is a product subalgebra of $\mathbf{P}(\mathsf{R})$. Notice that, since R is an Archimedean group, each element of the negative cone H^- can be written as $-n + r$, with $r \in C$ and $n \in \mathsf{N}$. The mapping

$$f\colon \mathbf{P}(\mathsf{R}) \to [0,1]_\Pi$$

defined by $f(x) = e^x$ for $x < 0$ and $f(\bot) = 0$ is indeed an isomorphism of product algebras, and therefore, $C^\star = \{e^{-n+r} \mid r \in C, n \in \mathsf{N}\} \cup \{0\}$ is the domain of a subalgebra of $[0,1]_\Pi$ isomorphic to $\mathbf{P}(\mathcal{H})$. Hence, we can consider the expanded logic $\Pi(\boldsymbol{C}^\star)$ and its canonical $\Pi(\boldsymbol{C}^\star)$-algebra $[0,1]_{\Pi(\boldsymbol{C}^\star)}$.

Therefore, we have seen that for each countable subalgebra $\boldsymbol{C}$ of the standard MV-algebra $[0,1]_{Ł}$, we can define a corresponding countable subalgebra $\boldsymbol{C}^{\star}$ of the real Π-algebra $[0,1]_{\Pi}$. Hence, we can associate to the canonical Ł$(\boldsymbol{C})$-chain the canonical $\Pi(\boldsymbol{C}^{\star})$-chain.

If $\boldsymbol{A}$ is a Ł$(\boldsymbol{C})$-algebra, then there is an ℓ-group $\mathcal{G}$, a sub-ℓ-group $\mathcal{L}$ and an order unit u of $\mathcal{G}$ such that $\boldsymbol{A} \cong \Gamma(\mathcal{G}, u)$ and $C^{\boldsymbol{A}} \cong \Gamma(\mathcal{L}, u)$. But $\Gamma(\mathcal{G}, u)$ is also isomorphic to the MV-algebra $\Gamma^{-}(\mathcal{G}, u)$ defined on the interval $[-u, 0]$ with the mirror operations. $\Gamma^{-}(\mathcal{L}, u)$ is analogously defined and it is also isomorphic to $\Gamma(\mathcal{L}, u)$. Since $C^{\boldsymbol{A}}$ is isomorphic to a subalgebra of the real MV-algebra, it follows that $\mathcal{L}$ is isomorphic to a sub-ℓ-group $\mathcal{H}$ of R, and since u is an order unit, all the elements of the negative cone L^{-} can be written as $-nu + \overline{r}^{\mathcal{A}}$, for $n \in \mathsf{N}$ and $r \in C$. Thus we can consider the product algebra $\mathcal{P}(\mathcal{G})$ as a $\Pi(\mathcal{C}^{\star})$-algebra, with $\overline{e^{-n+r}}^{\mathcal{P}(\mathcal{G})} = -nu + \overline{r}^{\mathcal{A}}$.

Let X be a finite subset of A. From now on, we identify $\boldsymbol{A}$ and $\Gamma(\mathcal{G}, u)$ (hence taking $\overline{0}^{\boldsymbol{A}} = 0_{\boldsymbol{G}}$ and $\overline{1}^{\boldsymbol{A}} = u$), and, without loss of generality, we can assume $u \in X$. Let $i \colon \Gamma(\mathcal{G}, u) \to \Gamma^{-}(\mathcal{G}, u)$ be defined by $i(x) = x - u$. By the partial embeddability property of Product logic with constants, the $\Pi(\boldsymbol{C}^{\star})$-chain $\mathbf{P}(G)$ is partially embeddable into the canonical $[0,1]_{\Pi(\boldsymbol{C}^{\star})}$. Therefore, considering $i(X)$, as a subset of the $\Pi(\boldsymbol{C}^{\star})$-chain $\mathbf{P}(G)$, there is a partial embedding from $i(X)$ into $[0,1]_{\Pi(\boldsymbol{C}^{\star})}$ such that $\overline{r}^{\boldsymbol{A}} - u = i(\overline{r}^{\boldsymbol{A}}) \mapsto e^{r-1}$, for each $\overline{r}^{\boldsymbol{A}} \in X$. In particular, $-u = i(\overline{0}^{\boldsymbol{A}}) \mapsto e^{-1}$ and $0_G = i(\overline{1}^{\boldsymbol{A}}) \mapsto e^{0} = 1$, thus all the elements of $i(X)$ go to the segment $[e^{-1}, 1]$. Applying natural logarithms, we obtain a partial embedding of $i(X)$ into $\Gamma^{-}(\mathsf{R}, 1)$ such that $i(\overline{r}^{\boldsymbol{A}}) \mapsto r - 1$ for each $\overline{r}^{\boldsymbol{A}} \in X$. Thus, composing i with this embedding and finally with the isomorphism from $\Gamma^{-}(\mathsf{R}, 1)$ to $\Gamma(\mathsf{R}, 1)$ mapping $r - 1 \mapsto r$, we obtain a partial embedding of $X \subset \boldsymbol{A}$ into the canonical Ł$(\boldsymbol{C})$-chain $[0,1]_{Ł(\boldsymbol{C})}$. This ends the proof. □

THEOREM 2.4.7. *Let $*$ be a continuous t-norm which is a finite ordinal sum and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. Then $\mathrm{L}_*(\boldsymbol{C})$ enjoys the partial embeddability property.*

Proof. Suppose that $[0,1]_* = \bigoplus_{i=1}^{n} \boldsymbol{A}_i$. We know that the subdirectly irreducible chains of $\mathbf{V}([0,1]_*)$ are members of

$$\mathbf{HSP}_U(\boldsymbol{A}_1) \cup (\mathbf{ISP}_U(\boldsymbol{A}_1) \oplus \mathbf{HSP}_U(\boldsymbol{A}_2)) \cup \cdots \cup (\textstyle\bigoplus_{i=1}^{n-1} \mathbf{ISP}_U(\boldsymbol{A}_i) \oplus \mathbf{HSP}_U(\boldsymbol{A}_n))$$

(see Chapter V). From this fact, we can use the previous results concerning expansions of G, Ł, and Π to prove the theorem. □

In the next three subsections, we describe different kinds of real completeness properties for the family of logics $\mathrm{L}_*(\boldsymbol{C})$, where $*$ and C satisfy the conditions $(\mathbf{C1})$ and $(\mathbf{C2})$. In the next subsection, we focus on (finite) strong completeness results, while in Section 2.6, we refine the results by determining which logics are canonical real complete. Finally, in Section 2.7, we study the completeness properties when we restrict to evaluated formulas.

2.5 About (finite) strong real completeness

The partial embeddability property allows us to prove both real completeness and conservativeness results.

THEOREM 2.5.1. *Let $*$ be a continuous t-norm and let $\boldsymbol{C}$ be a subalgebra of $[0,1]_*$. If $\mathrm{L}_*(\boldsymbol{C})$ satisfies the partial embeddability property, then $\mathrm{L}_*(\boldsymbol{C})$ has the* FS$\mathcal{R}$C. *In fact, for every finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_\mathcal{C}}$,*

$$\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbb{K}} \varphi,$$

where $\mathbb{K} = \{[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})} \mid F$ proper filter of $\boldsymbol{C}\}$.

Proof. It is a consequence of Theorem 2.4.7. □

PROPOSITION 2.5.2. $\mathrm{L}_*(\boldsymbol{C})$ *is a conservative expansion of* L_*.

Proof. Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ be arbitrary formulas and suppose that $\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$. Then, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$, and this implies that $\Gamma_0 \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \varphi$. Since the new truth-constants do not occur in $\Gamma_0 \cup \{\varphi\}$, we have $\Gamma_0 \models_{[0,1]_*} \varphi$, and by FS$\mathcal{R}$C of L_*, $\Gamma_0 \vdash_{\mathrm{L}_*} \varphi$, and hence $\Gamma \vdash_{\mathrm{L}_*} \varphi$. □

As a consequence of Theorem 2.5.1, we obtain that Ł$(\boldsymbol{C})$ enjoys the canonical FS$\mathcal{R}$C, because the algebra $\boldsymbol{C}$ is simple and so there is only one real algebra up to isomorphisms: the canonical one. In the case of expansions of Product logic $\Pi(\boldsymbol{C})$, $\boldsymbol{C}$ (if it has more than two elements) has only two proper filters $F_1 = \{1\}$ and $F_2 = \{r \in C \mid r > 0\}$. Let us denote by $[0,1]^\star_{\Pi(\boldsymbol{C})}$ the $\Pi(\boldsymbol{C})$-algebra $[0,1]^{F_2}_{\Pi((\boldsymbol{C})}$. Then an immediate consequence is the following proposition.

PROPOSITION 2.5.3. *For any $\Pi(\boldsymbol{C})$-formula φ and any finite set of $\Pi(\boldsymbol{C})$-formulas Γ, we have $\Gamma \vdash_{\Pi(\boldsymbol{C})} \varphi$ iff $\Gamma \models_{[0,1]_{\Pi(\boldsymbol{C})}} \varphi$ and $\Gamma \models_{[0,1]^\star_{\Pi(\boldsymbol{C})}} \varphi$.*

As for the S$\mathcal{R}$C, we obtain the following results.

THEOREM 2.5.4. L_* *has the* S$\mathcal{R}$C *if, and only if,* $\mathrm{L}_*(\boldsymbol{C})$ *has the* S$\mathcal{R}$C.

Proof. The right-to-left direction is a consequence of $\mathrm{L}_*(\boldsymbol{C})$ being a conservative expansion of L_* (Proposition 2.5.2). To prove the left-to-right direction, we follow an idea from [16, Lemma 3.4.4]. Assume L_* has the S$\mathcal{R}$C property and that $\Gamma \not\vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$. Let BK the set of instances of the book-keeping axioms over $\boldsymbol{C}$. Then, it is easy to check that over L_* (i.e. considering the truth-constants as fresh propositional variables) φ remains not provable from $\Gamma \cup \mathrm{BK}$, i.e. $\Gamma \cup \mathrm{BK} \not\vdash_{\mathrm{L}_*} \varphi$. Since L_* is S$\mathcal{R}$C, there is a real L_*-chain $\boldsymbol{A}$ and an evaluation e into $\boldsymbol{A}$ such that e is a model of $\Gamma \cup \mathrm{BK}$ and $e(\varphi) < 1^{\boldsymbol{A}}$. Now expand the signature of the algebra $\boldsymbol{A}$ with 0-ary operators $\overline{r}$, one for each $r \in C$, and set $\overline{r}^{\boldsymbol{A}} = e(\overline{r})$. The resulting algebra, call it $\boldsymbol{A}'$, is an $\mathrm{L}_*(\boldsymbol{C})$-chain, and e becomes an evaluation into $\boldsymbol{A}'$ such that it is a model of Γ but $e(\varphi) < 1^{\boldsymbol{A}} = 1^{\boldsymbol{A}'}$, and, consequently, $\Gamma \not\models_{\mathrm{L}_*(\boldsymbol{C})} \varphi$. □

As a consequence, G$(\boldsymbol{C})$ has the S$\mathcal{R}$C since Gödel logic has the S$\mathcal{R}$C. However, $\Pi(\boldsymbol{C})$ and Ł$(\boldsymbol{C})$ enjoy the FS$\mathcal{R}$C, but they fail to satisfy the S$\mathcal{R}$C. In general we obtain the following result.

	G($\boldsymbol{C}$)	Π($\boldsymbol{C}$)	Ł($\boldsymbol{C}$)	$\mathrm{L}_*(\boldsymbol{C})$
$\mathcal{R}$C	Yes	Yes	Yes	Yes
FS$\mathcal{R}$C	Yes	Yes	Yes	Yes
S$\mathcal{R}$C	Yes	No	No	No
Canonical FS$\mathcal{R}$C	No	No	Yes	No
Canonical S$\mathcal{R}$C	No	No	No	No

Table 1. Real completeness results for logics with truth-constants enjoying the partial embeddability property (where $*$ denotes a continuous t-norm which is a finite ordinal sum of at least two basic components).

THEOREM 2.5.5. *If* $*$ *is a continuous t-norm, then* $\mathrm{L}_*(\boldsymbol{C})$ *enjoys:*

(i) *the* S$\mathcal{R}$C *if, and only if,* $* = \min$*,*

(ii) *the* FS$\mathcal{R}$C *if* $* \in$ **CONT-fin***,*

(iii) *the canonical* FS$\mathcal{R}$C *if, and only if,* $*$ *is the Łukasiewicz t-norm.*

Proof. First of all, since L_* does not have the S$\mathcal{R}$C when $[0,1]_*$ contains as a component a copy of Łukasiewicz or product t-norms, it is clear that in such a case also the logic $\mathrm{L}_*(\boldsymbol{C})$ does not have the S$\mathcal{R}$C. Thus, (i) is proved.

Result (ii) is an obvious consequence of Theorem 2.4.7.

To prove (iii), we show that if $\boldsymbol{C}$ has a non-trivial proper filter, then the logic $\mathrm{L}_*(\boldsymbol{C})$ does not enjoy the canonical FS$\mathcal{R}$C. Namely, since $F \neq \{1\}$, there exists $r \in F$, $r \notin \{0,1\}$. Then, the following semantical deduction[6] is valid over the canonical real $\mathrm{L}_*(\boldsymbol{C})$-chain but not over $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$:

$$(p \to q) \to \overline{r} \models q \to p.$$

To prove it, take into account that for every evaluation e over the canonical real chain, $e((p \to q) \to \overline{r}) = 1$ iff $e(p \to q) \leq r < 1$, and this implies $e(q) < e(p)$: so, the deduction is valid. However, over the chain $\boldsymbol{A} = [0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$, the formula $(p \to q) \to \overline{r}$ is always satisfied (remember that $\overline{r}^{\boldsymbol{A}} = 1$), and thus the deduction is not valid. Therefore, $q \to p$ is not provable from $(p \to q) \to \overline{r}$ in the logic $\mathrm{L}_*(\boldsymbol{C})$, and, consequently, this logic does not have the canonical FS$\mathcal{R}$C. Taking into account that $\boldsymbol{C}$ is simple if, and only if, $*$ is the Łukasiewicz t-norm, Theorem 2.4.6 proves (iii). □

Notice that for a continuous t-norm $*$, $\mathrm{L}_*(\boldsymbol{C})$ does not have the canonical S$\mathcal{R}$C. Indeed, if $\mathrm{L}_*(\boldsymbol{C})$ had the canonical S$\mathcal{R}$C, then it would also enjoy the S$\mathcal{R}$C and the canonical FS$\mathcal{R}$C, which is impossible, as shown by the previous theorem.

All these completeness results are collected in Table 1.

[6]Actually there are simpler examples that could have been used in this proof, like $\overline{r} \models \overline{0}$, but the one chosen here will be useful later in Section 2.7.

2.6 About canonical real completeness

The logics $\mathrm{L}_*(\boldsymbol{C})$ considered in the last section do not have the canonical FS$\mathcal{R}$C, except for $\text{Ł}(\boldsymbol{C})$. However, some of them enjoy the weaker property of canonical $\mathcal{R}$C, i.e. their theorems are exactly the tautologies of their corresponding canonical algebra over $[0,1]$. In this section, we show which logics do have the canonical $\mathcal{R}$C.

THEOREM 2.6.1 ([30]). $\mathrm{G}(\boldsymbol{C})$ *has the canonical* $\mathcal{R}$C*.*

Proof. As usual, soundness is trivial. To prove completeness, suppose $\nvdash_{\mathrm{G}(\boldsymbol{C})} \varphi$. Then, by completeness of $\mathrm{G}(\boldsymbol{C})$ w.r.t. the $\mathrm{G}(\boldsymbol{C})$-chains, there exist a countable $\mathrm{G}(\boldsymbol{C})$-chain $\boldsymbol{A}$ and an evaluation e over $\boldsymbol{A}$ such that $e(\varphi) < \overline{1}^{\boldsymbol{A}}$. We have to show there is an evaluation e' on the real algebra $[0,1]_{\mathrm{G}(\boldsymbol{C})}$ such that $e'(\varphi) < 1$.

Let $s = \min\{r \in C \mid r = 1$ or $\overline{r}$ subformula of φ with $\overline{r}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}\}$. Clearly $s > 0$. Let $g\colon A \to [0,s]$ be an order-preserving injection such that $g(\overline{0}^{\boldsymbol{A}}) = 0$, $g(\overline{1}^{\boldsymbol{A}}) = s$ and $g(\overline{r}^{\boldsymbol{A}}) = r$ for $\overline{r}$ a subformula of φ with $r < s$. Then, we define a $\mathrm{G}(\boldsymbol{C})$-evaluation e' on the real $\mathrm{G}(\boldsymbol{C})$-algebra $[0,1]$ as follows: for all propositional variables p, $e'(p) = g(e(p))$. Then e' is extended to $\mathrm{G}(\boldsymbol{C})$-formulas as usual (of course with $e'(\overline{r}) = r$, for each $r \in \boldsymbol{C}$).

CLAIM 2.6.2. *For each* ψ *subformula of* φ*:*

(1) *if* $e(\psi) = \overline{1}^{\boldsymbol{A}}$ *then* $e'(\psi) \geq s$*,*

(2) *if* $e(\psi) < \overline{1}^{\boldsymbol{A}}$ *then* $e'(\psi) = g(e(\psi)) < s$*.*

Proof. The claim is clear for variables and for truth-constants $\overline{r}$ subformulas of φ. The induction step for $\wedge$ is trivial. Let us consider the case of $\to$. If $e(\gamma \to \delta) = e(\delta) < \overline{1}^{\boldsymbol{A}}$ then $e'(\delta) = g(e(\delta)) < s$. Now, if $e(\gamma) = \overline{1}^{\boldsymbol{A}}$ then $e'(\gamma) \geq s$ and $e'(\gamma \to \delta) = e'(\delta) < s$; and if $e(\gamma) < \overline{1}^{\boldsymbol{A}}$ then $e'(\gamma) = g(e(\gamma)) > g(e(\delta)) = e'(\delta)$, thus again $e'(\gamma \to \delta) = e'(\delta) < s$. On the other hand, assume $e(\gamma \to \delta) = \overline{1}^{\boldsymbol{A}}$, thus $e(\gamma) \leq e(\delta)$. If $e(\delta) = \overline{1}^{\boldsymbol{A}}$ then $e'(\gamma \to \delta) \geq e'(\delta) \geq s$. And if $e(\delta) < \overline{1}^{\boldsymbol{A}}$ then $e'(\gamma) = g(e(\gamma)) \leq g(e(\delta)) = e'(\delta)$ and $e'(\gamma \to \delta) = 1 \geq s$. This proves the claim. □

This also finishes the proof of the theorem; indeed, since $e(\varphi) < \overline{1}^{\boldsymbol{A}}$, then $e'(\varphi) < 1$ as required. □

THEOREM 2.6.3 ([78]). $\Pi(\boldsymbol{C})$ *has the canonical* $\mathcal{R}$C*.*

Proof. Let φ be a $\Pi(\boldsymbol{C})$ formula such that $\nvdash_{\Pi(\boldsymbol{C})} \varphi$. We can further assume that φ contains some truth constant $\overline{r}$ with $0 < r < 1$ as a subformula, otherwise the real completeness of product logic does the job. By general completeness, there is a linearly ordered $\Pi(\boldsymbol{C})$-algebra $\boldsymbol{A}$ and an evaluation e on $\boldsymbol{A}$ such that $e(\varphi) < \overline{1}^{\boldsymbol{A}}$. The task is to find an evaluation e' on the canonical real $\Pi(\boldsymbol{C})$-algebra $[0,1]_{\Pi(\boldsymbol{C})}$ such that $e'(\varphi) < 1$. Let $E = \{e(\psi) \mid \psi$ is a subformula of $\varphi\} \cup \{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$. We consider the following cases:

Case 1: $\boldsymbol{A}$ is of type I.

By applying Proposition 2.4.3 we obtain a partial embedding h of E into $[0,1]$. Now define a $[0,1]_{\Pi(\boldsymbol{C})}$-evaluation e' by putting

$$e'(p) = \begin{cases} h(e(p)) & \text{if } p \text{ is a propositional variable in } \varphi, \\ arbitrary & \text{otherwise.} \end{cases}$$

It is easy to check, by the properties of h, that $e'(\varphi) = h(e(\varphi)) < 1$.

Case 2: $\boldsymbol{A}$ is of type II.

By well-known results on Π-algebras (see [13]), there is a partial embedding f of E into the real Π-algebra $[0,1]_\Pi$ and the evaluation v on $[0,1]_\Pi$ defined as follows

$$v(p) = \begin{cases} f(e(p)) & \text{if } p \text{ is a propositional variable in } \varphi, \\ arbitrary & \text{otherwise,} \end{cases}$$

is such that $v(\varphi^*) < 1$, where φ^* is the Π-formula obtained from φ by replacing all truth-constants $\overline{r}$ with $0 < r$ by $\overline{1}$. Now, the evaluation v' on the real $\Pi(\boldsymbol{C})$-algebra of type II such that $v'(p) = v(p)$ for all propositional variables p satisfies $v(\varphi^*) = v'(\varphi) < 1$. Then, by Proposition 2.3.4, there is also an evaluation e' on the canonical real $\Pi(\boldsymbol{C})$-algebra $[0,1]_{\Pi(\boldsymbol{C})}$ such that $e'(\varphi) < 1$. This ends the proof of Case 2 and of the theorem as well. □

Notice that the canonical $\mathcal{R}$C is not valid in general for expansions of other logics of a continuous t-norm. First, we will show that the canonical $\mathcal{R}$C fails for a large family of logics giving a counterexample, i.e. showing a formula φ that is a tautology of the canonical real algebra but not of the algebra $[0,1]^F_{\text{L}_*(\boldsymbol{C})}$ for some proper filter F of $\boldsymbol{C}$. Suppose that the first component of $[0,1]_*$ is defined on the interval $[0,a]$.

(1) If the first component of the t-norm $*$ is a copy of Łukasiewicz t-norm (and $a \in C$), then, an easy computation shows that the formula

$$\overline{a} \to (\neg\neg p \to p)$$

is valid in the canonical real algebra but is not valid in the real chain defined by the filter $F = [a,1] \cap C$ (where $\overline{a}$ is interpreted as 1).

(2) If the first component of the t-norm $*$ is a copy of product t-norm, take b as any element of $C \cap (0,a)$. Then, an easy computation shows that the formula

$$\overline{b} \to \neg p \vee ((p \to p \,\&\, p) \to p)$$

is valid in the canonical real algebra but is not valid in the real chain defined by the filter $F = (0,1] \cap C$ (where $\overline{b}$ is interpreted as 1).

(3) If the first component is the minimum t-norm, take b as any element of $C \cap (0, a)$. Then, the formula
$$\overline{b} \to (p \to p \,\&\, p)$$
is valid in the canonical real algebra but is not valid in the real chain where $\overline{b}$ is interpreted as 1.

Observe that for a t-norm whose decomposition begins with two copies of the Łukasiewicz t-norm, the idempotent element a separating the two components must belong to the truth-constants subalgebra $\boldsymbol{C}$. Indeed, take into account that, by assumption, C must contain a-non idempotent element c of the second component, and for this element there exists a natural number n such that $c^n = a$ and thus $a \in C$. Hence, this case is subsumed in the above first item.

The remaining cases (when the first component is Łukasiewicz but its upper bound a does not belong to C) will be divided in two different groups:

(1) If $[0,1]_* = [0,a]_{Ł} \oplus [a,1]_{\mathrm{G}}$ or $[0,1]_* = [0,a]_{Ł} \oplus [a,1]_{\Pi}$, then the logic $\mathrm{L}_*(\boldsymbol{C})$ has the canonical $\mathcal{R}$C. Actually, in that case, the filters of $\boldsymbol{C}$ are the same as the filters of $C \cap [a,1]_{\mathrm{G}}$ or $C \cap [a,1]_{\Pi}$ respectively. Therefore, a modified version (given in the next two theorems) of the proof of the canonical $\mathcal{R}$C for G($\boldsymbol{C}$) and $\Pi(\boldsymbol{C})$ applies.

(2) If $[0,1]_*$ is an ordinal sum of three or more components, then $\mathrm{L}_*(\boldsymbol{C})$ does not have the canonical $\mathcal{R}$C, as the following examples show:

(2.1) If $[0,1]_* = [0,a]_{Ł} \oplus [a,b]_{\mathrm{G}} \oplus \boldsymbol{A}$, take $d \in F = (a,1] \cap C$ in the second component. Then the formula
$$\overline{d} \to (\neg\neg p \to p) \vee (p \to p \,\&\, p)$$
is a tautology of the canonical real algebra but not of $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$.

(2.2) If $[0,1]_* = [0,a]_{Ł} \oplus [a,b]_{\Pi} \oplus \boldsymbol{A}$, take $d \in F = (a,b] \cap C$ in the second component. Then the formula
$$\overline{d} \to [(\neg\neg p \to p) \vee (\neg\neg q \to q) \vee (p \to p \,\&\, p) \vee (q \to q \,\&\, q) \vee ((p \to p \,\&\, q) \to q)]$$
is a tautology of the canonical real algebra and not of $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$.[7]

The remaining cases enjoy the $\mathcal{R}$C.

THEOREM 2.6.4. *Let either* $[0,1]_* = [0,a]_{Ł} \oplus [a,1]_{\Pi}$ *or* $[0,1]_* = [0,a]_{Ł} \oplus [a,1]_{\mathrm{G}}$. *Then the logic* $\mathrm{L}_*(\boldsymbol{C})$ *has the canonical* $\mathcal{R}$C *if, and only if, the minimum element of the second component does not belong to* C.

[7] We thank Franco Montagna, who pointed out that the formula in [26], claimed to be a tautology of the canonical real algebra and not of $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$, does not satisfy the required conditions. Here, we provide a new formula satisfying the conditions and prove that the claimed result is true.

$[0,1]_*$		Canonical $\mathcal{R}$C for $L_*(\boldsymbol{C})$
$[0,1]_Ł$		Yes
$[0,1]_G$		Yes
$[0,1]_\Pi$		Yes
$[0,a]_Ł \oplus [a,1]_G$,	$a \notin C$	Yes
$[0,a]_Ł \oplus [a,1]_\Pi$,	$a \notin C$	Yes
$[0,a]_*$, for other $* \in$ **CONT-fin**		No

Table 2. Canonical real completeness results for logics $L_*(\boldsymbol{C})$ when $*$ is a finite ordinal sum of the three basic components.

Proof. The proof can easily be obtained by combining the proofs for $G(\boldsymbol{C})$, $\Pi(\boldsymbol{C})$ and $Ł(\boldsymbol{C})$ (see [26, Theorems 17 and 18] for the details). □

Summarizing (see Table 2), the canonical $\mathcal{R}$C holds for the expansion of the logic of a continuous t-norm $*$, which is a finite ordinal sum of the three basic ones, by a set of truth-constants if, and only if, $[0,1]_*$ is either one of the three basic algebras ($[0,1]_Ł$, $[0,1]_G$ or $[0,1]_\Pi$) or $[0,1]_* = [0,a]_Ł \oplus [a,1]_\Pi$ or $[0,1]_* = [0,a]_Ł \oplus [a,1]_G$ (with $a \notin C$).

2.7 Completeness results for evaluated formulas

This section deals with completeness results when we restrict to what we call *evaluated formulas*, i.e., formulas of type $\overline{r} \to \varphi$, where φ is a formula without new truth-constants. We denote by $\mathcal{R}C_{ev}$, $FS\mathcal{R}C_{ev}$ and $S\mathcal{R}C_{ev}$ the restriction of the properties we have been studying in the previous section to evaluated formulas. From the previous sections, we know that $FS\mathcal{R}C$ is true for the expansion of L_* with a subalgebra of truth-constants (not only for evaluated formulas), but the canonical $FS\mathcal{R}C$ is only true for expansions of Łukasiewicz logic. The next theorem states the canonical $FS\mathcal{R}C_{ev}$ for the expansions of Gödel and Product logics with truth-constants.

THEOREM 2.7.1. $G(\boldsymbol{C})$ *and* $\Pi(\boldsymbol{C})$ *have the canonical* $FS\mathcal{R}C_{ev}$*, i.e., for any formulas* $\varphi_1, \ldots, \varphi_n, \psi$ *and values* $r_1, \ldots, r_n, s \in C$*, and* $\Gamma = \{\overline{r}_i \to \varphi_i \mid 1 \leq i \leq n\}$*, we have:*

(i) $\Gamma \vdash_{G(\boldsymbol{C})} \overline{s} \to \psi$ *if, and only if,* $\Gamma \models_{[0,1]_{G(\boldsymbol{C})}} \overline{s} \to \psi$.

(ii) $\Gamma \vdash_{\Pi(\boldsymbol{C})} \overline{s} \to \psi$ *if, and only if,* $\Gamma \models_{[0,1]_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$.

Proof. (i) We start by stating the following previous result whose proof is not difficult (see [30] for the details).

CLAIM 2.7.2. *Let* $a \in (0,1]$ *and define a mapping* $f_a \colon [0,1] \to [0,1]$ *as follows:*

$$f_a(x) = \begin{cases} 1 & \text{if } x \geq a, \\ x & \text{otherwise.} \end{cases}$$

Then f_a *is a homomorphism of real Gödel chains. Therefore, if* e *is a* G*-evaluation of formulas, then* $e_a = f_a \circ e$ *is another* G*-evaluation.*

To prove the statement it is enough to show the following:

$$\Gamma \models_{[0,1]_{G(\boldsymbol{C})}} \overline{s} \to \psi \text{ iff } \models_{[0,1]_{G(\boldsymbol{C})}} (\bigwedge_{i=1}^{n} (\overline{r_i} \to \varphi_i)) \to (\overline{s} \to \psi).$$

One direction is easy. As for the non trivial one, it is enough to prove that if there is an evaluation e which is not a model of $(\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$, then we can find another evaluation e' that is a model of $\{\overline{r_1} \to \varphi_1, \ldots, \overline{r_n} \to \varphi_n\}$ and not of $\overline{s} \to \psi$.

So, let e be such that $e((\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)) < 1$. If e is a model of every $\overline{r_i} \to \varphi_i$ for $i = 1, \ldots, n$, then we can take $e' = e$ and the problem is solved. Otherwise, there exists some $1 \leq j \leq n$ for which $r_j > e(\varphi_j)$ and thus $e(\overline{r_j} \to \varphi_j) = e(\varphi_j) < 1$. Let $J = \{j \mid r_j > e(\varphi_j)\}$ and $a = e(\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) = \min\{e(\varphi_j) \mid j \in J\}$. Then, the $G(\boldsymbol{C})$-evaluation e' such that $e' = e_a$ over the propositional variables does the job. Namely, by Claim 2.7.2, over Gödel formulas we have $e' = e_a \geq e$, so e' is still model of those $\overline{r_i} \to \varphi_i$'s for $i \in \{1, \ldots, n\} \setminus J$. But now, $e'(\varphi_j) = 1$ for every $j \in J$, so e' is also a model of $(\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$. On the other hand, since $e((\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)) < 1$, it must be $s > e(\psi)$ and $a = e(\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) > e(\psi)$. Now, by the above claim, $e'(\psi) = e_a(\psi) = e(\psi)$, hence $e'(\overline{s} \to \psi)) = e(\overline{s} \to \psi) < 1$.

(ii) Due to Corollary 2.5.3, we only need to prove that if $\Gamma \models_{[0,1]_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$ then $\Gamma \models_{[0,1]^{\star}_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$. Without loss of generality we may assume $r_i > 0$ for all i and $s > 0$. Suppose $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]^{\star}_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$. Then, there exists a $[0,1]^{\star}_{\Pi(\boldsymbol{C})}$-evaluation e such that $e(\overline{r_1} \to \varphi_1) = \cdots = e(\overline{r_n} \to \varphi_n) = 1$ and $e(\overline{s} \to \psi) < 1$. Since $e(r_i) = e(s) = 1$ for all i, we also have $e(\varphi_1) = \cdots = e(\varphi_n) = 1$ and $e(\psi) < 1$.

Assume $e(\psi) = 0$. Then, letting e' be the $[0,1]_{\Pi(\boldsymbol{C})}$-evaluation defined by $e'(p) = e(p)$ for any propositional variable p, we have $1 = e'(\overline{r_1} \to \varphi_1) = \cdots = e'(\overline{r_n} \to \varphi_n)$ and $e'(\psi) = 0$, hence $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$.

Assume $e(\psi) > 0$. Let $\alpha \in \mathsf{R}^+$ so that $(e(\psi))^\alpha < s$.[8] Then, the $[0,1]_{\Pi(\boldsymbol{C})}$-evaluation e', where $e'(p) = (e(p))^\alpha$ for any propositional variable p, is such that $e'(\overline{r_i} \to \varphi_i) = 1$ for all i but $e'(\overline{s} \to \psi) < 1$, and thus $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$. □

One could wonder whether these restricted completeness results hold for formulas of type $\varphi \to \overline{r}$ such that φ does not contain a truth-constant different from $\overline{0}$ and $\overline{1}$. Actually, the situation is different for $G(\boldsymbol{C})$ and $\Pi(\boldsymbol{C})$:

- As for $G(\boldsymbol{C})$, the result does not hold. It is easy to check that

  $$\neg\neg p \to \overline{0.7} \models_{[0,1]_{G(\boldsymbol{C})}} p \to \overline{0.2},$$

  since the premise is only true if $e(p) = 0$, while

  $$\neg\neg p \to \overline{0.7} \not\vdash_{G(\boldsymbol{C})} p \to \overline{0.2}.$$

[8] Observe that $e(\psi) \in [0,1]$ and thus $(e(\psi))^\alpha$ is the usual α power of $e(\psi)$. Notice that, for any real α, the mappings $f(x) = x^\alpha$ are automorphisms of $[0,1]_\Pi$.

In fact, by the deduction-detachment theorem and the canonical $\mathcal{R}$C of the logic G($\boldsymbol{C}$) this is equivalent to show that

$$\not\models_{[0,1]_{G(\boldsymbol{C})}} (\neg\neg p \to \overline{0.7}) \to (p \to \overline{0.2}),$$

which is true, since, if $e(p) = c$ for $c > 0.2$, an easy computation shows that $e((\neg\neg p \to \overline{0.7}) \to (p \to \overline{0.2})) = 0.2$.

- As for $\Pi(\boldsymbol{C})$, the result holds true when the formulas $\varphi \to \overline{r}$ are such that $r > 0$ (see [78]), since in such a case these formulas are trivially satisfied in the non-canonical real $\Pi(\boldsymbol{C})$-algebra $[0,1]^F_{\Pi(\boldsymbol{C})}$ for $F = (0,1]$.

In any case, the result is not true if we allow formulas of both types together. Indeed, given $r \neq 1$, it is obvious that the semantical deduction (already used in the proof of Theorem 2.5.5)

$$(p \to q) \to \overline{r} \models \overline{1} \to (q \to p)$$

is valid over the canonical real chain but not over a real chain where $\overline{r}$ is interpreted as 1.

Now we will study the canonical $\mathcal{R}\mathrm{C}_{ev}$ and the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$ for other logics. Suppose that $*$ is a t-norm that is a non-trivial finite ordinal sum of the basic components, and suppose that the first component is defined on the interval $[0,a]$. For the following cases we can refute the canonical $\mathcal{R}\mathrm{C}_{ev}$ (and hence the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$ as well):

(1) The first component of the t-norm $*$ is a copy of the Łukasiewicz t-norm and $a \in C$.

