Conditional objects as Possibilistic Variables

Tommaso Flaminio[0000−0002−9180−7808] and Lluis Godo[0000−0002−6929−3126]

IIIA - CSIC, 08193 Bellaterra, Spain
{tommaso,godo}@iiia.csic.es

Abstract. The interpretation of basic conditionals as three-valued objects initiated by de Finetti has been mainly developed and extended by Gilio and Sanfilippo and colleagues, who look at (compound) conditionals as probabilistic random quantities. Recently, it has been shown that this approach ends up providing a Boolean algebraic structure for the set of conditional objects. In this paper, we show how that this probabilistic-based approach can also be developed within the possibilistic framework, where conditionals are attached with possibilistic variables instead: variables attached with a (conditional) possibility distribution on its domain of plain events. The possibilistic expectation of these variables now provides a means of extending the original possibility distribution on events to (compound) conditional objects. Our main result shows that this possibilistic approach leads to exactly the same underlying Boolean algebraic structure for the set of conditionals.

1 Introduction

Conditional objects are logical constructs very relevant in knowledge representation and reasoning. Conditional reasoning plays a prominent role in areas like non-monotonic reasoning [1,2,3,14,25,28,29], causal inference [27,33], and more generally reasoning under uncertainty [8,26,32] or conditional preferences [7,21,34].

Starting from an initial idea by de Finetti [11,12] (see also [31]), an approach to interpret both basic and compound conditionals as probabilistic random quantities have been developed mainly by Gilio and Sanfilippo, see e.g. [22,23,24]. In this approach, given a finite algebra of plain events $\mathbf{A}$, with Ω being its set of atoms, and a conditional probability space (Ω, P), a conditional $(a|b)$ is viewed as a three-valued quantity $X_{(a|b)}$ on the set of interpretations Ω such that $X_{(a|b)}(w) = P(a|b)$ if w falsifies b, besides taking value 1 when $w \models a \wedge b$ and value 0 when $w \models \neg a \wedge b$. It is shown that the expectation or prevision of the variable $\mathbb{P}(X_{(a|b)})$ coincides with the conditional probability $P(a|b)$. This idea has been recently formalised and extended in [16] to define a random quantity X_t for each compound conditional t in such a way that its prevision $\mathbb{P}(X_t)$ can be properly regarded as a probability on a Boolean algebra of conditionals $\mathcal{T}(\mathbf{A})$, built over the algebra of plain events $\mathbf{A}$, obtained by identifying conditionals t sharing the same random quantity X_t.

On the other hand, a pure algebraic setting for measure-free conditionals has been recently put forward in [18] and further developed in [17,20]. More precisely, in [18], given a finite Boolean algebra $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top)$ of events, another (much bigger but still finite) Boolean algebra $\mathcal{C}(\mathbf{A})$ is built, where *basic conditionals*, i.e. objects of the form $(a|b)$ with $a \in A$ and $b \in A' = A \backslash \{\bot\}$, can be freely combined with the usual Boolean operations, yielding compound conditional objects, while they are required to satisfy a set of natural properties. Moreover, the atoms of $\mathcal{C}(\mathbf{A})$ are fully identified and it is shown they are in a one-to-one correspondence with sequences of pairwise different atoms of $\mathbf{A}$ of maximal length. Finally, it is also shown that any positive probability P on the set of events from $\mathbf{A}$ can be *canonically* extended to a probability μ_P on the algebra of conditionals $\mathcal{C}(\mathbf{A})$ in such a way that the probability μ_P('$(a|b)$') of a basic conditional coincides with the conditional probability $P(a|b) = P(a \wedge b)/P(b)$. This is done by suitably defining the probability of each atom of $\mathcal{C}(\mathbf{A})$ as a certain product of conditional probabilities. A nice feature of the two approaches is that they lead to the same algebraic structure for conditionals, that is, both algebras $\mathcal{T}(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$ turn out to be isomorphic.

In this paper we show that the approach of [16] can also be developed within the possibilistic framework: each conditional t can be attached with a possibilistic variables X_t on Ω, where now the uncertainty on the values is governed by a (conditional) possibility on $\mathbf{A}$, and the possibilistic expectation of these variables now provides a means of extending the original possibility distribution on events to (compound) conditional objects. Our main result shows that this possibilistic approach leads to exactly the same underlying Boolean algebraic structure for the set of conditionals as those in the probabilistic setting.

2 Preliminaries

From now on we will consider a fixed *finite* Boolean algebra of ordinary events $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top)$. For an easier reading, for any $a, b \in A$, we will also write ab for $a \wedge b$ and $\bar{a}$ for $\neg a$, while we will keep denoting the disjunction by $a \vee b$.

