Annals of Operations Research manuscript No.
(will be inserted by the editor)

An analytic derivation of the efficient frontier in biobjective cash management and its implications for policies

Francisco Salas-Molina ·
David Pla-Santamaria ·
Juan A. Rodriguez-Aguilar

Received: date / Accepted: date

Abstract Cash managers who optimize returns and risk rely on biobjective optimization models to select the best policies according to their risk preferences. In the related portfolio selection problem, Merton (1972) provided the first analytical derivation of the efficient frontier with all non-dominated return and risk combinations. This first proposal was later extended to account for three or more criteria by other authors. However, the cash management literature needs an analytical derivation of the efficient frontier to help cash managers evaluate the implications of selecting policies and risk measures. In this paper, we provide three analytic derivations of the efficient frontier determining a closed-form solution for the expected returns and risk relationship using three different risk measures. We study its main properties and its theoretical implications for policies. Using the variance of returns as a risk measure imposes limitations due to invertibility reasons.

1 Introduction

The cash management problem (CMP) deals with balancing what the company holds in cash and what has been placed in short-term investments. Holding cash for precautionary motives implies an opportunity cost equivalent to the missed return on alternative investments. On the other hand, companies aim to minimize transaction costs associated with movements between cash and investment accounts. As a result, the CMP can be defined as an optimization problem whose goal is to find the best sequence of transactions (policy) over a given time horizon.

Francisco Salas-Molina
Universitat Politècnica de València, Ferrándiz y Carbonell, s/n, 03801 Alcoy, Spain
E-mail: frasamo@upv.es

David Pla-Santamaria
Universitat Politècnica de València, Ferrándiz y Carbonell, s/n, 03801 Alcoy, Spain
E-mail: dplasan@upv.es

Juan A. Rodriguez-Aguilar
IIIA-CSIC, Campus UAB, 08913 Cerdanyola, Catalonia, Spain
E-mail: jar@iiia.csic.es

The cash management problem was first addressed by Baumol (1952) in its deterministic form as an inventory control problem. Miller and Orr (1966) introduced a stochastic approach by considering a symmetric Bernoulli process and control bounds. This approach was later followed by many cash management works described in the surveys by da Costa Moraes et al (2015) and by Salas-Molina et al (2023). More recent works about the CMP include a multiobjective approach by Salas-Molina et al (2018b), selecting cash management models by Salas-Molina et al (2018c), theoretical results on cash management systems by Salas-Molina et al (2021), and the proposal of online algorithms to minimize the maximum regret by Schroeder and Kacem (2019) and Schroeder and Kacem (2020) based on the min-max regret criterion by Savage (1951).

Similarly to the portfolio selection problem initially formulated by Markowitz (1952), the CMP can be solved from a biobjective perspective by simultaneously optimizing both the expected returns and risk of alternative policies. A critical decision in the biobjective CMP is selecting the appropriate risk measure (Salas-Molina, 2019). In addition, when dealing with two or more criteria in an optimization problem, it is essential to know the analytical form of the efficient frontier with all non-dominated objective combinations. In the portfolio selection problem, Merton (1972) provided the first analytical derivation of the efficient frontier, showing that the mathematical function that maps the variance of returns to mean returns is a parabola. Later on, Qi et al (2017); Qi and Steuer (2020); Qi (2022, 2020) and Qi and Li (2020) extended the work by Merton (1972) to account for three or more criteria in the portfolio selection problem.

In this paper, we provide an analytical derivation of the efficient frontier in the biobjective CMP and analyze its implications for cash management policies. Following Qi et al (2017), we consider an analytical derivation as obtained by formal mathematical calculus as opposed to being computed by an optimization algorithm. The specific characteristics of the biobjective CMP introduce an additional level of complexity. For example, while the covariance matrix of returns for alternative assets in the portfolio selection problem is assumed to be known in advance, the variance of periodic returns for alternative policies in the biobjective CMP as a risk measure is not known because it depends on the solution. In this sense, the selection of a risk measure is critical in terms of its analytical properties. More precisely, we show that the selection of the variance of returns as a risk measure prevents us from obtaining an analytical derivation that is computable in practice due to the presence of non-invertible matrices in its formulation. To circumvent this problem, we propose two alternative risk measures that allow us to propose an analytical derivation of the efficient frontier with all non-dominated return and risk combinations. Both measures are based on the sum of deviations for a given reference. In one of them, this reference is set to zero. The approach to measure risk in cash management as a sum of deviations from a reference has been recently proposed by Salas-Molina et al (2018a) and Salas-Molina (2019, 2020).

The main advantage of using an analytical derivation of the efficient frontier is the possibility of developing formal analysis on efficient policies. As a result, the main contributions of this paper are as follows:

1. We propose three analytical derivations of the efficient frontier in a risk-returns space using different risk measures.

2. We show that the variance of returns as a risk measure imposes an essential limitation in practice.
3. We provide theoretical results derived from formal analysis of the mathematical expression of the efficient frontier.

In addition to this introduction, in Section 2 we describe the steps required to obtain three analytical derivations of the efficient frontier. In Section 3, we provide further insights and implications derived from the analytical derivations obtained. In Section 4, we illustrate our main results through numerical examples. Finally, we conclude this paper in Section 5, highlighting future lines of research.