(2) The first component of the t-norm $*$ is a copy of the product t-norm.

(3) The first component of the t-norm $*$ is a copy of the minimum t-norm.

(4) There are more than two components and the second component is a copy of the minimum t-norm.

(5) There are more than two components and the second component is a copy of the product t-norm.

Indeed, for all these cases we can use the same counterexample that was given in the previous section to show that the corresponding logics do not enjoy canonical $\mathcal{R}\mathrm{C}_{ev}$, because the counterexamples were actually evaluated formulas.

The following theorem deals with the remaining case of ordinal sums of two basic components. The case $[0,1]_* = [0,a]_{Ł} \oplus [a,1]_{Ł}$ is not considered here since in such a situation, under the working hypothesis that there exists $b \in (a,1]$ such that $b \in C$, $a \in C$ necessarily as well.

THEOREM 2.7.3. *The restriction to evaluated formulas of the logic* $\mathrm{L}_*(\boldsymbol{C})$*, when either* $[0,1]_* = [0,1]_{Ł} \oplus [0,1]_{G}$ *or* $[0,1]_* = [0,1]_{Ł} \oplus [0,1]_{\Pi}$ *and the minimum element of the second component does not belong to C, has the canonical* $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$.

Proof. The proof is an easy modification of the proofs given in [30] for $\mathrm{G}(\boldsymbol{C})$ and in [78] for $\Pi(\boldsymbol{C})$. Here, we only sketch the proof for $[0,1]_* = [0,1]_{\text{Ł}} \oplus [0,1]_\Pi$. Let $\Gamma = \{\overline{r_1} \to \varphi_1, \ldots, \overline{r_n} \to \varphi_n\}$. What we want to prove is:

$$\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \overline{s} \to \psi \quad \text{if, and only if,} \quad \Gamma \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$$

where φ_i and ψ are L_*-formulas, i.e., formulas not containing truth-constants different from $\overline{0}$ and $\overline{1}$. Actually, as always, one direction (soundness) is obvious. To prove the converse direction

$$\text{if } \Gamma \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi, \text{ then } \Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \overline{s} \to \psi$$

it is enough to combine the FS$\mathcal{R}$C of $\mathrm{L}_*(\boldsymbol{C})$ with the following result:

CLAIM 2.7.4. *If* $\Gamma \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$ *then* $\Gamma \models_{[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$*, where* $F = (a,1] \cap C$ *and* a *is the idempotent separating the first and second component of* $*$.

To prove it, without loss of generality, we may assume $r_i > 0$ for all i and $s > 0$. Suppose $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$. Then, there exists a $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$-evaluation e such that $e(\overline{r_1} \to \varphi_1) = \cdots = e(\overline{r_n} \to \varphi_n) = 1$ and $e(\overline{s} \to \psi) < 1$. Then we consider the following two cases:

(i) If $s \in (0,a]$, and hence $e(\overline{s}) = s$ and $e(\psi) < s$, take the evaluation e' over the canonical real chain defined by $e'(p) = e(p)$ for any propositional variable p. Notice that, since $e(\overline{r}) \geq e'(\overline{r})$ and $e(\varphi) = e'(\varphi)$, it is easy to compute that $e'(\overline{r_1} \to \varphi_1) = \cdots = e'(\overline{r_n} \to \varphi_n) = 1$ and $e'(\overline{s} \to \psi) = e(\overline{s} \to \psi) < 1$.

(ii) If $s \in (a,1]$, and hence $e(\overline{s}) = 1$ and $e(\psi) < 1$, we can assume $e(\psi) \geq s$, otherwise the above evaluation e' does the job. Then, take the family of evaluations e'_t (being t any natural number) over the canonical real chain defined by $e'_t(p) = k_t(e(p))$ for any propositional variable p, where $k_t \colon [0,1] \to [0,1]$ is the mapping

$$k_t(z) = \begin{cases} z & \text{if } z \in [0,a], \\ h^{-1}((h(z))^t) & \text{otherwise,} \end{cases}$$

where h is a bijection from $[a,1]$ to $[0,1]$ (e.g. the one defined by $h(x) = \frac{x-a}{1-a}$). By definition of k_t, it is easy to find a large enough t such that $a < e'_t(\psi) < s$, and hence $e'_t(\overline{s} \to \psi) < 1$. Moreover, it is easy to check that we still have $e'_t(\overline{r_1} \to \varphi_1) = \cdots = e'_t(\overline{r_n} \to \varphi_n) = 1$. Indeed, if $r_i \in (a,1]$, then $e(\overline{r}_i) = 1$ and $e(\varphi) = 1$, hence $e'_t(\varphi) = 1$ as well. If $r_i \in (0,a]$, then $e'_t(\overline{r_i}) = e(\overline{r_i}) = r_i$ and $e(\varphi_i) \geq r_i$. Now, if $e(\varphi_i) \leq a$ then $e'_t(\varphi_i) = e(\varphi_i)$, otherwise, if $e(\varphi_i) > a$ then $e'_t(\varphi_i) > a$ as well. In any case, $e'_t(\varphi_i) \geq r_i$, hence $e'_t(\overline{r_i} \to \varphi_i) = 1$.

Therefore in both cases $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$ and hence the claim and the theorem are proved. □

A final (partially negative) result that deserves some comments concerns the $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$ property. This property obviously fails for those logics $\mathrm{L}_*(\boldsymbol{C})$ such that L_* does not enjoy the S$\mathcal{R}$C: therefore it clearly makes sense to investigate what happens with the logics $\mathrm{G}(\boldsymbol{C})$. When $C = [0,1] \cap \mathsf{Q}$, it is easy to notice that all the logics $\mathrm{L}_*(\boldsymbol{C})$ under our scope fail to satisfy the $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$, as it can be seen with the following counterexample. Let $\Gamma = \{\overline{(\frac{n}{n+1})} \to \varphi \mid n \in \mathsf{N}\}$. For every logic $\mathrm{L}_*(\boldsymbol{C})$ we have $\Gamma \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \varphi$. If $\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$ then, since the logic is finitary, there would exist $n_0 \in \mathsf{N}$ such that $\overline{(\frac{n_0}{n_0+1})} \to \varphi \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$, hence, we would have $\overline{(\frac{n_0}{n_0+1})} \to \varphi \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \varphi$, i.e. a contradiction. An analogous counter-example works as well for the case when the algebra $\boldsymbol{C}$ has an accumulation point r that is the supremum of a strictly increasing sequence $(r_i)_{i\in\mathsf{N}}$ of points of $\boldsymbol{C}$. We call sup-*accessible* such an accumulation point r. Notice that for expansions $\mathrm{G}(\boldsymbol{C})$ where $\boldsymbol{C}$ does not have sup-accessible points, the $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$ holds [32, Theorem 6]: there the theorem is proved for rational semantics, but the same proof also works for the real semantics.

THEOREM 2.7.5. *The logic* $\mathrm{G}_*(\boldsymbol{C})$ *where* C *does not have* sup-*accessible points has the* $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$.

Proof. Soundness is obvious as usual. For completeness we have to prove that if a (possibly infinite) family of evaluated formulas $\{\overline{r}_i \to \varphi_i \mid i \in I\}$ does not prove an evaluated formula $\overline{s} \to \psi$ then there is an evaluation v over the canonical chain such that for every $i \in I$, $v(\overline{r}_i \to \varphi_i) = 1$ and $v(\overline{s} \to \psi) < 1$.

By the algebraizability of the logic with truth-constants, if the syntactical deduction is not valid there is a countable $\mathrm{G}(\boldsymbol{C})$-chain $\boldsymbol{A}$ and an evaluation e over it such that, for every $i \in I$, $e(\overline{r}_i \to \varphi_i) = \overline{1}^{\boldsymbol{A}}$ and $e(\overline{s} \to \psi) < \overline{1}^{\boldsymbol{A}}$. Suppose this is a chain of type F, that is, F is a filter of $\boldsymbol{C}$ such that for every $r \in F$, $\overline{r}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}$. Observe that, since the elements of C are not sup-accessible, for each point $r \in C$ there is an interval $I_r^- = (r-\delta, r)$ (with countably many elements) such that $I_r^- \cap C = \emptyset$. To build the desired evaluation v we need to study two cases:

(1) Suppose $s \in F$. In such a case, define the mapping $f\colon A \to [0,1]$ as follows: $f(\overline{1}^{\boldsymbol{A}}) = 1, f(\overline{0}^{\boldsymbol{A}}) = 0$ and f restricted to $A \setminus \{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$ is an embedding into I_s^-. An easy computation shows that f is a morphism of G-chains (without truth-constants). Define the $[0,1]_{\mathrm{G}(\boldsymbol{C})}$-evaluation v as $v(p) = f(e(p))$ for every propositional variable p. Such a v satisfies the required conditions since: if $r_i \in F$ then $v(\varphi_i) = e(\varphi_i) = 1 \geq r_i$, and if $r_i \notin F$ then $v(\varphi_i) \in \{1\} \cup I_s^-$, and thus $v(\varphi_i) \geq r_i$ as well. Moreover, since $e(\psi) < 1$, we have $v(\psi) \in I_s^- \cup \{0\}$ and thus $v(\psi) < s$.

(2) Suppose $s \notin F$. In such a case, define the mapping $f\colon A \to [0,1]$ as follows: $f(\overline{1}^{\boldsymbol{A}}) = 1, f(\overline{0}^{\boldsymbol{A}}) = 0$ and $f(\overline{s}^{\boldsymbol{A}}) = s$ and f restricted to $(\overline{s}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}})$ is an embedding into I_1^- and f restricted to $(\overline{0}^{\boldsymbol{A}}, \overline{s}^{\boldsymbol{A}})$ is an embedding into I_s^-. An easy computation shows that f is a morphism of G-chains (without truth-constants). Define the $[0,1]_{\mathrm{G}(\boldsymbol{C})}$-evaluation v as $v(p) = f(e(p))$ for every propositional variable p. Such a v satisfies the required conditions since: if $r_i \in F$, then $v(\varphi_i) = e(\varphi_i) = 1 \geq r_i$; if $r_i \notin F$, $r_i > s$, then $e(\varphi_i) > \overline{s}^{\boldsymbol{A}}$ and thus $v(\varphi_i) \in I_1^- \cup \{1\}$, which implies $v(\varphi_i) \geq r_i$; if there is some $r_i = s$, obviously $v(\varphi_i) \geq s$; if $r_i < s$ then $v(\varphi_i) \in \{1\} \cup I_1^- \cup I_s^-$, which implies $v(\varphi_i) \geq r_i$. Finally, since $e(\psi) < \overline{s}^{\boldsymbol{A}}$, we have $v(\psi) \in I_s^- \cup \{0\}$ and thus $v(\psi) < s$. □

$[0,1]_*$	$\mathrm{Can}\mathcal{R}\mathrm{C}_{ev}$	$\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$	$\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$
$[0,1]_Ł$	Yes	Yes	No
$[0,1]_G, C \notin SupAcc$	Yes	Yes	Yes
$[0,1]_G, C \in SupAcc$	Yes	Yes	No
$[0,1]_\Pi$	Yes	Yes	No
$[0,a]_Ł \oplus [a,1]_G, a \notin C$	Yes	Yes	No
$[0,a]_Ł \oplus [a,1]_\Pi, a \notin C$	Yes	Yes	No
other cases	No	No	No

Table 3. Canonical $\mathrm{S}\mathcal{R}\mathrm{C}_{ev}$ and $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$ results for logics $\mathrm{L}_*(\boldsymbol{C})$ when $*$ is a finite ordinal sum of the three basic components.

All the completeness results for evaluated formulas are summarized in Table 3, where we denote by $SupAcc$ the set of countable subalgebras of $[0,1]_*$ with sup-accessible accumulation points. Interestingly, it turns out that both the $\mathrm{Can}\mathcal{R}\mathrm{C}_{ev}$ and $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$ properties restricted to evaluated formulas become equivalent. Furthermore, comparing this table with Table 2, we realise that for a logic $\mathrm{L}_*(\mathcal{C})$ where $*$ is a finite ordinal sum of basic components, $\mathrm{Can}\mathcal{R}\mathrm{C}$ turns out to be equivalent to $\mathrm{Can}\mathcal{R}\mathrm{C}_{ev}$ (and to $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$).

2.8 Forcing the canonical interpretation of truth-constants: two approaches

In the previous subsections we have studied the logics $\mathrm{L}_*(\boldsymbol{C})$ obtained by adding truth-constants to logics of a continuous t-norms following Hájek's approach with the book-keeping axioms. One of the main drawbacks of these systems (with the exception of the expansions of Łukasiewicz logic) is the fact that different truth-constants can be interpreted to the same value. In this section, we introduce two approaches that overcome this problem and force the canonical interpretation of truth-constants (modulo an isomorphism).

2.8.1 Using the $\triangle$ operator

One possible solution is to further expand the logics with the Monteiro–Baaz $\triangle$ operator. Indeed, for every continuous t-norm $*$, we can consider the expansion of the logic L_* with $\triangle$, denoted $\mathrm{L}_{*\triangle}$. The reader may consult Chapter I for more details: there, $\mathrm{L}_{*\triangle}$ is shown to be a conservative expansion of L_*. For these expansions the following results hold (see for instance [16, 46]):

(i) The logics $\mathrm{L}_{*\triangle}$ enjoy the $\mathrm{FS}\mathcal{R}\mathrm{C}$;

(ii) The logics $\mathrm{L}_{*\triangle}$ enjoy the $\mathrm{S}\mathcal{R}\mathrm{C}$ if, and only if, $* = \min$.

Now, we will consider expansions with truth-constants for these logics with $\triangle$. Given a continuous t-norm $*$ and a countable subalgebra $\boldsymbol{C} \subseteq [0,1]_*$, we define the logic $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ as the expansion of $\mathrm{L}_{*\triangle}$ in the language $\mathcal{L}_C$ obtained by adding the following book-keeping axioms:

$$\overline{r} \,\&\, \overline{s} \leftrightarrow \overline{r * s} \qquad (\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow_* s} \qquad \triangle\overline{r} \leftrightarrow \overline{\triangle(r)}$$

for every $r, s \in C$. Here, we use the symbol $\triangle$ to denote the truth function on $\boldsymbol{C}$, i.e. $\triangle(1) = 1$ and $\triangle(r) = 0$ for each $r \in C \setminus \{1\}$.

Since $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ is an expansion of $\mathrm{L}_{*\triangle}$ with no new rules of inference then, by [21], $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ is a $\triangle$-core fuzzy logic. As a consequence, any $\mathrm{L}_{*\triangle}(\boldsymbol{C})$-algebra is a subdirect product of chains, and so the logic $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ is complete not only with respect to the full variety of $\mathrm{L}_{*\triangle}(\boldsymbol{C})$-algebras, but also with respect to the class of chains of the variety.

PROPOSITION 2.8.1. *For every continuous t-norm $*$ and every countable subalgebra $\boldsymbol{C} \subseteq [0,1]_*$, the logic $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ is a conservative expansion of $\mathrm{L}_{*\triangle}$.*

Proof. It is analogous to the proof of Proposition 2.5.2. □

LEMMA 2.8.2. *Let $\boldsymbol{A}$ be a non-trivial $\mathrm{L}_{*\triangle}(\boldsymbol{C})$-chain. Then, for every $r, s \in C$ such that $r < s$, we have $\overline{r}^{\boldsymbol{A}} < \overline{s}^{\boldsymbol{A}}$.*

Proof. If $r < s$ and $\overline{r}^{\boldsymbol{A}} = \overline{s}^{\boldsymbol{A}}$, then $\overline{1}^{\boldsymbol{A}} = \triangle\overline{1}^{\boldsymbol{A}} = \triangle(\overline{s \to r}^{\boldsymbol{A}}) = \overline{\triangle(s \to r)}^{\boldsymbol{A}} = \overline{0}^{\boldsymbol{A}}$, a contradiction. □

Therefore, if $*$ is a finite ordinal sum of Łukasiewicz and Product components, there is only one (up to isomorphism) real chain, the canonical one, that we denote by $[0,1]_{\mathrm{L}_{*\triangle}(\boldsymbol{C})}$. The result is not true for a continuous t-norm containing a Gödel component, as the counterexample at the end of Section 2.3 shows (with the obvious changes). Nevertheless, similar to the case of $\mathrm{L}_*(\boldsymbol{C})$ (without $\triangle$), the following results hold.

THEOREM 2.8.3. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. If $\mathrm{L}_*(\boldsymbol{C})$ has the partial embeddability property,*[9] *then $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ has the canonical* $\mathrm{FS}\mathcal{R}\mathrm{C}$.

Proof. Take an arbitrary $\mathrm{L}_{*\triangle}(\boldsymbol{C})$-chain $\boldsymbol{A}$. Then, its $\mathcal{L}_C$-reduct is partially embeddable into $[0,1]_{\mathrm{L}_*(\boldsymbol{C})}$, thus obviously $\boldsymbol{A}$ is partially embeddable into $[0,1]_{\mathrm{L}_{*\triangle}(\boldsymbol{C})}$ as well. □

PROPOSITION 2.8.4. *Let $*$ be a continuous t-norm and $\boldsymbol{C}$ a countable subalgebra of $[0,1]_*$ such that $\mathrm{L}_*(\boldsymbol{C})$ satisfies the partial embeddability property. Then, $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ is a conservative expansion of $\mathrm{L}_*(\boldsymbol{C})$ if, and only if, $\mathrm{L}_*(\boldsymbol{C})$ enjoys the canonical* $\mathrm{FS}\mathcal{R}\mathrm{C}$.

Proof. One direction is again analogous to the proof of Proposition 2.5.2. For the converse, suppose that $\mathrm{L}_*(\boldsymbol{C})$ does not enjoy the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}$. Then, there is a finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_\mathcal{C}}$ such that $\Gamma \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \varphi$ and $\Gamma \not\vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$. But then, $\Gamma \models_{[0,1]_{\mathrm{L}_{*\triangle}(\boldsymbol{C})}} \varphi$ and hence $\Gamma \vdash_{\mathrm{L}_{*\triangle}(\boldsymbol{C})} \varphi$, by the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}$ of $\mathrm{L}_{*\triangle}(\boldsymbol{C})$. Therefore, $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ is not a conservative expansion of $\mathrm{L}_*(\boldsymbol{C})$. □

THEOREM 2.8.5. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. $\mathrm{L}_{*\triangle}$ has the $\mathcal{SR}\mathrm{C}$ if, and only if, $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ has the* $\mathcal{SR}\mathrm{C}$.

Proof. The proof is analogous to the one of Theorem 2.5.4, taking into account that $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ is a conservative expansion of $\mathrm{L}_{*\triangle}$ (Proposition 2.8.1). □

COROLLARY 2.8.6. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ enjoys the $\mathcal{SR}\mathrm{C}$ if, and only if, $* = \min$.*

[9] In particular if $\mathrm{L}_*(\boldsymbol{C})$ satisfies conditions $(\mathbf{C1})$ and $(\mathbf{C2})$.

2.8.2 Introducing additional inference rules

An alternative approach to the use of the $\triangle$ operator, in order to force truth-constants to be interpreted in their intended values, is proposed in [6]. Given a logic of a continuous t-norm L_* and a countable subalgebra $\boldsymbol{C}$ of $[0,1]_*$, one defines the logic $\overline{\mathrm{L}}_*(\boldsymbol{C})$ as the extension of $\mathrm{L}_*(\boldsymbol{C})$ with the following inference rule for each $r \in C$ such that $r < 1$:

$$\text{from} \quad \varphi \vee \bar{r} \quad \text{infer} \quad \varphi.$$

The algebraic counterpart of these logical systems is the class of $\overline{\mathrm{L}}_*(\boldsymbol{C})$-algebras, which are defined in the natural way, i.e. as $\mathrm{L}_*(\boldsymbol{C})$-algebras satisfying the following quasiequations for $r \in C \setminus \{1\}$: if $x \vee \bar{r} = 1$ then $x = 1$. It is clear then that the class of $\overline{\mathrm{L}}_*(\boldsymbol{C})$-algebras forms a quasivariety. Since the new inference rule is closed under $\vee$-forms, the logic $\overline{\mathrm{L}}_*(\boldsymbol{C})$ turns out to be a semilinear logic (see Chapter II), and is therefore complete with respect to the class of $\overline{\mathrm{L}}_*(\boldsymbol{C})$-chains of the quasivariety. Moreover, every algebra of the quasivariety is a subdirect product of chains of the quasivariety.

The presence of the new inference rule has as a consequence that (like in the case of expansions of $\mathrm{L}_{*\triangle}$ with truth-constants) the interpretation of truth-constants in a $\overline{\mathrm{L}}_*(\boldsymbol{C})$-chain is one-to-one.

LEMMA 2.8.7. *Let $\boldsymbol{A}$ be a non-trivial $\overline{\mathrm{L}}_*(\boldsymbol{C})$–chain. Then, for every $r, s \in C$ such that $r < s$, we have $\bar{r}^{\boldsymbol{A}} < \bar{s}^{\boldsymbol{A}}$.*

Proof. Suppose $r < s$ and $\bar{r}^{\boldsymbol{A}} = \bar{s}^{\boldsymbol{A}}$. Let $t = s \Rightarrow r$. It is clear that $t < 1$ but $\bar{t}^{\boldsymbol{A}} = \bar{s}^{\boldsymbol{A}} \to_{\boldsymbol{A}} \bar{r}^{\boldsymbol{A}} = \bar{1}^{\boldsymbol{A}}$, which contradicts the fulfillment of the rule. □

Therefore, analogously to what happens in the variety of $\mathrm{L}_{*\triangle}(\boldsymbol{C})$-algebras, if $*$ is a continuous t-norm that is a finite ordinal sum of Łukasiewicz and Product components, in the quasivariety of $\overline{\mathrm{L}}_*(\boldsymbol{C})$-algebras there is only one (up to isomorphism) real chain over $[0,1]_*$, the canonical one, denoted as $[0,1]_{\mathrm{L}_*(\boldsymbol{C})}$. Again, this is not true if $*$ contains a Gödel component, as the counterexample at the end of Section 2.3 also shows.

Moreover the logical system $\overline{\mathrm{L}}_*(\boldsymbol{C})$ is also a conservative expansion of L_*.

PROPOSITION 2.8.8. *For every continuous t-norm $*$ and every countable subalgebra $\boldsymbol{C} \subseteq [0,1]_*$, the logic $\overline{\mathrm{L}}_*(\boldsymbol{C})$ is a conservative expansion of* L_*.

Proof. It is analogous to the proof of Proposition 2.5.2. □

The partial embeddability property also applies, in this setting, for continuous t-norms satisfying conditions $(\mathbf{C1})$ and $(\mathbf{C2})$. The proof is completely analogous to the proof for the case of $\mathrm{L}_*(\boldsymbol{C})$, and, so, it is left to the reader. This implies the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}$.

THEOREM 2.8.9. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. If $\overline{\mathrm{L}}_*(\boldsymbol{C})$ has the partial embeddability property,[10] then $\overline{\mathrm{L}}_*(\boldsymbol{C})$ has the canonical* $\mathrm{FS}\mathcal{R}\mathrm{C}$.

Finally the following result fully characterizes the logics satisfying the $\mathrm{S}\mathcal{R}\mathrm{C}$.

[10] In particular for any $*$ satisfying conditions $(\mathbf{C1})$ and $(\mathbf{C2})$.

THEOREM 2.8.10. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra.* $\bar{\mathrm{L}}_*(\boldsymbol{C})$ *enjoys the* $\mathrm{S}\mathcal{R}\mathrm{C}$ *if, and only if,* $* = \min$.

Proof. The proof is the same as the one given for $\mathrm{G}(\boldsymbol{C})$. □

Notice that the canonical $\mathrm{S}\mathcal{R}\mathrm{C}$ is not true even for $\overline{\mathrm{G}}(\boldsymbol{C})$. The above mentioned counterexample at the end of Section 2.3, with the obvious changes, proves that no countable $\overline{\mathrm{G}}(\boldsymbol{C})$-chain is embeddable into the canonical $\overline{\mathrm{G}}(\boldsymbol{C})$-chain. In fact, the completeness results for these logics coincide with those for the expansions of the logics with $\triangle$ described in the previous section.

2.9 Some open questions

In the previous subsections, we have provided a complete description of completeness results for the expansions of logics of continuous t-norms with a set of truth-constants $\{\overline{r} \mid r \in C\}$, for a suitable countable $C \subseteq [0,1]$, when (i) the t-norm is a finite ordinal sum of basic components and (ii) the set of truth-constants *covers* all the unit interval, in the sense that each component of the t-norm contains at least one value of C different from the bounds of the component. From a practical point of view, it seems that these cases are the most interesting ones for fuzzy logic-based systems, since they usually consider a set of truth values spread all over the real unit interval, and it is natural to assume that there are elements of C in each component of the t-norm. All those cases where at least one of the above two conditions (i) and (ii) is not satisfied remain to be studied. It seems that for these remaining cases (i.e., when either the t-norm has infinitely many components, or the set C does not cover $[0,1]$), a methodology similar to the one used in this section could be applied. In fact, there is a multitude of cases to be considered and the need of new definitions and tools seems unavoidable. Let us show a couple of illustrative examples: the first when the set C does not cover $[0,1]$ and the second when the t-norm has infinitely many components.

EXAMPLE 2.9.1. Let $[0,1]_* = [0,a]_\Pi \oplus [a,1]_\Pi$ and let $C = \{0,1\} \cup \{b^n \mid n \in \mathsf{N}\}$ for some $b < a$. Obviously, there are only two proper filters of C, $F_1 = \{1\}$ and $F_2 = C \setminus \{0\}$, but there are (up to isomorphism) three real $\mathrm{L}_*(\boldsymbol{C})$-chains. One, of type F_2, in the sense used in this paper, is the $\mathrm{L}_*(\boldsymbol{C})$-chain over $[0,1]_*$ where the constants different from $\overline{0}$ are interpreted as 1, and $\overline{0}$ is interpreted as 0. The other two are of type F_1. They are both $\mathrm{L}_*(\boldsymbol{C})$-chains over $[0,1]_*$, where all the constants are interpreted as different elements, either as powers of an element of the first product component or as powers of an element of the second product component. Of course, these two algebras are not isomorphic. This example shows that, in general, there is not a bijection between proper filters and real algebras and, even though it seems possible to have the partial embedding property, the notion and treatment of real chains should be modified in the case that C does not cover all components.

EXAMPLE 2.9.2. Let $[0,1]_* = \bigoplus_{n\in\mathsf{N}}[a_n, a_{n+1}]_{\text{Ł}}$, where $a_n = n/(n+1)$, be an infinite ordinal sum of Łukasiewicz components where the idempotent elements form an increasing sequence with limit 1. For a given $k > 2$, let C_i be the carrier of the k-element subalgebra of $[a_i, a_{i+1}]_{\text{Ł}}$, and denote its elements as $r_{1i} = a_i, r_{2i}, \ldots, r_{ki} = a_{i+1}$. Take $C = \cup_{i\in\mathsf{N}} C_i \cup \{1\}$. It is clear that C covers all the components but there are real algebras where the interpretations of the truth-constants do not cover all the components. Indeed, let f be any strictly increasing mapping $f\colon \mathsf{N} \to \mathsf{N}$ such that $f(1) = 1$. One real $\mathrm{L}_*(\boldsymbol{C})$-algebra is the chain over $[0,1]_*$, where $\overline{r_{ij}}$ is interpreted as $r_{f(i)j}$. An easy computation shows that this interpretation defines a real $\mathrm{L}_*(\boldsymbol{C})$-chain where the interpretations of truth-constants do not cover the real unit interval. In fact, if $f(i+1)$ is not the successor of $f(i)$ (there are some natural numbers in between), the corresponding components contain no interpretations of truth-constants.

The general case of adding truth-constants to the logic of a left-continuous t-norm $*$ is only studied in the particular case of $*$ corresponding a weak nilpotent minimum t-norm in [30, 33]. Moreover, in [32], the completeness problem of these logics with truth-constants is studied for some distinguished semantics, especially for rational and finite semantics.

2.10 First-order fuzzy logics expanded with truth-constants

The expansion of a first-order t-norm based fuzzy logic with truth-constants, in principle, could be introduced in two different ways:

- Given a left-continuous t-norm $*$ and a countable subalgebra $\boldsymbol{C} \subseteq [0,1]_*$, consider the logic $\mathrm{L}_*(\boldsymbol{C})$ and take its first-order extension $\mathrm{L}_*(\boldsymbol{C})\forall$.
- Given a left-continuous t-norm $*$, consider its associated propositional logic L_*. Take its first-order extension $\mathrm{L}_*\forall$ and now (by enhancing the language with the constants and adding the book-keeping axioms) define its expansion $\mathrm{L}_*\forall(\boldsymbol{C})$ with truth-constants from a countable algebra $\boldsymbol{C} \subseteq [0,1]_*$.

However, these two methods turn out to define the same logic. To set the notation, we will use the second one: $\mathrm{L}_*\forall(\boldsymbol{C})$.

As in the propositional case, we are interested in completeness properties of these logics and even though there are some results for $*$ being a left-continuous t-norm, we restrict ourselves to the case of continuous t-norms. We will find again some positive and some negative results. For the negative ones, we can note that the failure of a completeness property in a weaker logic implies the failure in the stronger one. To make use of this observation, an interesting result is to show that adding truth-constants to a first-order logic $\mathrm{L}_*\forall$ results into a conservative expansion. This is done in the next section.

2.10.1 Conservativeness results

In the case of Łukasiewicz t-norm, Hájek *et al.* already proved in [52] that $\mathrm{RPL}\forall$ (Rational Pavelka predicate logic[11]) is a conservative expansion of Ł$\forall$. Actually, from the proofs in [52], we can extract the following result:

LEMMA 2.10.1 ([52]). *Let $\boldsymbol{C}$ be a subalgebra of $[0,1]_{\text{Ł}}^{\mathsf{Q}}$, $\boldsymbol{A}$ be a countable* MV*-chain and* $\mathbf{M}$ *be an $\boldsymbol{A}$-safe structure in a predicate language for* Ł$\forall$. *Then, there is a divisible*[12] MV*-chain $\boldsymbol{A}'$, such that $\boldsymbol{A}$ is σ-embeddable into $\boldsymbol{A}'$, and the truth-constants from $\boldsymbol{C}$ are interpretable in $\boldsymbol{A}'$ in such a way that $\mathbf{M}$ is also an $\boldsymbol{A}'$-safe structure for* Ł$\forall(\boldsymbol{C})$.

[11] In our notation $\mathrm{RPL}\forall$ corresponds to Ł$\forall(\boldsymbol{C})$ when $C = [0,1] \cap \mathsf{Q}$.

[12] An MV-chain $\boldsymbol{A}$ is called *divisible* if for every natural m and every $x \in A$ there exists $y \in A$ such that $y \oplus \overset{m}{\ldots} \oplus y = x$ and $y \,\&\, (y \oplus \overset{m-1}{\ldots} \oplus y) = 0$.

This embeddability result is also valid for the predicate logics of the remaining basic continuous t-norms, i.e. Gödel and product logics. In fact it is also valid for the predicate logic of any SBL t-norm.[13]

LEMMA 2.10.2. *Let $*$ be an* SBL *t-norm, $\boldsymbol{C}$ be a countable subalgebra of $[0,1]_*$ and $\mathbf{M}$ be an $\boldsymbol{A}$-safe structure in a predicate language for* $\mathrm{L}_*\forall$. *Then, the truth-constants from $\boldsymbol{C}$ are interpretable in $\boldsymbol{A}$ in such a way that $\mathbf{M}$ is also an $\boldsymbol{A}$-safe structure for* $\mathrm{L}_*\forall(\boldsymbol{C})$.

Proof. For every $r \in C \setminus \{0\}$, interpret $\overline{r}$ as $\overline{1}^{\boldsymbol{A}}$, and $\overline{0}$ as $\overline{0}^{\boldsymbol{A}}$. This turns $\boldsymbol{A}$ into a chain for the expanded language. It is clear that $\mathbf{M}$ is also $\boldsymbol{A}$-safe in this language, since the interpretation of the constants does not give any new value. □

From these lemmas, we obtain conservativeness results for logics based on continuous t-norms. In the proof of next theorem, we use the fact that an SBL t-norm is an ordinal sum either with a first component that is not a Łukasiewicz component or without a first component (see [24]).

THEOREM 2.10.3. *Let $*$ be a continuous t-norm and $\boldsymbol{C}$ a countable subalgebra of $[0,1]_*$ such that, if $*$ is not an* SBL*-t-norm, the truth-constants in the Łukasiewicz first component of the decomposition correspond to rational numbers. Then,* $\mathrm{L}_*\forall(\boldsymbol{C})$ *is a conservative expansion of* $\mathrm{L}_*\forall$.

Proof. Let $\Gamma \cup \{\varphi\}$ be a set of $\mathrm{L}_*\forall$-formulas such that $\Gamma \not\vdash_{\mathrm{L}_*\forall} \varphi$. We must show that $\Gamma \not\vdash_{\mathrm{L}_*\forall(\boldsymbol{C})} \varphi$. By hypothesis, there is some safe $\mathrm{L}_*\forall$-structure $\langle \mathbf{M}, \boldsymbol{A}\rangle$ such that $\langle \mathbf{M}, \boldsymbol{A}\rangle \models \Gamma$ and $\langle \mathbf{M}, \boldsymbol{A}\rangle \not\models \varphi$, where $\boldsymbol{A}$ is a countable L_*-chain. If $*$ is an SBL-t-norm, then $\boldsymbol{A}$ is an SBL-chain and applying Lemma 2.10.2 the problem is solved. If $*$ is not an SBL-t-norm, then $*$ is the ordinal sum of a Łukasiewicz component and a hoop $\boldsymbol{B}$. Then, by [24, Proposition 3], $\boldsymbol{A}$ must be a chain of $\mathbf{HSP}_U([0,1]_{\text{Ł}}) \cup (\mathbf{ISP}_U([0,1]_{\text{Ł}}) \oplus \mathbf{HSP}_U(\boldsymbol{B}))$. Then, $\boldsymbol{A}$ is either an MV-chain or the ordinal sum (in the sense of hoops) of an MV-chain $\boldsymbol{A}_1$ and a hoop $\boldsymbol{A}_2$ of $\mathbf{HSP}_U(\boldsymbol{B})$. Take $\boldsymbol{A}'$ as the ordinal sum of the divisible hull $\boldsymbol{A}_1'$ of $\boldsymbol{A}$ (as done in Lemma 2.10.1) and $\boldsymbol{A}_2$. Thus, we obtain a BL-chain $\boldsymbol{A}'$ belonging to $\mathbf{V}([0,1]_*)$ (again by [24, Proposition 3]). Then, we define an $\mathrm{L}_*(\boldsymbol{C})$-chain over $\boldsymbol{A}'$ interpreting the truth-constants from the Łukasiewicz first component of $\boldsymbol{C}$ as the corresponding truth-values of $\boldsymbol{A}_1'$ and the remaining truth-constants as $\overline{1}^{\boldsymbol{A}'}$. By the previous lemmas, $\boldsymbol{A}$ is σ-embeddable into $\boldsymbol{A}'$ as $\mathrm{L}_*(\boldsymbol{C})$-chains. Therefore, we have obtained a chain $\boldsymbol{A}'$ in the expanded language, such that $\mathbf{M}$ is an $\boldsymbol{A}'$-safe structure, and, consequently, $\langle \mathbf{M}, \boldsymbol{A}'\rangle \models \Gamma$, while $\langle \mathbf{M}, \boldsymbol{A}'\rangle \not\models \varphi$. So, the theorem is proved. □

2.10.2 Completeness results

Using the results of previous subsection along with the fact that, for every continuous t-norm $*$ different from Gödel t-norm, the $\mathcal{R}$C fails for $\mathrm{L}_*\forall$, we have that $\mathcal{R}$C also fails for their expansions with truth-constants. However, for the minimum-t-norm based logic, we can give positive answers to some completeness problems.