The set of the atoms $\mathrm{at}(\mathbf{A})$ of $\mathbf{A}$ is identifiable with the set Ω of interpretations for $\mathbf{A}$, i.e. the set of homomorphisms $w : \mathbf{A} \to \{0, 1\}$. Thanks to this identification, we will say that an event $a \in \mathbf{A}$ is *true* (resp. *false*) under an interpretation (or possible world) $w \in \Omega$ when $w(a) = 1$ (resp. $w(a) = 0$), also denoted as $w \models a$ (resp. $w \not\models a$).

We will be interested in conditional events like "if b then a", or "a given b", where a and b are events from $\mathbf{A}$ with b different from $\bot$. These objects are denoted by $(a|b)$. Let $A|A' = \{(a|b) : a \in A, b \in A'\}$, where $A' = A \backslash \{\bot\}$, be the set of all conditionals that can be built from $\mathbf{A}$, that will also be called *basic conditionals*. By *compound conditionals* we will understand Boolean-style combinations of basic conditionals. More formally, they will be elements of $\mathbb{T}(A)$, the term algebra of type $(\wedge, \vee, \neg, \bot, \top)$ over $A|A'$, so that $\mathbb{T}(A)$ contains arbitrary terms generated from elements of $A|A'$ (taken as variables) that are freely combined with the operations from the signature, without any specific properties. For instance, if $a, c, e \in A$ and $b, d, f \in A'$, then $(a|b) \wedge (c|d)$ or $(a|b) \vee ((\bar{c}|d) \wedge \neg(e|f))$ are compound conditionals from $\mathbb{T}(A)$.

In the rest of this section we recall from [16] a reduction procedure for compound conditionals from $\mathbb{T}(A)$ given an interpretation. The idea of the reduction is to partially evaluate conditionals by classical evaluations in accordance with de Finetti's three-valued semantics. Under this semantics, a conditional $(a|b)$ is deemed to be *true* in w when $w \models a$ and $w \models b$, *false* when $w \models b$ and $w \not\models a$, and *undefined* if $w \not\models b$. In other words, an interpretation $w : \mathbf{A} \to \{0, 1\}$ partially extends to $A|A'$ as follows:

$$w(a|b) = \begin{cases} 1, & \text{if } w(a) = w(b) = 1 \\ 0, & \text{if } w(a) = 0, w(b) = 1 \\ \text{undefined}, & \text{if } w(b) = 0 \end{cases}$$

Although some of the basic components of a compound conditional may remain undefined for a given interpretation w, we can sometimes provide a definite evaluation or at at least a simplified form of the conditional, assuming a Boolean behaviour of the operations. For instance if w is such that $w \models \bar{a}bc\bar{d}$, then $w((a|b) \wedge (c|d)) = 0 \wedge w(c|d) = 0$, while $w((c|b) \wedge (a|d)) = 1 \wedge w(a|d) = w(a|d)$. So, from the point of view of w, we can reduce $(a|b) \wedge (c|d)$ to $\perp$ (the conditional that always evaluate to false), while $(c|b) \wedge (a|d)$ can be reduced to $(a|d)$.

More formally, for every $t \in \mathbb{T}(A)$, let us write $Cond(t) = \{(a_1|b_1), \ldots, (a_n|b_n)\}$ for the set of basic conditionals appearing in t, and let us denote by $\mathbf{b}(t) = b_1 \vee \ldots \vee b_n$ the disjunction of the antecedents in $Cond(t)$.

Definition 1. *Let $w \in \Omega$ be a classical interpretation and let $t \in \mathbb{T}(A)$ be a term. The* w-reduct *of t, denoted t^w, is the term in $\mathbb{T}(A)$, obtained as follows:*

(1) replace each $(a_i|b_i) \in Cond(t)$ by $\top$ if $w \models a_i b_i$, and by $\perp$ if $w \models \bar{a}_i b_i$,
(2) apply the following reduction rules to subterms of t until no further reduction is possible: for every subterm r of t

$$\neg\top := \perp,\ \neg\perp := \top,\ r \wedge \top = \top \wedge r := r,\ r \wedge \perp = \perp \wedge r := \perp,$$
$$r \vee \top = \top \vee r := \top,\ r \vee \perp = \perp \vee r := r.$$

This symbolic reduction procedure has some interesting properties.