2 An analytic derivation of the risk-returns efficient frontier

Consider the common two assets framework in cash management shown in Figure 1 in which account 1 is a regular cash account and account 2 is an investment account. It can be reasonably assumed that the excess in returns between accounts 2 and 1 is the holding cost of keeping idle cash in account 1. Let h be the difference in returns obtained per money unit in account 2 with respect to the returns obtained per money unit in account 1. In other words, h is excess returns obtained per money unit between accounts 2 and 1. In addition, there is a transaction cost γ for transferring money which is proportional to the amount transferred x_t at each time step t. Then, the CMP can be defined as an optimization problem whose goal is to find the best sequence of transactions from account 1 to account 2 denoted by $\boldsymbol{x} \in \mathbb{R}^n_{\geq 0}$:

$$\boldsymbol{x} = [x_1, x_2 \ldots, x_n]^T ,$$

what is called a policy that optimizes some performance function $f(\boldsymbol{x})$ over a time horizon of n time steps, usually days. For simplicity, we do not impose any additional restrictions on transaction vector $\boldsymbol{x}$ provided that cash balances in accounts 1 and 2 are non-negative to avoid shortage costs. Cash managers may consider additional constraints such as minimum and maximum transaction values.

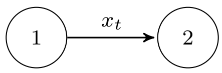

Fig. 1 The common two-assets framework in which money flows from cash account 1 to investment account 2 through transaction x_t.

Within the cash management problem for a single bank account, Constantinides and Richard (1978) pointed out the necessary condition for transferring money from account 1 to account 2. These results were later generalized to the context of multiple bank accounts by Salas-Molina et al (2021). In words, the cost γ of transferring one money unit through any transaction must be smaller than the excess in returns h between the target account and the source account. These results imply that transaction x_t is recommended to go from 1 to 2 or from 2 to 1, but not in both directions except for transactions to avoid negative cash balances usually charged with high penalty costs. Note that the cash management system described in Figure 1 is a directed graph in which all edges have a direction. Then, a non-negative transaction from account 2 to account 1 can be represented by a directed arc starting from account 2 and ending in account 1. In what follows, we use the cash management system depicted in Figure 1 as a basic structure to derive an analytic expression of the biobjective efficient frontier.

Let us first assume that account 1 is regularly endowed with some amount of money as a cash inflow to this account, allowing cash managers to transfer money to account 2 to achieve a higher return. By constructing an n-dimensional vector $\boldsymbol{\mu} \in \mathbb{R}^n_{\geq 0}$ with net unitary returns derived from transferring money from account 1 to account 2:

$$\boldsymbol{\mu} = [h - \gamma, h - \gamma, \ldots, h - \gamma]^T , \tag{1}$$

we compute the global expected returns $z_1(\boldsymbol{x})$ of policy $\boldsymbol{x}$ as follows:

$$z_1(\boldsymbol{x}) = \boldsymbol{\mu}^T \boldsymbol{x} = \sum_{t=1}^{n} (h - \gamma) x_t = \sum_{t=1}^{n} r x_t.$$

Our first goal is the expected returns function $z_1(\boldsymbol{x})$. However, there is a need for an additional goal to consider the biobjective CMP as the simultaneous optimization of returns and risk.

2.1 Variance of returns as a risk measure

Salas-Molina et al (2018b) proposed using the variance of daily costs to measure the risk of alternative policies. Daily costs are the opposite of daily returns, and the variance of daily costs is equal to that of daily returns. Then, our goal here is to maximize the difference between total returns as a profits measu and (one-half of) the variance of returns as a risk measure, including a risk aversion parameter $\lambda > 0$. This parameter reflects how many units of return an investor requires to accept an increase of one unit of risk. The larger, the more conservative the investor. Given an n-dimensional vector $\boldsymbol{c}$ with returns $c_t = r x_t$ for all $t \in \{1, 2, \ldots, n\}$, we can compute the variance (σ^2) of its elements as follows:

$$\sigma^2(\boldsymbol{c}) = \mathbb{E}\left[\boldsymbol{c}^2\right] - \mathbb{E}^2\left[\boldsymbol{c}\right].$$

Then, we can compute the variance of returns as follows:

$$z_2(\boldsymbol{x}) = \frac{1}{n} \boldsymbol{x}^T V \boldsymbol{x} - \frac{1}{n^2} (\boldsymbol{\mu}^T \boldsymbol{x})^2$$

where matrix V is a diagonal matrix with elements V_{ij} set to r^2 when $i = j$, zero otherwise. Now we consider the following objective function to maximize the difference between expected returns and one-half of the variance of returns:

$$\max\ \boldsymbol{\mu}^T \boldsymbol{x} - \frac{\lambda}{2} \left(\frac{1}{n} \boldsymbol{x}^T V \boldsymbol{x} - \frac{1}{n^2} (\boldsymbol{\mu}^T \boldsymbol{x})^2 \right).$$

The derivative of a squared function of vector $\boldsymbol{x}$ is:

$$\frac{\partial}{\partial \boldsymbol{x}} (\boldsymbol{\mu}^T \boldsymbol{x})^2 = 2(\boldsymbol{\mu}^T \boldsymbol{x}) \frac{\partial}{\partial x} (\boldsymbol{\mu}^T \boldsymbol{x}) = 2(\boldsymbol{\mu}^T \boldsymbol{x}) \boldsymbol{\mu}.$$

Then, by setting the derivative of the objective function to zero, we obtain the following expression:

$$\boldsymbol{\mu} = \frac{\lambda}{n} V\boldsymbol{x} - \frac{\lambda}{n^2}\boldsymbol{\mu}^T\boldsymbol{x}\boldsymbol{\mu}$$

where recall that V is a diagonal matrix.