THEOREM 2.10.4. *If $* = \min$, the logic* $\mathrm{L}_*\forall(\boldsymbol{C})$ *enjoys the* S$\mathcal{R}$C.

[13]An SBL t-norm is a continuous t-norm $*$ such that, for all $x \in [0,1]$, it holds that $\min\{x, x \Rightarrow_* 0\} = 0$.

Proof. In [46] it was proved that every countable G-chain $\boldsymbol{A}$ is σ-embeddable into the real G-chain $\boldsymbol{B}$. Denote by $f\colon A \to [0,1]$ one of these σ-embeddings. Assume, in addition, that $\boldsymbol{A}$ is a $\mathrm{G}(\boldsymbol{C})$-chain. For every $r \in C$, interpret $\overline{r}$ in B as $f(\overline{r}^A)$: this gives a real $\mathrm{G}(\boldsymbol{C})$-chain. Thus, we obtain the $\mathrm{S}\mathcal{R}\mathrm{C}$ for $\mathrm{G}\forall(\boldsymbol{C})$. □

Moreover, $\mathrm{G}\forall(\boldsymbol{C})$ enjoys canonical completeness.

THEOREM 2.10.5. *The logic* $\mathrm{G}\forall(\boldsymbol{C})$ *enjoys the* $\mathrm{Can}\mathcal{R}\mathrm{C}$.

Proof. Soundness is obvious, as usual. For the other direction, we will argue by contraposition, i.e. we will prove that if $\nvdash_{\mathrm{G}\forall(\boldsymbol{C})} \varphi$ for some formula φ, then there is a $\mathrm{G}\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}, [0,1]_{\mathrm{G}(\boldsymbol{C})}\rangle$ such that $\langle \mathbf{M}, [0,1]_{\mathrm{G}(\boldsymbol{C})}\rangle \not\models \varphi$.

If $\nvdash_{\mathrm{L}_*\forall(\boldsymbol{C})} \varphi$, then there exists an $\mathrm{L}_*\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}, \boldsymbol{A}\rangle$ over a countable L_*-chain $\boldsymbol{A}$ and an evaluation v such that $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} < \overline{1}^{\boldsymbol{A}}$. As in Theorem 2.6.1, take $s = \min(\{r \in C \mid \overline{r}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}, r \text{ appears in } \varphi\} \cup \{1\})$ and define an order-preserving injection $g\colon A \to [0,1]$, also preserving existing suprema and infima, and such that $g(\overline{0}^{\boldsymbol{A}}) = 0$, $g(\overline{1}^{\boldsymbol{A}}) = s$ and $g(\overline{r}^{\boldsymbol{A}}) = r$, for every truth-constant appearing in φ such that $\overline{r}^{\boldsymbol{A}} \neq \overline{1}^{\boldsymbol{A}}$. If $\mathbf{M} = \langle M, \langle P_{\mathbf{M}}\rangle_{P\in Pred}, \langle f_{\mathbf{M}}\rangle_{f\in Funct}\rangle$, using the mapping g, we produce a structure

$$\langle \mathbf{M}', [0,1]_{\mathrm{L}_*(\boldsymbol{C})}\rangle,$$

where $\mathbf{M}' = \langle M, \langle P_{\mathbf{M}'}\rangle_{P\in Pred}, \langle f_{\mathbf{M}}\rangle_{f\in Funct}\rangle$, with $P^{\mathbf{M}'}\colon M^{ar(P)} \to [0,1]$ defined as $P_{\mathbf{M}'} = g \circ P_{\mathbf{M}}$. Therefore, for every evaluation of variables e on M one has

$$\|P(t_1, t_2, \ldots, t_n)\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',e} = g(\|P(t_1, t_2, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{M},e})$$

for each predicate symbol P and terms $t_1, t_2, \ldots, t_n$.

Now, we will prove by induction that given any $\mathbf{M}$ and e and their associated $\mathbf{M}'$ and e', the following statements are true for every subformula ψ of φ:

(a) if $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},e} = \overline{1}^{\boldsymbol{A}}$, then $\|\psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e} \geq s$,

(b) if $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},e} \neq \overline{1}^{\boldsymbol{A}}$, then $\|\psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e} = g(\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},e}) < s$.

The inductive steps for $\psi = \overline{r}$, $\psi = P(t_1, t_2, \ldots, t_n)$, $\psi = \alpha \,\&\, \beta$ and $\psi = \alpha \to \beta$ are proved as in the propositional case in Theorem 2.6.1. Therefore, we are left only with the steps involving quantifiers. We start with $\psi = (\forall x)\alpha$. Let $V(e)$ denote the set of evaluations v of variables such that $e(y) = v(y)$ for all variables y, except x. Recall that $\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e} = \inf\{\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid v \in V(e)\}$.

If $\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e} = \overline{1}^{\boldsymbol{A}}$, then for every such $v \in V(e)$ we have $\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$, and hence $\|\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',v} \geq s$, which implies that $\|(\forall x)\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',e} \geq s$.

If $\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e} \neq \overline{1}^{\boldsymbol{A}}$, it suffices to consider the infimum over the set $V^+(e)$ of evaluations v such that $\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} \neq \overline{1}^{\boldsymbol{A}}$, i.e. $\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e} = \inf\{\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid v \in V^+(e)\} \neq \overline{1}^{\boldsymbol{A}}$. Then, since g preserves all the existing infima, we have: $s > g(\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e}) =$

$g(\inf\{\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid v \in V^+(e)\}) = \inf\{g(\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v}) \mid v \in V^+(e)\} = \inf\{\|\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',v} \mid v \in V^+(e)\} = \inf\{\|\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',v} \mid v \in V(e)\} = \|(\forall x)\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',e}$.

The reasoning in the case $\psi = (\exists x)\alpha$ is similar to the previous one (now it uses that g preserves existing suprema). □

Therefore, we have solved all the real completeness problems for first-order logics under our scope, since in the remaining cases the properties obviously do not hold as they already fail for the corresponding propositional logics with truth-constants. Table 4 collects these results.

Logic	$\mathcal{R}$C, FS$\mathcal{R}$C, S$\mathcal{R}$C	Can$\mathcal{R}$C	CanFS$\mathcal{R}$C
$\mathrm{L}_*\forall(\boldsymbol{C})$, $* \in \mathbf{CONT\text{-}fin} \setminus \{*_{\mathrm{G}}\}$	No	No	No
$\mathrm{G}\forall(\boldsymbol{C})$	Yes	Yes	No

Table 4. Real completeness properties for first-order t-norm based logics with truth-constants.

2.10.3 The case of evaluated formulas

In this section we restrict the completeness properties of our first-order logics to evaluated formulas in the hope of improving the completeness results we have obtained in general. These completeness properties are straightforwardly refuted in many cases. Namely, for each $* \in \mathbf{CONT\text{-}fin} \setminus \{*_{\mathrm{G}}\}$, there is a constant-free formula φ such that $\nvdash_{\mathrm{L}_*\forall} \varphi$ and $\models_{[0,1]_*} \varphi$, and hence, since φ is equivalent to the evaluated formula $\overline{1} \to \varphi$ and $\mathrm{L}_*\forall(C)$ is a conservative expansion of $\mathrm{L}_*\forall$, we also have a counterexample to the $\mathcal{R}\mathrm{C}_{ev}$ of $\mathrm{L}_*\forall(C)$.

In addition, the completeness properties for evaluated formulas are also refuted in those cases where they already fail at the propositional level (and hence also including the failure of $\mathrm{CanS}\mathcal{R}\mathrm{C}$ for the cases 1–5 listed before Theorem 2.7.3).

There are, nonetheless, several positive results. Regarding canonical completeness properties, the only cases that remain to be checked are those corresponding to the logics $\mathrm{G}\forall(\boldsymbol{C})$. In the rest of this section, we show that $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$ always holds for these logics, while we provide only some partial (positive) results in the case of $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$.

THEOREM 2.10.6. *The logics* $\mathrm{G}\forall(\boldsymbol{C})$ *enjoy the* $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$.

Proof. We have to show that for every formulas $\varphi_1, \ldots, \varphi_k, \psi$ in the language of G∀ and positive constants $\overline{r_1}, \ldots, \overline{r_k}, \overline{s}$:

$\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \vdash_{\mathrm{G}\forall(\boldsymbol{C})} \overline{s} \to \psi$ if, and only if,
$\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \overline{s} \to \psi$.

The proof is analogous to the one of (i) of Theorem 2.7.1 with the obvious changes. Still, we include it for the sake of readability. By the deduction theorem and the canonical standard completeness for $\mathrm{G}\forall(\boldsymbol{C})$, a finite deduction of type $\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \vdash_{\mathrm{G}\forall(\boldsymbol{C})} \overline{s} \to \psi$ is equivalent to $\models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$. Thus, what we need to prove is the semantical version of the deduction theorem for

$\text{Ł}_*\forall(\mathcal{C})$, i.e. the equivalence between $\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \overline{s} \to \psi$ and $\models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$.

From right to left the implication is obvious. We prove the other direction by contraposition. If $\not\models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$, there must exist (by the previous theorem) a $\mathrm{G}\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}, [0,1]_{\mathrm{G}(\boldsymbol{C})}\rangle$ and an evaluation e such that

$$\|\&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} < 1.$$

We have to build a $\mathrm{G}\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}', [0,1]_{\mathrm{G}(\boldsymbol{C})}\rangle$ and an evaluation of variables e' such that $\|\&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} = 1$ and $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} < 1$. Observe first that the previous inequality implies that $\|\&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} > \|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e}$ and thus $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} = \|\psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} < 1$. We follow the proof by cases:

(i) If $\|\overline{r_i} \to \varphi_i\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} = 1$ for every $i \in \{1, \ldots, k\}$, then we just take $\mathbf{M}' = \mathbf{M}$ and $e' = e$.

(ii) Suppose there exists a non-empty set of indexes $J \subseteq \{1, \ldots, k\}$ such that for all $j \in J$, $\|\overline{r_j} \to \varphi_j\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} = \|\varphi_j\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} < 1$. Let $a = \min\{\|\varphi_j\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} \mid j \in J\}$. Define (like in Theorem 2.7.1) f_a as the endomorphism of $[0,1]_{\mathrm{G}(\boldsymbol{C})}$ given by $f_a(x) = 1$ for every $x \geq a$ and by an order preserving bijection between $[0,a)$ and $[0,1)$ preserving existing suprema and infima. Now, we consider a structure $\mathbf{M}'$ over the same domain as $\mathbf{M}$ with the same interpretation of functional symbols, with the same evaluation of variables $e' = e$, and we will just change the interpretation of the predicate symbols. Indeed, for every n-ary predicate P and arbitrary elements of the domain $m_1, \ldots, m_n$, we define $P_{\mathbf{M}'}(m_1, \ldots, m_n) = f_a(P_{\mathbf{M}}(m_1, \ldots, m_n))$. Then, since f is a homomorphism that preserves existing suprema and infima, it is obvious that for every $\mathrm{G}\forall$-formula φ we have $\|\varphi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e} = f_a(\|\varphi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e})$. An easy computation shows that $\|\&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} = 1$, while $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} < 1$. □

Regarding the properties of $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$, as already mentioned above, it remains to check the cases of logics $\mathrm{G}\forall(\boldsymbol{C})$ when the algebra of truth-constants C has no positive sup-accessible points, i.e. for each $r \in C$ there exists an open interval $(r - \epsilon, r)$ containing no element of C (otherwise $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$ already fails in the propositional case). Two paradigmatic particular examples of algebras of truth-constants satisfying this condition are the case when C is finite (which is obvious) and the case when $C \setminus \{0\}$ is a strictly decreasing sequence with limit 0 (addressed in next theorem).

THEOREM 2.10.7. *Let $\boldsymbol{C}$ be such that $C \setminus \{0\} = \{t_n \mid n \in \mathsf{N}\}$, where $\langle t_n\rangle_{n\in\mathsf{N}}$ is a strictly decreasing sequence with limit* 0*. Then, the logic* $\mathrm{G}\forall(\boldsymbol{C})$ *enjoys* $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$.

Proof. We have to show that for every set of formulas $\{\varphi_i \mid i \in I\} \cup \{\psi\}$, in the language of $\mathrm{G}\forall$ and positive constants $\{\overline{r_i} \mid i \in I\} \cup \{\overline{s}\}$:

$\{\overline{r_i} \to \varphi_i \mid i \in I\} \vdash_{\mathrm{G}\forall(\boldsymbol{C})} \overline{s} \to \psi$ if, and only if, $\{\overline{r_i} \to \varphi_i \mid i \in I\} \models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \overline{s} \to \psi$.

In one direction the implication is obvious. We prove the other one by contraposition. If $\{\overline{r_i} \to \varphi_i \mid i \in I\} \nvdash_{\mathrm{G}\forall(\boldsymbol{C})} \overline{s} \to \psi$, there must exist a countable G$\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}, \boldsymbol{A}\rangle$, and an evaluation e over $\boldsymbol{A}$ such that $\|\overline{r_i} \to \varphi_i\|^{\boldsymbol{A}}_{\mathbf{M},e} = \overline{1}^{\boldsymbol{A}}$ for all $i \in I$ and $\|\overline{s} \to \psi\|^{\boldsymbol{A}}_{\mathbf{M},e} < \overline{1}^{\boldsymbol{A}}$. We have to build a G$\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}', [0,1]_{\mathrm{G}(\boldsymbol{C})}\rangle$ and an evaluation of variables e' such that $\|\overline{r_i} \to \varphi_i\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} = 1$ for all $i \in I$ and $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} < 1$.

The proof will consist in taking the same domain of individuals $\mathbf{M}' = \mathbf{M}$, the same evaluation $e' = e$, and defining for every n-ary predicate P and arbitrary elements of the domain $m_1, \dots, m_n$, $P_{\mathbf{M}'}(m_1, \dots, m_n) = f(P_{\mathbf{M}}(m_1, \dots, m_n))$, where f is a σ-embedding of $\mathcal{A}$ as G-algebra into $[0,1]_{\mathrm{G}}$ satisfying:

(i) $f(\overline{r_i}^{\mathcal{A}}) \geq r_i$ for all $i \in I$,

(ii) $f(\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},e}) < s$.

Notice that such a mapping f solves our problem, since being a σ-embedding it holds that, for any G$\forall$-formula φ, $\|\varphi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e} = f(\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},e})$, and so by (i) we obtain that $\|\varphi_i\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e} \geq r_i$ for all $i \in I$, and (ii) gives us $\|\psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e} < s$. Therefore, the rest of the proof is devoted to building the σ-embedding f.

Since $\boldsymbol{A}$ is a G$(\boldsymbol{C})$-chain, it defines a filter $F_A = \{r \in C \mid \overline{r}^A = \overline{1}^A\}$ of $\mathcal{C}$ such that $\overline{p}^A < \overline{q}^A$ for any $p, q \notin F_A$ and $p < q$. We consider the following cases:

(1) $s \in F_A$ and $\inf_n \overline{t_n}^A = \overline{0}^A$.

Let t_m be the greatest element of $C \setminus F_A$. We split the construction of f in two parts. The restriction of f to the interval $[\overline{0}^A, \overline{t_m}^A]$ is taken as any σ-embedding into $[0, t_m]$ such that $f(\overline{t_k}^A) = t_k$ for each $k \geq m$. On the other hand, if $\|\psi\|^A_{\mathbf{M},e} \leq \overline{t_m}^A$, the restriction of f to $[\overline{t_m}^A, \overline{1}^A]$ is taken as any σ-embedding into $[t_m, 1]$. Otherwise, let $\delta \in [0,1]$ be such that $\delta < s$ and $[\delta, s) \cap C = \emptyset$. Then the restriction of f to $[\overline{t_m}^A, \overline{1}^A]$ is taken as any σ-embedding into $[t_m, 1]$ such that $f(\|\psi\|^A_{\mathbf{M},e}) = \delta$.

(2) $s \in F_A$ and there exists $\overline{0}^A < \alpha \in A$ such that $\overline{t_n}^A > \alpha$ for each n.

The construction of the restriction of f to $[\overline{t_m}^A, \overline{1}^A]$ is exactly the same as in (1). Now, the restriction of f to $[\overline{0}^A, \overline{t_m}^A]$ is defined as any σ-embedding into $[0, t_m]$ such that $f(\alpha) = t_{m-1}$. In this case, it holds that $f(\overline{t_k}^A) \geq t_k$ for $k \geq m$.

(3) $s \notin F_A$.

In this case, the restriction of f to $[\overline{s}^A, \overline{1}^A]$ can be taken as any σ-embedding into $[s, 1]$ such that $f(\overline{t_i}^A) = t_i$ for all $t_i \notin F_A$ and $t_i \geq s$ (there are finitely many). The restriction of f to $[\overline{0}^A, \overline{s}^A]$ depends on whether $\inf_n \overline{t_n}^A = \overline{0}^A$ or there exists $\overline{0}^A < \alpha \in A$ such that $\overline{t_n}^A > \alpha$ for each n. Taking t_m as s, the restriction of f to $[\overline{0}^A, \overline{s}^A]$ in the former case is defined as in (1) and in the latter case as in (2). □

The proof of the above theorem can be easily adapted to the cases considered in the next corollary, and, thus, we omit the proofs.

COROLLARY 2.10.8. *The logic* $\mathrm{G}\forall(\boldsymbol{C})$ *also enjoys the* $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$ *in the following cases:*

- C *is finite,*
- $C \setminus \{0\} = \{t_n \mid n \in \mathsf{N}\}$*, where* $\langle t_n \rangle_{n \in \mathsf{N}}$ *is a strictly decreasing sequence without limit in* C*,*
- $C \setminus \{0\} = \{t_n \mid n \in \mathsf{N}\} \cup \{\alpha\}$*, where* $\langle t_n \rangle_{n \in \mathsf{N}}$ *is a strictly decreasing sequence with limit* $\alpha \in C$*.*

However, it is still unknown whether these positive results also hold for the general case of $\boldsymbol{C}$ having no positive sup-accessible points. In Table 5, we summarize the available results about canonical completeness properties. We do not include there the results about non-canonical completeness for evaluated formulas since, as already discussed in the beginning of this section, they turn out to be the same as for arbitrary formulas.

Logic	$\mathrm{Can}\mathcal{R}\mathrm{C}_{ev}$, $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$	$\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$
$\mathrm{L}_*\forall(\boldsymbol{C})$, $* \in \textbf{CONT-fin} \setminus \{*_\mathrm{G}\}$	No	No
$\mathrm{G}\forall(\boldsymbol{C})$ C^+ has sup-accessible points	Yes	No
$\mathrm{G}\forall(\boldsymbol{C})$ C^+ has no sup-accessible points	Yes	?

Table 5. Canonical real and rational completeness properties for first-order t-norm based logics with truth-constants restricted to evaluated formulas.

3 Expansions with truth-stressing and truth-depressing hedges

Typical examples of fuzzy truth-values in the sense of Zadeh (see [87]) are "very true", "quite true", "more or less true", "slightly true", etc. They are represented in fuzzy logic in narrow sense as fuzzy subsets on the set of truth values, typically the real unit interval. In order to cope with these fuzzy truth values in the setting of mathematical fuzzy logic, Hájek proposed in [47] to understand them as truth functions of new unary connectives called either *truth-stressing* or *truth-depressing hedges* (depending on whether they reinforce or weaken the truth value). The intuitive interpretation of a truth-stressing (resp. depressing) hedge like *very true* (resp. *slightly true*) on a chain of truth-values is a subdiagonal (resp. superdiagonal) non-decreasing function preserving 0 and 1. From now on, such functions will be called *hedge functions*. Notice that the well-known globalization operator $\triangle$ (introduced independently first by Monteiro in the context of intuitionistic logic [72] and later by Baaz in the context of Gödel–Dummett logics [2]) is a limit case of a truth-stresser, since, over a chain, it maps 1 to 1 and all the other elements to 0, and its intuitive interpretation would correspond to *definitely true*.

Hájek [47] and Vychodil [86] proposed an axiomatization of truth-stressing and depressing hedges respectively as expansions of BL (and of some of its prominent extensions, like Łukasiewicz, Product or Gödel logics) by new unary connectives vt, for *very true*, and st, for *slightly true*, respectively. The logics they define are shown to be algebraizable and to enjoy completeness with respect to the classes of chains of their corresponding varieties. However the axiomatization proposed by Hájek (also used by Vychodil) is quite restrictive, since not any BL-chain expanded with a hedge function is a model of the proposed logic, as one would expect from the traditional use of hedges in fuzzy logic in a wide sense. Moreover, the defined logics are not proved to enjoy general standard completeness, except for the case of logics expanding Gödel logic. One of the main reasons behind both problems is the presence in the axiomatizations of the well-known modal axiom K for the vt connective, which puts quite a lot of constraints on the hedges to be models of these logics without a natural algebraic interpretation.

Next (based on the preliminary paper [34]) we show simple and general axiomatizations with very intuitive properties and nice completeness results based on the abstract logical approach to fuzzy logic (in the sense of semilinear residuated logics) fully described in Chapter II of this handbook.

3.1 The logic L_S of truth-stressing hedges

Let L be a core fuzzy logic, and let L_S be the expansion of L with a new unary connective s (for *stresser*) defined by the following additional axioms:

(VTL1) $\quad s\varphi \to \varphi$
(VTL2) $\quad s\overline{1}$

and the following additional inference rule:

(MON) $\quad$ from $(\varphi \to \psi) \vee \chi$ infer $(s\varphi \to s\psi) \vee \chi$.

If we denote by $\vdash_{\mathrm{L}_S}$ the notion of deduction defined as usual from the above axioms and rules, one can easily show the following:

LEMMA 3.1.1. *The following deductions are valid in* L_S*:*

(i) $\vdash_{\mathrm{L}_S} \neg s\overline{0}$

(ii) $\varphi \to \psi \vdash_{\mathrm{L}_S} s\varphi \to s\psi$

(iii) $\psi \vdash_{\mathrm{L}_S} s\psi$

(iv) $s\varphi, \varphi \to \psi \vdash_{\mathrm{L}_S} s\psi$.

Proof. (i) follows directly from (VTL1) taking $\varphi = \overline{0}$.
(ii) follows directly from (MON) taking $\chi = \overline{0}$.
(iii) follows directly from (ii) taking $\varphi = \overline{1}$ and using (VTL2).
(iv) is easily derivable using (ii) and *modus ponens*. □

Notice that (iv) is a kind of stronger version of *modus ponens:* if φ implies ψ and φ is s-true (for instance "very true"), then one can derive that ψ is s-true (very true) as well. On the other hand, (ii) shows that s satisfies the congruence property (see Section 3.3 in Chapter I). Therefore, the logic L_S is Rasiowa-implicative and its equivalent algebraic semantics is the class of L_S-algebras. An algebra $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, s, \overline{0}, \overline{1}\rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ is an L_S-algebra if it is an L-algebra expanded by a unary operator $s\colon A \to A$ (truth-stressing hedge) that satisfies, for all $x, y, z \in A$,

(1) $s(\overline{1}) = \overline{1}$,

(2) $s(x) \leq x$,

(3) if $(x \to y) \vee z = \overline{1}$ then $(s(x) \to s(y)) \vee z = \overline{1}$.

It is clear that the class of L_S-algebras forms a quasivariety (call it $\mathbb{L}_S$). Notice that if $\langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a totally ordered L-algebra and $s\colon A \to A$ is any non-decreasing mapping such that $s(\overline{1}) = \overline{1}$ and $s(a) \leq a$ for any $a \in A$, then the expanded structure $\langle A, \&, \to, \wedge, \vee, s, \overline{0}, \overline{1}\rangle$ is an L_S-chain. In other words, in L_S-chains the quasiequation (3) turns out to be equivalently expressed by this simplified form: if $x \to y = \overline{1}$ then $s(x) \to s(y) = \overline{1}$, and this condition simply expresses that s is non-decreasing.

Moreover, since the rule (MON) is closed under $\vee$-forms we know that $\vee$ keeps being a disjunction in the expanded logic. On the other hand, since $(\varphi \to \psi) \vee (\psi \to \varphi)$ was already valid in L, we obtain that L_S is also semilinear and hence it is complete with respect to the semantics given by all L_S-chains (see Chapter II, Section 3.2), for the role of disjunction in semilinear logics).

THEOREM 3.1.2. *L_S is strongly complete with respect to the class of all L_S-chains, that is, L_S is $\mathrm{S}\mathbb{K}\mathrm{C}$, with $\mathbb{K}$ being the class of L_S-chains.*

COROLLARY 3.1.3. *The following deductions are valid in L_S:*

(v) $\vdash_{\mathrm{L}_S} s(\varphi \vee \psi) \leftrightarrow s\varphi \vee s\psi$

(vi) $\vdash_{\mathrm{L}_S} s(\varphi \wedge \psi) \leftrightarrow s\varphi \wedge s\psi$.

Proof. Both properties can be easily seen to hold on L_S-chains. □

One might wonder whether one or both corresponding equations for the monotonicity of s (i.e. $s(x \wedge y) = s(x) \wedge s(y)$ and/or $s(x \vee y) = s(x) \vee s(y)$) may substitute the quasiequation (3) in the definition of L_S-algebras. Notice first that over algebras satisfying (1) and (2), the two equations for monotonicity are not equivalent,[14] as the following examples show.

EXAMPLE 3.1.4. Let $\boldsymbol{A}$ be the 5-element Gödel algebra $\{0, a, b, c, 1\}$, where 0 is the bottom, a is an atom, $b \wedge c = a$, $b \vee c = 1$ and 1 is the top element.

- Take s as $s(b) = s(c) = s(a) = 0$. Monotonicity is satisfied for the infimum but not for the supremum since $s(c \vee b) = s(1) = 1$ and $s(c) \vee s(b) = 0 \vee 0 = 0$.

[14] We thank Franco Montagna for pointing out this fact to us.

- Take s as the identity operator except for $s(a) = 0$. This mapping satisfies the monotonicity for the supremum but not for the infimum since $s(c \wedge b) = s(0) = 0$ and $s(c) \wedge s(b) = c \wedge b = a$.

By Corollary 3.1.3, the monotonicity equations are both satisfied in L_S. Hence, the right question is whether the two monotonicity equations may substitute the quasiequation (3). In other words, does the quasivariety $\mathbb{L}_s$ coincide with the variety $\mathbb{V}$ of expansions of L-algebras satisfying the equations (1), (2) and the two monotonicity equations of s? The answer is negative as shown by the following example.

EXAMPLE 3.1.5. Let $\boldsymbol{A}$ be the same Gödel algebra as in Example 3.1.4 and define s as the truth-stresser given by $s(1) = s(b) = 1$ and $s(a) = s(c) = s(0) = 0$. Take the filter $F = \{c, a, b, 1\}$. An easy computation shows that (A, F) is a model of the logic defined by (1), (2) and the monotonicity equations, but the rule (3), even in its simplified form (from $\varphi \to \psi$ deduce $s(\varphi) \to s(\psi)$), is not sound for this model since $a \to b, b \to a \in F$ and $s(b) \to s(a) = 1 \to 0 = 0 \notin F$.

Thus, $\mathbb{V}$ and $\mathbb{L}_s$ coincide over chains but they are different. While L_s is semilinear due to the rule (MON), the logic associated to $\mathbb{V}$ is not. This also shows that in the presentation of L_S, (MON) cannot be substituted by the simpler rule: from $\varphi \to \psi$ infer $s\varphi \to s\psi$ (which, as we have just seen, is sound in L_S-chains but not for all L_S-algebras).

Similarly, inspired by the well-known presentation of logics with $\triangle$, one might also ask whether (MON) could be substituted by the globalization rule: from φ infer $s\varphi$. The answer is again negative.

EXAMPLE 3.1.6. Let $\boldsymbol{C}$ be the finite MTL-chain defined over $C = \{0, 1, 2, 3, 4, 5\}$ with the natural order and the following monoidal operation:

&	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	1	1	2	2
3	0	1	1	1	2	3
4	0	1	2	2	4	4
5	0	1	2	3	4	5

Take the MTL-filter $F = \{4, 5\}$ and the following unary operation s:

x	0	1	2	3	4	5
$s(x)$	0	1	1	3	4	5

It is clear that s is subdiagonal, maps the top element to itself and is non-decreasing. Moreover, for every $x \in F$, $s(x) \in F$, i.e. it is sound w.r.t. the globalization rule. However, it is not sound w.r.t. (MON): indeed, $3 \to 2 = 4 \in F$, while $s(3) \to s(2) = 3 \to 1 = 3 \notin F$.

We consider now the issue of completeness of L_S with respect to distinguished semantics of L_S-chains. One can prove that if L has the finite strong real completeness property (FS$\mathcal{R}$C), then L_S has it as well. As usual, this can be done by showing that any L_S-chain is partially embeddable into a standard L_S-chain.

THEOREM 3.1.7 (Finite strong real completeness). *If* L *is a finite strong real complete (*FS$\mathcal{R}$C*) core fuzzy logic, then the logic* L_S *is finite strong real complete as well.*

Proof. Assume that L has the FS$\mathcal{R}$C. Take any L_S-chain $\boldsymbol{A} = \langle A, \&, \rightarrow, \wedge, \vee, s, \overline{0}, \overline{1}\rangle$, and let B be a finite partial subalgebra of $\boldsymbol{A}$. We have to show that there exist a real L_S-chain $\langle [0,1], \wedge, \vee, *, \Rightarrow, s', 0, 1\rangle$ and a mapping $f\colon B \rightarrow [0,1]$ preserving the existing operations. By assumption, the s-free reduct of $\boldsymbol{A}$ is partially embeddable into a real L-chain $\langle [0,1], \wedge, \vee, *, \Rightarrow, 0, 1\rangle$. Denote this embedding by f, and consider any non-decreasing and subdiagonal function $s'\colon [0,1] \rightarrow [0,1]$ satisfying $s'(f(x)) = f(s(x))$ for every $x \in B$ such that $s(x) \in B$. There are obviously many such functions s' interpolating the set of points $P = \{\langle f(x), f(s(x))\rangle \mid x, s(x) \in B\}$ (a linear interpolant, for instance). Another interpolant can be defined as follows: let $0 = z_1 < \cdots < z_n < 1$ be the set of elements of $[0,1]$ such that $\langle z_i, \cdot\rangle \in P$ and define $s'(1) = 1$ and, for all $z \in [0,1)$,

$$s'(z) = f(s(x_i)), \text{ if } z_i \leq z < z_{i+1}$$

where $x_i \in B$ is such that $z_i = f(x_i)$. In any case, s' makes $\langle [0,1], \wedge, \vee, *, \Rightarrow, s', 0, 1\rangle$ an L_S-chain and f a partial embedding of L_S-chains. □

Actually, this theorem can be generalized to arbitrary classes of L-chains and their s-expansions, proved in a completely analogous way, and yielding a more general result.

COROLLARY 3.1.8. *Let* L *be a core fuzzy logic,* $\mathbb{K}$ *a class of* L*-chains, and* $\mathbb{K}_S$ *the class of the* L_S*-chains whose* s*-reducts are in* $\mathbb{K}$*. If* L *has the* FS$\mathbb{K}$C*, then* L_S *has the* FS$\mathbb{K}_\mathrm{S}$C *as well.*

THEOREM 3.1.9 (Strong real completeness). *If* L *is a strong real complete (*S$\mathcal{R}$C*) core fuzzy logic, then the logic* L_S *is strong real complete as well.*

Proof. Let L have the S$\mathcal{R}$C. We have to show that any countable L_S-chain can be embedded into a standard L_S-chain. Let $\boldsymbol{A}$ be a countable L_S-chain. By the assumption, the s-free reduct of $\boldsymbol{A}$ is embeddable into a standard L-chain $\boldsymbol{B} = \langle [0,1], \wedge, \vee, *, \Rightarrow, 0, 1\rangle$. Denote this embedding by f and define $s'\colon B \rightarrow B$ in the following way: for each $z \in [0,1]$, $s'(z) = \sup\{f(s(x)) \mid x \in A, f(x) \leq z\}$. So defined, s' is a non-decreasing and subdiagonal function such that $s'(f(x)) = f(s(x))$ for any $x \in A$. Therefore, $\boldsymbol{B}$ expanded with s' is a standard L_S-chain where $\boldsymbol{A}$ can be embedded. □

Observe that the proof of the previous theorem can be repeated whenever the linear order of the chains is complete. Therefore we obtain the following corollary.

COROLLARY 3.1.10. *Let* L *be a core fuzzy logic,* $\mathbb{K}$ *a class of completely ordered* L*-chains, and* $\mathbb{K}_S$ *the class of the* L_S*-chains whose* s*-reducts are in* $\mathbb{K}$*. If* L *has the* S$\mathbb{K}$C*, then* L_S *has the* S$\mathbb{K}_S$C.

3.2 On the logics L_S and their associated quasivarieties

In the previous subsection, we have seen that if L is a core fuzzy logic, then L_S is a semilinear logic (complete with respect to chains of the associated quasivariety). However, this does not imply that it has either a global or local deduction-detachment theorem (denoted, from now on, GDDT and LDDT respectively). In this subsection, we present two families of logics L_S that enjoy the GDDT and one family enjoying the LDDT. Moreover, we prove that the quasivarieties associated to these families of logics are, in fact, varieties.

3.2.1 The case of L being the logic of a finite BL-chain

The first family we consider is that of the logics L_S where L is the logic of a finite BL-chain $\boldsymbol{A}$ having n elements, i.e. $\boldsymbol{A}$ is an ordinal sum of copies of finite MV-chains ($\boldsymbol{\L}_k$) and finite Gödel chains ($\boldsymbol{G}_r$).

PROPOSITION 3.2.1. *If* L *is the logic of a finite* BL*-chain* $\boldsymbol{A}$*, then:*

(1) *The chains of the variety generated by* $\boldsymbol{A}$ *are the subalgebras of* $\boldsymbol{A}$*.*

(2) *Given a* BL*-filter* F *of* $\boldsymbol{A}$ *, the congruence defined by it,* $\equiv_F$*, is defined by:* $x \equiv_F y$ *iff either* $x = y$ *or* $x, y \in F$*, i.e. the congruence classes are* F *and the singletons* $\{x\}$ *for any* $x \notin F$*.*

(3) *The set of* L_S*-filters of* $\boldsymbol{A}$ *coincides with the set of* L*-filters that are closed under* s*.*

Proof. The first claim is a consequence of [24, Theorem 1], taking into account that every finite BL-chain is subdirectly irreducible and the fact that any chain belonging to the variety generated by a finite Gödel or MV-chain is a subalgebra of it.

The proof of the second claim is easy since if $x \geq y$, then $x \equiv_F y$ iff $x \to y \in F$. The filters of $\boldsymbol{A}$ are the principal filters defined by an element a that either belongs to a Gödel component or is the bottom of an MV component. Thus, an easy computation shows that $x \to y \in F$ iff either $x = y$ or $x, y \in F$.