Fact 1 *(1) If $w \models \bar{\mathbf{b}}(t)$, that is w does not satisfy no antecedent of the conditionals in t, then no reduction is possible and hence $t^w = t$.*

(2) The reduction commutes with the operation symbols, in the following sense: for every terms $t, s \in \mathbb{T}(A)$ and for every $w \in \Omega$: (i) $(\neg t)^w = \neg t^w$; (ii) $(t \wedge s)^w = t^w \wedge s^w$; and (iii) $(t \vee s)^w = t^w \vee s^w$.

In the following, we will denote by $Red(t) = \{t^w | w \in \Omega\}$ the set of w-reducts of t, and by $Red^0(t) = Red(t) \backslash \{t\}$, the set of its *proper w-reducts*.

Example 1. Let $t = (a|b) \wedge ((c|d) \vee \neg(e|f))$ and let w such that $w(a) = 1, w(b) = 0, w(c) = 0, w(d) = 0, w(e) = 1, w(f) = 1$, i.e. $w \models a\bar{b}\bar{c}\bar{d}ef$. Then

$$t^w = (a|b) \wedge ((c|d) \vee \neg\top) = (a|b) \wedge ((c|d) \vee \perp) = (a|b) \wedge (c|d).$$

Let w' such that $w' \models abc\bar{d}ef$. Then $t^{w'} = \top \wedge ((c|d) \vee \neg \top) = (c|d) \vee \bot = (c|d)$. In fact, one can check that

$$Red^0(t) = \{\top, \bot, (a|b), (c|d), \neg(e|f), (a|b) \wedge (c|d), (a|b) \wedge \neg(e|f), (c|d) \vee \neg(e|f)\}. \ \square$$

3 Possibilistic variables and their expectations

We first recall the notion of conditional possibility measures. Coletti and colleagues proposed an axiomatic approach to the notion of conditional possibility, similar to the case of conditional probability that is a primitive notion, not derived from a (unconditional) possibility, see e.g. [4,9,10]. The following definition is basically from [4].

Definition 2. *Given a (continuous) t-norm $\odot$, a $\odot$-conditional possibility*[1] *measure on* $\mathbf{A}$ *is a binary mapping* $\Pi(\cdot|\cdot) : A \times A' \to [0,1]$, *where* $A' = A \backslash \{\bot\}$, *satisfying the following conditions:*

- $(C\Pi 1)$ $\Pi(a|b) = \Pi(a \wedge b|b)$, *for all* $a \in A, b \in A'$
- $(C\Pi 2)$ $\Pi(\cdot|b)$ *is a possibility measure for each* $b \in A'$
- $(C\Pi 3)$ $\Pi(a \wedge b|c) = \Pi(b|a \wedge c) \odot \Pi(a|c)$, *for all* $a, b, c \in A$ *such that* $a \wedge c \in A'$.

We will call the pair $(\mathbf{A}, \Pi)$ *a* $\odot$*-conditional possibility space.*

In what follows, given a $\odot$-conditional possibility $\Pi : A \times A' \to [0,1]$, for any event $a \in A$, we will write $\Pi(a)$ to denote $\Pi(a|\top)$, without danger of confusion. Note that $\Pi(\cdot) = \Pi(\cdot|\top)$ is indeed a possibility measure.

Also, whenever it is clear by the context, we will simply say that Π is a conditional possibility without explicitly referring to the t-norm $\odot$.

Let $(\mathbf{A}, \Pi)$ be a given finite conditional possibility space, and let Ω be the set of atoms of $\mathbf{A}$. By a *possibilistic variable* (or quantity) we mean a function $X : \Omega \to [0,1]$, that propagates the possibilistic uncertainty on Ω to the values of X. Indeed the possibility that X takes value in a subset $S \subseteq [0,1]$, conditional to an event $b \in A$, is naturally defined as

$$\Pi(X \in S \mid b) = \max\{\Pi(w|b) \mid X(w) \in S\}.$$

This can be interpreted as a sort of possibilistic counterpart of the notion of random variable particularised to our framework of conditional possibility spaces.

Notation 1 *In the following, for any event* $a \in A$, *we will denote by* X_a *the indicator function of* a *in* Ω, *that is, for all* $w \in \Omega$, $X_a(w) = 1$ *if* $w \models a$, *and* $X_a(w) = 0$ *otherwise. Accordingly,* $X_\top$ *is the constant function of value 1 (also denoted* $\mathbf{1}$*) and* $X_\bot$ *is the constant function of value 0 (also denoted* $\mathbf{0}$*). Also, if* $\lambda \in [0,1]$, *by* $\lambda \odot X$ *we will denote the variable such that* $(\lambda \odot X)(w) = \lambda \odot X(w)$ *for all* $w \in \Omega$. *Finally, if* X *and* Y *are variables, sometimes we will denote by* $X \wedge Y$ *and* $X \vee Y$ *the variables such that, for all* $w \in \Omega$, $(X \wedge Y)(w) = \min(X(w), Y(w))$ *and* $(X \vee Y)(w) = \max(X(w), Y(w))$ *respectively.*

[1] Called T-conditional possibility in [9,10].