From the definition of vectors $\boldsymbol{x}$ and $\boldsymbol{\mu}$, we rewrite the previous expression as follows:

$$\boldsymbol{\mu} = \frac{\lambda}{n} V\boldsymbol{x} - \frac{\lambda}{n^2} W\boldsymbol{x} \tag{2}$$

where W is an $n \times n$ matrix with all elements set to r^2. Then, optimal policy $\boldsymbol{x}$ has the following form in which we maintain notation $\boldsymbol{x}$ instead of the more appropriate $\boldsymbol{x}^*$ for ease of notation:

$$\boldsymbol{x} = \frac{n^2}{\lambda}\left(nV - W\right)^{-1}\boldsymbol{\mu} = \frac{n^2}{\lambda}A^{-1}\boldsymbol{\mu}. \tag{3}$$

Premultiplying equation (3) by $\boldsymbol{\mu}^T$, we obtain $z_1(\boldsymbol{x})$ as a function of λ:

$$z_1(\boldsymbol{x}) = \boldsymbol{\mu}^T\boldsymbol{x} = \frac{n^2}{\lambda}\boldsymbol{\mu}^T A^{-1}\boldsymbol{\mu} = a_1\lambda^{-1}$$

where $a_1 = n^2\boldsymbol{\mu}^T A^{-1}\boldsymbol{\mu}$ allows us to simplify notation in the rest of the paper.

Premultiplying equation (2) by $\boldsymbol{x}^T$, we obtain the relation between $z_1(\boldsymbol{x})$ and $z_2(\boldsymbol{x})$ when policy $\boldsymbol{x}$ is optimal:

$$\boldsymbol{x}^T\boldsymbol{\mu} = \frac{\lambda}{n}\boldsymbol{x}^T V\boldsymbol{x} - \frac{\lambda}{n^2}\boldsymbol{x}^T W\boldsymbol{x} = \frac{\lambda}{n}\boldsymbol{x}^T V\boldsymbol{x} - \frac{\lambda}{n^2}\boldsymbol{x}^T\boldsymbol{\mu}^T\boldsymbol{x}\boldsymbol{\mu}$$

$$z_1(\boldsymbol{x}) = \frac{\lambda}{n}\boldsymbol{x}^T V\boldsymbol{x} - \frac{\lambda}{n^2}(\boldsymbol{\mu}^T\boldsymbol{x})^2 = \lambda z_2(\boldsymbol{x})$$

$$z_2(\boldsymbol{x}) = \lambda^{-1} z_1(\boldsymbol{x}) = \frac{z_1^2(\boldsymbol{x})}{a_1}. \tag{4}$$

A necessary condition for deriving the analytical expression of risk in terms of returns in practice is that matrix A is invertible. By observing the construction of matrix A, it is easy to prove that $A \cdot \mathbf{1} = \mathbf{0}$ because the sum of $n-1$ and $n-1$ times -1 is always zero.

$$A = nV - W = r^2 \begin{bmatrix} (n-1) & -1 \dots & -1 \\ -1 & (n-1) \dots & -1 \\ \vdots & \vdots \dots & \vdots \\ -1 & -1 \dots & (n-1) \end{bmatrix}. \tag{5}$$

As a result, matrix A is non-invertible and the analytic efficient frontier using the variance of returns as a risk measure is not computable in practice. Even though an optimization algorithm can compute it, there is a need to find alternative risk measures that do not impose this limitation, as we next propose.

2.2 Sum of squared returns as a risk measure

In this case, we measure risk as the sum of squared returns using the following expression:

$$z_2(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T V \boldsymbol{x} = \frac{r^2}{2}\sum_{t=1}^{n} x_t^2.$$

Our goal now is to maximize $z_1(\boldsymbol{x}) - \lambda z_2(\boldsymbol{x})$, where $\lambda > 0$ is again a risk aversion parameter:

$$\max \ \boldsymbol{\mu}^T \boldsymbol{x} - \frac{\lambda}{2}\boldsymbol{x}^T V \boldsymbol{x}. \tag{6}$$

Deriving this expression with respect to $\boldsymbol{x}$, we obtain the first-order condition for an optimal policy:

$$\boldsymbol{\mu} = \lambda V \boldsymbol{x} \tag{7}$$

and the optimal policy has the following form:

$$\boldsymbol{x} = (\lambda V)^{-1}\boldsymbol{\mu}. \tag{8}$$

Premultiplying equation (8) by $\boldsymbol{\mu}^T$, we obtain the expression of $z_1(\boldsymbol{x})$ and the inverse of λ:

$$z_1(\boldsymbol{x}) = \boldsymbol{\mu}^T \boldsymbol{x} = \lambda^{-1}(\boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu})$$

$$\lambda^{-1} = z_1(\boldsymbol{x})(\boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu})^{-1} = a_2 z_1(\boldsymbol{x}) \tag{9}$$

where $a_2 = (\boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu})^{-1}$ allows us to simplify notation.