In order to prove the third claim observe first that, if F is an L_S-filter of $\boldsymbol{A}$, then, it is closed under s, since if $\bar{a} \in F$, then $1 \to \bar{a} \in F$, and thus $1 \to s(\bar{a}) = s(\bar{a}) \in F$. On the other hand, suppose that F is a BL-filter closed under s. Then F is a L_S-filter. Remember that if F is a BL-filter over a finite BL-chain then $a \equiv_F b$ iff $a = b$ or $a, b \in F$. Therefore, if F is closed under s, then $s(a) \equiv_F s(b)$. ☐

LEMMA 3.2.2. *Let* L *be the logic of a finite* BL*-chain* $\boldsymbol{L}$*, and let* L_S *be the expansion of* L *with a truth-stressing hedge as defined in Section 3.1. Then, in any* L_S*-algebra* $\boldsymbol{A}$*, the* L_S*-filter* $F(\bar{a})$ *generated by an element* $\bar{a} \in \boldsymbol{A}$ *is principal, i.e. there is an element* $t(\bar{a})$ *such that* $F(\bar{a}) = [t(\bar{a}), \bar{1}] \cap A$*.*

Proof. If $\boldsymbol{A}$ is an L_S-algebra, then $\boldsymbol{A}$ can be embedded into a direct product $\prod_{i\in I} \boldsymbol{L}$ (remember that any L_S-chain is a subalgebra of $\boldsymbol{L}$, and suppose that $\boldsymbol{L}$ has n elements, k components and m is the maximum length of an MV component). Given an element $\bar{a} \in A$, take the element $t(\bar{a}) = (s^n(\overset{k}{\ldots} s^n(\bar{a}^m))^m \ldots)^m$. An easy computation shows that $t(\bar{a})$ is idempotent and it is a fixed point by s. Then, we will prove that $F(\bar{a})$ is the principal filter defined by $t(\bar{a})$. The proof follows from the following facts:

(i) $t(\bar{a}) \in F(\bar{a})$,

(ii) if $t(\bar{a})_i$ is the i-projection of $t(\bar{a})$, then $F(t(\bar{a})_i) = \{x \in L \mid x \geq t(\bar{a})_i\}$ is the filter of L generated by $t(\bar{a})_i$, and

(iii) $F(t(\bar{a})) = A \cap \prod_{i \in I} F(t(\bar{a})_i)$, by definition. □

THEOREM 3.2.3. *Let* L *be the logic of a finite* BL*-chain* $\boldsymbol{L}$ *(with* n *elements,* k *components and with* m *being the maximum length of an* MV *component), and let the logic* L_S *be its expansion with a truth-stressing hedge as defined in Section 3.1. Then, the logic* L_S *enjoys the GDDT, i.e. given a set* $\Gamma \cup \{\varphi, \psi\}$ *of formulas, there is a formula* $t(\varphi) = (s^n(\overset{k}{\ldots} s^n(\varphi^m))^m \ldots)^m$ *such that,*

$$\Gamma, \varphi \vdash_{\mathrm{L}_S} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathrm{L}_S} t(\varphi) \to \psi.$$

Proof. The right-to-left direction follows easily from the observation that $\varphi \vdash_{\mathrm{L}_S} t(\varphi)$. Let us prove the other direction by reasoning semantically, using completeness: i.e., we assume $\Gamma, \varphi \models_{\mathbb{L}_s} \psi$, and we show $\Gamma \models_{\mathbb{L}_s} t(\varphi) \to \psi$. Take any L_S-algebra $\boldsymbol{A}$ and any $\boldsymbol{A}$-evaluation e such that $e[\Gamma] \subseteq \{\bar{1}^A\}$. Consider the matrix L_S-model $\langle \boldsymbol{A}, Fe[\Gamma], e(\varphi)\rangle$. By soundness $e(\psi) \in F(e[\Gamma], e(\varphi))$, i.e. $e(\psi) \in F(e(\varphi)) = [t(e(\varphi)), \bar{1}^A]$. Then, $t(e(\varphi)) \leq e(\psi)$, and so $e(t(\varphi) \to \psi) = \bar{1}^{\boldsymbol{A}}$. □

From GDDT the following result is obvious.

COROLLARY 3.2.4. *The quasivariety associated to the logic of a finite* BL*-chain is a variety.*

Some remarks are in order here:

- The results in this section are valid for any logic of a finite MTL-chain with the condition that L_S-filters on L_S-chains coincide with MTL-filters closed under s.
- A sufficient condition for an MTL-filter on an L_S-chain closed under s to be an L_S-filter is the fact that $a \equiv_F b$ iff either $a = b$ or $a, b \in F$. For example, any finite WNM-chain $\boldsymbol{L}$ (with n elements) satisfies this condition, and so the logic L_S enjoys the GDDT (with the formula $t(\varphi) = (s^n(\varphi))^2$), and hence the quasivariety corresponding to the logic of a finite WNM-chain with a truth-stresser is a variety.
- The following example proves that there are finite MTL-chains with MTL-filters closed under s that are not L_S filters.

EXAMPLE 3.2.5. Take a 6-element chain $\boldsymbol{A}$ such that $(1 > a > b > c > d > 0)$, and define the operation $*$ by (assuming that $*$ is determined when one value is 0 or 1) $a * a = a$, and $x * y = d$ otherwise. Then the MTL-filters are $\{1\}, \{1, a\}, \{1, a, b, c, d\}$ and A itself. Define the operator s by (the values of 0 and 1 are determined) $s(a) = a, s(b) = b, s(c) = s(d) = 0$. It is obvious that the MTL-filters closed under s are $\{1\}, \{1, a\}$ and A. But $\{1, a\}$ is not an L_S-filter since $b \to c = a$ and $s(b) \to s(c) = b \to 0 = 0 \notin \{1, a\}$.

3.2.2 The case of logics L_S where the operator $\triangle$ is definable

The second family we consider is the family of logics L_S where the Monteiro–Baaz $\triangle$ operator is definable. In such a case it is obvious that the $\triangle$ detachment-deduction theorem (that is global) is valid. Then, having L_S a GDDT, by a general result of algebraic logic, the quasivariety of L_S-algebras enjoys the congruence extension property, and, consequently, the class of L_S-algebras forms a variety.

Indeed, if $\triangle$ is definable in L_S, then the (MON) inference rule in L_s can equivalently be replaced by the axiom

$$(\mathrm{MON}_\triangle) \quad \triangle(\varphi \to \psi) \to (s\varphi \to s\psi)$$

and so the quasivariety $\mathbb{L}_S$ is in fact defined by a family of equations and thus it is a variety.

Core fuzzy logics L where $\triangle$ is definable include e.g. the n-valued Łukasiewicz logic $\mathbf{Ł}_n$ or the axiomatic extensions of MTL by the axiom $\neg(\varphi)^n \vee \varphi$, called $\mathrm{S}_n\mathrm{MTL}$. In these cases, $\triangle\varphi$ is defined as φ^n. In both cases, we have a sequence of nested logics, $\mathrm{Boolean} = Ł_2 \subset Ł_3 \subset \cdots \subset Ł_n \subset \cdots$ and $\mathrm{Boolean} = \mathrm{S}_2\mathrm{MTL} \subset \mathrm{S}_3\mathrm{MTL} \subset \cdots \subset \mathrm{S}_n\mathrm{MTL} \subset \cdots$ respectively. On the other hand, given a core fuzzy logic L, one can also consider the family of axiomatic extensions of L_S with the axiom $\neg(s^n(\overset{n}{\ldots}(s^n(\varphi^n))^n \ldots))^n \vee \varphi$, where $\triangle$ is also definable. Of course, these logics, denoted $\mathrm{S_nL_S}$, are parameterized by n, and, hence, we obtain again a sequence of nested logics $\mathrm{S_2L_S} \subset \mathrm{S_3L_S} \subset \cdots \subset \mathrm{S_nL_S} \subset \cdots$. In all these logics, $\triangle$ is definable by $\triangle\varphi := (s^n(\overset{n}{\ldots}(s^n(\varphi^n))^n \ldots))^n$.

3.2.3 The case of L_S satisfying the modal axiom K

The third family we consider consists of the logics L_{SK} defined over any core fuzzy logic L (as Hájek did in [47] over any axiomatic extension of BL) by adding a unary (truth-stressing) connective s satisfying the axioms,

$$(\mathrm{VE1}) \quad s\varphi \to \varphi$$
$$(\mathrm{VE2}) \quad s(\varphi \to \psi) \to (s\varphi \to s\psi)$$
$$(\mathrm{VE3}) \quad s(\varphi \vee \psi) \to (s\varphi \vee s\psi)$$

with *modus ponens* and necessitation for s (from φ derive $s(\varphi)$) as inference rules.

Axiom $(\mathrm{VE3})$ is a formula that is derivable in the logic L_S. Axiom $(\mathrm{VE2})$ is the well-known axiom K of modal logics for the truth-stresser s. In our setting it means that if both φ and $\varphi \to \psi$ are "very true" then so is ψ. Moreover, it also implies that the interpretation of s over an MTL-chain is a non-decreasing mapping, as it is in our general system studied in this chapter. However, axiom $(\mathrm{VE2})$ is not always sound in our general framework, i.e. the L_S logic. Take for example the L_S-chain defined over the standard MV-chain $[0,1]_Ł$ by an operator s such that it is non-decreasing, $s(0) = 0$, $s(1) = 1$, $s(x) \leq x$ (it is a truth-stressing hedge), and suppose there are two elements $a, b \in [0,1]_Ł$ such that $a > b$ and $s(a) < a$ and $s(b) = b$. Then, $s(a) \to s(b) = 1 - s(a) + s(b) > 1 - a + b = a \to b \geq s(a \to b)$ in contradiction with $(\mathrm{VE2})$.

Next, we prove that L_{SK} is an axiomatic extension of L_S.

LEMMA 3.2.6.

(1) *The following formulas are provable in* L_{SK}*:*

(i) $\neg s\bar{0}$

(ii) $(s\varphi \,\&\, s\psi) \to s(\varphi \,\&\, \psi)$

(iii) $s(\varphi \vee \psi) \leftrightarrow (s\varphi \vee s\psi)$.

(2) *The rule of inference* (MON) *is derivable in* L_{SK}*:*

$$\textit{from } (\varphi \to \psi) \vee \chi \textit{ infer } (s(\varphi) \to s(\psi)) \vee \chi.$$

Proof. By (VE1) $\vdash_{L_{SK}} s\bar{0} \to \bar{0}$ and so (i) is proved. From $\vdash_{L_{SK}} \varphi \to (\psi \to (\varphi \& \psi))$, applying necessitation and (VE2), we obtain $\vdash_{L_{SK}} s\varphi \to (s\psi \to s(\varphi \& \psi))$. Therefore, (ii) is proved as well. Clearly $\vdash_{L_{SK}} s\varphi \to (s\varphi \vee s\psi)$ and $\vdash_{L_{SK}} s\psi \to (s\varphi \vee s\psi)$, then $\vdash_{L_{SK}} (s\varphi \vee s\psi) \to (s\varphi \vee s\psi)$, and, taking into account (VE3), (iii) is proved.

Finally from $(\varphi \to \psi) \vee \chi$, using necessitation and (ii) of this lemma, we infer $s(\varphi \to \psi) \vee s\chi$, and, by (VE1), $s(\varphi \to \psi) \vee \chi$ and, by (VE3), we infer $(s\varphi \to s\psi) \vee \chi$. Consequently, (2) is also proved. □

COROLLARY 3.2.7. L_{SK} *is the axiomatic extension of* L_S *by adding the axiom* (VE2).

Now, following Hájek in [47], we prove a deduction-like theorem (similar to the one proved for $\triangle$). We will need an auxiliary notation: $\tau\varphi$ stands for $s(\varphi \,\&\, \varphi)$ and $\tau^n\varphi$ stands for $\tau(\ldots\tau(\tau\varphi)\ldots)$.

LEMMA 3.2.8. *In* L_{SK} *the following formulas are provable:*

(i) $\tau^{n+1}\varphi \to \tau^n\varphi$,

(ii) $\tau\varphi \to s\varphi$, $\tau\varphi \to \varphi \,\&\, \varphi$,

(iii) $\tau(\varphi \vee \psi) \leftrightarrow (\tau\varphi \vee \tau\psi)$.

THEOREM 3.2.9 (LDDT). *Let T be a theory and let* φ, ψ *be formulas. Then:* $T \cup \{\varphi\} \vdash_{L_{SK}} \psi$*, iff, for some n,* $T \vdash_{L_{SK}} \tau^n\varphi \to \psi$.

Proof. As usual, let us check the deduction rules. If $T \vdash_{L_{SK}} \tau^n\varphi \to \alpha$ and also $T \vdash_{L_{SK}} \tau^n\varphi \to (\alpha \to \beta)$, then $T \vdash_{L_{SK}} (\tau^n\varphi \& \tau^n\varphi) \to \beta$, thus $T \vdash_{L_{SK}} \tau^{n+1}\varphi \to \beta$. Similarly, if $T \vdash_{L_{SK}} \tau^n\varphi \to \beta$, then $T \vdash_{L_{SK}} s(\tau^n\varphi) \to s\beta$, thus $T \vdash_{L_{SK}} \tau^{n+1}\varphi \to s\beta$. □

The corresponding algebraic structures are the L_{SK}-algebras. An algebra $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, s, 0, 1\rangle$ is an L_{SK}-algebra if it is an L-algebra expanded with a unary operator s (truth-stressing hedge) that satisfies, for all $x, y \in A$,

(ve1) $s(x) \leq x$
(ve2) $s(x \to y) \leq (s(x) \to s(y))$
(ve3) $s(x \vee y) \leq (s(x) \vee s(y))$
(ve4) $s(1) = 1$.

From the above remarks, an L_{SK}-algebra is just an L_S-algebra also satisfying the property $(\mathrm{ve2})$. In this case, it is obvious that L_{SK}-algebras form a variety (recall that, like in the expansion with $\triangle$, the inference rules of the logic are MP and necessitation). On the other hand, as usual, for each left-continuous t-norm $*$, the chain obtained by adding to $[0,1]_*$ a truth-stressing hedge s satisfying the above properties is an L_{SK}-chain called a real chain.

Next, we give some examples of truth-stressers on real chains $[0,1]_*$ satisfying axiom $(\mathrm{VE2})$. We will call them K-truth-stressers.

EXAMPLE 3.2.10. (1) *The function $s(x) = x * \overset{n}{\ldots} * x$ (x^n for short) is a K-truth-stressing function over $[0,1]_*$ for any left-continuous t-norm $*$. Obviously, this truth-stressing function is continuous if so is the t-norm, and it is the identity if the t-norm corresponds to the minimum.*

(2) *The function $s(x) = x \cdot x$ (product of reals) is also a K-truth-stressing function for the three basic continuous t-norms. Observe that this function coincides with the one of the previous example whenever $*$ is the product t-norm and $n = 2$.*

(3) *The function defined by the Łukasiewicz t-norm as $s(x) = x*x = \max\{0, 2x-1\}$ is a K-truth-stressing function for Łukasiewicz and minimum t-norms but not for the product. This function coincides with the first example for the Łukasiewicz t-norm and $n = 2$.*

(4) *For any $k \in [0,1]$, the function $s(x) = k \cdot x$ for $x < 1$ and $s(1) = 1$ is a K-truth-stressing function for the three basic continuous t-norms. Observe that when $k = 0$, this is the $\triangle$ operator.*

Since it is an axiomatic extension of L_S, the logic L_{SK} is semilinear and so it is complete with respect to the quasivariety of L_{SK}-algebras and with respect to the class of L_{SK}-chains. The problem of standard completeness for the logics L_{SK} is far from being solved. When L is the logic of a Gödel chain (for continuous t-norms) or a WNM-chain (for the general MTL-chains) the problem is easy, since we have the following result.

PROPOSITION 3.2.11. *Let* L *be the logic of a given* WNM*-chain.*[15] *Then the* L_{SK} *logic coincides with the logic* L_S.

Proof. It is only necessary to prove that axiom $(\mathrm{VE2})$ is valid over each L_S-chain. This is easy, because, if $a \leq b$, then $s(a \to b) = 1 = s(a) \to s(b)$, and, if $a > b$, then either $s(a) = s(b)$ and then $s(a \to b) \leq s(a) \to s(b) = 1$, or $s(a \to b) = s(\neg a \vee b) = s(\neg a) \vee s(b) \leq \neg s(a) \vee s(b) = s(a) \to s(b)$ (take into account that $s(\neg a) \leq \neg a \leq \neg s(a)$). □

Applying Corollary 3.1.8 we obtain the following result.

COROLLARY 3.2.12. *Let* L *be the logic of a given* WNM*-chain. Then* L_{SK} *is (finite) strong real complete whenever* L *is (finite) strong real complete.*

The only logic of a continuous t-norm that satisfies $(\mathrm{VE2})$ is Gödel logic, and thus G_{SK} is strong real complete. For the rest of logics L of continuous t-norms, the problem of real completeness, both for the general case L_{SK} and for $Ł_{SK}$ or Π_{SK}, is still open.

[15] Notice that a Gödel chain is a particular case of a WNM-chain.

3.3 The case of truth-depressers

Similar to the case of truth-stressers, we can proceed to define an axiomatization for truth-depressers just by replacing axioms (VTL1) and (VTL2) with their dual versions (STL1) and (STL2) (for *slightly true*). Namely, given a core fuzzy logic L, we define L_D as the expansion of L with a new unary connective d (for *depresser*), the following additional axioms

(STL1) $\varphi \to d\varphi$
(STL2) $\neg d\overline{0}$

and the following additional inference rule

(MON) from $(\varphi \to \psi) \vee \chi$ infer $(d\varphi \to d\psi) \vee \chi$.

Since L_D is a kind of dual version of L_S, many properties are proved in a completely analogous way.

LEMMA 3.3.1. *The following deductions are valid in* L_D*:*

(i) $\vdash_{\mathrm{L}_D} d\overline{1}$

(ii) $\varphi \to \psi \vdash_{\mathrm{L}_D} d\varphi \to d\psi$

(iii) $\neg\varphi \vdash_{\mathrm{L}_D} \neg d\varphi$

(iv) $\vdash_{\mathrm{L}_D} \neg d\varphi \to \neg\varphi$

(v) $d\varphi, \varphi \to \psi \vdash_{\mathrm{L}_D} d\psi$.

Proof. (i) follows directly from (STL1) taking $\varphi = \overline{0}$.
(ii) follows directly from (MON) taking $\chi = \overline{0}$.
(iii) follows from (ii) for $\psi = \overline{0}$ and (STL2).
(iv) follows directly from (STL1) using the fact that $\vdash_{\mathrm{MTL}} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$.
(v) is very easy using (ii) and *modus ponens*. □

Notice that (v) is a kind of weaker or modified version of *modus ponens:* if φ implies ψ and φ is slightly true, then one can derive that ψ is slightly true as well.

Again, (ii) shows that the congruence condition is satisfied for the new unary connective too. Therefore, the logic L_D is Rasiowa-implicative (see Chapter II, Section 2), and its equivalent algebraic semantics is the class of L_D-algebras. An algebra $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, d, \overline{0}, \overline{1}\rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ is an L_D-algebra if it is an L-algebra expanded with a unary operator $d\colon A \to A$ (truth-depressing hedge) that satisfies, for all $x, y, z \in A$,

$(1')$ $d(0) = 0$,

$(2')$ $x \leq d(x)$,

$(3')$ if $(x \to y) \vee z = \overline{1}$ then $(d(x) \to d(y)) \vee z = \overline{1}$.

Also, since the lattice disjunction still satisfies the (PCP) in the expanded logic, L_D is semilinear and hence complete with respect to the semantics of all L_D-chains. As a straightforward consequence, we have:

LEMMA 3.3.2. *The following deductions are valid in* L_D*:*

(vi) $\vdash_{\mathrm{L}_D} d(\varphi \vee \psi) \leftrightarrow d\varphi \vee d\psi$

(vii) $\vdash_{\mathrm{L}_D} d(\varphi \wedge \psi) \leftrightarrow d\varphi \wedge d\psi.$

Taking $s(a,b) = \langle a \vee b, a \vee b \rangle$, Example 3.1.5 shows that in the context of truth-depressers the rule (MON) cannot be substituted by simple monotonicity. Similarly, Example 3.1.6 can be modified by taking the function d:

x	0	1	2	3	4	5
$d(x)$	0	1	2	4	4	5

showing that in the presentation of L_D the rule (MON) cannot be substituted by the following rule: from $\neg\varphi$ infer $\neg d\varphi$. Indeed, d is superdiagonal, maps the bottom element to itself, is non-decreasing, and satisfies $\neg d(x) \in F$ whenever $\neg x \in F$. However, $3 \to 2 = 4 \in F$, while $d(3) \to d(2) = 4 \to 2 = 3 \notin F$.

Finally, analogous proofs makes it possible to prove this theorem about preservation of completeness properties:

THEOREM 3.3.3 ((Real) completeness properties). *Let* L *be a core fuzzy logic,* $\mathbb{K}$ *a class of* L*-chains and* $\mathbb{K}_D$ *the class of* L_D*-chains whose* d*-free reducts are in* $\mathbb{K}$*. Then:*

(i) *If* L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$*, then* L_D *has the* $\mathrm{FS}\mathbb{K}_D\mathrm{C}$.

(ii) *If* L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$ *and all the chains in* $\mathbb{K}$ *are completely ordered, then* L_D *has the* $\mathrm{S}\mathbb{K}_D\mathrm{C}$.

4 Expansions with an involutive negation

In all the t-norm based fuzzy logics studied in the previous chapters, the negation connective $\neg$ is defined from the implication $\to$ and the truth constant $\bar{0}$, namely $\neg\varphi$ is $\varphi \to \bar{0}$. However, this negation may behave quite differently in different varieties of algebras. Indeed, for instance, the associated negation function is involutive in any IMTL chain (in particular in algebras associated to Łukasiewicz logic), but it may not be involutive outside the variety of IMTL-algebras. The most paradigmatic cases are the chains of the variety of SMTL-algebras, where $\neg$ is interpreted as the so-called Gödel's negation n_G, defined by:

$$n_G(x) = x \Rightarrow 0 = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this section, we will define and study the expansion of any axiomatic extension of $\mathrm{MTL}_\triangle$ with an independent involutive negation. Some particularly interesting cases are those of $\mathrm{SMTL}_{\sim}$ and its axiomatic extensions $\mathrm{G}_{\sim}$ and $\Pi_{\sim}$, where $\triangle$ is definable as a composition of the two negations (residuated and involutive) defined there.

Notice that having an involutive negation in the logic enriches, in a non-trivial way, the expressive power of the logical language. For instance, in the enriched language:

- a strong disjunction $\varphi \veebar \psi$ is definable as $\sim(\sim\varphi \ \& \sim\psi)$, thus having a truth function in real algebras $[0,1]_*$ defined by the *dual t-conorm* $\oplus$ given by $x \oplus y = n(n(x) * n(y))$;
- a contrapositive implication $\varphi \hookrightarrow \psi$ is definable as $\sim\varphi \veebar \psi$, thus having a truth function corresponding to the *strong implication* function $\stackrel{c}{\Rightarrow}$ defined as $x \stackrel{c}{\Rightarrow} y = \sim x \oplus y$.

Although these new connectives are interesting for future developments and are already present in early fuzzy logic papers (see for example, [3, 83, 88]), we shall make no further use of them in the rest of the chapter.

4.1 Expanding a $\triangle$-core fuzzy logic with an involutive negation

Following [35], we define the expansion of a $\triangle$-core fuzzy logic with an involutive negation as follows.

DEFINITION 4.1.1. *Let* L *be a* $\triangle$*-core fuzzy logic. Then the logic* $\mathrm{L}_{\sim}$ *is the axiomatic expansion of* L *obtained adding a new unary connective* $\sim$ *satisfying the following two additional axioms:*

(~ 1)	$(\sim\sim\varphi) \leftrightarrow \varphi$	*(Involution)*
(~ 2)	$\triangle(\varphi \to \psi) \to (\sim\psi \to \sim\varphi)$	*(Order Reversing)*

LEMMA 4.1.2. *In* $\mathrm{L}_{\sim}$ *the following inference rule is derivable and the following formulas are provable:*

(AMON)	*from* $\varphi \to \psi$ *infer* $\sim\psi \to \sim\varphi$	*(Antimonotonicity)*
(DM1)	$\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$	*(De Morgan law for* $\wedge$*)*
(DM2)	$\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$	*(De Morgan law for* $\vee$*)*

Proof. As for the inference rule (AMON), from $\varphi \to \psi$, by the necessitation rule for $\triangle$, $\mathrm{L}_{\sim}$ proves $\triangle(\varphi \to \psi)$, and, by axiom (~ 2), it also proves $\sim\psi \to \sim\varphi$. Let us prove (DM1). Clearly, L proves $\varphi \wedge \psi \to \varphi$ and $\varphi \wedge \psi \to \psi$. By (AMON), $\mathrm{L}_{\sim}$ proves $\sim\varphi \to \sim(\varphi \wedge \psi)$ and $\sim\psi \to \sim(\varphi \wedge \psi)$, and thus it proves $(\sim\varphi \vee \sim\psi) \to \sim(\varphi \wedge \psi)$ as well. Analogously, we can prove $\sim(\varphi \vee \psi) \to (\sim\varphi \wedge \sim\psi)$. Substituting φ and ψ by $\sim\varphi$ and $\sim\psi$ in the last formula we have $\sim(\sim\varphi \vee \sim\psi) \to \varphi \wedge \psi$, and by (AMON) we infer $\sim(\varphi \wedge \psi) \to (\sim\varphi \vee \sim\psi)$. This ends the proof of (DM1). The proof for (DM2) is analogous. □

It is very easy to show that $\mathrm{L}_{\sim}$ is itself a $\triangle$-core fuzzy logic.

THEOREM 4.1.3. *Let* L *be a* $\triangle$*-core fuzzy logic. Then* $\mathrm{L}_{\sim}$ *is itself a* $\triangle$*-core fuzzy logic, that is, the following conditions are satisfied:*

(1) $\mathrm{L}_\sim$ *has the congruence property for* $\sim$*, i.e. for any formulas* φ, ψ *of* $\mathrm{L}_\sim$ *it holds:*

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}_\sim} \sim\varphi \leftrightarrow \sim\psi.$$

(2) $\mathrm{L}_\sim$ *satisfies the* $\triangle$*-deduction theorem, i.e. for each theory* T *over* $\mathrm{L}_\sim$ *and formula* φ *of* $\mathrm{L}_\sim$*, it holds:*

$$T \cup \{\varphi\} \vdash_{\mathrm{L}_\sim} \psi \text{ iff } T \vdash_{\mathrm{L}_\sim} \triangle\varphi \to \psi.$$

Proof. The congruence property is an immediate consequence of the (AMON) inference rule. Also the $\triangle$-deduction theorem for $\mathrm{L}_\sim$ is a direct consequence of the fact that $\mathrm{L}_\sim$ is in fact an axiomatic expansion of L, i.e. no new inference rules are added. □

The corresponding algebraic semantics for $\mathrm{L}_\sim$ is given by the class of $\mathrm{L}_\sim$-algebras, defined in the natural way.

DEFINITION 4.1.4. *Let* L *be a* $\triangle$*-core fuzzy logic. An* $\mathrm{L}_\sim$*-algebra is an* L*-algebra expanded with a unary operation* $\sim$ *satisfying the following conditions:*

$(A_\sim 1)$ $\sim\sim x = x$,

$(A_\sim 2)$ *if* $x \leq y$*, then* $\sim y \leq \sim x$.

Since $\mathrm{L}_\sim$ is a $\triangle$-core fuzzy logic, then we know that the class of $\mathrm{L}_\sim$-algebras is in fact a variety (see e.g. Section 3.2 in Chapter I), since $(A_\sim\ 2)$ can be equivalently expressed as an equation using $\triangle$. Moreover, we get also for free that $\mathrm{L}_\sim$-algebras are representable as subdirect products of $\mathrm{L}_\sim$-chains and that $\mathrm{L}_\sim$ is strongly complete with respect to the class of $\mathrm{L}_\sim$-chains.

Since, by definition, the $\sim$-free reducts of $\mathrm{L}_\sim$-chains are L-chains, chain completeness readily yields that for each $\triangle$-core fuzzy logic L, $\mathrm{L}_\sim$ is a conservative expansion of L.

PROPOSITION 4.1.5. *Let* L *be any* $\triangle$*-core fuzzy logic. Then the logic* $\mathrm{L}_\sim$ *is a conservative expansion of* L.

As usual, the $\mathrm{L}_\sim$-chains over the unit real interval, that will be called *real* $\mathrm{L}_\sim$-chains, are especially interesting. If $\boldsymbol{A}$ is a real $\mathrm{L}_\sim$-chain, then it is the expansion of its L-reduct with a strong negation function $n\colon [0,1] \to [0,1]$, that is a strictly decreasing function n such that $n(0) = 1$ and such that $n(n(x)) = x$ for all $x \in [0,1]$.

REMARK 4.1.6. *It is well known that all strong negation functions on* $[0,1]$ *are isomorphic to each other (see [81]), that is, if* n *and* n' *are strong negation functions, there is a strictly increasing mapping* $h\colon [0,1] \to [0,1]$*, with* $h(0) = 0$ *and* $h(1) = 1$*, such that* $n'(x) = h^{-1}(n(h(x)))$ *for all* $x \in [0,1]$*. In particular, all strong negation functions are isomorphic to the so-called* standard negation*, defined as* $n_s(x) = 1 - x$*. Accordingly, real* $\mathrm{L}_\sim$*-chains having* n_s *as involutive negation will be called* standard $\mathrm{L}_\sim$*-chains. Notice that if* $\boldsymbol{A}$ *is a real* $\mathrm{L}_\sim$*-chain, there always exists a standard* $\mathrm{L}_\sim$*-chain* $\boldsymbol{A}'$ *which is isomorphic to* $\boldsymbol{A}$*. Indeed, if* h *is the mapping such that* $n_s = h^{-1} \circ n \circ h$*, where* n *is the involutive negation in* $\boldsymbol{A}$*, then the operations of* $\boldsymbol{A}'$ *are obtained applying the same* *transformation, i.e., for instance if $\star$ is a binary operation in $\boldsymbol{A}$, the corresponding operation in $\boldsymbol{A}'$ is defined as $\star' = h^1 \circ \star \circ (h \times h)$. Therefore, when talking later about different kinds of completeness properties of logics $\mathrm{L}_\sim$ with respect to the whole class of real $\mathrm{L}_\sim$-chains, like $\mathrm{S}\mathcal{R}\mathrm{C}$ or $\mathrm{FS}\mathcal{R}\mathrm{C}$, we can always restrict ourselves to the subclass of real chains with the standard negation.*

Some comments are in order here. In [28] the authors give an axiomatization of $\mathrm{SBL}_\sim$, and of their main axiomatic extensions $\mathrm{G}_\sim$ and $\Pi_\sim$, as expansions of SBL, G and Π, respectively, with an involutive negation. In these logics the initial negation $\neg$ is Gödel negation and the operator $\triangle$ is definable as the composition of the two negations, i.e. $\triangle\varphi$ is defined as $\neg{\sim}\varphi$, but the axiomatization needs the addition of the necessitation rule for $\triangle$. Hence, even though $\triangle$ is definable, in some sense, the logic is an expansion of a logic with $\triangle$.

The axiomatization of $\mathrm{L}_\sim$ for a $\triangle$-core fuzzy logic L presented in this section makes heavily use of the $\triangle$ operator. An interesting question is the possibility of obtaining an axiomatization without $\triangle$. An approach, suggested in [35], would be to take the axiom $({\sim}1)$ together with the previously mentioned rule (AMON). However, this axiomatization produces a logic which is not semilinear as the following example shows.

EXAMPLE 4.1.7. Let $\boldsymbol{B}_4^\sim$ be the algebra obtained by expanding the four element Boolean algebra $\boldsymbol{B}_4$ with the involutive negation $\sim$ defined by ${\sim}0 = 1$, ${\sim}1 = 0$, ${\sim}a = a$, ${\sim}b = b$. Let $\mathrm{L}_{\boldsymbol{B}_4^\sim}$ be the finitary logic given by the matrix $\langle \boldsymbol{B}_4^\sim, \{1\}\rangle$. This logic is an expansion of Classical logic that is not semilinear. Indeed, the $\vee$-form of the (AMON) inference rule, i.e.

$$\text{from } \ (\varphi \to \psi) \vee \chi \quad \text{infer} \quad ({\sim}\psi \to {\sim}\varphi) \vee \chi$$

is not sound. Namely, take for instance φ, ψ, χ to be three different propositional variables and an evaluation e such that $e(\varphi) = 1, e(\psi) = b$ and $e(\chi) = a$. Then we have $(1 \to b) \vee a = b \vee a = 1$, while $({\sim}b \to {\sim}1) \vee a = a \vee a = a$. Thus, the logic is not semilinear (see Chapter II).

On the other hand, another possibility considered in the same paper amounts to axiomatizing $\sim$ with axiom $({\sim}1)$ and the axiom

$$({\sim}3) \quad (\varphi \to \psi) \to ({\sim}\psi \to {\sim}\varphi).$$

However, the authors explicitly mention that this solution might not be completely satisfactory. Indeed, for any element a of an algebra of the variety corresponding to this logic, the new axiom $({\sim}3)$ implies $\neg a = a \to 0 = 1 \to {\sim}a = {\sim}a$. So, axiom $({\sim}3)$ forces the two negations to coincide, and thus $\neg$ becomes involutive as well. Therefore, the expansion of a core fuzzy logic L with an additional negation $\sim$ together with the axioms $({\sim}1)$ and $({\sim}3)$ turns out to be equivalent to the axiomatic extension of L with the involutiveness axiom

$$(\neg\neg) \quad \neg\neg\varphi \to \varphi$$

for the residual negation $\neg$ of L. Thus L must be an axiomatic extension of an IMTL logic.

4.2 Real completeness

As shown in [35], the Jenei-Montagna method for embedding a MTL-chain into a real MTL-chain [59]) can be extended to the case of MTL-chains with an involutive negation. Based on this result we can show the following general completeness result.