Likewise, the possibilistic counterpart of the notion of expected value for a random value will be played here by a *generalized* Sugeno integral [13,19].

Definition 3. *Let* $(\mathbf{A}, \Pi)$ *be a finite* $\odot$*-conditional possibility space and let* $X : \Omega \to [0,1]$ *be a possibilistic random variable. Then, the* possibilistic expectation of X *is defined as the following generalised Sugeno integral of* X *w.r.t. the possibility distribution* $\pi : \Omega \to [0,1]$ *defined as* $\pi(w) = \Pi(w|\top)$*, that is:*

$$\mathbb{E}(X) = \max_{w \in \Omega} X(w) \odot \pi(w).$$

Analogously, the conditional possibilistic expectation of X given an event $b \in A'$ *is defined as the generalised Sugeno integral of* X *w.r.t. the possibility distribution* $\pi(\cdot|b) : \Omega \to [0,1]$ *defined as* $\pi(w|b) = \Pi(w|b)$*, namely:*

$$\mathbb{E}(X|b) = \max_{w \in \Omega} X(w) \odot \pi(w|b) = \max_{w \in \Omega : w \models b} X(w) \odot \pi(w|b).$$

Unsurprisingly, we recover the unconditional expectation when we take $b = \top$, namely $\mathbb{E}(X|\top) = \mathbb{E}(X)$. Also as expected, we recover the conditional possibility Π from $\mathbb{E}$ when applied over indicator functions, in fact, for any $a \in A$, $\mathbb{E}(X_a|b) = \Pi(a|b)$.

It is worth pointing out that the case of non-conditional expectations have been studied in [15] under the name of *extended generalised possibility measures* (see also [6]), whereas $\odot$-conditional possibilistic expectations have been formally introduced in [5], under the name of T*-conditional possibilistic previsions*, where the authors show they satisfy the following properties for every $b \in A'$:

- $\mathbb{E}(\mathbf{1}|b) = \mathbb{E}(X_b|b) = 1$
- $\mathbb{E}(\mathbf{0}|b) = 0$
- $\mathbb{E}(X|b) = \mathbb{E}(X \odot X_b|b)$
- $\mathbb{E}(X_1 \vee X_2|b) = \max(\mathbb{E}(X_1|b), \mathbb{E}(X_2|b))$
- $\mathbb{E}(\lambda \odot X|b) = \lambda \odot \mathbb{E}(X|b)$, for every $\lambda \in [0,1]$
- $\mathbb{E}(X_a \odot X|b) = \mathbb{E}(X_a|b) \odot \mathbb{E}(X|a \wedge b)$, for every $a \in A'$

Actually, these properties characterise them, as implicitly understood in [5]. We provide here a proof for the sake of completeness

Proposition 1. *Let* $\mathbf{A}$ *be a finite Boolean algebra,* Ω *be the set of its atoms, and let* $\mathbb{E}(\cdot|\cdot) : [0,1]^{\Omega} \times A' \to [0,1]$ *be a mapping. Then* $\mathbb{E}$ *satisfies the following properties for any* $b \in A'$*:*

(i) $\mathbb{E}(\mathbf{1}|b) = 1$
(ii) $\mathbb{E}(X_1 \vee X_2|b) = \max(\mathbb{E}(X_1|b), \mathbb{E}(X_2|b))$
(iii) $\mathbb{E}(\lambda \odot X|b) = \lambda \odot \mathbb{E}(X|b)$*, for every* $\lambda \in [0,1]$
(iv) $\mathbb{E}(X_a \odot X|b) = \mathbb{E}(X_a|b) \odot \mathbb{E}(X|a \wedge b)$*, for every* $a \in A'$

if, and only if, there exists a (normalised) $\odot$*-conditional possibility distribution* $\pi : \Omega \times A' \to [0,1]$ *such that* $\mathbb{E}(X|b) = \max_{w \in \Omega} X(w) \odot \pi(w|b)$*.*