Premultiplying equation (7) by $\boldsymbol{x}^T/2$, we derive the efficient frontier that maps optimal return $z_1(\boldsymbol{x})$ to optimal risk $z_2(\boldsymbol{x})$:

$$\frac{1}{2}\boldsymbol{x}^T \boldsymbol{\mu} = \frac{\lambda}{2}\boldsymbol{x}^T V \boldsymbol{x}$$

$$z_1(\boldsymbol{x}) = 2\lambda z_2(\boldsymbol{x}). \tag{10}$$

Finally, introducing equation (9) in equation (10), we derive the efficient frontier that maps optimal expected returns $z_1(\boldsymbol{x})$ to optimal risk $z_2(\boldsymbol{x})$:

$$z_2(\boldsymbol{x}) = \frac{\lambda^{-1}}{2} z_1(\boldsymbol{x}) = \frac{a_2}{2} z_1^2(\boldsymbol{x}). \tag{11}$$

In this case, we only have to guarantee the invertibility of V. By definition, V is a diagonal matrix with positive values in its main diagonal and zero otherwise. Matrix V is positive definite, and then invertible, because for any vector $\boldsymbol{v} \neq \boldsymbol{0}$, we have that $\boldsymbol{v}^T V \boldsymbol{v} > 0$. Indeed, the inverse of diagonal matrix V equals a diagonal matrix with the elements of the main diagonal set to $1/r^2$.

Equation (11) shows that the form of the efficient frontier using the sum of squared returns is a parabola. However, it is usual in the literature of portfolio selection to represent this frontier in a risk-returns space in which risk is measured by the horizontal axis and the vertical axis represents expected returns.To this end, we need to express $z_1(\boldsymbol{x})$ as a function of $z_2(\boldsymbol{x})$:

$$z_1(\boldsymbol{x}) = \sqrt{\frac{2z_2(\boldsymbol{x})}{a_2}}. \tag{12}$$

For illustrative purposes, in Figure 2, we represent the case $z_1(\boldsymbol{x}) = \sqrt{z_2(\boldsymbol{x})}$ and observe that a diminishing rate of returns can be obtained by increasing risk as it is also usual in portfolio selection.

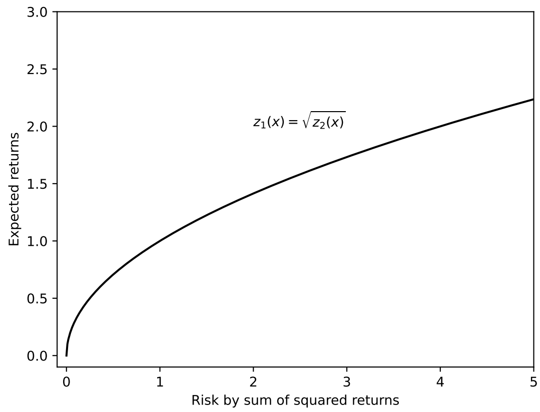

Fig. 2 The efficient frontier in the risk-returns space using the sum of squared deviations of returns as a risk measure.

2.3 Sum of squared deviations of returns for a given reference as a risk measure

Cash managers may be interested in setting a return reference to minimize the sum of squared deviations. Let us denote $\boldsymbol{p}$ as an n-dimensional vector with all elements set to p as a percentage of the returns r established as a reference for optimization purposes. Then, we define the following risk measure:

$$z_2(\boldsymbol{x}) = \frac{1}{2}\sum_{t=1}^{n}(rx_t - rp)^2 = \frac{1}{2}\sum_{t=1}^{n} r^2(x_t - p)^2 = \frac{1}{2}(\boldsymbol{x}-\boldsymbol{p})^T V(\boldsymbol{x}-\boldsymbol{p})$$

where recall that V is a diagonal matrix with diagonal entries set to r^2, zero otherwise. By minimizing the sum of deviations around a given reference, the goal is to smooth returns. Then, we aim to maximize the following objective function:

$$\max \ \boldsymbol{\mu}^T\boldsymbol{x} - \frac{\lambda}{2}(\boldsymbol{x}-\boldsymbol{p})^T V(\boldsymbol{x}-\boldsymbol{p})$$

where λ is a non-negative risk aversion parameter. Deriving with respect to $\boldsymbol{x}$, we obtain the first-order condition for an optimal policy:

$$\boldsymbol{\mu} = \lambda V(\boldsymbol{x}-\boldsymbol{p}) = \lambda V\boldsymbol{x} - \lambda V\boldsymbol{p}. \tag{13}$$

The form of the optimal policy is as follows:

$$\boldsymbol{x} = (\lambda V)^{-1}(\boldsymbol{\mu} + \lambda V \boldsymbol{p}). \tag{14}$$