THEOREM 4.2.1. *Let* L *be the expansion with* $\triangle$ *of an axiomatic extension* L' *of* MTL*. If the Jenei-Montagna completion method provides a way to embed any countable* L'*-chain into a real* L'*-chain, then the logic* $\mathrm{L}_\sim$ *has the* $\mathrm{S}\mathcal{R}\mathrm{C}$*.*

Proof. The proof is an easy extension of the Jenei-Montagna method (a particular case of the embedding given in Lemma 4.1.4 of Chapter IV; see the original construction in [59] and its adaptation the involutive case in [25]). Let $\boldsymbol{C}$ be a countable $\mathrm{L}_\sim$, let $\boldsymbol{C}'$ be the completion of its L-reduct given by the Jenei-Montagna completion method, and let $\boldsymbol{D}$ be the real L-chain where $\boldsymbol{C}'$ embeds. Call this last embedding h. Then we can define an involutive negation n on $\boldsymbol{C}'$ as follows: for each $\langle s, q\rangle \in C'$ (with $s \in C$ and $q \in (0,1] \cap \mathsf{Q}$),

$$n(s,q) = \begin{cases} \langle \sim s, 1\rangle & \text{if } q = 1, \\ \langle succ(\sim s), 1-q\rangle & \text{otherwise,} \end{cases}$$

where $\sim$ denotes the involutive negation in the original $\mathrm{L}_\sim$-chain $\boldsymbol{C}$, and $succ(s)$ denotes the successor of s, if it exists, otherwise $succ(s) = s$. The expansion of $\boldsymbol{C}'$ with n makes it an $\mathrm{L}_\sim$-chain. Then, it is easy to check that the embedding of $\boldsymbol{C}$ into $\boldsymbol{C}'$, defined by $s \mapsto \langle s, 1\rangle$, is indeed also a morphism with respect to the involutive negations, and hence $\boldsymbol{C}$ embeds into the dense chain $\boldsymbol{C}'$ also as $\mathrm{L}_\sim$-chains. We can extend n over an involution on the real unit interval $\bar{n}$ by defining $\bar{n}(x) = \inf\{h(n(z)) \mid z \in \boldsymbol{C}', h(z) \leq x\}$. Again, expanding $\boldsymbol{D}$ with $\bar{n}$ makes it a real $\mathrm{L}_\sim$-chain where $\boldsymbol{C}'$ embeds (as $\mathrm{L}_\sim$-chains). Finally, from the compound embedding of the original chain $\boldsymbol{C}$ into $\boldsymbol{D}$ as $\mathrm{L}_\sim$-chains, the $\mathrm{S}\mathcal{R}\mathrm{C}$ of $\mathrm{L}_\sim$ immediately follows (see Chapter II, Section 3.4). □

As a direct consequence of this theorem we get the $\mathrm{S}\mathcal{R}\mathrm{C}$ for all the logics $\mathrm{L}_\sim$ with $\mathrm{L} \in \{\mathrm{MTL}_\triangle, \mathrm{SMTL}_\triangle, \mathrm{IMTL}_\triangle, \mathrm{WNM}_\triangle, \mathrm{NM}_\triangle, \mathrm{G}_\triangle\}$.

For the case of L being $\mathrm{G}_\triangle$, there is a stronger result. Indeed, in this case the logic $\mathrm{L}_\sim$, that we will denote by $\mathrm{G}_\sim$, is not only strongly complete with respect to the class of real $\mathrm{G}_\sim$-chains but also with respect the *standard* $\mathrm{G}_\sim$-chain, i.e. with respect to the real $\mathrm{G}_\sim$-chain where the involutive negation is the standard one, $n_s(x) = 1-x$. This is due to the fact that, as already mentioned in Remark 4.1.6, all involutive negations on $[0,1]$ are isomorphic to each other and there is only one real G-chains that is in fact the standard G-chain $[0,1]_G$.

PROPOSITION 4.2.2. *The logic* $\mathrm{G}_\sim$ *is strongly standard complete, i.e. strongly complete w.r.t. the standard* $\mathrm{G}_\sim$*-chain.*

Theorem 4.2.1 says nothing about expansions of other logics like $\Pi\mathrm{MTL}_\sim$, $\mathrm{BL}_\sim$, $\Pi_\sim$, or $\mathrm{SBL}_\sim$ since on these logics the Jenei-Montagna method does not apply. Nevertheless, all the expansions $\mathrm{L}_\sim$ where L is the $\triangle$-expansion of the logic of a continuous t-norm enjoy the $\mathrm{FS}\mathcal{R}\mathrm{C}$. In fact we have the following more general result.

THEOREM 4.2.3. *Let* L *be the* $\triangle$*-core fuzzy logic with a finite language enjoying the* $\mathrm{FS}\mathcal{R}\mathrm{C}$*. Then the logic* $\mathrm{L}_\sim$ *has the* $\mathrm{FS}\mathcal{R}\mathrm{C}$ *as well.*

Proof. Taking into account that, under the assumption of finite language, $\mathrm{FS}\mathcal{R}\mathrm{C}$ is equivalent to the partial embeddability property into real chains [18], it is enough to prove that this property extends from L to $\mathrm{L}_\sim$. Let X be any finite subset of an $\mathrm{L}_\sim$-chain $\boldsymbol{A}$ and let $Y = X \cup \{0,1\} \cup \{\sim x \mid x \in X\}$. Being Y finite, there exists a partial embedding of Y into a real L-chain $\boldsymbol{A}'$, call it h. We can always define an involution n on $\boldsymbol{A}'$ coinciding with $\sim$ over $h[Y]$, i.e. such that $n(h(x)) = h(\sim x)$ for every $x \in Y$. □

A direct consequence of this result is that e.g. $\mathrm{SBL}_\sim$ and $\Pi_\sim$ enjoy the $\mathrm{FS}\mathcal{R}\mathrm{C}$.

As we have already pointed out, all real $\mathrm{G}_\sim$-chains are isomorphic to each other. However, this does not hold in general, e.g. this is not true for $\Pi_\sim$. Indeed, in [28], in order to show that $\Pi_\sim$ is not standard complete, i.e. it is not complete with respect to the standard $\Pi_\sim$-chain $[0,1]_{\Pi_\sim} = \langle [0,1], \max, \min, *_\Pi, \Rightarrow_\Pi, n_s, 0, 1\rangle$, the authors check that the formula $(\sim\varphi\&\varphi) \to (\sim(\sim\varphi\&\varphi))^3$ (where ψ^3 means $\psi\&\psi\&\psi$) is a 1-tautology over $[0,1]_{\Pi_\sim}$ but it is not a 1-tautology in some real $\Pi_\sim$-chain with a strong negation different from n_s. In fact, $\mathrm{SBL}_\sim$ is not complete with respect to only one real chain.

4.3 The lattice of subvarieties of $\Pi_\sim$-algebras and of SBL$_\sim$-algebras

The fact that the logics $\mathrm{SBL}_\sim$ and $\Pi_\sim$ are only $\mathrm{FS}\mathcal{R}\mathrm{C}$ complete, i.e. complete with respect to all the real $\mathrm{SBL}_\sim$- and $\Pi_\sim$-chains respectively, makes the study of the logics of families of these chains (and equivalently, the subvarieties generated by families of these chains) interesting.

We start by studying the lattice of subvarieties generated by real $\Pi_\sim$-chains. To do so, the next proposition highlights the important role played by the set $\mathcal{I}$ of the $\Pi_\sim$-chains defined over the standard Π-chain by adding an involutive negation with $\frac{1}{2}$ as its fixed point. Since a $\Pi_\sim$-chain $\langle [0,1], *_\Pi, \Rightarrow_\Pi, \min, \max, n, 0, 1\rangle$ of $\mathcal{I}$ is determined by the involutive negation n, in what follows we will denote it by $[0,1]_{\Pi,n}$.

PROPOSITION 4.3.1. *Any real* $\Pi_\sim$*-chain is isomorphic to a* $\Pi_\sim$*-chain from* $\mathcal{I}$*.*

Proof. Let $\boldsymbol{D}$ be a real $\Pi_\sim$-chain with n being its involutive negation. As recalled in Remark 4.1.6, any real Π-chain is isomorphic to the standard one. Let f be the isomorphism between the Π-chain reduct of $\boldsymbol{D}$ and the standard Π-chain. Then $\boldsymbol{D}$ is isomorphic to the $\Pi_\sim$-chain defined over the standard Π-chain adding the involutive negation $\bar{n} = f^{-1} \circ n \circ f$. Denote by s the fixed point of $\bar{n}$. On the other hand, any automorphism g of the standard Π-chain is of the form $x \to x^a$ for a fixed $a \in \mathsf{R}^+$. Let g be the automorphism defined by taking an a such that $s^a = \frac{1}{2}$. Then, $g \circ f$ gives the desired isomorphism, since g transforms the involutive negation $\bar{n}$ into a new involutive negation with fixed point $\frac{1}{2}$. □

Thus, in order to study the subvarieties generated by real $\Pi_\sim$-chains, one only needs to consider as generators the chains belonging to $\mathcal{I}$. The main result in this section is stated in the following theorem.

THEOREM 4.3.2. *The lattice of subvarieties generated by real* $\Pi_\sim$*-chains has infinite height and infinite (uncountable) width.*

The result is really surprising if we take into account that the lattice of subvarieties of Π-algebras contains Boolean algebras as the only proper subvariety and, thus, the addition of an involutive negation gives rise to a continuum of subvarieties.

The proof of the next theorem is based on results from [38].[16] Actually, in that paper only the infinite (uncountable) width result is proved, while the infinite height result is proved in [19]. Next, we follow [38] for the proof of the uncountable width result, and from there we provide a new proof of the infinite height result.

THEOREM 4.3.3. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two real $\Pi_\sim$-chains. The variety generated by $\boldsymbol{C}$ is comparable with the variety generated by $\boldsymbol{D}$ if, and only if, $\boldsymbol{C}$ and $\boldsymbol{D}$ are isomorphic.*

By the previous result we can restrict ourselves to chains in $\mathcal{I}$. Obviously, any two chains from $\mathcal{I}$ are isomorphic to each other if, and only if, they are the same chain. Thus, the theorem says that two different chains in $\mathcal{I}$ generate incomparable subvarieties. To prove this statement we need several lemmas.

LEMMA 4.3.4. *If $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, then $n(\frac{1}{2^k}) = \eta(\frac{1}{2^k})$ for all $k \in \mathsf{N}$.*

Proof. Consider for any $k, l, m \in \mathsf{N}$, the equations,[17]

$$(\sim((x \vee \sim(x))^k))^l \leq (y \vee \sim(y))^m \tag{1}$$
$$(\sim((x \wedge \sim(x))^k))^l \geq (y \wedge \sim(y))^m. \tag{2}$$

Equation (1) is valid over $[0,1]_{\Pi,n}$ if the inequality holds for any $a, b \in [0,1]$ which is equivalent to

$$\max_{a\in[0,1]} (n(a \vee n(a))^k)^l \leq \min_{b\in[0,1]} (b \vee n(b))^m.$$

It is obvious that these extreme values are obtained at $\frac{1}{2}$, the fixed point of the negations. Thus (1) holds in $[0,1]_{\Pi,n}$ if, and only if,

$$n\left(\frac{1}{2^k}\right) \leq \left(\frac{1}{2}\right)^{\frac{m}{l}}.$$

Similarly, we can prove that (2) holds in $[0,1]_{\Pi,n}$ if, and only if,

$$n\left(\frac{1}{2^k}\right) \geq \left(\frac{1}{2}\right)^{\frac{m}{l}}.$$

If $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, the inequalities that hold for $[0,1]_{\Pi,n}$ also must hold for $[0,1]_{\Pi,\eta}$, and being the set $\{(\frac{1}{2})^{\frac{m}{l}} \mid l, m \in \mathsf{N}\}$ dense in the real unit interval, we conclude that for each $k \in \mathsf{N}$, $n(\frac{1}{2^k}) = \eta(\frac{1}{2^k})$. $\square$

Now for any involutive negation n with fixed point $\frac{1}{2}$, define the set

$$M(n) = \left\{ \left(n\left(\frac{1}{2^k}\right)\right)^l \mid k, l \in \mathsf{N} \right\}.$$

[16] The paper studies logics of strict De Morgan triples, $\Pi_\sim$-chains without residuated implication, but the result is also valid when the implication is included.

[17] Remember that $a \leq b$ is equivalent to $a \wedge b = a$.

LEMMA 4.3.5. *For any involutive negation n with fixed point $\frac{1}{2}$, $M(n)$ is dense in the real unit interval.*

Proof. Obviously $\left\{n(\frac{1}{2^k}) \mid k \in \mathsf{N}\right\}$ is an increasing sequence with limit 1, and thus for any $\epsilon > 0$ there is k_0 such that $1 - n(\frac{1}{2^{k_0}}) < \epsilon$. But $1 - b < \epsilon$ implies $b^m - b^{m+1} = b^m(1-b) < (1-b) < \epsilon$. Therefore, it follows that for each element of the real unit interval there is an element of the sequence $\left\{n(\frac{1}{2^k}))^l \mid k, l \in \mathsf{N}\right\}$ whose difference from it is at most ϵ. □

LEMMA 4.3.6. *If $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, then n and η coincide on $M(n)$.*

Proof. If $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, from Lemma 4.3.4, we know that for all $k, l \in \mathsf{N}$, $n((\frac{1}{2^k}))^l = \eta((\frac{1}{2^k}))^l$. Thus, the sets $M(n)$ and $M(\eta)$ coincide. Now, consider the inequalities:

$$(\sim((\sim((x \wedge \sim(x))^k))^l))^r \leq (y \vee \sim(y))^m \tag{3}$$

and

$$(\sim((\sim((x \vee \sim(x))^k))^l))^r \geq (y \wedge \sim(y))^m. \tag{4}$$

By an argument similar to the one used in Lemma 4.3.4, we obtain that (3) holds in $[0,1]_{\Pi,n}$ if, and only if,

$$n\left(\left(n\left(\frac{1}{2^k}\right)\right)^l\right) \leq \left(\frac{1}{2}\right)^{\frac{m}{r}},$$

and that (4) holds in $[0,1]_{\Pi,n}$ if, and only if,

$$n\left(\left(n\left(\frac{1}{2^k}\right)\right)^l\right) \geq \left(\frac{1}{2}\right)^{\frac{m}{r}}.$$

The same conditions are valid for η and thus, reasoning as in Lemma 4.3.4, we obtain that for all $a \in M(n) = M(\eta)$, $n(a) = \eta(a)$. □

We have shown that if $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, then n and η agree on a dense set, and since involutive negations are continuous functions, they coincide over the whole real unit interval. This ends the proof of Theorem 4.3.3.

The two families of equations used in the proofs above have been considered separately. However the first family is a special case of the second. Namely, taking $k = 1$, equation (3) becomes (1), and (4) becomes (2) as an easy computation shows. Thus, in fact, we only have one family of equations used to separate the subvarieties. From Theorem 4.3.3 it follows that there are as many incomparable subvarieties as there are involutive negations with $\frac{1}{2}$ as a fixed point. Of course there are uncountable many of the latter. Summarizing, we have the following result.

COROLLARY 4.3.7. *The set of subvarieties of the variety generated by a single real $\Pi_\sim$-chain contains an uncountable set of pairwise incomparable subvarieties. Furthermore, these subvarieties are separated by the following family of equations:*

$$(\sim((\sim((x \wedge \sim(x))^k))^l))^r \leq (y \vee \sim(y))^m.$$

Now, we will prove the infinite height part of Theorem 4.3.2. Set $k_0, m_0 \in \mathsf{N}$ and a strictly increasing sequence of naturals $\{l_i\}_{i\in\mathsf{N}}$. Then, define the sequence $\{T_i\}_{i\in\mathsf{N}}$ of subsets of $\mathcal{I}$ as

$$T_i = \left\{ [0,1]_{\Pi,n} \in \mathcal{I} \mid n\left(\frac{1}{2^{k_0}}\right) \leq \left(\frac{1}{2}\right)^{\frac{m_0}{l_i}} \right\}.$$

Since $\{\frac{m_0}{l_i}\}_{i\in\mathsf{N}}$ is a decreasing sequence with limit 0, for all $i \in \mathsf{N}$, $T_i \subset T_{i+1}$, and the same inclusions hold true for the varieties generated by these families. Finally, an easy observation shows that these inclusions are proper, since the equation (1) for k_0, m_0, and l_i is valid for T_i but not for T_{i+1}. Thus, we have an infinite sequence of strict inclusions of subvarieties, and so the height of the lattice of subvarieties is, at least, countable.

In [19] and [55], the authors offer further insight into the subvarieties of $\mathrm{SBL}_\sim$- and $\Pi_\sim$-algebras. In order to separate the subvarieties, Cintula et al. [19] use a different family of equations. They define for each natural n, the equation[18]

$$\sim((\sim(x^n))^n) = x \qquad (\mathrm{D}_n)$$

and prove, using these equations, that the lattice of subvarieties of $\mathrm{SBL}_\sim$ and $\Pi_\sim$-algebras contain a sublattice isomorphic to the lattice of natural numbers $\langle \mathsf{N}, \preceq\rangle$, with the order $\preceq$ defined by: $1 \preceq n$ for all $n \in \mathsf{N}$ and $n \preceq m$ if there is a natural k such that $n^k = m$. It is clear that, under this definition, $\langle \mathsf{N}, \preceq\rangle$ has infinite width and infinite height.

Haniková and Savický [55] generalize Theorem 4.3.3 from real $\Pi_\sim$-chains to SBL-chains defined by ordinal sums with a finite number of components in the following way. Let $[0,1]_{*,n}$ denote the SBL-chain defined by a strict Archimedean t-norm $*$ and an involutive negation n.

THEOREM 4.3.8 ([55]). *Let $*$ be a t-norm with a finite number of idempotents. Then, if $*$ is of either of type Π, or $\Pi \oplus j.$Ł, or $\Pi \oplus i.$Ł $\oplus \Pi \oplus j.$Ł (where $\oplus$ is interpreted as the ordinal sum, and $i.$Ł means the ordinal sum of i copies of a Łukasiewicz component) then the following two conditions hold for arbitrary involutive negations n_1 and n_2:*

(1) *the varieties generated by $[0,1]_{*,n_1}$ and $[0,1]_{*,n_2}$ coincide iff $[0,1]_{*,n_1}$ is isomorphic to $[0,1]_{*,n_2}$;*

(2) *if the varieties generated by $[0,1]_{*,n_1}$ and $[0,1]_{*,n_2}$ do not coincide then they are incomparable.*

Otherwise, if $*$ is of type $\Pi \oplus i.$Ł $\oplus \Pi$ or it contains at last three product components, then (1) does not hold for $*$.

Finally one interesting question is to know whether or not there is an axiomatization with finitely-many axiom schemes for the logic that is complete with respect to a chain $[0,1]_{\Pi,n}$, or equivalently, whether there is a finite equational basis for the subvariety generated by $[0,1]_{\Pi,n}$. The question makes sense because in each of the cited papers [20, 38, 55] the authors give different sets of separating equations, i.e. defining different

[18] We have applied our notation to the equation.

subvarieties, but in each case they need an infinite number of equations to axiomatize the subvariety generated by each one of the real chains.[19] As far as we know, only the case of the standard $\Pi_\sim$ chain, the one defined by the standard negation $n_s(x) = 1-x$, has been proved to be finitely axiomatizable. The proof of this result is not trivial and is based on the study of the logic $Ł\Pi$ [29] (see Section 5.2 for details). The original definition of $Ł\Pi$ was given in a language with four basic connectives, i.e. the Łukasiewicz and product conjunctions and implications, and it was shown to be complete with respect to the standard $Ł\Pi$-chain $[0,1]_{Ł\Pi} = \langle [0,1], *_Ł, \Rightarrow_Ł, *_\Pi, \Rightarrow_\Pi, \max, \min, 0, 1\rangle$. However, a very nice result due to Cintula [14] proves that $Ł\Pi$ is also complete with respect to the standard $\Pi_\sim$-chain (modulo term equivalence). Indeed, he observes that, over the standard $\Pi_\sim$-chain, the standard Łukasiewicz conjunction and implication operations are definable as follows:

(I) $x \Rightarrow_Ł y = n_s(x *_\Pi n_s(x \Rightarrow_\Pi y))$

(C) $x *_Ł y = x *_\Pi n_s(x \Rightarrow_\Pi n_s(y))$

and, following this idea, he proves that $Ł\Pi$ can be defined as an axiomatic extension of the logic $\Pi_\sim$ by adding the axiom:

$$(\varphi \to_Ł \psi) \to_Ł ((\psi \to_Ł \chi) \to_Ł (\varphi \to_Ł \psi)).$$

This result was later complemented by Vetterlein [84], who showed that one could alternatively add the axiom:

$$\varphi \,\&_Ł\, \psi \to_Ł \psi \,\&_Ł\, \varphi.$$

Therefore, the logic that is (standard) complete with respect to $[0,1]_{\Pi_\sim}$ is in fact $Ł\Pi$, and hence it is finitely axiomatizable (since $Ł\Pi$ is). A final remark is that if in the above definitions (C) and (I) one takes an involutive negation different from n_s, the resulting operation in (C) is no longer commutative, and, analogously, the resulting function in (I) is not transitive anymore.

5 Expansions of Łukasiewicz logic

Some of the most remarkable properties of Łukasiewicz logic and MV-algebras come from their connection with ordered Abelian groups (see Chapter VI). Chang's Completeness Theorem and McNaughton's Theorem are clear examples of results deriving from this connection. It is then natural to investigate whether the addition of new connectives to Łukasiewicz logic, and new operators to MV-algebras, can lead to finding similar relations to richer and well-known structures like rings and fields.

Apart from purely technical motivations, the interest in exploring expansions of Łukasiewicz logic also comes from the fact that the addition of new connectives significantly increases the expressive power of the logic. This allows to notably enhance the spectrum of definable functions in the algebras over the reals associated to the expansions, yielding richer and more complex logical systems.

[19] In [55], the authors prove that this infinite set of equations does not axiomatize the logic due to the fact that the logic is finitary, and, consequently, there are tautologies that cannot be derived from a finite number of axioms.

A first step is obtained by expanding Łukasiewicz logic with divisibility connectives δ_n, defining the logic RL. The linearly ordered algebras related to RL, called DMV-chains, are related to ordered divisible Abelian groups the same way MV-chains are related to ordered Abelian groups.

Even richer systems come from adding to Łukasiewicz logic the product connective and the product implication. The logics $\mathrm{PŁ}, \mathrm{PŁ}', \mathrm{Ł\Pi}$ and $\mathrm{Ł\Pi}\frac{1}{2}$ are the results of this expansion. As imagined, the algebras related to these logics, i.e. $\mathrm{PMV}, \mathrm{PMV}^+, \mathrm{Ł\Pi}$ and $\mathrm{Ł\Pi}\frac{1}{2}$ algebras bear a strong relation with certain rings, integral domains, and fields.

Among the logics mentioned so far, $\mathrm{Ł\Pi}\frac{1}{2}$ is the system with greater expressive power. In fact, the first-order theory of real numbers can be interpreted within the equational theory of $\mathrm{Ł\Pi}\frac{1}{2}$, making $\mathrm{Ł\Pi}\frac{1}{2}$ a powerful framework for the interpretation of other logical systems.

The purpose of this part of the chapter is to explore the above mentioned expansions, providing the basic notions and results, and making their relation with groups, rings and fields explicit.

5.1 Rational Łukasiewicz logic

Rational Łukasiewicz logic RL is an expansion of Łukasiewicz logic obtained by adding the unary connectives δ_n, for each $n \geq 1$, plus the following axioms:

(D1) $n(\delta_n\varphi) \leftrightarrow \varphi$

(D2) $\neg\delta_n\varphi \oplus \neg(n-1)(\delta_n\varphi)$,

with $n\psi := \underbrace{\psi \oplus \cdots \oplus \psi}_{n}$. As in the case of Łukasiewicz logic, other connectives are definable as follows:

$$\begin{array}{rcl@{\qquad}rcl}
\varphi \,\&\, \psi & := & \neg(\varphi \to \neg\psi) & \varphi \oplus \psi & := & \neg(\neg\varphi \,\&\, \neg\psi) \\
|\varphi - \psi| & := & (\varphi \ominus \psi) \oplus (\psi \ominus \varphi) & \varphi \vee \psi & := & \varphi \oplus (\psi \ominus \varphi) \\
\varphi \wedge \psi & := & \varphi \ominus (\varphi \ominus \psi) & \varphi \ominus \psi & := & \neg(\neg\varphi \oplus \psi) \\
\varphi \leftrightarrow \psi & := & (\varphi \to \psi) \,\&\, (\psi \to \varphi) & \overline{1} & := & \varphi \to \varphi.
\end{array}$$

The algebraic semantics of RL is given by DMV-*algebras* (Divisible MV-algebras), i.e. structures $\boldsymbol{A} = \langle A, \oplus, \neg, \{\delta_n\}_{n\in\mathsf{N}}, 0\rangle$ of type $\langle 2, 1, 1, 0\rangle$ such that $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra and the following equations hold for all $x \in A$ and $n \geq 1$:

$$(\delta_n 1)\ \ n(\delta_n x) = x, \qquad (\delta_n 2)\ \ \delta_n x \odot (n-1)(\delta_n x) = 0,$$

with $nx := \underbrace{x \oplus \cdots \oplus x}_{n}$. As in the case of MV-algebras, other operations can be defined as follows:

$$\begin{array}{rcl@{\qquad}rcl}
x \to y & := & \neg x \oplus y & x \odot y & := & \neg(x \to \neg y) \\
x \ominus y & := & \neg(\neg x \oplus y) & |x - y| & := & (x \ominus y) \oplus (y \ominus x) \\
x \wedge y & := & x \ominus (x \ominus y) & x \vee y & := & x \oplus (y \ominus x) \\
x \leftrightarrow y & := & (x \to y) \odot (y \to x). & & &
\end{array}$$

The class $\mathbb{DMV}$ of DMV-algebras is a variety.

$[0,1]_{\mathrm{DMV}}$ denotes the standard DMV-chain over the real unit interval $[0,1]$, and its operations are defined as follows:

$$\begin{array}{rclrcl} x \to y & = & \min\{1-x+y,1\} & \neg x & = & 1-x \\ x \oplus y & = & \min\{x+y,1\} & x \odot y & = & \max\{x+y-1,0\} \\ x \ominus y & = & \max\{x-y,0\} & |x-y| & = & \max\{x-y,y-x\} \\ x \vee y & = & \max\{x,y\} & x \wedge y & = & \min\{x,y\} \\ x \leftrightarrow y & = & 1-|x-y| & \delta_n x & = & \frac{n}{x}. \end{array}$$

An evaluation e of RL-formulas into a DMV-algebra is simply an extension of an evaluation for Łukasiewicz logic (see Chapter VI) for the connectives δ_n, so that $e(\delta_n\varphi) = \delta_n(e(\varphi))$. Notice that in RL, all rationals in $[0,1]$ are definable as truth-constants in the following way:

– $\frac{1}{n}$ is definable as $\delta_n\overline{1}$, and

– $\frac{m}{n}$ is definable as $m(\delta_n\overline{1})$,

since for every evaluation e into the real unit interval $[0,1]$,

$$e(\delta_n\overline{1}) = \frac{1}{n} \qquad \text{and} \qquad e(m(\delta_n\overline{1})) = m\left(\frac{1}{n}\right) = \frac{m}{n}.$$

The definition of 'ideal' for DMV-algebras coincides with the one for MV-algebras.

DEFINITION 5.1.1. *Given an* MV*-algebra* $\boldsymbol{A}$*, a non-empty set* $I \subseteq A$ *is an* ideal *whenever the following properties are satisfied:*

(1) $a \leq b$ *and* $b \in I$ *imply* $a \in I$*,*

(2) $a, b \in I$ *implies* $a \oplus b \in I$*.*

As a consequence, the proof of the next theorem is almost identical to the proof of Chang's Representation Theorem (see Chapter VI).

THEOREM 5.1.2. *Every* DMV*-algebra is isomorphic to a subdirect product of linearly ordered* DMV*-algebras.*

Every MV-chain is well-known to be isomorphic to the MV-chain defined over the unit interval of an ordered Abelian group with strong unit. A similar result holds from DMV-chains w.r.t. ordered divisible Abelian groups with strong unit.

LEMMA 5.1.3. *Let* $\boldsymbol{A}$ *be a* DMV*-chain. Then there exists an ordered divisible Abelian group* $\boldsymbol{G}$ *with a strong unit* u *such that* $\boldsymbol{A} \cong \Gamma(\boldsymbol{G}, u)$*.*

Proof. Let $\boldsymbol{A}$ be a DMV-chain and $\boldsymbol{A}'$ its MV-reduct. Clearly, $\boldsymbol{A}'$ is an MV-chain isomorphic to $\Gamma(\boldsymbol{G}_{A'}, (1,0))$ (see Chapter VI). Let $u = (1,0)$. For any $a \in G_{A'}$, there exists an $n \in \mathsf{N}$ such that $nu \leq_{G_{A'}} a \leq_{G_{A'}} (n+1)u$. Let $b = a - nu$. Since $b, u \in [0,u]$, for any $m \in \mathsf{N}$, there exist $c, d \in [0,u]$ such that $b = mc$ and $u = md$, respectively. Then $a = nu + b = n(md) + mc = m(nd+c)$, which means that $\mathbf{G}_{A'}$ is indeed divisible. $\square$

The fact that each DMV-chain is definable over the unit interval of an ordered divisible Abelian group with strong unit and that ordered divisible Abelian groups are *elementarily equivalent* to each other, i.e. they satisfy the same first-order sentences in the language of ordered groups $\langle +, -, 0, < \rangle$, make it possible to prove the next theorems.

THEOREM 5.1.4. $\mathbb{DMV}$ *is generated by the standard* DMV*-chain* $[0, 1]_{\mathrm{DMV}}$.

Proof. Let $\boldsymbol{A}$ be a DMV-algebra and $\varphi(\overline{x})$ an equation not valid in $\boldsymbol{A}$. $\boldsymbol{A}$ is a subdirect product of DMV-chains $\boldsymbol{B}_i$, so there is at least one $\boldsymbol{B}_i$ in which $\varphi(\overline{x})$ fails. $\boldsymbol{B}_i$ is isomorphic to the DMV-algebra of an ordered divisible Abelian group $\boldsymbol{G}_{\boldsymbol{B}_i}$. Ordered divisible Abelian groups are elementarily equivalent to each other, and to the group of reals R [8], in particular. Since the operations of a DMV-algebra are definable in the related ordered group (see Chapter VI), this implies that $\varphi(\overline{x})$ does not hold in the reals. □

Moreover, we have:

THEOREM 5.1.5. RL *has the* FS$\mathcal{R}$C.

Proof. We just need to show that every DMV-chain $\boldsymbol{A}$ is embeddable into an ultrapower of $[0, 1]_{\mathrm{DMV}}$ (see Chapter I). All ordered divisible Abelian groups are elementarily equivalent to each other, and so are all DMV-chains, being structures defined over the interval of ordered divisible Abelian groups. This means that every DMV-chain $\boldsymbol{A}$ is elementarily equivalent to $[0, 1]_{\mathrm{DMV}}$, and so, by Frayne's Theorem , it can be embedded into an ultrapower of $[0, 1]_{\mathrm{DMV}}$ (see [8]). □

5.2 Expansions with the product connective

We are now going to study expansions of Łukasiewicz logic that include the product conjunction. We will study some of their basic algebraic properties and their relationship to certain classes of ordered commutative rings to provide completeness results. We will devote special attention to the logic $Ł\Pi\frac{1}{2}$, that combines both Product and Łukasiewicz logics in a unified framework, whose algebraic semantics bears a strong relation to ordered fields.

5.2.1 The axiomatic systems and their algebraic semantics

The language of the logic PŁ (Product Łukasiewicz) is the language of Łukasiewicz logic, i.e. $\{\rightarrow, \neg\}$, plus the Product connective $\&_\Pi$. All the connectives definable in Łukasiewicz logic are obviously definable in PŁ (see Chapter VI and Section 5.1 above). The axioms of the logic PŁ are those of Łukasiewicz logic, plus the following axioms:

(P1) $(\varphi \,\&_\Pi\, \psi) \ominus (\varphi \,\&_\Pi\, \chi) \leftrightarrow \varphi \,\&_\Pi\, (\psi \ominus \chi)$
(P2) $\varphi \,\&_\Pi\, (\psi \,\&_\Pi\, \chi) \leftrightarrow (\varphi \,\&_\Pi\, \psi) \,\&_\Pi\, \chi$
(P3) $\varphi \rightarrow (\varphi \,\&_\Pi\, \overline{1})$
(P4) $(\varphi \,\&_\Pi\, \psi) \rightarrow \varphi$
(P5) $(\varphi \,\&_\Pi\, \psi) \rightarrow (\psi \,\&_\Pi\, \varphi)$.

The only deduction rule is *modus ponens* for $\&$ and $\rightarrow$.

The logic $\mathrm{P\L}'$ is obtained from $\mathrm{P\L}$ by adding the deduction rule:

(ZD) $\neg(\varphi \mathbin{\&_\Pi} \varphi) \vdash_{\mathrm{P\L}'} \neg\varphi$.

The logic $\L\Pi$ is defined from $\mathrm{P\L}$ by adding the product implication connective $\to_\Pi$ to the language, along with the following axioms:

(ŁΠ1) $((\varphi \mathbin{\&_\Pi} \psi) \to_\Pi \chi) \leftrightarrow (\varphi \to_\Pi (\psi \to_\Pi \chi))$

(ŁΠ2) $(\varphi \mathbin{\&_\Pi} (\varphi \to_\Pi \psi)) \leftrightarrow (\varphi \wedge \psi)$

(ŁΠ3) $\varphi \to_\Pi \varphi$

(ŁΠ4) $(\varphi \to_\Pi (\varphi \wedge \psi)) \leftrightarrow (\varphi \to_\Pi \psi)$.

Other definable connectives are the following:

$$\neg_\Pi\varphi := \varphi \to_\Pi 0 \qquad \triangle\varphi := \neg_\Pi\neg\varphi \qquad \nabla\varphi := \neg\neg_\Pi\varphi.$$

The logic $\L\Pi\frac{1}{2}$ is an expansion of $\L\Pi$ obtained by adding the constant $\overline{\frac{1}{2}}$ and the following axiom:

(ŁΠ5) $\overline{\frac{1}{2}} \leftrightarrow \neg\overline{\frac{1}{2}}$.

The following Deduction Theorem is easily seen to hold (see Chapter I).

THEOREM 5.2.1.

(1) *Let Γ be a theory over $\mathrm{P\L}$ and φ, ψ be formulas. Then $\Gamma \cup \{\varphi\} \vdash_{\mathrm{P\L}} \psi$ iff there is an n such that $\Gamma \vdash_{\mathrm{P\L}} \varphi^n \to \psi$.*

(2) *Let L denote either $\L\Pi$ or $\L\Pi\frac{1}{2}$, Γ be a theory over L, and φ, ψ be L-formulas. Then $\Gamma \cup \varphi \vdash_{\mathrm{L}} \psi$ iff $\Gamma \vdash_{\mathrm{L}} \triangle\varphi \to \psi$.*

We will see later (Corollary 5.3.21) that $\mathrm{P\L}'$ does not satisfy the same Deduction Theorem as $\mathrm{P\L}$.

We now introduce the classes of $\mathrm{PMV}, \mathrm{PMV}^+, \L\Pi$, and $\L\Pi\frac{1}{2}$ algebras that constitute the algebraic semantics for $\mathrm{P\L}, \mathrm{P\L}', \L\Pi$, and $\L\Pi\frac{1}{2}$, respectively.