Premultiplying equation (14) by $\boldsymbol{\mu}^T$, we obtain the expression of $z_1(\boldsymbol{x})$ and the inverse of λ:

$$z_1(\boldsymbol{x}) = \boldsymbol{\mu}^T \boldsymbol{x} = \lambda^{-1}\boldsymbol{\mu}^T V^{-1}\boldsymbol{\mu} + \boldsymbol{\mu}^T \boldsymbol{p} = \lambda^{-1} a_3 + b \tag{15}$$

where $a_3 = \boldsymbol{\mu}^T V^{-1}\boldsymbol{\mu}$ and $b = \boldsymbol{\mu}^T \boldsymbol{p}$ allow us to simplify notation. Then, it follows that:

$$\lambda^{-1} = \frac{z_1(\boldsymbol{x}) - b}{a_3}. \tag{16}$$

Premultiplying equation (13) by $(\boldsymbol{x}-\boldsymbol{p})^T/2$, we obtain the expression of $z_2(\boldsymbol{x})$:

$$\frac{1}{2}(\boldsymbol{x} - \boldsymbol{p})^T \boldsymbol{\mu} = \frac{\lambda}{2}(\boldsymbol{x} - \boldsymbol{p})^T V(\boldsymbol{x} - \boldsymbol{p}) = \lambda z_2(\boldsymbol{x})$$

$$\frac{1}{2}\boldsymbol{x}^T \boldsymbol{\mu} - \frac{1}{2}\boldsymbol{p}^T \boldsymbol{\mu} = \frac{\lambda}{2}(\boldsymbol{x} - \boldsymbol{p})^T V(\boldsymbol{x} - \boldsymbol{p}) = \lambda z_2(\boldsymbol{x}).$$

Rearranging the terms and using the expression of λ^{-1} in equation (16), we finally derive the efficient frontier that maps optimal expected returns $z_1(\boldsymbol{x})$ to optimal risk $z_2(\boldsymbol{x})$:

$$z_2(\boldsymbol{x}) = \frac{1}{2}\lambda^{-1}(z_1(\boldsymbol{x}) - b) = \frac{(z_1(\boldsymbol{x}) - b)^2}{2a_3}. \tag{17}$$

Again, the inverse of diagonal matrix V is equal to a diagonal matrix with the elements of the main diagonal set to $1/r^2$. As a result, the analytical derivation of the efficient frontier using the sum of squared deviations of returns as a risk measure described in equation (17) has the form of a parabola. However, to represent this frontier in the risk-returns space, we need to reorganize its terms as follows:

$$z_1(\boldsymbol{x}) = \sqrt{2z_2(\boldsymbol{x})a_3} + b. \tag{18}$$

For illustrative purposes, Figure 3 shows the efficient frontier for the case $z_1(\boldsymbol{x}) = \sqrt{z_2(\boldsymbol{x})} + 0.5$ derived from the use of the sum of squared deviations of returns as a risk measure. By comparing frontiers in Figures 2 and 3, one may have the impression that setting a reference implies an increase in efficiency. However, we must warn against this impression. At least in theory, every cash manager can set a reference of returns to follow as a target. The higher the reference, the larger the returns, even when the rate of returns remains unaltered. This reasoning fails in practice when the endowment due to cash inflows is limited. However, the analytical derivation using the sum of squared deviations of returns as a risk measure is accurate from a theoretical perspective.

3 Insights and implications

In this section, we elaborate on the insights and implications derived from the particular form of the biobjective efficient frontier with all non-dominated return and risk combinations.

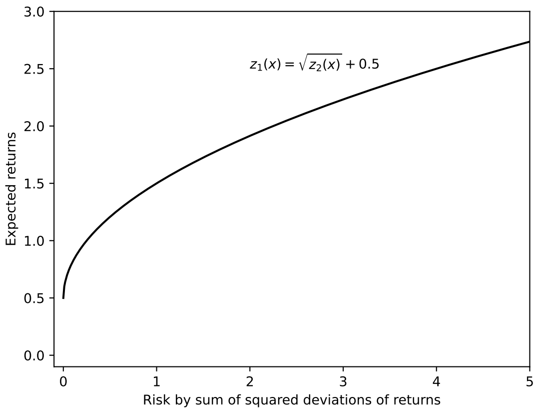

Fig. 3 The efficient frontier in the risk-returns space.

3.1 The selection of the risk measure is critical

As expected, the selection of the risk measure results in a different form of the efficient frontier. However, similar results are obtained when these measures have points in common as in the case of the three different risk measures considered in this paper. From this analysis, we elaborate on the necessary condition that any risk measure must satisfy to obtain an analytic derivation of the efficient frontier.

1. Let us assume that we aim to maximize the following objective function

$$\max \ z_1(\boldsymbol{x}) - \lambda z_2(\boldsymbol{x})$$

   where $z_1(\boldsymbol{x})$ is a general return measure and $z_2(\boldsymbol{x})$ is a general risk measure.
2. Provided that $z_1(\boldsymbol{x})$ and $z_2(\boldsymbol{x})$ are differentiable and non-null, the first-order condition for an optimal policy is:

$$\lambda = \frac{dz_1(\boldsymbol{x})/d\boldsymbol{x}}{dz_2(\boldsymbol{x})/d\boldsymbol{x}}.$$

3. Then, a necessary condition to obtain an analytical derivation is that the inverse of $dz_2(\boldsymbol{x})/d\boldsymbol{x}$ exists. In other words, that optimal policy $\boldsymbol{x}$ can be expressed in terms of a non-singular matrix.

The main implication derived from the previous reasoning is that not all risk measures are suitable to compute an analytical derivation of the efficient frontier in practice. Indeed, the variance of returns as a risk measure presents the inconvenience of not being computable in practice due to the presence of a singular matrix in the expression of optimal policy $\boldsymbol{x}$. This fact implies that we must find alternative risk measures that lead to an analytical derivation that is computable in practice such as the sum of squared returns and the sum of squared deviations for a given return reference.