DEFINITION 5.2.2. *A PMV-algebra $\boldsymbol{A} = \langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a structure of type $\langle 2, 1, 2, 0, 0\rangle$, where $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra, $\langle A, \cdot, 1\rangle$ is a commutative monoid, and the following equation holds:*

$$(x \cdot\ y) \ominus (x \cdot z) = x \cdot (y \ominus z). \tag{1}$$

A PMV^+-algebra $\boldsymbol{A} = \langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a PMV-algebra in which the following quasiequation holds:

$$\textit{if } x \cdot x = 0, \textit{ then } x = 0. \tag{2}$$

An $\L\Pi$-algebra $\boldsymbol{A} = \langle A, \oplus, \neg, \cdot, \to_\Pi, 0, 1\rangle$ is a structure of type $\langle 2, 1, 2, 2, 0, 0\rangle$, where $\langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a PMV-algebra, and the following equations hold

$$(x \cdot y) \to_\Pi z = x \to_\Pi (y \to_\Pi z), \tag{3}$$

$$x \cdot (x \to_\Pi y) = x \wedge y, \tag{4}$$

$$x \to_\Pi x = 1, \tag{5}$$

$$x \to_\Pi (x \wedge y) = x \to_\Pi y. \tag{6}$$

An Ł$\Pi\frac{1}{2}$*-algebra* $\boldsymbol{A}=\langle A, \oplus, \neg, \cdot, \rightarrow_\Pi, 0, 1, \frac{1}{2}\rangle$ *is a structure of type* $\langle 2,1,2,2,0,0,0\rangle$*, where* $\langle A, \oplus, \neg, \cdot, \rightarrow_\Pi, 0, 1\rangle$ *is an* ŁΠ*-algebra, and the following equation holds*

$$\neg\tfrac{1}{2} = \tfrac{1}{2}. \tag{7}$$

In the classes of structures introduced above, the operations

$$x \to y \qquad x \odot y \qquad x \ominus y \qquad |x-y| \qquad x \wedge y \quad x \vee y \qquad x \leftrightarrow y$$

are defined as in the case of MV-algebras (see Chapter VI and Section 5.1). Other operations can be defined as follows:

$$\neg_\Pi x := x \rightarrow_\Pi 0 \qquad \triangle x := \neg_\Pi \neg x \qquad \nabla x := \neg\neg_\Pi x.$$

The following proposition makes the connection between ŁΠ and PMV^+ algebras explicit, and will be (implicitly) used throughout the rest of the chapter.

PROPOSITION 5.2.3. *The quasiequation 'if* $x \cdot x = 0$*, then* $x = 0$*' holds in every* ŁΠ*-algebra.*

Proof. Suppose $x \cdot x = 0$. Then, $(x\cdot x)\cdot\neg_\Pi(x\cdot x) \le 0$, and so $\neg_\Pi(x\cdot x) \le x \rightarrow_\Pi \neg_\Pi x$. We show that $x \rightarrow_\Pi \neg_\Pi x \le \neg_\Pi x$.

From $x \rightarrow_\Pi \neg_\Pi x \le \neg_\Pi\neg_\Pi x \rightarrow_\Pi \neg_\Pi x$ we derive $\neg_\Pi\neg_\Pi x \le (x \rightarrow_\Pi \neg_\Pi x) \rightarrow_\Pi \neg_\Pi x$, and so $\neg_\Pi x \le (x \rightarrow_\Pi \neg_\Pi x) \rightarrow_\Pi \neg_\Pi x$. From the fact that $\neg_\Pi x \vee \neg_\Pi\neg_\Pi x = 1$ holds in every ŁΠ-algebra (easy to check), we obtain that $x \rightarrow_\Pi \neg_\Pi x \le \neg_\Pi x$.

So we have $\neg_\Pi(x\cdot x) \le \neg_\Pi x$, and since $\neg\neg x = x$, $\triangle\neg(x\cdot x) \le \neg_\Pi x$. From the assumption $x \cdot x = 0$, we derive $\triangle\neg(x\cdot x) = 1$, which means that $\neg_\Pi x = 1$, and so $x = 0$. □

The notions of evaluation, model, and proof for the above logics are defined as usual. In particular, the operations $x\cdot y$, $x \rightarrow_\Pi y$, $\neg_\Pi x$, $\triangle x$, ∇x have the following interpretation over the real unit interval $[0,1]$:

$$x \cdot y = xy \qquad\qquad x \rightarrow_\Pi y = \begin{cases} 1 & \text{if } x \le y \\ \frac{y}{x} & \text{if } x > y \end{cases}$$

$$\neg_\Pi x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases} \qquad\qquad \triangle x = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases}$$

$$\nabla x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$[0,1]_{\mathrm{PMV}}$ denotes the standard PMV-algebra over the real unit interval $[0,1]$, where the operations of its MV-reduct are obviously interpreted as in the standard MV-chain, and $\cdot$ is interpreted as the product of reals. Clearly, $[0,1]_{\mathrm{PMV}}$ is also a PMV^+-algebra. The standard ŁΠ and Ł$\Pi\frac{1}{2}$ algebras over $[0,1]$ are denoted by $[0,1]_{\text{Ł}\Pi}$ and $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$, respectively, and their operations correspond to those given for $[0,1]_{\mathrm{PMV}}$ along with the interpretation of the product implication $\rightarrow_\Pi$ given above as the residuum of the product t-norm.

The classes of PMV, ŁΠ, and Ł$\Pi\frac{1}{2}$ algebras are varieties, denoted by $\mathbb{PMV}$, $\mathbb{LP}$, and $\mathbb{LP}\frac{1}{2}$, respectively, while the class of PMV^+-algebras $\mathbb{PMV}^+$ is a quasivariety. At the end of the next section, we will see that $\mathbb{PMV}^+$ cannot constitute a variety.

5.3 Completeness results

The goal of this section is to prove several completeness results for $\mathrm{P}Ł$, $\mathrm{P}Ł'$, $Ł\Pi$, and $Ł\Pi\frac{1}{2}$, including both completeness w.r.t. evaluations into the related class of linearly ordered algebras and evaluations into the related algebra over the real unit interval. Linearly ordered PMV, PMV^{+}, and $Ł\Pi$ algebras will be shown to be structures definable on certain classes of ordered rings, and $Ł\Pi\frac{1}{2}$-chains will be shown to be definable on ordered fields. These connections will be exploited in order to prove completeness w.r.t. real evaluations for the related logics.

It is fairly easy to see that $\mathrm{P}Ł$, $\mathrm{P}Ł'$, $Ł\Pi$, and $Ł\Pi\frac{1}{2}$ are algebraizable logics in the sense of Blok and Pigozzi [4]. As an immediate consequence, we obtain:

THEOREM 5.3.1. *Let* L *denote any among the logics* $\mathrm{P}Ł$, $\mathrm{P}Ł'$, $Ł\Pi$, *and* $Ł\Pi\frac{1}{2}$. *Let* $\mathbb{K}$ *denote any among the classes* $\mathbb{PMV}$, $\mathbb{PMV}^{+}$, $\mathbb{LP}$, *and* $\mathbb{LP}\frac{1}{2}$. *Then* L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$.

COROLLARY 5.3.2. $\mathrm{P}Ł$ *and* $\mathrm{P}Ł'$ *are conservative extensions of Łukasiewicz logic.*

Proof. Let φ be a formula of Łukasiewicz logic which is a theorem of $\mathrm{P}Ł$. Then, $e(\varphi) = 1$ for every evaluation e into any PMV-algebra $\boldsymbol{A}$, and, in particular, for each evaluation into $[0,1]_{\mathrm{PMV}}$. Since φ is a formula of Łukasiewicz logic, it is also a tautology w.r.t. evaluations into the MV-algebra over $[0,1]$. Using the fact that Łukasiewicz logic has the $\mathcal{R}\mathrm{C}$ we conclude that φ is a theorem of Łukasiewicz logic.

The same holds for $\mathrm{P}Ł'$. □

We are now going to prove that each member of the classes of PMV-, $Ł\Pi$-, and $Ł\Pi\frac{1}{2}$-algebras is a subdirect product of chains of the related class.[20]

DEFINITION 5.3.3 ([44]). *Let* $\boldsymbol{A}$ *be any algebra having a constant* 0. *A* 0*-ideal of* $\boldsymbol{A}$ *is a subset* J *of* A *for which there is a congruence* θ *of* $\boldsymbol{A}$ *such that* $J = \{a \in A \mid a\,\theta\,0\}$.

In the rest of this chapter, we will refer to 0-ideals simply as "ideals".

LEMMA 5.3.4. *Let* $\boldsymbol{A}$ *be a* PMV*-algebra and* $\boldsymbol{B}$ *its underlying* MV*-algebra. Then:*

(1) $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *have the same congruences.*[21]

(2) $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *have the same ideals, and for every ideal* J *of* $\boldsymbol{A}$*, there is exactly one congruence* θ *of* $\boldsymbol{A}$ *such that* J *is the congruence class of* 0 *with respect to* θ*. This congruence is defined by* $x\ \theta\ y$ *iff* $|x-y| \in J$.

Proof. To prove the first claim it is sufficient to prove that for every congruence θ of $\boldsymbol{B}$, and for all $x, y, u, v \in A$, if $x\ \theta\ y, u\ \theta\ v$, then $(x\cdot u)\ \theta\ (y\cdot v)$. Given the assumptions above, and since $|x-y| = (x\vee y)\ominus(x\wedge y)$ and $|u-v| = (u\vee v)\ominus(u\wedge v)$, we have $|x-y|\ \theta\ 0$ and $|u-v|\ \theta\ 0$. It follows that

$$\begin{aligned} |x\cdot u - y\cdot v| &\leq ((x\vee y)\cdot(u\vee v))\ominus((x\wedge y)\cdot(u\wedge v)) \leq \\ &\leq (((x\vee y)\cdot(u\vee v))\ominus((x\vee y)\cdot(u\wedge v))) \\ &\qquad \oplus (((x\vee y)\cdot(u\wedge v))\ominus((x\wedge y)\cdot(u\wedge v))) = \end{aligned}$$

[20] Notice that for PMV the result follows from the fact that $\mathrm{P}Ł$ is a core fuzzy logic, while $Ł\Pi$ and $Ł\Pi\frac{1}{2}$ are $\triangle$-core fuzzy logics (see Chapter I). Still, we are going to provide a direct proof of these facts.

[21] See Chapter VI for the definition of "ideal" and "congruence" for MV-algebras.

$$= ((x \vee y) \cdot ((u \vee v) \ominus (u \wedge v))) \oplus ((u \wedge v) \cdot ((x \vee y) \ominus (x \wedge y))) \leq$$
$$\leq ((u \vee v) \ominus (u \wedge v)) \oplus ((x \vee y) \ominus (x \wedge y)) =$$
$$= |u - v| \oplus |x - y|.$$

Since $(|u - v| \oplus |x - y|)\ \theta\ (0 \oplus 0)$, we have $|x \cdot u - y \cdot v|\ \theta\ 0$, and, consequently, $(x \cdot u)\ \theta\ (y \cdot v)$, as desired.

The second claim: From (1) and Definition 5.3.3, we have that $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same ideals. Let θ be any congruence of $\boldsymbol{A}$, and let J be the congruence class of 0 modulo θ. If $x\ \theta\ y$, then $(x \ominus y)\ \theta\ (y \ominus y)$, which means that $(x \ominus y)\ \theta\ 0$. By a similar argument, we have that $y \ominus x\ \theta\ 0$. So, if $x\ \theta\ y$, then $|x - y|\ \theta\ 0$. Conversely, if $|x-y|\ \theta\ 0$, and since $((x \ominus y) \wedge |x-y|)\ \theta\ ((x \ominus y) \wedge 0)$ and $(x \ominus y) = (x \ominus y) \wedge |x-y|$, we have that $x \ominus y\ \theta\ 0$. Similarly, we obtain $(y \ominus x)\ \theta\ 0$, and so $(x \vee y)\ \theta\ x$, given that $x \vee y = (x \oplus (y \ominus x))$ and $(x \oplus (y \ominus x))\ \theta\ (x \oplus 0)$. From $(x \vee y)\ \theta\ y$ we get $x\ \theta\ y$. In conclusion, for all $x, y \in A$ we have $x\ \theta\ y$ iff $|x - y| \in J$. □

LEMMA 5.3.5. *Every* PMV*-algebra is isomorphic to a subdirect product of a family of* PMV*-chains.*

Proof. Let $\boldsymbol{A}$ be any subdirectly irreducible PMV-algebra. Since, every PMV-algebra is isomorphic to a subdirect product of subdirectly irreducible PMV-algebras, it is sufficient to prove that every subdirectly irreducible PMV-algebra is linearly ordered. The congruence lattice of $\boldsymbol{A}$ has a minimum non-zero element, i.e. the monolith, so there is a minimum non-zero ideal J. Clearly, J is generated by a single element $c > 0$. Suppose by contradiction that there are $a, b \in A$ such that neither $a \leq b$ nor $b \leq a$ holds. Thus, $a \ominus b > 0$ and $b \ominus a > 0$. It follows that c belongs both to the ideal $I_{a \ominus b}$ generated by $a \ominus b$ and to the ideal $I_{b \ominus a}$ generated by $b \ominus a$. By Lemma 5.3.4, there is $n \in \mathsf{N}$ such that $c \leq n(a \ominus b)$ and $c \leq n(b \ominus a)$. So, $c \leq n(a \ominus b) \wedge n(b \ominus a)$. Since $(a \ominus b) \wedge (b \ominus a) = 0$, we conclude that $c \leq n(a \ominus b) \wedge n(b \ominus a) = 0$, which clearly is a contradiction. □

The same result can be proven for $\mathbb{LP}$ and $\mathbb{LP}\frac{1}{2}$ by using a similar argument.

LEMMA 5.3.6. *Let J be a subset of an* ŁΠ*-algebra $\boldsymbol{A}$. J is an ideal of $\boldsymbol{A}$ iff it is an ideal of the underlying* MV*-algebra and is closed under* $\triangledown$*.*

Proof. Let θ be any congruence of $\boldsymbol{A}$, and let $J = \{x \in A \mid x\ \theta\ 0\}$. If $x \in J$, then $x\ \theta\ 0$, therefore $\neg_\Pi x\ \theta\ \neg_\Pi 0$, and $\triangledown x\ \theta\ 0$. So, $\triangledown x \in J$, and J is closed under $\triangledown$. Conversely, let J be an ideal of the underlying MV-algebra which is closed under $\triangledown$, and define: $x\ \theta\ y$ iff $|x - y| \in J$. By Lemma 5.3.4, θ is a congruence of the underlying PMV-algebra. Now suppose that $x\ \theta\ y$ and $u\ \theta\ v$. Then, $|x - y| \in J$, $|u - v| \in J$. So, $\triangledown(|x - y|) \in J$, and $\triangledown(|u - v|) \in J$. It follows that $\triangledown(|x - y|) \vee \triangledown(|u - v|) \in J$, and $|(u \to x) - (v \to y)| \in J$. In fact, note that $|(u \to x) - (v \to y)| \leq \triangledown(|x - y|) \vee \triangledown(|u - v|)$ holds in all PMV-algebras (see [66]). Thus, $|(u \to x) - (v \to y)|\ \theta\ 0$, and $(u \to x)\ \theta\ (v \to y)$. So, θ is a congruence of $\boldsymbol{A}$, and clearly $J = \{x \in A \mid x\ \theta\ 0\}$. □

LEMMA 5.3.7. *Let $\boldsymbol{A}$ be an* ŁΠ*-algebra and $a \in A$ an arbitrary element. Then the ideal J_a generated by a is given by $J_a = \{x \in A \mid x \leq \triangledown(a)\}$.*

Proof. Let $I = \{x \in A \mid x \leq \nabla a\}$. Clearly, I is a lattice ideal. It is easily seen that $\nabla a \oplus \nabla a = \nabla a \vee \nabla a = \nabla a$, therefore I is closed under $\oplus$. Finally, it is easy to see that if $x \leq \nabla a$, then $\nabla x \leq \nabla a$. So, I is closed under ∇. Therefore, I is an ideal of $\boldsymbol{A}$. Conversely, if J is an ideal and $a \in J$, then $\nabla a \in J$, therefore $I \subseteq J$. □

LEMMA 5.3.8. *Every* $\text{Ł}\Pi$*-algebra* (*every* $\text{Ł}\Pi\frac{1}{2}$*-algebra*) *is isomorphic to a subdirect product of a family of* $\text{Ł}\Pi$*-chains* ($\text{Ł}\Pi\frac{1}{2}$*-chains*).

Proof. We show that every subdirectly irreducible ŁΠ algebra is linearly ordered. Suppose that an ŁΠ-algebra $\boldsymbol{A}$ is subdirectly irreducible but not linearly ordered. Let J be a minimal non-zero ideal. By Lemma 5.3.7 and by the minimality of J, there is $c \neq 0$ such that $J = \{x \in A \mid x \leq \nabla c\}$. Now, suppose that for some $a, b \in A$, neither $a \leq b$ nor $b \leq a$. Then, both $a \ominus b$ and $b \ominus a$ generate non-trivial ideals $I_{a\ominus b}$ and $I_{b\ominus a}$ respectively. Both ideals contain J, and so, by Lemma 5.3.7, $\nabla c \leq \nabla(a \ominus b)$, and $\nabla c \leq \nabla(b \ominus a)$. Consequently, $c \leq \nabla c \leq \nabla(a \ominus b) \wedge \nabla(b \ominus a)$. Notice that, in all ŁΠ-algebras, $x \wedge y = 0$ implies $\nabla x \wedge \nabla y = 0$. Since $(a \ominus b) \wedge (b \ominus a) = 0$, we have that $c \leq \nabla(a \ominus b) \wedge \nabla(b \ominus a) = 0$, which is a contradiction. □

As an immediate consequence of the above results (see Chapter I), we have:

THEOREM 5.3.9. *Let* L *denote any among the logics* $\text{PŁ}, \text{Ł}\Pi, \text{Ł}\Pi\frac{1}{2}$. *Then* L *has the* $\text{S}\mathbb{K}\text{C}$*, where* $\mathbb{K}$ *denotes the class of chains of the corresponding class of algebras.*

We are now going to prove that some of the varieties introduced above are generated by their algebra over the reals (up to isomorphism). As a consequence, the related logics will be shown to have $\text{FS}\mathcal{R}\text{C}$. These results will be obtained by exploiting the connection between the linearly ordered members of the above varieties and certain classes of ordered rings.

An *ordered ring* is a structure $\mathbf{R} = \langle R, +, -, \cdot, 0, 1, \leq\rangle$ where $\langle R, +, -, \cdot, 0, 1\rangle$ is a commutative ring with unit, and $\langle R, \leq\rangle$ is a totally ordered set such that, for all $x, y, z \in R$, if $x \leq y$ then $x + z \leq y + z$; and if $0 \leq x$ and $0 \leq y$, then $0 \leq x \cdot y$. The notions of *ordered integral domain* and *ordered field* are analogously defined.

Now, given an ordered ring $\mathbf{R} = \langle R, +, -, \cdot, 0, 1, \leq\rangle$, the following structure is a PMV-chain:

$$\boldsymbol{A}_{\mathbf{R}} = \langle [0,1]_R, \oplus, \neg, \cdot, 0, 1\rangle,$$

where $[0,1]_R = \{x \in R \mid 0 \leq x \leq 1\}$, and

$$x \oplus y = \min\{x + y, 1\},$$
$$\neg x = 1 - x,$$
$$x \cdot y = x \cdot y.$$

Given an ordered integral domain $\mathbf{D} = \langle D, +, -, \cdot, 0, 1, \leq\rangle$, the following structure (defined in the same way as $\boldsymbol{A}_{\mathbf{R}}$) is a PMV^+-chain:

$$\boldsymbol{A}_{\mathbf{D}} = \langle [0,1]_D, \oplus, \neg, \cdot, 0, 1\rangle.$$

Given an ordered field $\mathbf{F} = \langle F, +, -, \cdot, 0, 1, \leq\rangle$, define the structure

$$\boldsymbol{A}_{\mathbf{F}} = \langle [0,1]_F, \oplus, \neg, \cdot, \rightarrow_\Pi, 0, 1\rangle,$$

where $\langle [0,1]_F, \oplus, \neg, \cdot, 0, 1\rangle$ is defined as above and

$$x \to_\Pi y = \begin{cases} 1 & \text{if } x \leq y, \\ y \cdot x^{-1} & \text{otherwise,} \end{cases}$$

where x^{-1} corresponds to the multiplicative inverse of x.

Similarly, from an ordered field $\mathbf{F}$, we can define the structure

$$\boldsymbol{A}'_{\mathbf{F}} = \langle [0,1]_F, \oplus, \neg, \cdot, \to_\Pi, 0, 1, \tfrac{1}{2}\rangle,$$

where $\frac{1}{2} = 2^{-1}$. It is easily seen that $\boldsymbol{A}_{\mathbf{F}}$ and $\boldsymbol{A}'_{\mathbf{F}}$ are an ŁΠ and an ŁΠ$\frac{1}{2}$-chain respectively.

The structures above defined are called *interval chains*, being defined over the unit interval of certain ordered rings.

LEMMA 5.3.10.

(1) *Every* PMV*-chain is isomorphic to the interval* PMV*-chain of an ordered ring.*

(2) *Every* PMV$^+$*-chain is isomorphic to the interval* PMV$^+$*-chain of an ordered integral domain.*

(3) *Every* ŁΠ*-chain with more than two elements is an* ŁΠ$\frac{1}{2}$*-chain (modulo an extension by definition), and contains an isomorphic copy of the interval* ŁΠ*-chain of the field of rationals* Q.

(4) *Every* ŁΠ$\frac{1}{2}$*-chain is isomorphic to the interval* ŁΠ$\frac{1}{2}$*-chain of an ordered field.*

(5) *Every* ŁΠ*-chain is isomorphic either to the interval* ŁΠ*-chain of an ordered field or to the interval* ŁΠ*-chain of* Z.

Proof. (1) Take any PMV-chain $\boldsymbol{A} = \langle A, \oplus, \neg, \cdot, 0, 1\rangle$ and define a structure

$$\mathbf{R}_{\boldsymbol{A}} = \langle R_A, +, -, \cdot, 0_{\mathbf{R}_{\boldsymbol{A}}}, \leq_{\mathbf{R}_{\boldsymbol{A}}}\rangle,$$

where $R_A = \{\langle n, x\rangle \mid n \in \mathsf{Z}, x \in A\backslash\{1\}\}$, and $0_{\mathbf{R}_{\boldsymbol{A}}} = \langle 0, 0\rangle$,

$$\langle n,x\rangle + \langle m,y\rangle = \begin{cases} \langle n+m, x\oplus y\rangle & \text{if } x\oplus y < 1, \\ \langle n+m+1, \neg(\neg x \oplus \neg y)\rangle & \text{if } x \oplus y = 1, \end{cases}$$

$$-\langle n,x\rangle = \begin{cases} \langle -n, 0\rangle & \text{if } x = 0, \\ \langle -(n+1), \neg x\rangle & \text{if } 0 < x < 1, \end{cases}$$

$$\langle n,x\rangle \cdot \langle m,y\rangle = \langle nm, x\cdot y\rangle + m\langle 0,x\rangle + n\langle 0,y\rangle,$$

$$\langle n,x\rangle \leq_{R_A} \langle m,y\rangle \quad \text{if} \quad n < m, \text{ or } n = m \text{ and } x \leq y.$$

$\langle R_A, +, -, 0_{\mathbf{R}_{\boldsymbol{A}}}, \leq_{\mathbf{R}_{\boldsymbol{A}}}\rangle$ corresponds to the Chang group built from the MV-chain $\langle A, \oplus, \neg, 0, 1\rangle$ (see Chapter VI). It is fairly easy to see $\mathbf{R}_{\boldsymbol{A}}$ is indeed an ordered ring.

(2) We just need to prove that $\mathbf{R}_{\boldsymbol{A}}$ has no zero-divisors. From $\langle n,x\rangle \cdot \langle m,y\rangle = \langle 0,0\rangle$ we can deduce that either $n=0$ or $m=0$. If $n=m=0$, then $x \cdot y = 0$, and either $x=0$ or $y=0$, since $\boldsymbol{A}$ is a linearly ordered PMV^+-algebra. Now suppose $n=0$ and $m \neq 0$. Then $\langle n,x\rangle \cdot \langle m,y\rangle = m\langle 0,x\rangle + \langle 0, x\cdot y\rangle$. Suppose $x \neq 0$. Let, for $\langle n,x\rangle \in R_A$, $|\langle n,x\rangle|$ denote $\langle n,x\rangle$ if $\langle n,x\rangle \geq_{\boldsymbol{R_A}} \langle 0,0\rangle$ and $-\langle n,x\rangle$ otherwise. We obtain:

$$|n\langle 0,x\rangle + \langle 0,x\cdot y\rangle| \geq_{\mathbf{R}_{\boldsymbol{A}}} |\langle 0, x \ominus (x\cdot y)\rangle| = |\langle 0, x\cdot \neg y\rangle|.$$

Now, $y \neq 1$, and therefore $\neg y \neq 0$, $x \cdot \neg y \neq 0$ (being $\boldsymbol{A}$ a linearly ordered PMV^+-algebra). So, we obtain a contradiction. The case where $m=0$ and $n \neq 0$ is symmetric.

(3) Let $\boldsymbol{A}$ be any linearly ordered ŁΠ algebra with more than two elements. By taking its $\to_\Pi$-free reduct, we can safely assume that $\boldsymbol{A}$ is the interval algebra of an ordered integral domain $\mathbf{D}_{\boldsymbol{A}}$. Let $a \in A$ be such that $0 < a < 1$, and let $b = \min\{a, 1-a\}$. Clearly, $0 < b < b+b = b \oplus b \leq 1$. Let $c = (b\oplus b) \to_\Pi b$. It is easy to check that: $b = b \wedge (b \oplus b) = ((b\oplus b) \to_\Pi b)\cdot(b\oplus b)$. So, in $\mathbf{D}_{\boldsymbol{A}}$ we have: $b = c\cdot b + c \cdot b$, and $b\cdot(1-(c+c)) = 0$. Then, it follows that $1-(c+c) = 0$, $c = 1-c$, and finally $c = \neg c$. Clearly, there is a unique c such that $c = \neg c$: call it $\frac{1}{2}$. Thus, $\boldsymbol{A}$ is an ŁΠ$\frac{1}{2}$-algebra.

We show that $\boldsymbol{A}$ contains an isomorphic copy of the ŁΠ-chain of rationals Q. We have shown that we already have the element $\frac{1}{2}$. For $n \geq 2$, we define

$$\frac{1}{n+1} = \left((n+1)\left(\frac{1}{2}\cdot\frac{1}{n}\right)\right) \to_\Pi \left(\frac{1}{2}\cdot\frac{1}{n}\right).$$

Define the map h from $\mathsf{Q}\cap[0,1]$ into A as: $h(0)=0$; $h(\frac{n}{m}) = n\frac{1}{m}$ if $0 < n \leq m$. We check that h is a one-to-one homomorphism from the interval ŁΠ-algebra of Q into $\boldsymbol{A}$. It is sufficient to show that

$$\text{for every positive } n \in \mathsf{N}, n\cdot\frac{1}{n} = 1. \tag{8}$$

Indeed, from (8) and distributivity, we easily obtain that for all $x \in D_A$,

$$\underbrace{x\cdot\frac{1}{n} + \cdots + x\cdot\frac{1}{n}}_{n} = x.$$

It follows that the group $\boldsymbol{G}$ underlying $\mathbf{D}_{\boldsymbol{A}}$ is Abelian, torsion-free and divisible. Let $\boldsymbol{G}'$ be the smallest divisible subgroup of $\boldsymbol{G}$ containing 1. Let $\frac{x}{n}$ denote 0 if $x=0$, and the unique y such that $ny = x$ otherwise. Then, $\boldsymbol{G}'$ consists of 0 plus all elements of the form $\pm\frac{m}{n}$ with $n,m > 0$. We have that the map h' defined by $h'(0) = 0$, and $h'(\pm\frac{m}{n}) = \pm\frac{m}{n}$ if $n,m>0$ is an isomorphism from the additive group of Q onto $\boldsymbol{G}'$. Using distributivity, we can also see that h' is compatible with the product $\cdot$. Finally, h' is order-preserving, and so it is an embedding of the ordered ring Q into $\mathbf{D}_{\boldsymbol{A}}$. The claim follows from the fact that h is a restriction of h' to the interval ŁΠ$\frac{1}{2}$-algebra of Q.

(4) If $\boldsymbol{A}$ is any linearly ordered ŁΠ$\frac{1}{2}$-algebra, then the algebra $\boldsymbol{A}'$ obtained from $\boldsymbol{A}$ by omitting the interpretations of $\rightarrow_\Pi$ and $\frac{1}{2}$ is the interval algebra of an ordered integral domain, $\mathbf{D}_{\boldsymbol{A}}$. Let $\boldsymbol{F}$ be the fraction field of $\mathbf{D}_{\boldsymbol{A}}$. It suffices to prove that every $c \in F \cap [0,1]$ is in $\boldsymbol{A}$. Writing, for $z \in \mathsf{Z}$ and for $\alpha \in A$, $z + \alpha$ instead of (z, α), we can represent any $c \in F \cap [0,1]$ as $c = \frac{z+\alpha}{y+\beta}$, where $\alpha, \beta \in A, \alpha < 1, \beta < 1, z, y \in \mathsf{Z}, z \geq 0, y \geq 0$, if $y = 0$, then $\beta \neq 0$, and either $z < y$, or $z = y$ and $\alpha \leq \beta$. Now, if $y = 0$, then $z = 0$, and $c = \beta \rightarrow_\Pi \alpha \in A$. Otherwise, let

$$d = \left(\frac{1}{2} \oplus \left(\frac{1}{2} \cdot \frac{1}{y} \cdot \beta\right)\right) \rightarrow_\Pi \left(\frac{1}{2} \cdot \frac{1}{y}\right).$$

It is easy to check that $c = zd \oplus (\alpha \cdot d) \in A$.

(5) Let $\boldsymbol{A}$ be an ŁΠ-chain. If $\boldsymbol{A}$ has more than two elements, it is an ŁΠ$\frac{1}{2}$-chain by (3). If that is not the case, then $\boldsymbol{A}$ is obviously isomorphic to the interval ŁΠ$\frac{1}{2}$-algebra of Z. □

The above results show the strong connection between ŁΠ$\frac{1}{2}$-chains and ordered fields. We are now going to see that each ŁΠ$\frac{1}{2}$-chain can be formally defined within the related ordered field.

From now on, we will denote ŁΠ$\frac{1}{2}$-algebras by $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, while we will use $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ for ordered fields. A subscript will make explicit the relation between an ŁΠ$\frac{1}{2}$-chain and an ordered field, i.e.: given an ŁΠ$\frac{1}{2}$-chain $\boldsymbol{A}$, the related field is denoted by $\mathbf{F}_{\boldsymbol{A}}$; conversely, given an ordered field $\mathbf{F}$, the related ŁΠ$\frac{1}{2}$-chain is denoted by $\boldsymbol{A}_{\mathbf{F}}$.

Let Th denote a first-order classical theory in some language $\mathcal{L}$. We use $\sqcap$, $\sqcup$, $\backsim$, and $\Longrightarrow$ for classical conjunction, disjunction, negation, and implication, respectively. Denote by $\mathsf{Th}(\mathrm{OF})$ the first-order theory of ordered fields in the language

$$\langle +, \cdot, -, <, 0, 1\rangle$$

(see [57]). Denote by $\mathsf{Th}(\text{Ł}\Pi\frac{1}{2})$ the first-order theory of linearly ordered ŁΠ$\frac{1}{2}$-algebras in the language

$$\langle \oplus, \neg, \cdot, \rightarrow_\Pi, 0, 1, \tfrac{1}{2}, <\rangle.$$

$\mathsf{Th}(\text{Ł}\Pi\frac{1}{2})$ is axiomatized by the universal closure of the equations defining the variety of ŁΠ$\frac{1}{2}$-algebras plus the sentence defining the linearity of the order relation $<$. We are going to show that $\mathsf{Th}(\text{Ł}\Pi\frac{1}{2})$ can be interpreted into $\mathsf{Th}(\mathrm{OF})$.

Let $\mathcal{L}$ be a signature of the form $\langle <, f_1, \ldots, f_n, c_1, \ldots, c_m\rangle$, where each f_i is a function symbol and each c_j is a constant symbol. $\mathcal{L}$ will be assumed to include no relation symbol but $<$ (and, of course, =). By an unnested atomic formula in $\mathcal{L}$ we mean one of the following formulas:

- $x = y, x < y$;
- $x = c$, $c = x$, $x < c$, $c < x$, for some constant symbol $c \in \mathcal{L}$;
- $f(\overline{x}) = y$, $y = f(\overline{x})$, $f(\overline{x}) < y$, $y < f(\overline{x})$, for some function symbol $f \in \mathcal{L}$.

A formula is called unnested if all its atomic subformulas are unnested. Then, it is straightforward to prove that (see [57]):

LEMMA 5.3.11. *For a first-order language* $\mathcal{L} = \langle <, f_1, \ldots, f_n, c_1, \ldots, c_m \rangle$*, every formula is equivalent to an unnested formula.*

The following definition sets what it means for a theory Th_1 in the language $\mathcal{L}_1$ to be interpretable in a theory Th_2 in the language $\mathcal{L}_2$.

DEFINITION 5.3.12. *Let* Th_1 *and* Th_2 *be two theories in the languages* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, respectively.* Th_1 *is* interpretable *in* Th_2 *if*

(i) *there exists an* $\mathcal{L}_2$*-formula* $\Xi(z)$*,*

(ii) *there exists a map* $\sharp$ *from the set of unnested atomic* $\mathcal{L}_1$*-formulas into the set of* $\mathcal{L}_2$ *formulas,*

(iii) *there exists a map* $\star$ *from the set of models of* Th_1 *into the set of models of* Th_2*,*

such that, for every $\mathbf{M} \models \mathsf{Th}_1$ *(i.e., for every model* $\mathbf{M}$ *of* Th_1*),*

(1) *there exists a bijection* $h_{\mathbf{M}} \colon M \to \{a \mid \mathbf{M}^\star \models \Xi(a)\}$ *from the domain of* $\mathbf{M}$ *into the set defined by* $\Xi(z)$ *over the domain of* $\mathbf{M}^\star$*;*

(2) *for all* $\bar{b} \in M$ *and each unnested atomic* $\mathcal{L}_1$*-formula* Φ

$$\mathbf{M} \models \Phi(\bar{b}) \quad \textit{iff} \quad \mathbf{M}^\star \models \Phi^\sharp(h_{\mathbf{M}}(\bar{b})).$$

$\mathbf{M}^\star$ *is called the* complementary model of $\mathbf{M}$.

The above definition together with Lemma 5.3.11 yields that the interpretation of Th_1 into Th_2 can be extended to arbitrary formulas.

LEMMA 5.3.13. *Let* Th_1 *and* Th_2 *be two theories in the languages* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, respectively. Suppose that* Th_1 *is interpretable in* Th_2*. Then, the mapping* $\sharp$ *can be applied to the whole set of* $\mathcal{L}_1$*-formulas. In other words, for each* $\mathcal{L}_1$*-formula* $\Phi(\overline{x})$ *there exists an* $\mathcal{L}_2$*-formula* $\Phi^\sharp(\overline{x})$ *so that, for every* $\mathbf{M} \models \mathsf{Th}_1$ *and all* $\bar{b} \in M$

$$\mathbf{M} \models \Phi(\bar{b}) \quad \textit{iff} \quad \mathbf{M}^\star \models \Phi^\sharp(h_{\mathbf{M}}(\bar{b})).$$

Proof. This can be simply proved by induction on the complexity of the formulas.[22] By Lemma 5.3.11, every formula in the language $\mathcal{L}_1$ is equivalent to an unnested formula, and so all its atomic subformulas are unnested. Definition 5.3.12 sets the case of unnested atomic formulas. For the case of compound formulas, define:

$$\begin{aligned}
(\backsim\Phi)^\sharp &:= \backsim(\Phi^\sharp),\\
(\Phi \sqcap \Psi)^\sharp &:= \Phi^\sharp \sqcap \Psi^\sharp,\\
(\forall x\ \Phi)^\sharp &:= \forall x\ \Xi(x) \Longrightarrow \Phi^\sharp,\\
(\exists x\ \Phi)^\sharp &:= \exists x\ \Xi(x) \sqcap \Phi^\sharp,
\end{aligned}$$

where $\Xi(x)$ is the formula defining the domain of each $\mathbf{M} \models \mathsf{Th}_1$ into the related complementary model $\mathbf{M}^\star$. □

[22] The proof is basically the same as the proof of Theorem 5.3.2 in [57].