3.2 Efficient policies

By focusing on the form of the analytical derivations described in Section 2.2 for the case of the sum of squared returns as a risk measure (Case 2), and in Section 2.3 for the case of the sum of squared deviations around a reference as a risk measure (Case 3), we develop the following theoretical results.

Theorem 1 *Given* $z_1(\boldsymbol{x}) = \boldsymbol{\mu}^T\boldsymbol{x}$ *as a return measure and* $z_2(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T V\boldsymbol{x}$ *as a risk measure, the efficient policy of minimum risk is no transaction.*

Proof It follows directly from objective function (6) by setting $\boldsymbol{x}$ to zero (no transaction) that results in minimum risk as graphically shown in Figure 2.

Theorem 2 *Let* $z_1(\boldsymbol{x}) = \boldsymbol{\mu}^T\boldsymbol{x}$ *be a return measure. Given* $z_2(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T V\boldsymbol{x}$ *and* $z_2'(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{p})^T V(\boldsymbol{x} - \boldsymbol{p})$ *as two alternative risk measures, the returns in the efficient frontier are proportional to the square root of risk, being the constant of proportionality equal to* $\sqrt{2n}$*, where* n *is the planning horizon.*

Proof Note first that $a_3 = 1/a_2$ and consider the elements of vector $\boldsymbol{\mu}$ from definition in equation (1), and matrix V from expression (5), we compute the value of a_3 and $1/a_2$:

$$a_3 = \frac{1}{a_2} = \boldsymbol{\mu}^T V^{-1}\boldsymbol{\mu} = n \tag{19}$$

because the inverse of diagonal matrix V is equal to a diagonal matrix with the elements of the main diagonal set to $1/r^2$:

$$a_3 = [r \;\; r \ldots r] \begin{bmatrix} \frac{1}{r^2} & 0 \ldots & 0 \\ 0 & \frac{1}{r^2} \ldots & 0 \\ \vdots & \vdots \ldots & \vdots \\ 0 & 0 \ldots & \frac{1}{r^2} \end{bmatrix} \begin{bmatrix} r \\ r \\ \vdots \\ r \end{bmatrix} = n.$$

Replacing a_2 and a_3 with their respective values, equation (12) and (18) become:

$$z_1(\boldsymbol{x}) = \sqrt{\frac{2z_2(\boldsymbol{x})}{a_2}} = \sqrt{2nz_2(\boldsymbol{x})} \tag{20}$$

$$z_1(\boldsymbol{x}) = \sqrt{2z_2'(\boldsymbol{x})a_3} + b = \sqrt{2nz_2'(\boldsymbol{x})} + b. \tag{21}$$

The previous theorem leads to the next corollary:

Corollary 1 *Let* $z_1(\boldsymbol{x}) = \boldsymbol{\mu}^T\boldsymbol{x}$ *be a return measure. Given* $z_2(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T V\boldsymbol{x}$ *and* $z_2'(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x}-\boldsymbol{p})^T V(\boldsymbol{x}-\boldsymbol{p})$ *as two alternative risk measures, the efficient frontier in equation* (20) *for the first risk measure is equal to the efficient frontier in equation* (21) *for the second risk measure when* $b = 0$*, that is when the reference vector* $\boldsymbol{p}$ *is set to zero.*

Theorem 3 *Let* $z_1(\boldsymbol{x}) = \boldsymbol{\mu}^T\boldsymbol{x}$ *and* $z_2(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{p})^T V(\boldsymbol{x} - \boldsymbol{p})$ *be return and risk measures, respectively. Then, the policy of minimum risk and returns is point* $(0, p \cdot r \cdot n)$*.*

Proof It follows directly from the definition of b in equation (15):

$$b = \boldsymbol{\mu}^T \boldsymbol{p} = \sum_{i=1}^{n} p \cdot r = p \cdot r \cdot n. \tag{22}$$

3.3 The returns vs the cost perspective

Instead of a returns reference, cash managers may be interested in setting a cost reference to minimize the sum of squared deviations. A straightforward representation of costs is the negative of expected returns. This representation leads us to the following theoretical result.

Theorem 4 *The efficient frontier from the cost perspective is a shifted version of the efficient frontier from the returns perspective. The shift equals $2b$ in the expected returns (cost) axis.*

Proof Setting $z_1(\boldsymbol{x}) = -\boldsymbol{\mu}^T \boldsymbol{x}$, we aim to maximize the following objective function:

$$\max \quad -\boldsymbol{\mu}^T \boldsymbol{x} - \frac{\lambda}{2}(\boldsymbol{x} - \boldsymbol{p})^T V(\boldsymbol{x} - \boldsymbol{p}).$$

The risk measure remains unaltered because the sum of squared deviations for a given cost (negative of returns) reference equals the sum of squared deviations for a return reference. Deriving with respect to $\boldsymbol{x}$, we obtain the first-order condition for an optimal policy:

$$\boldsymbol{\mu} = -\lambda V(\boldsymbol{x} - \boldsymbol{p}) = \lambda V \boldsymbol{p} - \lambda V \boldsymbol{x}. \tag{23}$$

The form of the optimal policy is as follows:

$$\boldsymbol{x} = (\lambda V)^{-1}(\lambda V \boldsymbol{p} - \boldsymbol{\mu}). \tag{24}$$