THEOREM 5.3.14. $\mathsf{Th}(\text{Ł}\Pi\frac{1}{2})$ *is interpretable into* $\mathsf{Th}(\mathrm{OF})$.

Proof. Let $\Phi(\bar{x})$ be any formula in the language $\langle \oplus, \neg, \cdot, \to_\Pi, 0, 1, \frac{1}{2}, < \rangle$, and let $\boldsymbol{A}$ be an $\text{Ł}\Pi\frac{1}{2}$-chain.

Any unnested atomic formula of $\mathsf{Th}(\text{Ł}\Pi\frac{1}{2})$ can be translated into a formula in the language of ordered fields as follows (the translation of inequalities is similar):

$$\begin{aligned}
x = 0 &\quad\mapsto\quad x = 0 \\
x = 1 &\quad\mapsto\quad x = 1 \\
x = \tfrac{1}{2} &\quad\mapsto\quad x = \tfrac{1}{2} \\
x \oplus y = z &\quad\mapsto\quad ((x + y \leq 1) \sqcap (x + y = z)) \sqcup ((x + y \geq z) \sqcap (z = 1)) \\
\neg x = y &\quad\mapsto\quad 1 - x = y \\
x \cdot y = z &\quad\mapsto\quad x \cdot y = z \\
x \to_\Pi y = z &\quad\mapsto\quad ((x \leq y) \sqcap (z = 1)) \sqcup ((x > y) \sqcap (y = z \cdot x)).
\end{aligned}$$

The formula $\Xi(x) := (0 \leq x) \sqcap (x \leq 1)$ obviously defines over $\mathbf{F}_{\boldsymbol{A}}$ an order-isomorphic copy of the domain of $\boldsymbol{A}$. By Lemma 5.3.13, the above translation, and the fact that every $\text{Ł}\Pi\frac{1}{2}$-chain is the interval algebra of an ordered field, we conclude that there exists a formula $\Phi^\sharp(\overline{x})$ in the language of ordered fields such that, for all $\bar{a} \in A$:

$$\boldsymbol{A} \models \Phi(\overline{a}) \quad \text{iff} \quad \mathbf{F}_{\boldsymbol{A}} \models \Phi^\sharp(\overline{a}). \qquad \square$$

A *real closed field* $\overline{\mathbf{F}} = \langle F, +, -, \cdot, \leq, 0, 1 \rangle$ is a field with a unique ordering whose positive cone $\{x \mid x \geq 0\}$ is the set of squares of F, and every polynomial of $F[X]$, of odd degree, has a root in F. In other words, real closed fields are ordered fields satisfying the following sentences for each $n \geq 0$:

$$\forall x_1 \ldots \forall x_n \, (x_1^2 + \cdots + x_n^2 + 1) \neq 0,$$
$$\forall x \exists y \, ((y^2 = x) \sqcup (y^2 + x = 0)),$$
$$\forall x_0 \ldots \forall x_{2n} \exists y \, \left(y^{2n+1} + \sum_{i=0}^{2n} x_i y^i \right) = 0.$$

Given an ordered field $\mathbf{F}$, the *real closure* of $\mathbf{F}$ is an algebraic extension $\overline{\mathbf{F}}$ which is a real closed field and with a unique ordering extending the ordering of $\mathbf{F}$. The field of real numbers R is a real closed field, while the field of rational numbers Q is not. The real closure of Q is the field of real algebraic numbers A, i.e. the real roots of polynomials

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$$

with integer coefficients $a_0, \ldots, a_m$. The first-order theory of real closed fields, denoted as $\mathsf{Th}(\mathrm{RCF})$, admits *quantifier elimination* in the language $\langle +, -, \cdot, <, 0, 1 \rangle$, Tarski [80], i.e. any first-order formula in the language of ordered fields is equivalent to a quantifier-free formula in the same language. As a consequence, the theory of real closed fields is complete, i.e. for every formula Φ, either $\mathsf{Th}(\mathrm{RCF}) \vdash \Phi$ or $\mathsf{Th}(\mathrm{RCF}) \vdash \backsim\Phi$, and decidable (see [8]). Moreover, $\mathsf{Th}(\mathrm{RCF})$ is *model-complete*, i.e. every embedding between any of its models is elementary, i.e.: for any $\mathbf{F}, \mathbf{G} \models \mathsf{Th}(\mathrm{RCF})$, any embedding $f\colon \mathbf{F} \to \mathbf{G}$, any formula $\Phi(x_1, \ldots, x_m)$, and $a_1, \ldots, a_m \in F$,

$$\mathbf{F} \models \Phi(a_1, \ldots, a_m) \quad \text{iff} \quad \mathbf{G} \models \Phi(f(a_1), \ldots, f(a_m)).$$

A is (elementarily) embeddable into every real closed field, and therefore all real closed field are elementarily equivalent to A, and, thus, to each other.

We are now ready to prove that ŁΠ and ŁΠ$\frac{1}{2}$ are generated by their related structure over the real unit interval $[0, 1]$.

THEOREM 5.3.15.

(1) $\mathbb{LP}$ *is generated by* $[0,1]_{Ł\Pi}$.

(2) $\mathbb{LP}\frac{1}{2}$ *is generated by* $[0,1]_{Ł\Pi\frac{1}{2}}$.

Proof. We prove the case of $\mathbb{LP}$. The proof for $\mathbb{LP}\frac{1}{2}$ is almost identical. Let $\boldsymbol{A}$ be any ŁΠ-algebra, and let $\phi(\overline{x})$ be an equation not valid in $\boldsymbol{A}$. $\boldsymbol{A}$ is a subdirect product of ŁΠ-chains $\boldsymbol{B}_i$, and so there is at least one $\boldsymbol{B}_i$ in which $\phi(\overline{x})$ fails. By Lemma 5.3.10, $\boldsymbol{B}_i$ is either the interval algebra of Z or the interval algebra of an ordered field $\mathbf{F}_{\boldsymbol{B}}$. If $\boldsymbol{B}_i$ is the interval algebra of Z, then it can be trivially embedded into the interval algebra of Q. In both cases, Q and $\mathbf{F}_{\boldsymbol{B}}$ can be embedded into their respective (unique) real closure. This means that $\boldsymbol{B}_i$ is embeddable into the interval ŁΠ-algebra of a real closed field. Real closed fields are elementarily equivalent to each other, and to R, in particular. By Theorem 5.3.14, we know that every ŁΠ-chain is definable into its related ordered field. This means that $\phi(\overline{x})$ does not hold in $[0,1]_{Ł\Pi}$. □

Recall that, given a set A, a discriminator function on A is the function $d\colon A^3 \to A$ defined by

$$d(a,b,c) = \begin{cases} a & \text{if } a \neq b, \\ c & \text{if } a = b. \end{cases}$$

A ternary term $d(x,y,z)$ representing the discriminator function on a structure $\boldsymbol{A}$ is called a *discriminator term* for $\boldsymbol{A}$. Let $\mathbb{K}$ be a class of algebras with a common discriminator term $d(x,y,z)$. The variety generated by $\mathbb{K}$ is called a *discriminator variety* (see [7]). Both $\mathbb{LP}$ and $\mathbb{LP}\frac{1}{2}$ are discriminator varieties, with discriminator term

$$d(x,y,z) := (\triangle(x \leftrightarrow y) \wedge y) \vee \neg((\triangle(x \leftrightarrow y)) \wedge x).$$

We use the above fact to show that:

THEOREM 5.3.16.

(1) $\mathbb{LP}$ *is generated as a quasivariety by* $[0,1]_{Ł\Pi}$.

(2) $\mathbb{LP}\frac{1}{2}$ *is generated as a quasivariety by* $[0,1]_{Ł\Pi\frac{1}{2}}$.

Proof. We use the Theorem 5.3.15 and the fact that the variety $\mathbb{V}$ generated by any discriminator algebra $\boldsymbol{A}$ is generated by $\boldsymbol{A}$ as a quasivariety (see [7] for the details). We give an explicit proof, for the sake of completeness.

Being $\mathbb{LP}$ a discriminator variety means that it also is congruence-distributive, and, by Jónsson's Lemma [7], every subdirectly irreducible algebra in $\mathbb{LP}$ is a homomorphic image of a subalgebra $\boldsymbol{B}$ of an ultraproduct of $[0,1]_{\text{Ł}\Pi}$. $\boldsymbol{B}$ is a discriminator algebra and, therefore, it is simple. Thus, every subdirectly irreducible member of $\mathbb{LP}$ embeds into an ultraproduct of $[0,1]_{\text{Ł}\Pi}$. It follows that every member of $\mathbb{LP}$ is isomorphic to a subdirect product of subalgebras of ultraproducts of $[0,1]_{\text{Ł}\Pi}$, i.e.: $\mathbb{LP}$ is generated by $[0,1]_{\text{Ł}\Pi}$ as a quasivariety.

The same argument applies to $\mathbb{LP}^{\frac{1}{2}}$. □

The above results now allow us to prove that $\mathbb{PMV}^+$ is a quasivariety generated by $[0,1]_{\mathrm{PMV}}$.

LEMMA 5.3.17.

(1) $\mathbb{PMV}^+$ *coincides with the quasivariety generated by* $[0,1]_{\mathrm{PMV}}$. *Therefore, every quasiequation* Φ *is true in* $[0,1]_{\mathrm{PMV}}$ *iff it is true in all* PMV^+*-algebras.*

(2) PMV^+*-chains are precisely the isomorphic images of subalgebras of ultrapowers of* $[0,1]_{\mathrm{PMV}}$. *Therefore, every universal formula* Ψ *is true in* $[0,1]_{\mathrm{PMV}}$ *iff it is true in all* PMV^+*-chains.*

Proof. First of all, every PMV^+-chain $\boldsymbol{A}$ is the interval algebra of an ordered integral domain $\mathbf{D}_{\boldsymbol{A}}$ which is embeddable in an ordered field $\boldsymbol{F}$. The interval algebra $\boldsymbol{A}_{\mathbf{F}}$ of $\mathbf{F}$ is an ŁΠ-chain that contains $\boldsymbol{A}$ as a subreduct. So, every linearly ordered PMV^+-algebra is a subreduct of a linearly ordered ŁΠ-algebra.

(1) $\mathbb{PMV}^+$ is a quasivariety containing $[0,1]_{\mathrm{PMV}}$, therefore it contains the quasivariety generated by $[0,1]_{\mathrm{PMV}}$. Conversely, by Theorem 5.3.16, the variety of ŁΠ-algebras is generated as a quasivariety by $[0,1]_{\text{Ł}\Pi}$. Hence, every PMV^+-algebra is a subreduct of a direct product of ultrapowers of $[0,1]_{\text{Ł}\Pi}$; therefore, it is a subalgebra of a direct product of ultrapowers of $[0,1]_{\mathrm{PMV}}$.

(2) Every linearly ordered PMV^+-algebra $\boldsymbol{A}$ is a subreduct of a linearly ordered ŁΠ-algebra $\boldsymbol{B}$. The ordered field $\mathbf{F}_{\boldsymbol{B}}$ related to $\boldsymbol{B}$ (see Lemma 5.3.10) embeds into its real closure $\overline{\mathbf{F}_{\boldsymbol{B}}}$, which is elementarily equivalent to R. By Frayne's Theorem (see [8]), $\overline{\mathbf{F}_{\boldsymbol{B}}}$ embeds into an ultrapower R^* of R. Consequently, $\boldsymbol{B}$ embeds into the interval algebra $\boldsymbol{C}_{\mathsf{R}^*}$ of R^*, which is isomorphic to an ultrapower of $[0,1]_{\text{Ł}\Pi}$. Clearly, $\boldsymbol{A}$ is an isomorphic image of subalgebras of ultrapowers of $[0,1]_{\mathrm{PMV}}$, and every universal formula Ψ is true in $[0,1]_{\mathrm{PMV}}$ iff it is true in all PMV^+-chains. □

As an immediate consequence of the above lemma, we have:

THEOREM 5.3.18. $\mathbb{PMV}^+$ *is generated as a quasivariety by the class of* PMV^+*-chains and by* $[0,1]_{\mathrm{PMV}}$.

THEOREM 5.3.19. $\mathbb{PMV}^+$ *is not a variety.*

Proof. Let $[0,1]^*$ be a non-trivial ultraproduct of $[0,1]_{Ł\Pi}$, and let ϵ be a (strictly) positive infinitesimal, i.e., a non-zero element of $[0,1]^*$ such that for every positive natural number n, one has: $(n)\epsilon \leq 1-\epsilon$. Call $[0,1]^-$ the PMV-reduct of $[0,1]^*$, and let J be a subset of $[0,1]^-$ consisting of all z for which there is a natural number n such that $z \leq (n)\epsilon^2$. J is an MV-ideal, therefore, by Lemma 5.3.4, it determines a congruence θ of $[0,1]^-$. Now let $[0,1]^-/\theta$ denote the quotient of $[0,1]^-$ modulo θ, and for $a \in [0,1]^-$, let a_θ denote the equivalence class of a modulo θ. Then, in $[0,1]^-/\theta$ we have that $\epsilon_\theta^2 = 0$ and $\epsilon_\theta \neq 0$. On the other hand, the quasi identity $\forall x\ (x^2 = 0) \implies (x = 0)$ is true in $[0,1]_{Ł\Pi}$, and in every PMV^+-algebra. Consequently, $[0,1]^-/\theta$ cannot be a PMV^+-algebra. Indeed, $\mathbb{PMV}^+$ is not closed under homomorphic images, and, so, it does not constitute a variety. □

As an immediate consequence of Theorem 5.3.16 and Theorem 5.3.18, we now have:

THEOREM 5.3.20. $\mathrm{PŁ}'$*,* $\mathrm{Ł\Pi}$*, and* $\mathrm{Ł\Pi}\frac{1}{2}$ *have the* $\mathrm{FS}\mathcal{R}\mathrm{C}$*.*

We can now show a negative result about the Deduction Theorem for $\mathrm{PŁ}'$.

COROLLARY 5.3.21. $\mathrm{PŁ}'$ *does not satisfy the same Deduction Theorem as* $\mathrm{PŁ}$*.*

Proof. Suppose that the Deduction Theorem is valid. Since $\neg(\varphi \,\&_\Pi\, \varphi) \vdash \neg\varphi$, we have that for some n, $(\neg(\varphi \,\&_\Pi\, \varphi))^n \to \neg\varphi$ is a theorem of $\mathrm{PŁ}'$. This means that $(\neg(\varphi \,\&_\Pi\, \varphi))^n \to \neg\varphi$ is a tautology for evaluations into the reals, i.e., there is an n such that $(\neg(x \cdot x))^n \leq \neg x$ for each $x \in [0,1]$. Notice that the derivatives of $(\neg(x \cdot x))^n$ and $\neg x$ at the point 0 are equal to 0 and -1, respectively. This implies that for each n, there exists an x such that $(\neg(x \cdot x))^n > \neg x$, which leads to a contradiction. □

5.4 The expressive power of $\mathrm{Ł\Pi}\frac{1}{2}$

$\mathrm{Ł\Pi}\frac{1}{2}$ is certainly one of the most expressive ($\triangle$-)core fuzzy logics. We are going to make this even clearer by proving that the first-order theory $\mathsf{Th}(\mathrm{OF})$ of ordered fields can be interpreted into the theory $\mathsf{Th}(\mathrm{Ł\Pi}\frac{1}{2})$ of $\mathrm{Ł\Pi}\frac{1}{2}$-chains. As a consequence we will obtain that the theory of the reals in interpretable into the equational theory of $\mathrm{Ł\Pi}\frac{1}{2}$. Therefore, functions definable in the theory of the reals, in the language of ordered fields, can be defined by means of $\mathrm{Ł\Pi}\frac{1}{2}$-equations (in a sense that will be made clear later on). This will allow us to show that many ($\triangle$-)core fuzzy logics can be interpreted within $\mathrm{Ł\Pi}\frac{1}{2}$.

Let **F** be any ordered field, and let $(0,1)_{\mathbf{F}} = F \cap (0,1)$ ($[0,1]_{\mathbf{F}} = F \cap [0,1]$) denote the open (closed) unit interval of **F**. Let $\sigma\colon (0,1)_{\mathbf{F}} \to F$ be the following strictly increasing surjective mapping, continuous w.r.t. the order topology:

$$\sigma(x) = \begin{cases} \frac{(2x-1)}{2x} & \text{if } 0 < x \leq \frac{1}{2}, \\ \frac{1-2x}{2(x-1)} & \text{if } \frac{1}{2} \leq x < 1, \end{cases}$$

whose inverse is

$$\sigma^{-1}(y) = \begin{cases} \frac{1}{2(1-y)} & \text{if } -\infty < y \leq 0, \\ \frac{2y+1}{2y+2} & \text{if } 0 \leq y < +\infty. \end{cases}$$

We use the function h to define an isomorphic copy of $\mathbf{F}$ over $(0,1)_{\mathbf{F}}$. This will allow, in turn, to define an interpretation of $\mathbf{F}$ over $\boldsymbol{A}_{\mathbf{F}}$. More precisely, we define:

$$\begin{aligned} x +_0 y &= \sigma^{-1}(\sigma(x) + \sigma(y)) \\ x \cdot_0 y &= \sigma^{-1}(\sigma(x) \cdot \sigma(y)) \\ -_0 x &= \sigma^{-1}(-\sigma(x)) \\ 0_0 &= \sigma^{-1}(0) = \tfrac{1}{2} \\ 1_0 &= h^{-1}(1) \\ x \leq_0 y &\text{ iff } \sigma(x) \leq \sigma(y). \end{aligned}$$

Clearly, σ is an isomorphism from

$$\mathbf{F}_0 = \langle (0,1) \cap F, +_0, \cdot_0, -_0, \leq_0, 0_0, 1_0 \rangle \text{ onto } \mathbf{F} = \langle F, +, \cdot, -, \leq, 0, 1 \rangle.$$

LEMMA 5.4.1. $\mathsf{Th}(\mathrm{OF})$*is interpretable into* $\mathsf{Th}(\text{Ł}\Pi\tfrac{1}{2})$.

Proof. First, let $\star$ be the map associating each ordered field $\mathbf{F}$ to its interval Ł$\Pi\frac{1}{2}$-chain $\boldsymbol{A}_{\mathbf{F}}$. Let $\Xi(x)$ be the Ł$\Pi\frac{1}{2}$-formula $(0 < x) \sqcap (x < 1)$. Then, clearly, $\sigma^{-1} \colon F \to \{a \mid \boldsymbol{A}_{\mathbf{F}} \models \chi(a)\}$. We need to show that for all $\overline{b} \in F$ and unnested atomic formulas $\Phi(\overline{x})$

$$\mathbf{F} \models \Phi(\overline{b}) \quad \text{iff} \quad \boldsymbol{A}_{\mathbf{F}} \models \Phi^{\sharp}(\sigma^{-1}(\overline{b})).$$

Now,

- if $\Phi(\overline{x})$ is $x = y$ or $x < y$, then $\Phi^{\sharp}(\overline{x})$ is $x = y$ or $x < y$, respectively;
- if $\Phi(\overline{x})$ is $x = 0$, $x = 1$, $x < 0$, or $x < 1$, then $\Phi^{\sharp}(\overline{x})$ is $x = 0$, $x = \frac{3}{4}$, $x < 0$, or $x < \frac{3}{4}$, respectively (the other cases are similar);
- if $\Phi(\overline{x})$ is $x = -y$, then $\Phi^{\sharp}(\overline{x})$ is $x = \neg y$;
- if $\Phi(\overline{x})$ is either $z = x + y$, or $z = x \cdot y$, we can find, by using the above defined functions σ and σ^{-1}, the equivalent definitions of $x +_0 y$ and $x \cdot_0 y$ over $\mathbf{F}_0$. An easy but tedious calculation shows that there exist Ł$\Pi\frac{1}{2}$-formulas $\Phi^{+_0}(x,y,z)$ and $\Phi^{\cdot_0}(x,y,z)$,[23] such that for all $a, b, c \in F$

$$\begin{aligned} \mathbf{F} \models a = b + c \quad &\text{iff} \quad \boldsymbol{A}_{\mathbf{F}} \models \Phi^{+_0}(\sigma^{-1}(a), \sigma^{-1}(b), \sigma^{-1}(c)), \\ \mathbf{F} \models a = b \cdot c \quad &\text{iff} \quad \boldsymbol{A}_{\mathbf{F}} \models \Phi^{\cdot_0}(\sigma^{-1}(a), \sigma^{-1}(b), \sigma^{-1}(c)). \end{aligned}$$ □

The next lemma shows that each quantifier-free Ł$\Pi\frac{1}{2}$-formula is equivalent to an Ł$\Pi\frac{1}{2}$-equation.

LEMMA 5.4.2. *Let* $\Phi(\overline{x})$ *be a quantifier-free formula in the language of* Ł$\Pi\frac{1}{2}$*-chains. Then there exists an* Ł$\Pi\frac{1}{2}$*-term* $t(\overline{x})$ *such that, for every* Ł$\Pi\frac{1}{2}$*-chain* $\boldsymbol{A}$*, and all* $b \in A$*:*

$$\boldsymbol{A} \models \Phi(\overline{b}) \quad \textit{iff} \quad \boldsymbol{A} \models t(\overline{b}) = 1.$$

Proof. The formula $x \leq y$ is interpreted by the term $\triangle(x \to y)$, which will be denoted as $t^{\leq}(x,y)$. The formula $x = y$ is translated by $\triangle(x \leftrightarrow y)$ (denoted as $t^{=}(x,y)$). Then, $x < y$ is interpreted by $t^{<}(x,y) = t^{\leq}(x,y) \wedge \neg(t^{=}(x,y))$.

[23]See [62] for an example of how a similar construction works.

We define for every quantifier-free formula Φ in the language of $Ł\Pi\frac{1}{2}$-chains, a term t^Φ in the following inductive way:

- If Φ is $x = y$, then $t^\Phi := t^=(x, y)$.
- If Φ is $x < y$, then $t^\Phi := t^<(x, y)$.
- If Φ is $\Psi \sqcup \Lambda$ ($\Psi \sqcap \Lambda$, $\backsim\Psi$ respectively), then $t^\Phi := t^\Psi \vee t^\Lambda$ ($t^\Psi \wedge t^\Lambda$, $\neg(t^\Psi)$ respectively).

The claim easily follows from the above construction. □

The following corollary is an easy consequence of Lemma 5.4.1 and Lemma 5.4.2.

COROLLARY 5.4.3. *Let* $\mathbf{F}$ *be an ordered field, and let* $\Phi(x_1, \ldots, x_n)$ *be a quantifier-free formula in the language of ordered fields with coefficients in* Q*. Then, there exists an* $Ł\Pi\frac{1}{2}$*-term* $t(x_1, \ldots, x_n)$ *such that, for all* $a_1, \ldots, a_n \in F$*, the following are equivalent:*

(1) $\mathbf{F} \models \Phi(a_1, \ldots, a_n)$.

(2) $\boldsymbol{A}_\mathbf{F} \models t(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)) = 1$.

From the above translation, we immediately obtain the following result:

THEOREM 5.4.4. *Let* $\overline{\mathbf{F}}$ *be a real closed field, and let* $\Phi(x_1, \ldots, x_n)$ *be any formula in the language of ordered fields with coefficients in* Q*. Then, there exists an* $Ł\Pi\frac{1}{2}$*-term* $t(x_1, \ldots, x_n)$ *such that, for all* $a_1, \ldots, a_n \in F$*:*

$$\overline{\mathbf{F}} \models \Phi(a_1, \ldots, a_n) \quad \textit{iff} \quad \boldsymbol{A}_{\overline{\mathbf{F}}} \models t(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)) = 1.$$

Proof. The theory of real closed fields enjoys the elimination of quantifiers in the language of ordered fields, thus $\Phi(x_1, \ldots, x_n)$ is equivalent to a quantifier-free formula $\Psi(x_1, \ldots, x_n)$. The result follows from Corollary 5.4.3. □

Given a real closed field $\overline{\mathbf{F}} = \langle F, +, \cdot, -, \leq, 0, 1\rangle$, a *semialgebraic set* is a subset of F^n of the form

$$\bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in F^n \mid f_{i,j}(x) \circledcirc_{i,j} 0\} \qquad (\natural)$$

where $f_{i,j}(x) \in F[X_1, \ldots, X_n]$ and $\circledcirc_{i,j}$ is either $<$ or $=$, for $i = 1, \ldots, s$, and $j = 1, \ldots, r_i$. It is easy to see that semialgebraic subsets of F are exactly finite unions of points and open intervals. In particular, every semialgebraic subset of F^n can be written as a finite union of semialgebraic sets of the form:

$$\{x \in F^n \mid f_1(x) = \cdots = f_l(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\},$$

where $f_1, \ldots, f_l, g_1, \ldots, g_m \in F[X_1, \ldots, X_n]$. In other words, semialgebraic sets are subsets of a real closed field defined by a finite Boolean combination of polynomial equations and inequalities.

We call a set $S \subseteq \mathsf{R}^n$ Q-*semialgebraic* if it has the form ($\natural$), where the $f_{i,j}(x)$ are polynomials with rational coefficients (see also Chapter IX).

Recall that a set $S \subseteq \mathsf{R}^n$ is said to be *definable* in R, in the language of ordered fields, if there is a first-order formula $\Phi(x_1, \ldots, x_n)$ such that

$$S = \{\langle a_1, \ldots, a_n\rangle \mid \mathsf{R} \models \Phi(a_1, \ldots, a_n)\}\,.$$

A function is said to be *definable* in R iff its graph is definable in R.

DEFINITION 5.4.5.

(1) *A function* $g\colon [0,1]^n \to [0,1]$ *is said to be* term-definable *in* $\text{Ł}\Pi\frac{1}{2}$ *if there is a term* $t(x_1, \ldots, x_n)$ *of* $\text{Ł}\Pi\frac{1}{2}$*-algebras such that for all* $a_1, \ldots, a_n \in [0,1]$*:*

$$t(a_1, \ldots, a_n) = g(a_1, \ldots, a_n).$$

(2) *A set* $X \subseteq [0,1]^n$ *is said to be* definable *in* $\text{Ł}\Pi\frac{1}{2}$ *if its characteristic function is term-definable in* $\text{Ł}\Pi\frac{1}{2}$.

(3) *A function* f *is said to be* implicitly definable *in* $\text{Ł}\Pi\frac{1}{2}$ *if its graph is definable in* $\text{Ł}\Pi\frac{1}{2}$.

Recall that an $\text{Ł}\Pi\frac{1}{2}$-*hat* over $[0,1]^n$ is a function $h\colon [0,1]^n \to [0,1]$ such that there exist a Q-semialgebraic set $S \subseteq [0,1]^n$ and polynomials $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n) \in \mathsf{Q}[X_1, \ldots, X_n]$ such that $g(x_1, \ldots, x_n)$ has no zeros on S, $h = \frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)}$ on S, and $h = 0$ on $[0,1]^n \backslash S$ (see Chapter IX). A function $h\colon [0,1]^n \to [0,1]$ is said to be *piecewise rational* if it is the supremum of finitely many $\text{Ł}\Pi\frac{1}{2}$-hats.

The next theorem, whose proof can be found in Chapter IX, characterizes term-definable functions.

THEOREM 5.4.6. *A function* $h\colon [0,1]^n \to [0,1]$ *is term-definable in* $\text{Ł}\Pi\frac{1}{2}$ *iff it is a piecewise rational function.*

Clearly, it follows that functions as $\sqrt{x}$ or $\sqrt{1-x^2}$ cannot be defined by terms in $\text{Ł}\Pi\frac{1}{2}$.

The next theorem gives a characterization of definable sets and, therefore, of implicitly definable functions in $\text{Ł}\Pi\frac{1}{2}$.

THEOREM 5.4.7. *A set* $S \subseteq [0,1]^n$ *is definable in* $\text{Ł}\Pi\frac{1}{2}$ *iff it is definable in* R *by a formula with rational coefficients iff it is* Q*-semialgebraic. Thus, a function* $f\colon [0,1]^n \to [0,1]$ *is implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$ *iff its graph is* Q*-semialgebraic.*

Proof. If $S \subseteq [0,1]^n$ is definable in $\text{Ł}\Pi\frac{1}{2}$, then, by Theorem 5.3.14, there exists a formula in the language of ordered fields that defines S over R, and so S is obviously Q-semialgebraic.

Conversely, if $S \subseteq [0,1]^n$ is Q-semialgebraic, it is defined by a Boolean combination of polynomial equalities and inequalities over R, and, consequently, by Theorem 5.4.4, it is definable in $\text{Ł}\Pi\frac{1}{2}$.

It is then obvious that a function $f\colon [0,1]^n \to [0,1]$ is implicitly definable in $\text{Ł}\Pi\frac{1}{2}$ iff its graph is Q-semialgebraic. $\square$

The fact that functions definable in the theory of the real numbers can be implicitly defined in $\text{Ł}\Pi\frac{1}{2}$ can be used to show that some ($\triangle$-)core fuzzy logics have a faithful interpretation in the equational theory of $\text{Ł}\Pi\frac{1}{2}$, as shown below.

DEFINITION 5.4.8. *Let* L *be any* ($\triangle$-)*core fuzzy logic whose equivalent algebraic semantics is a variety generated by a structure whose lattice reduct is the real unit interval* $[0,1]$. L *is said to be* definable in $\text{Ł}\Pi\frac{1}{2}$ *if the interpretation of each* L*-connective over* $[0,1]$ *corresponds to an implicitly definable function.*

THEOREM 5.4.9. *Let* L *be any* ($\triangle$-) *core fuzzy logic definable in* $\text{Ł}\Pi\frac{1}{2}$. *Then, for every* L*-formula* ϕ *there exists an* $\text{Ł}\Pi\frac{1}{2}$*-formula* $\phi^\bullet$ *such that*

$$\models_{\mathrm{L}} \phi \quad \textit{iff} \quad \models_{\text{Ł}\Pi\frac{1}{2}} \phi^\bullet.$$

Proof. Let $C = \{\lambda_i\}_{1 \leq i \leq n}$ be the set of basic connectives of L. By definition, the graph of each λ_i is term-definable by a formula ψ_{λ_i} in $\text{Ł}\Pi\frac{1}{2}$.

Now, let ϕ be any L-formula, and let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be the set of subformulas of ϕ. Next, to each γ_j associate a variable v_j (different variables for different subformulas). For each λ_i, let

$$\Sigma_{\lambda_i} = \{(v_\sigma, v_{\sigma_{1_i}}, \ldots, v_{\sigma_{t_i}}) \mid v_\sigma = \lambda_i(v_{\sigma_{1_i}}, \ldots, v_{\sigma_{t_i}})\},$$

where each v_{σ_j} is a variable associated to a subformula.

For each $(v_\sigma, v_{\sigma_{1_i}}, \ldots, v_{\sigma_{t_i}}) \in \Sigma_{\lambda_i}$, introduce the formulas $\psi_{\lambda_i}(v_\sigma, v_{\sigma_{1_i}}, \ldots, v_{\sigma_{t_i}})$ for each basic connective λ_i. Each ψ_{λ_i} defines the graph of λ_i.

For each $\lambda_i \in F$, denote by Θ_{λ_i} the conjunction of all the above formulas. Let $\phi^\bullet$ be the following formula:

$$\phi^\bullet := \left(\bigwedge \Theta_{\lambda_i}\right) \to v_m,$$

where v_m is the variable associated to the whole formula ϕ.

It can be to checked from the construction that

$$\models_{\mathrm{L}} \phi \quad \text{iff} \quad \models_{\text{Ł}\Pi\frac{1}{2}} \phi^\bullet. \qquad \square$$

The previous theorem shows that $\text{Ł}\Pi\frac{1}{2}$'s expressive power allows to faithfully interpret several logical systems. As an example, we are going to show that the logic associated to any continuous t-norm representable as a finite ordinal sum is definable (see Chapter I for the background notions on t-norms and ordinal sums).

THEOREM 5.4.10. *Let* $*$ *be a continuous t-norm. The following are equivalent:*

(1) *Up to isomorphism,* $*$ *is implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$.

(2) *Up to isomorphism,* $*$ *is term-definable in* $\text{Ł}\Pi\frac{1}{2}$.

(3) $*$ *is representable as a finite ordinal sum of Łukasiewicz and product t-norms.*

Proof. If $*$ is term-definable it clearly also is implicitly definable. Thus, (2) implies (1).

Assume $*$ is representable as a finite ordinal sum of Łukasiewicz and Product t-norms. Without any loss of generality we can suppose that the cut points in the ordinal sum are rationals. Then, the graph of the function

$$x * y = \begin{cases} a_i + (b_i - a_i) \cdot \left(\frac{x-a_i}{b_i-a_i} *_i \frac{y-a_i}{b_i-a_i} \right) & \text{if } x, y \in (a_i, b_i]^2, \\ \min\{x, y\} & \text{otherwise} \end{cases}$$

is obviously a Q-semialgebraic set, and therefore $*$ is implicitly definable in $Ł\Pi\frac{1}{2}$, by Theorem 5.4.7. Moreover, an easy inspection shows that $*$ is a piecewise rational function, and therefore it is term-definable, by Theorem 5.4.6. Thus (3) implies both (1) and (2).

We show that (1) implies (3). Suppose that $*$ is an infinite ordinal sum of Product and Łukasiewicz components. The set Id_* of idempotent elements of $*$ is definable as $Id_* = \{x \mid x{*}x = x\}$. However, Id_* cannot be a Q-semialgebraic set, since it is not a finite union of points. This clearly implies that the graph of $*$ cannot be Q-semialgebraic, and, as a consequence, $*$ is not implicitly definable. This concludes the proof of the theorem. □

Now, we can prove:

THEOREM 5.4.11. *Let* L_* *be the logic of a continuous t-norm* $*$ *representable as a finite ordinal sum. Then* L_* *is definable in* $Ł\Pi\frac{1}{2}$.

Proof. It is easy to see that the residuum of any implicitly definable left-continuous t-norm is implicitly definable. Indeed, if $*$ is implicitly definable in $Ł\Pi\frac{1}{2}$, then its graph is definable in the theory of reals by a quantifier-free formula $\Phi(x, y, z)$, and so is the graph of its residuum $\Rightarrow_*$ by means of the first-order formula

$$\forall u \forall v (\Phi(u, x, v) \Longrightarrow (u \leq z \Longleftrightarrow v \leq y)).$$

The claim now follows by Theorem 5.4.4 and Theorem 5.4.10. □

6 Historical remarks and further reading

6.1 Expansions with truth-constants

When one is interested in explicitly representing and reasoning with intermediate degrees of truth, a convenient and elegant way is by introducing truth-constants into the language. In fact, if one introduces in the language new constant symbols $\overline{r}$ for suitable values $r \in [0, 1]$ and stipulates that $e(\overline{r}) = r$ for all truth-evaluations, then a formula of the kind $\overline{r} \to \varphi$ becomes 1-true under any evaluation e whenever $r \leq e(\varphi)$. The first formal treatment of this kind of system is due to Pavelka [75], who built a propositional many-valued logical system, which turned out to be equivalent to the expansion of Łukasiewicz logic by adding into the language a truth-constant $\overline{r}$ for each *real* $r \in [0, 1]$, together with a number of additional axioms. The resulting system was shown to be complete in a non-standard sense, later known as Pavelka-style completeness (see Section 2.1). Novák extended Pavelka's approach to Łukasiewicz first-order logic [73].

Later, Hájek [46] proved that Pavelka's logic could be significantly simplified by showing that it is enough to expand the language only with a countable set of truth-constants, one for each *rational* in $[0, 1]$, and by adding to the logic the so-called *book-keeping axioms* dealing with truth-constants. He called this new system Rational Pavelka logic (RPL), and proved it is standard complete for finite theories in the usual sense. He also defined the logic $\mathrm{RPL}\forall$, the first-order expansion of RPL, and showed that $\mathrm{RPL}\forall$ enjoys the same Pavelka-style completeness.

Several expansions à la Pavelka with truth-constants of fuzzy logics different from Łukasiewicz have also been studied, mainly related to the other two outstanding continuous t-norm based logics, namely Gödel and product logic. We may cite [46] where an expansion of $\mathrm{G}_\triangle$ with a finite number of rational truth-constants was studied, [28] where the authors define logical systems obtained by adding (rational) truth-constants to $\mathrm{G}_\sim$ (Gödel logic with an involutive negation) and to Π (product logic) and $\Pi_\sim$ (product logic with an involutive negation). In the case of the rational expansions of Π and $\Pi_\sim$ an infinitary inference rule (from $\{\varphi \to \overline{r} \mid r \in \mathsf{Q} \cap (0, 1]\}$ infer $\varphi \to \overline{0}$) is introduced in order to get Pavelka-style completeness.

Following the same line, Cintula gives in [16] a definition of what he calls *Pavelka-style extension* of a particular fuzzy logic. He considers the Pavelka-style extensions of the most popular fuzzy logics, and for each one of them he defines an axiomatic system with infinitary rules (to overcome discontinuities like in the case of Π explained above) which is proved to be Pavelka-style complete. Moreover he also considers the first-order versions of these extensions and provides necessary conditions for them to satisfy Pavelka-style completeness.

A difficulty concerning Pavelka-style completeness is that it cannot be obtained for logics different from Łukasiewicz without the introduction of infinitary rules, since Łukasiewicz logic is the only fuzzy logic whose truth-functions (conjunction and implication) are continuous functions. Due to this fact, a more general approach has been developed in a series of papers [12, 26, 30–33, 78] where, rather than Pavelka-style completeness, the authors have focused on the usual notion of completeness of a logic.

In all these works, special attention has been paid to formulas of the kind $\overline{r} \to \varphi$, where $\overline{r}$ denotes the truth-constant r and φ is a formula without any additional truth-constants. Actually, this kind of formulas has been extensively considered in other frameworks for reasoning with partial degrees of truth, like in Novák's evaluated syntax formalism based on Łukasiewicz logic (see e.g. [74]) or in fuzzy logic programming (see e.g. [85]). In particular, these formulas can be seen as a special kind of Novák's *evaluated* formulas, which are expressions a/A where a is a truth value (from a given algebra) and A is a formula that may contain truth-constants again, and whose interpretation is that the truth-value of A is at least a. Hence, our formulas $\overline{r} \to \varphi$ would be expressed as r/φ in Novák's evaluated syntax. On the other hand, formulas $\overline{r} \to \varphi$, when φ is a Horn-like rule of the form $b_1 \,\&\, \ldots \,\&\, b_n \to h$, also correspond to typical fuzzy logic programming rules $(b_1 \,\&\, \ldots \,\&\, b_n \to h, r)$, where r specifies a lower bound for the validity of the rule. Finally, truth-degrees in the syntax also appear in the Gerla's framework of abstract fuzzy logics [40], which is based on the notion of fuzzy consequence operators over fuzzy sets of formulas, where the membership degree of formulas are, again, interpreted as lower bounds of their truth-degrees.

6.2 Expansions with truth-stressing and truth-depressing hedges

There are two main references when talking about the formalization of truth-stressing hedges within the framework of mathematical fuzzy logic. The first one is Hájek's paper [47], already referred to in the previous sections, where he axiomatizes a logic for the hedge *very true* over BL. The second one is the paper by Vychodil [86], where the author extends Hájek's analysis to truth-depressing hedges.

A relevant further study of logics with truth-stressers can be found in the paper by Ciabattoni et al. [9], that makes significant contributions in various aspects. The authors basically consider expansions of MTL with a unary modality (i.e. a unary operator that satisfies axiom K and the necessitation rule), they consider three possible additional axioms to be added to Hájek axiomatics, and they develop proof systems for the new logics and study their algebraic and completeness properties. Given a logic L that is an extension of MTL, they consider the following logics particularly relevant for our purposes:

$$\text{L-KT}^r = \mathrm{L} + (\mathrm{VE1}) + (\mathrm{VE2}) + (\mathrm{VE3}) + \mathrm{NEC},$$
$$\text{L-S4}^r = \text{L-KT}^r + (\mathrm{VE4})\ s\varphi \to s(s\varphi).$$

Axiom $(\mathrm{VE4})$, together with axiom $(\mathrm{VE1})$, forces the truth-stressing hedges to be closed over their image, i.e. $s\varphi$ has to be equivalent to $s(s\varphi)$ (hence s becomes a closure operator like in some previous work; see [50], for instance).

Notice that Hájek's logic BL_{SK} (called BL_{vt} in his paper) is nothing but the logic BL-KT^r. Moreover, Ciabattoni et al. prove in [9] standard completeness of the L-S4^r logics for different choices for L, namely MTL, SMTL, C_nMTL, IMTL, and $\mathrm{C}_n\mathrm{IMTL}$. Finally, observe that after adding the axiom $s\varphi \vee \neg s\varphi$ to L-KT^r, s turns to be equivalent to the well-know Monteiro–Baaz projection connective $\triangle$.

Other papers dealing with particular types of truth-stressers are:

- The paper [50], a pioneering work in the setting of truth-stressing hedges, which proves that the Yashin *strong future tense operator* can be interpreted, in our framework, as a hedge over G that is a closure operator and satisfies axiom K.

- The paper [48], which defines the logical system $\mathrm{BL}^!_{LU}$ obtained by adding two unary connectives, L and U, (for truth stresser and depresser) to $\mathrm{BL}_\triangle$ that are required to be idempotent with respect to the monoidal operation, among other technical properties. The paper contains an interesting result about the undecidability of $*$-tautologies.

- In the paper [51] the authors introduce in $\mathrm{BL}\forall$ a new unary connective At, interpreted as *almost true*, in order to analyze the *sorites* paradox in the setting of mathematical fuzzy logic. It turns out that the axioms proposed for this new connective are $(\mathrm{STL1})$ and the new axiom

  $$(\varphi \to \psi) \to (At\varphi \to At\psi)$$

  which is stronger than (MON). However, the axiom $(\mathrm{STL2})$ is not required.

- The paper [68] studies the system obtained by adding to a fuzzy logic L a unary connective called *storage operator* which has some analogies with Girard's exponentials and behaves as an idempotent truth-stresser closed over its image (it is in fact an interior operator).

Despite the undoubtable theoretical interest of these papers, truth-hedges that are either closure operators, satisfy axiom K, or are idempotent, have a quite limited behavior and can account only for some very special cases of truth-stressers.

As for truth-depressers, Vychodil [86] first introduces a logic combining both a truth-stresser and a truth-depresser. His logic, called $\mathrm{BL}_{vt,st}$, is defined as an expansion of Hájek's BL_{vt} logic with a new unary connective "slightly true" d and with the following additional axioms:

(ST1) $\varphi \to d\varphi$
(ST2) $d\varphi \to \neg s \neg \varphi$
(ST3) $s(\varphi \to \psi) \to (d\varphi \to d\psi)$

This logic is proved to be complete with respect to the class of all linearly-ordered $\mathrm{BL}_{vt,st}$-algebras (defined in the obvious way). Note that axioms (ST1) and (ST2) put into relation both connectives s and d. Vychodil also proposes two slightly different axiomatizations (systems I and II) for the truth-depressing hedge *slightly true* alone. They are defined again as expansions of BL with the unary connective d. Namely, the system (I) has the following set of additional axioms:

(DH1) $\varphi \to d\varphi$
(DH2) $\neg d(\bar{0})$
(DH3) $d(\varphi \to \psi) \to (d\varphi \to d\psi)$

while the system (II) includes the axioms (DH1), (DH2), and

(DH4) $(\varphi \to \psi) \to (d\varphi \to d\psi)$

Both systems also have the following inference rule:

(RN_d) from $\neg\varphi$ infer $\neg d\varphi$

Chain-completeness for both systems is proved, but, again, the issue of real completeness is left open.

Notice that axioms (DH1) and (DH2) correspond exactly to (STL1) and (STL2) of the logic L_D, and that the inference rule (RN_d) is derivable from the rule (MON) using axiom (STL2). So, again, the main difference between Vychodil's logics and the logics L_D is the presence of the K-like axioms (DH3) and (DH4), which do not appear in the logics L_D.[24]

[24] In fact for both Vychodil's systems over any axiomatic extension L of *Involutive* MTL logic IMTL the associated real chains are real L-chains taking the identity function Id as a truth-depresser. In fact, if d is a truth-depresser such that $d \neq Id$, then (DH3) and (DH4) are not satisfied. Namely, if $d \neq Id$, there exists an element $x \in (0,1)$ such that $d(x) > x$, and thus $d(x \Rightarrow 0) \geq (x \Rightarrow 0) = \neg x > \neg d(x) = d(x) \Rightarrow 0$. As a consequence, the only function d over an IMTL-chain that satisfies the axioms of either system (I) or (II) is the identity function.

Following [34], in Section 3, we have presented a more general approach to fuzzy logics with truth-stressing (depressing) hedges. The main advantage of the proposed systems with respect to the previously proposed ones is that we can show standard completeness, i.e. completeness with respect to the class of chains over the real unit interval expanded by arbitrary (stressing and depressing) hedges. The price paid in this process is that the class of corresponding algebras cannot be shown in general to be a variety any longer, but only a quasivariety. Actually it remains as an open problem to prove or disprove whether they form in fact a variety in the general case. It is only proved that L_S-algebras form a variety if either L is the logic of a finite BL-chain or it is the case that $\triangle$ is definable in L_S or when axiom $(\mathrm{VE2})$ for the s is derivable in L_S. All these cases enjoy a local or global deduction-detachment theorem.

6.3 Expansions with an involutive negation

Heyting algebras endowed with an involution were introduced by Moisil [65] already in 1942, as the algebraic models of an expansion of intuitionistic propositional calculus by means of a De Morgan negation. These algebras have been extensively investigated by Monteiro under the name of *symmetric Heyting algebras* [72]. They were also considered by Sankappanavar [77] independently from the previous work. Recently, in [11], the authors go a step further in the algebraic study of symmetric residuated lattices, in particular focusing on the properties of the combination of the two negations.

In the setting of fuzzy logic, early papers about fuzzy connectives were interested in the so-called De Morgan triples, i.e., triples formed by a t-norm, an involutive negation and the dual t-conorm (see for instance [1, 82]). In this tradition, Gehrke et al. [38] study De Morgan triples, their associated logics and the subvarieties generated by De Morgan triples over the real unit interval, with special attention to the case of De Morgan triples based on strict t-norms. In fact, the logics studied in [38] are implication-free fragments of the logics $\Pi_\sim$ and $\mathrm{SBL}_\sim$ described in this section.

In the more formal setting of mathematical fuzzy logic, the first paper on expansions with an involutive negation was [28], where Esteva et al., also independently from previous work, defined expansions of the logic SBL and their main axiomatic extensions, G and Π, with an involution. The key observation in [28] was that Monteiro–Baaz's $\triangle$ operator is definable as $\triangle\varphi := \neg{\sim}\varphi$ when $\neg$ is Gödel negation (common to SBL, G and Π) and $\sim$ is an involutive negation, and hence one can define, e.g., the expanded logic $\mathrm{SBL}_\sim$ as it was over the logic $\mathrm{SBL}_\triangle$, which makes the axiomatization much easier. The same approach works for G and Π. This line of research was continued by Flaminio and Marchioni in [35], where the more general case of adding an involution to $\mathrm{MTL}_\triangle$ and their axiomatic extensions is defined. On the other hand, Cintula et al. investigated in [19, 20] the lattice of subvarieties generated by $\mathrm{SBL}_\sim$-chains and $\Pi_\sim$-chains, while Haniková and Savický [55] went further in the study of subvarieties generated by $\mathrm{SBL}_\sim$-chains investigating isomorphisms between pairs formed by a (SBL) t-norm and different involutive negations. The main result is the characterization of families of such pairs such that they are either pairwise isomorphic or they generate incomparable subvarieties.

6.4 Expansions of Łukasiewicz logic

Section 5 covers the most important notions regarding expansions of Łukasiewicz logic and MV-algebras. We review the basic literature on the topics introduced above (not necessarily in strict chronological order), and mention where the interested reader can also find several complementary and advanced results.

Rational Łukasiewicz logic and DMV-algebras were introduced and studied by Gerla in [39], where the author also proved the basic completeness results. The category of DMV-algebras was shown to be equivalent to the category of divisible Abelian ℓ-groups with strong unit (both with homomorphisms). Moreover, the satisfiability problem for RL was proved to be $\mathbf{NP}$-complete.

Hájek, Godo, and Esteva [49] were the first to approach the problem of expanding Łukasiewicz logic with the product connective. In fact, they introduced an expansion of Rational Pavelka logic with product in order to define a logic to represent simple and conditional probability (see Section 2.1).

Later, Riečan [76] was the first to present an expansion of MV-algebras with the product operation, with the goal of defining product measures taking values in MV-algebras. A first algebraic study of MV-algebras with product was given by Di Nola and Dvurečenskij in [23]. In that work, however, the reduct $\langle A, \cdot, 1\rangle$ is not necessarily a commutative monoid. They proved a categorical equivalence result between this class of MV-algebras with product and non-commutative lattice-ordered rings.

PMV-algebras were introduced by Montagna in [66], where the author proved that the related variety is generated by the class of chains and that each PMV-chain is the interval algebra of an ordered commutative ring. Montagna explored in [69] several algebraic properties of subreducts of MV-algebras with the product conjunction and the product implication. In particular, he showed that $\mathbb{PMV}^+$ is not a variety and that it is a quasivariety generated by $[0,1]_{\mathrm{PMV}}$. The logics PŁ and PŁ$'$ were introduced by Horčik and Cintula in [58] and shown to have finite strong completeness w.r.t. to the class of chains of the related variety and quasivariety.

ŁΠ and its related algebras were first introduced by Esteva and Godo in [27]. Montagna in [66], and Esteva, Godo, and Montagna [29] further investigated ŁΠ and introduced ŁΠ$\frac{1}{2}$ making their relation w.r.t. ordered fields explicit, and proving finite strong completeness w.r.t. evaluations into the reals. Montagna also proved in [69] that both the variety of ŁΠ and ŁΠ$\frac{1}{2}$ algebras are generated as quasivarieties by $[0,1]_{\text{ŁΠ}}$ and $[0,1]_{\text{ŁΠ}\frac{1}{2}}$, respectively. Cintula provided different equivalent axiomatizations for ŁΠ and ŁΠ$\frac{1}{2}$ in [14], and for their algebras in [17], and also studied their first-order expansion in [15].

An in-depth categorical investigation of the classes of PMV, PMV^+, ŁΠ, and ŁΠ$\frac{1}{2}$ algebras was carried out by Montagna in [66, 67, 69], where the author showed that the categories of PMV, PMV^+, ŁΠ, and ŁΠ$\frac{1}{2}$ algebras, all with homomorphisms, are equivalent to the categories of commutative lattice-ordered f-rings with strong unit, commutative lattice-ordered f-integral domains with strong unit, Q-f-semifields and f-semifields, with homomorphisms, respectively.

Montagna and Panti [70] gave a functional characterization of free ŁΠ and ŁΠ$\frac{1}{2}$ algebras in terms of piecewise rational functions (see also Chapter IX). A similar characterization for free PMV^+-algebras is still unavailable and is strictly related to the long-standing Pierce-Birkhoff conjecture in semialgebraic geometry [56].

Vetterlein gave in [84] a comprehensive study of the connections between certain classes of effect algebras, lattice-ordered rings and expansions of MV-algebras with product. Vetterlein explicitly showed the one-to-one correspondence between the class of f-product effect algebras, torsion-free f-product effect algebras, torsion-free f-product effect algebras with strict compatibility, and divisible torsion-free f-product effect algebras with strict compatibility, and the class of PMV, PMV^+, ŁΠ, and Ł$\Pi\frac{1}{2}$ algebras, respectively.

Basic definitions and completeness results concerning expansions with rational truth-constants and their related bookkeeping axioms for PŁ and PŁ$'$ can be found in [58], and in [29] for ŁΠ and Ł$\Pi\frac{1}{2}$. Expansions of PMV and PMV^+ algebras with $\triangle$ (and their related logics) are extensively studied in [58, 66, 67, 70], where completeness is shown along with categorical representations and functional characterizations.

The tautology problem for Ł$\Pi\frac{1}{2}$ was shown to be in $\mathbf{PSPACE}$ by Hájek and Tulipani in [54] by relying on a polynomial-time translation into the universal theory of the field of reals.

The definability in Ł$\Pi\frac{1}{2}$ of logics based on continuous t-norms representable as finite ordinal sums was first studied by Cintula in [14], who showed that such t-norms are term-definable in Ł$\Pi\frac{1}{2}$. Marchioni and Montagna investigated functional definability issues within the equational theory of Ł$\Pi\frac{1}{2}$-algebras in [62, 63], studying the definability of Q-semialgebraic sets, and triangular norms and uninorms. In particular they gave a complete characterization of term-definable and implicitly definable continuous t-norms and weak nilpotent minimum t-norms. Marchioni and Montagna also showed that the universal theory of real closed fields is definable into the equational theory of Ł$\Pi\frac{1}{2}$-algebras and they both share the same computational complexity. Moreover, they proved that the logic associated to any implicitly definable uninorm is in $\mathbf{PSPACE}$, while the logic associated to any class of implicitly definable uninorms is decidable.

Marchioni investigated in [61] the lattice of subvarieties of Ł$\Pi\frac{1}{2}$-algebras showing that it has the cardinality of the continuum. [61] also contains a brief study of the basic model-theoretic properties of the theory of Ł$\Pi\frac{1}{2}$-chains that are interval algebras of real closed fields.

Other expansions of PMV-algebras were introduced exploiting their expressive power. The operations of the MV-algebra over the reals are continuous functions, but that is not the case for ŁΠ and Ł$\Pi\frac{1}{2}$ algebras, since the product implication is obviously not continuous. The quasivariety of ŁΠ_q-algebras was introduced for this reason in [71], by Spada and Montagna, expanding the language of PMV-algebras with the operator $\to_q$, interpreted as a continuous approximation of the product implication.

Spada introduced in [79] μŁΠ-algebras and their logic. μŁΠ-algebras are an expansion of ŁΠ-algebras with fixed point operators $\mu x_{t(x,\overline{y})}$ for each term $t(x,\overline{y})$ not containing the product implication. Spada gave completeness results, showing that μŁΠ-chains are exactly the interval algebras of real closed fields, and that the category of μŁΠ-algebras with homomorphisms is equivalent to the category of real closed f-semifields with homomorphisms. A characterization of free μŁΠ-algebras and other model-theoretic results were given by Marchioni and Spada in [64].

Extending the approach initiated in [49] over Rational Pavelka logic, $Ł\Pi\frac{1}{2}$ has also been used for the logical representation of uncertainty measures. Esteva, Godo, and Hájek in [41, 42] defined a fuzzy modal logic over $Ł\Pi\frac{1}{2}$ to represent conditional probability and belief functions, Flaminio and Montagna defined in [36] an expansion of $Ł\Pi\frac{1}{2}$, called SŁΠ, to represent non-standard probabilities. Also, Godo and Marchioni in [43] for coherent conditional probability and Marchioni in [60] for conditional possibility, relied on $Ł\Pi\frac{1}{2}$ for building a logic to represent such classes of measures.

Finally, we mention the work [22] by Ciucci and Flaminio, where the authors use $Ł\Pi\frac{1}{2}$ to define inner and outer approximations of fuzzy sets.

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BIBLIOGRAPHY

[1] Claudi Alsina, Enric Trillas, and Llorenç Valverde. Some logical connectives for fuzzy set theory. *Journal of Mathematical Analysis and Applications*, 93(1):15–26, 1983.

[2] Matthias Baaz. Infinite-valued Gödel logic with 0-1-projections and relativisations. In Petr Hájek, editor, *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics*, volume 6 of *Lecture Notes in Logic*, pages 23–33. Springer-Verlag, Brno, 1996.

[3] Wyllis Bandler and Ladislav J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4(1):13–30, 1980.

[4] Willem J. Blok and Don L. Pigozzi. *Algebraizable Logics*, volume 396 of *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, RI, 1989. Freely downloadable from `http://orion.math.iastate.edu/dpigozzi/`.

[5] Félix Bou, Francesc Esteva, Josep Maria Font, Àngel Gil, Lluís Godo, Antoni Torrens, and Ventura Verdú. Logics preserving degrees of truth from varieties of residuated lattices. *Journal of Logic and Computation*, 19(6):1031–1069, 2009.

[6] Félix Bou, Francesc Esteva, Lluís Godo, and Ricardo O. Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *Journal of Logic and Computation*, 21(5):739–790, 2011.

[7] Stanley Burris and H.P. Sankappanavar. *A Course in Universal Algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, 1981.

[8] Chen Chung Chang and H. Jerome Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, Amsterdam, 1973.

[9] Agata Ciabattoni, George Metcalfe, and Franco Montagna. Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions. *Fuzzy Sets and Systems*, 161(3):369–389, 2010.

[10] Roberto Cignoli, Itala M.L. D'Ottaviano, and Daniele Mundici. *Algebraic Foundations of Many-Valued Reasoning*, volume 7 of *Trends in Logic*. Kluwer, Dordrecht, 1999.

[11] Roberto Cignoli and Francesc Esteva. Commutative integral bounded residuated lattices with an added involution. *Annals of Pure and Applied Logic*, 161(2):150–160, 2009.

[12] Roberto Cignoli, Francesc Esteva, and Lluís Godo. On Łukasiewicz logic with truth constants. In Oscar Castillo, Patricia Melin, Oscar Montiel Ross, Roberto Sepúlveda Cruz, Witold Pedrycz, and Janusz Kacprzyk, editors, *Theoretical Advances and Applications of Fuzzy Logic and Soft Computing*, volume 42/2007 of *Advances in Soft Computing*, pages 869–875. Springer, Berlin/Heidelberg, 2007.

[13] Roberto Cignoli and Antoni Torrens. An algebraic analysis of product logic. *Multiple-Valued Logic*, 5(1):45–65, 2000.

[14] Petr Cintula. An alternative approach to the ŁΠ logic. *Neural Network World*, 11(6):561–572, 2001.

[15] Petr Cintula. Advances in the ŁΠ and ŁΠ$\frac{1}{2}$ logics. *Archive for Mathematical Logic*, 42(5):449–468, 2003.

[16] Petr Cintula. *From Fuzzy Logic to Fuzzy Mathematics*. PhD thesis, Czech Technical University, Faculty of Nuclear Sciences and Physical Engineering, Prague, 2005.

[17] Petr Cintula. A note to the definition of the ŁΠ-algebras. *Soft Computing*, 9(8):575–578, 2005.

[18] Petr Cintula, Francesc Esteva, Joan Gispert, Lluís Godo, Franco Montagna, and Carles Noguera. Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies. *Annals of Pure and Applied Logic*, 160(1):53–81, 2009.

[19] Petr Cintula, Erich Peter Klement, Radko Mesiar, and Mirko Navara. Residuated logics based on strict t-norms with an involutive negation. *Mathematical Logic Quarterly*, 52(3):269–282, 2006.

[20] Petr Cintula, Erich Peter Klement, Radko Mesiar, and Mirko Navara. Fuzzy logics with an additional involutive negation. *Fuzzy Sets and Systems*, 161(3):390–411, 2010.

[21] Petr Cintula and Carles Noguera. Implicational (semilinear) logics I: A new hierarchy. *Archive for Mathematical Logic*, 49(4):417–446, 2010.

[22] Davide Ciucci and Tommaso Flaminio. Generalized rough approximations in ŁΠ$\frac{1}{2}$. *International Journal of Approximate Reasoning*, 48(2):544–558, 2008.

[23] Antonio Di Nola and Anatolij Dvurečenskij. Product MV-algebras. *Multiple-Valued Logic*, 6:193–215, 2001.

[24] Antonio Di Nola, Francesc Esteva, Lluís Godo, and Franco Montagna. Varieties of BL-algebras. *Soft Computing*, 9(12):875–888, 2005.

[25] Francesc Esteva, Joan Gispert, Lluís Godo, and Franco Montagna. On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic. *Studia Logica*, 71(2):199–226, 2002.

[26] Francesc Esteva, Joan Gispert, Lluís Godo, and Carles Noguera. Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results. *Fuzzy Sets and Systems*, 158(6):597–618, 2007.

[27] Francesc Esteva and Lluís Godo. Putting together Łukasiewicz and product logic. *Mathware and Soft Computing*, 6(2–3):219–234, 1999.

[28] Francesc Esteva, Lluís Godo, Petr Hájek, and Mirko Navara. Residuated fuzzy logics with an involutive negation. *Archive for Mathematical Logic*, 39(2):103–124, 2000.

[29] Francesc Esteva, Lluís Godo, and Franco Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[30] Francesc Esteva, Lluís Godo, and Carles Noguera. On rational Weak Nilpotent Minimum logics. *Journal of Multiple-Valued Logic and Soft Computing*, 12(1–2):9–32, 2006.

[31] Francesc Esteva, Lluís Godo, and Carles Noguera. First-order t-norm based fuzzy logics with truth-constants: Distinguished semantics and completeness properties. *Annals of Pure and Applied Logic*, 161(2):185–202, 2009.

[32] Francesc Esteva, Lluís Godo, and Carles Noguera. Expanding the propositional logic of a t-norm with truth-constants: Completeness results for rational semantics. *Soft Computing*, 14(3):273–284, 2010.

[33] Francesc Esteva, Lluís Godo, and Carles Noguera. On expansions of WNM t-norm based logics with truth-constants. *Fuzzy Sets and Systems*, 161(3):347–368, 2010.

[34] Francesc Esteva, Lluís Godo, and Carles Noguera. Fuzzy logics with truth hedges revisited. In Sylvie Galichet, Javier Montero, and Gilles Mauris, editors, *Proceedings of the Joint 7th Conference of EUSFLAT and LFA-2011*, pages 146–152, Amsterdam, 2011. Atlantis Press.

[35] Tommaso Flaminio and Enrico Marchioni. T-norm based logics with an independent involutive negation. *Fuzzy Sets and Systems*, 157(4):3125–3144, 2006.

[36] Tommaso Flaminio and Franco Montagna. A logical and algebraic treatment of conditional probability. *Archive for Mathematical Logic*, 44(2):245–262, 2005.

[37] Josep Maria Font, Angel Gil, Antoni Torrens, and Ventura Verdú. On the infinite-valued Łukasiewicz logic that preserves degrees of truth. *Archive for Mathematical Logic*, 45(7):839–868, 2006.

[38] Mai Gehrke, Carol Walker, and Elbert Walker. Fuzzy logics arising from strict de Morgan systems. In Steve Rodabaugh and Erich Peter Klement, editors, *Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets*, pages 257–276. Kluwer, Dordrecht, 2003.

[39] Brunella Gerla. *Many-Valued Logics of Continuous T-Norms and Their Functional Representation*. PhD thesis, University of Milano, Milano, 2002.

[40] Giangiacomo Gerla. *Fuzzy Logic—Mathematical Tool for Approximate Reasoning*, volume 11 of *Trends in Logic*. Kluwer and Plenum Press, New York, 2001.

[41] Lluís Godo, Francesc Esteva, and Petr Hájek. Reasoning about probability using fuzzy logic. *Neural Network World*, 10(5):811–823, 2000. Special issue on SOFSEM 2000.

[42] Lluís Godo, Petr Hájek, and Francesc Esteva. A fuzzy modal logic for belief functions. *Fundamenta Informaticae*, 57(2–4):127–146, 2003.

[43] Lluís Godo and Enrico Marchioni. Coherent conditional probability in a fuzzy logic setting. *Logic Journal of the Interest Group of Pure and Applied Logic*, 14(3):457–481, 2006.

[44] H. Peter Gumm and Aldo Ursini. Ideals in universal algebra. *Algebra Universalis*, 19(1):45–54, 1984.

[45] Petr Hájek. Fuzzy logic and arithmetical hierarchy. *Fuzzy Sets and Systems*, 73(3):359–363, 1995.

[46] Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998.

[47] Petr Hájek. On very true. *Fuzzy Sets and Systems*, 124(3):329–333, 2001.

[48] Petr Hájek. Some hedges for continuous t-norms logics. *Neural Network World*, 12(2):159–164, 2002.

[49] Petr Hájek, Lluís Godo, and Francesc Esteva. Fuzzy logic and probability. In *Proceedings of the 11th Annual Conference on Uncertainty in Artificial Intelligence UAI '95*, pages 237–244, Montreal, 1995.

[50] Petr Hájek and Dagmar Harmancová. A hedge for Gödel fuzzy logic. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 8(4):495–498, 2000.

[51] Petr Hájek and Vilém Novák. The sorites paradox and fuzzy logic. *International Journal of General Systems*, 32(4):373–383, 2003.

[52] Petr Hájek, Jeff Paris, and John C. Shepherdson. Rational Pavelka logic is a conservative extension of Łukasiewicz logic. *Journal of Symbolic Logic*, 65(2):669–682, 2000.

[53] Petr Hájek and David Švejda. A strong completeness theorem for finitely axiomatized fuzzy theories. *Tatra Mountains Mathematical Publications*, 12(3):213–219, 1997.

[54] Petr Hájek and Sauro Tulipani. Complexity of fuzzy probability logics. *Fundamenta Informaticae*, 45(3):207–213, 2001.

[55] Zuzana Haniková and Petr Savický. Distinguishing standard SBL-algebras with involutive negations by propositional formulas. *Mathematical Logic Quarterly*, 54(6):579–596, 2008.

[56] Melvin Henriksen and John R. Isbell. Lattice-ordered rings and function rings. *Pacific Journal of Mathematics*, 12:533–565, 1962.

[57] Wilfried Hodges. *Model Theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Ox, Cambridge, 1993.

[58] Rostislav Horčík and Petr Cintula. Product Łukasiewicz logic. *Archive for Mathematical Logic*, 43(4): 477–503, 2004.

[59] Sándor Jenei and Franco Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[60] Enrico Marchioni. Possibilistic conditioning framed in fuzzy logics. *International Journal of Approximate Reasoning*, 43(2):133–165, 2006.

[61] Enrico Marchioni. Ordered fields and $\text{Ł}\Pi\frac{1}{2}$-algebras. *Soft Computing*, 13(6):559–564, 2009.

[62] Enrico Marchioni and Franco Montagna. Complexity and definability issues in $\text{Ł}\Pi\frac{1}{2}$. *Journal of Logic and Computation*, 17(2):311–331, 2007.

[63] Enrico Marchioni and Franco Montagna. On triangular norms and uninorms definable in $\text{Ł}\Pi\frac{1}{2}$. *International Journal of Approximate Reasoning*, 47(2):179–201, 2008.

[64] Enrico Marchioni and Luca Spada. Advances in the theory of $\mu\text{Ł}\Pi$ algebras. *Logic Journal of the Interest Group of Pure and Applied Logic*, 19(3):476–489, 2011.

[65] Grigore Constantin Moisil. Logique modale. *Disquisitiones Mathematicae et Physicae*, 2:3–98, 1942. (In French).

[66] Franco Montagna. An algebraic approach to propositional fuzzy logic. *Journal of Logic, Language and Information*, 9(1):91–124, 2000.

[67] Franco Montagna. Functorial representation theorems for MV_Δ-algebras with additional operators. *Journal of Algebra*, 238(1):99–125, 2001.

[68] Franco Montagna. Storage operators and multiplicative quantifiers in many-valued logics. *Journal of Logic and Computation*, 14(2):299–322, 2004.

[69] Franco Montagna. Subreducts of MV-algebras with product and product residuation. *Algebra Universalis*, 53(1):109–137, 2005.

[70] Franco Montagna and Giovanni Panti. Adding structure to MV-algebras. *Journal of Pure and Applied Algebra*, 164(3):365–387, 2001.

[71] Franco Montagna and Luca Spada. Continuous approximations of product implication in MV-algebras with product. *Soft Computing*, 9(3):149–154, 2005.

[72] António Monteiro. Sur les algèbres de Heyting symétriques. *Portugaliae Mathematica*, 39(1–4):1–239, 1980.

[73] Vilém Novák. On the syntactico-semantical completeness of first-order fuzzy logic part I (syntax and semantic), part II (main results). *Kybernetika*, 26:47–66, 134–154, 1990.

[74] Vilém Novák, Irina Perfilieva, and Jiří Močkoř. *Mathematical Principles of Fuzzy Logic*. Kluwer, Dordrecht, 2000.

[75] Jan Pavelka. On fuzzy logic I, II, III. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25:45–52, 119–134, 447–464, 1979.

[76] Beloslav Riečan. On the Product MV algebras. *Tatra Mountains Mathematical Publications*, 16(1): 143–149, 1999.

[77] H.P. Sankappanavar. Heyting algebras with a dual lattice endomorphism. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 33(6):565–573, 1987.

[78] Petr Savický, Roberto Cignoli, Francesc Esteva, Lluís Godo, and Carles Noguera. On product logic with truth constants. *Journal of Logic and Computation*, 16(2):205–225, 2006.

[79] Luca Spada. ŁΠ logic with fixed points. *Archive for Mathematical Logic*, 47(7–8):741–763, 2008.

[80] Alfred Tarski. *A Decision Method for Elementary Algebra and Geometry*. University of California Press, Berkeley, CA, 1951.

[81] Enric Trillas. Sobre funciones de negación en la teoría de conjuntos difusos. *Stochastica*, 3(1):47–60, 1979.

[82] Enric Trillas, Claudi Alsina, and Llorenç Valverde. Do we need max, min and $1-j$ in fuzzy set theory? In Ronald R. Yager, editor, *Fuzzy Set and Possibility Theory: Recent Developments*, pages 275–287. Pergamon Press, New York, 1982.

[83] Enric Trillas and Llorenç Valverde. An inquiry into indistinguishability operators. In Heinz J. Skala, Settimo Termini, and Enric Trillas, editors, *Aspects of Vagueness*, pages 231–256. Reidel, Dordrecht, 1984.

[84] Thomas Vetterlein. Partial algebras for Łukasiewicz logic and its extensions. *Archive for Mathematical Logic*, 44(7):913–933, 2005.

[85] Petr Vojtáš. Fuzzy logic programming. *Fuzzy Sets and Systems*, 124(3):361–370, 2001.

[86] Vilém Vychodil. Truth-depressing hedges and BL-logic. *Fuzzy Sets and Systems*, 157(15):2074–2090, 2006.

[87] Lotfi A. Zadeh. A fuzzy set theoretic interpretation of linguistic hedges. *Journal of Cybernetics*, 2(3): 4–34, 1972.

[88] Lotfi A. Zadeh. Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Transactions on Systems, Man, and Cybernetics*, 3(1):28–44, 1973.

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