Premultiplying equation (24) by $-\boldsymbol{\mu}^T$, we obtain the expression of $z_1(\boldsymbol{x})$ and the inverse of λ:

$$z_1(\boldsymbol{x}) = -\boldsymbol{\mu}^T \boldsymbol{x} = \lambda^{-1} \boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu} - \boldsymbol{\mu}^T \boldsymbol{p} = \lambda^{-1} a_3 - b$$

$$\lambda^{-1} = \frac{z_1(\boldsymbol{x}) + b}{a_3}. \tag{25}$$

Premultiplying equation (23) by $(\boldsymbol{x} - \boldsymbol{p})^T/2$, we obtain the expression of $z_2(\boldsymbol{x})$:

$$\frac{1}{2}(\boldsymbol{x} - \boldsymbol{p})^T \boldsymbol{\mu} = -\frac{\lambda}{2}(\boldsymbol{x} - \boldsymbol{p})^T V(\boldsymbol{x} - \boldsymbol{p}) = -\lambda z_2(\boldsymbol{x})$$

$$-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{\mu} + \frac{1}{2}\boldsymbol{p}^T \boldsymbol{\mu} = \frac{\lambda}{2}(\boldsymbol{x} - \boldsymbol{p})^T V(\boldsymbol{x} - \boldsymbol{p}) = \lambda z_2(\boldsymbol{x}).$$

Rearranging the terms and using the expression of λ^{-1} in equation (25), we finally derive the efficient frontier that maps $z_1(\boldsymbol{x})$ to optimal risk $z_2(\boldsymbol{x})$:

$$z_2(\boldsymbol{x}) = \frac{1}{2}\lambda^{-1}(z_1(\boldsymbol{x}) + b) = \frac{(z_1(\boldsymbol{x}) + b)^2}{2a_3}.$$

Because of equation (19), this expression reduces to:

$$z_1(\boldsymbol{x}) = \sqrt{2a_3 z_2(\boldsymbol{x})} - b = \sqrt{2n z_2(\boldsymbol{x})} - b. \tag{26}$$

Graphically, equation (26) is a shifted version of equation (12). The magnitude of displacement $|D|$ along the $z_1(\boldsymbol{x})$ axis is computed by subtracting both equations

$$|D| = \sqrt{2a_3 z_2(\boldsymbol{x})} - b - (\sqrt{2a_3 z_2(\boldsymbol{x})} - b) = 2b.$$

From the cost perspective, the goal is minimizing cost and risk.

Corollary 2 *Let $z_1(\boldsymbol{x}) = -\boldsymbol{\mu}^T\boldsymbol{x}$ and $z_2(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x}-\boldsymbol{p})^T V(\boldsymbol{x}-\boldsymbol{p})$ be cost and risk measures, respectively. Then, there is a policy whose cost is zero when the risk is $z_2(\boldsymbol{x}) = b^2/2n$.*

Proof It follows directly from setting $z_1(\boldsymbol{x}) = 0$ in equation (26).

4 Numerical examples

In this section, we illustrate with numerical examples the main contributions of this paper. From a particular case of return r and planning horizon n, we obtain the three analytical derivations of the efficient frontier as a function of expected returns $z_1(\boldsymbol{x})$ expressed in terms of three different risk measures. First, we show that using the variance of returns as a risk measure prevents us from computing the analytical derivation in practice due to invertibility. Second, we obtain the analytical derivation of the efficient frontier when using the sum of squared returns and the sum of squared deviations of returns from a reference as two alternative risk measures.

Let us assume that $r = 0.001$ is the daily net expected return from transferring money from cash account 1 to investment account 2. By considering a planning horizon of $n = 5$ days, we obtain vector $\boldsymbol{\mu}$ and matrix A required to derive the efficient frontier when using the variance of returns as a risk measure:

$$\boldsymbol{\mu} = [0.001, 0.001, 0.001, 0.001, 0.001]^T$$

$$A = 10^{-6}\begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}.$$

To obtain the analytical derivation in equation (4), we need to obtain the inverse of matrix A to find parameter a_1. However, matrix A is singular because the sum of 4 and 4 times -1 is always zero, hence $A \cdot \mathbf{1} = \mathbf{0}$. Then, matrix A is not computable and we cannot obtain the analytical derivation using the variance of returns as a risk measure.

To obtain the analytical derivation in equation (12) using the sum squared of returns as a risk measure, we need to obtain matrix V and a_2 as follows:

$$V = \begin{bmatrix} 10^{-6} & 0 & 0 & 0 & 0 \\ 0 & 10^{-6} & 0 & 0 & 0 \\ 0 & 0 & 10^{-6} & 0 & 0 \\ 0 & 0 & 0 & 10^{-6} & 0 \\ 0 & 0 & 0 & 0 & 10^{-6} \end{bmatrix}$$

$$a_2 = (\boldsymbol{\mu}^T V^{-1} \boldsymbol{\mu})^{-1} = 0.2.$$

As a result, the analytical derivation of the efficient frontier using the sum squared of returns as a risk measure is the following:

$$z_1(\boldsymbol{x}) = \sqrt{10 z_2(\boldsymbol{x})}.$$

Finally, to obtain the analytical derivation in equation (18) using the sum squared deviations for a given reference of returns p set, for instance, to 10 times the return r as a risk measure, we need to obtain parameters a_3 and b as follows:

$$\boldsymbol{p} = [0.01, 0.01, 0.01, 0.01, 0.01]^T$$

$$a_3 = 1/a_2 = 5$$

$$b = \boldsymbol{\mu}^T \boldsymbol{p} = 0.00005.$$

As a result, the analytical derivation of the efficient frontier using the sum of squared deviations for a given reference is the following:

$$z_1(\boldsymbol{x}) = \sqrt{10 z_2(\boldsymbol{x})} + 0.00005.$$

After determining the analytical derivations for the sum of squared returns and the sum of squared deviations in our numerical example, we find that the difference between them is slight according to equation (22) ($b = p \cdot r \cdot n$) and the particular values used in this example. Larger values for reference p, return r, and planning horizon n would lead to more prominent differences between the two analytical derivations.

5 Concluding remarks

An analytical derivation of the efficient frontier in cash management from a biobjective perspective implies the possibility of obtaining several theoretical insights. We focus on a biobjective space and leave the consideration of more than two objectives for further research. In this paper, we show that any analytical derivation depends on the definition of the return and risk measures. We also show that some standard risk measures, such as the variance of returns, impose an important limitation in practice. Although an analytical expression can be derived, it cannot be computed in practice due to the impossibility of inverting the matrix required in this analytical derivation. As a result, our first conclusion is that the selection of both the return and risk measures are critical aspects in deriving an analytical expression.

Along the lines of the previous remark, finding alternative measures of desired objectives that may lead to other efficient frontier forms is an interesting future line of research. In addition, this paper paves the way to deriving further theoretical results by focusing on the conditions for alternative objective measures that allow an analytical expression of the efficient frontier.

An additional advantage of using an analytical derivation of the efficient frontier is the possibility of developing formal analysis on efficient policies. In this paper, we use the analytical derivation to present novel theoretical results on a) minimum risk policies (Theorem 1); b) the relationship established between returns in the efficient frontier and the planning horizon (Theorem 2 and Theorem 3); and c) the comparison between the returns perspective and the cost perspective. Finally, obtaining further insights from using the analytical derivation for other risk measures represents an interesting future line of research.

References

Baumol WJ (1952) The transactions demand for cash: An inventory theoretic approach. The Quarterly Journal of Economics 66(4):545–556

Constantinides GM, Richard SF (1978) Existence of optimal simple policies for discounted-cost inventory and cash management in continuous time. Operations Research 26(4):620–636

da Costa Moraes MB, Nagano MS, Sobreiro VA (2015) Stochastic cash flow management models: A literature review since the 1980s. In: Guarnieri, P (Editor), Decision Models in Engineering and Management, Springer International Publishing, pp 11–28

Markowitz H (1952) Portfolio selection. The Journal of Finance 7(1):77–91

Merton RC (1972) An analytic derivation of the efficient portfolio frontier. Journal of Financial and Quantitative Analysis 7(4):1851–1872

Miller MH, Orr D (1966) A model of the demand for money by firms. The Quarterly Journal of Economics 80(3):413–435

Qi Y (2020) Parametrically computing efficient frontiers of portfolio selection and reporting and utilizing the piecewise-segment structure. Journal of the Operational Research Society 71(10):1675–1690

Qi Y (2022) Classifying the minimum-variance surface of multiple-objective portfolio selection for capital asset pricing models. Annals of Operations Research 311(2):1203–1227

Qi Y, Li X (2020) On imposing ESG constraints of portfolio selection for sustainable investment and comparing the efficient frontiers in the weight space. SAGE Open 10(4):1–17

Qi Y, Steuer RE (2020) On the analytical derivation of efficient sets in quad-and-higher criterion portfolio selection. Annals of Operations Research 293(2):521–538

Qi Y, Steuer RE, Wimmer M (2017) An analytical derivation of the efficient surface in portfolio selection with three criteria. Annals of Operations Research 251(1-2):161–177

Salas-Molina F (2019) Selecting the best risk measure in multiobjective cash management. International Transactions in Operational Research 26(3):929–945

Salas-Molina F (2020) Risk-sensitive control of cash management systems. Operational Research 20(2):1159–1176

Salas-Molina F, Pla-Santamaria D, Rodríguez-Aguilar JA (2018a) Empowering cash managers through compromise programming. In: Masri, H, Perez-Gladish, B, and Zopounidis, C (Editors), Financial decision aid using multiple criteria, Springer, pp 149–173

Salas-Molina F, Pla-Santamaria D, Rodriguez-Aguilar JA (2018b) A multi-objective approach to the cash management problem. Annals of Operations Research 267(1):515–529

Salas-Molina F, Rodriguez-Aguilar JA, Díaz-García P (2018c) Selecting cash management models from a multiobjective perspective. Annals of Operations Research 261(1):275–288

Salas-Molina F, Rodriguez-Aguilar JA, Pla-Santamaria D, García-Bernabeu A (2021) On the formal foundations of cash management systems. Operational Research 21(2):1081–1095

Salas-Molina F, Rodríguez-Aguilar JA, Guillen M (2023) A multidimensional review of the cash management problem. Financial Innovation 9(67):1–35

Savage LJ (1951) The theory of statistical decision. Journal of the American Statistical Association 46(253):55–67

Schroeder P, Kacem I (2019) Optimal cash management with uncertain, interrelated and bounded demands. Computers & Industrial Engineering 133:195–206

Schroeder P, Kacem I (2020) Competitive difference analysis of the cash management problem with uncertain demands. European Journal of Operational Research 283(3):1183–1192