Proof. Suppose E satisfies (i), (ii) and (iii). Since everything is finite, we can write $X = \max_{w\in\Omega} \overline{X(w)} \odot X_w$, where $\overline{X(w)}$ is the constant function of value $X(w)$ and X_w is the characteristic function of w, i.e. for everything $w' \in \Omega$, $X_w(w') = 1$ if $w' = w$ and $X_w(w') = 0$ otherwise. Therefore, for any $b \in A'$, by (ii) and (iii), we have $E(X|b) = \max_{w\in\Omega} E(\overline{X(w)} \odot X_w|b) = \max_{w\in\Omega} X(w) \odot E(X_w|b)$. Finally, by defining $\pi(\cdot|b) : \Omega \to [0,1]$ as $\pi(w|b) = E(X_w|b)$ we get that $E(X|b) = \max_{w\in\Omega} X(w) \odot \pi(w|b)$. Now, let us define the $\Pi(\cdot|\cdot) : A \times A' \to [0,1]$ by letting $\Pi(a|b) = \max_{w\models a} \pi(w|b) = \max_{w\models a} E(X_w|b) = E(X_a|b)$. Finally, we are led to check that Π is a $\odot$-conditional possibility:

($C\Pi1$) : it holds by definition of $\pi(w|b)$.

($C\Pi2$) : by (i) and (i), it follows that $\Pi(\cdot|b)$ is a normalised possibility measure for each $b \in A'$.

($C\Pi3$) : let $w \in \Omega$ such that $w \leqslant a \wedge b$, then (iv) gives $\mathbb{E}(X_w|b) = \mathbb{E}(X_a \odot X_w|b) = \mathbb{E}(X_a|b) \odot \mathbb{E}(X_w|a \wedge b)$, that is, $\pi(w|b) = \Pi(a|b) \odot \pi(w|a \wedge b)$. Therefore, we have $\Pi(a \wedge b|c) = \max_{w\models a\wedge b} \pi(w|c) = \max_{w\models a\wedge b} \pi(w|a \wedge c) \odot \Pi(a|c) = \Pi(a|c) \odot \max_{w\models a\wedge b} \pi(w|a\wedge c) = \Pi(a|c) \odot \Pi(a\wedge b|a\wedge c) = \Pi(a|c) \odot \Pi(b|a\wedge c)$.

□

4 Conditionals and their associated possibilistic variables

In this section, following the idea in [16], we associate a possibilistic variable to every compound conditional $t \in \mathbb{T}(A)$ and study basic properties of these variables and of their possibilistic expectations.

Definition 4. *Let $(\mathbf{A}, \Pi)$ be a finite conditional possibility space. For every term t in $\mathbb{T}(A)$, we define the variable $X_t : \Omega \to [0,1]$ as follows: for every $w \in \Omega$,*

$$X_t(w) := \mathbb{E}(X_{t^w}|\mathbf{b}(t^w)).$$

If $t^w = \top$ or $t^w = \bot$, we define $\mathbf{b}(t^w) = \top$, and hence we take $X_\top$ and $X_\bot$ as the constant functions of value 1 and 0 respectively.

Let us show that the above definition captures the intuition by analysing the most basic cases. We start by considering the case $t = (a|\top)$. Here we have $t^w = \top$ if $w \models a$, $t^w = \bot$ otherwise, and $\mathbf{b}(t^w) = \top$ in either case. Therefore, $X_t(w) = \mathbb{E}(X_\top|\mathbf{b}(t^w)) = 1$ when $w \models a$ and $X_t(w) = 0$ when $w \models \bar{a}$; in other words, $X_{(a|\top)}$ is nothing but the characteristic or indicator function of the event a. From now on, we will simply write X_a for $X_{(a|\top)}$. Moreover, the expectation of X_a is $\mathbb{E}(X_a) = \max_{w\in\Omega} X_a(w) \odot \Pi(w) = 1 \odot \max_{w\models a} \Pi(w) = \Pi(a)$.

Let us consider now the case $t = (a|b)$. By the above definition, we get

$$t^w = \begin{cases} \top, & \text{if } w \models ab \\ \bot, & \text{if } w \models \bar{a}b \\ (a|b), & \text{if } w \models \bar{b} \end{cases}, \quad \mathbf{b}(t^w) = \begin{cases} \top, \text{ if } w \models ab \\ \top, \text{ if } w \models \bar{a}b \\ b, \text{ if } w \models \bar{b} \end{cases}$$

and thus we have:

$$X_{(a|b)}(w) = \mathbb{E}(X_{t^w}|\mathbf{b}(t^w)) = \begin{cases} \mathbb{E}(X_\top|\top) = 1, \text{ if } w \models ab \\ \mathbb{E}(X_\bot|\top) = 0, \text{ if } w \models \bar{a}b \\ \mathbb{E}(X_{(a|b)}|b), \quad \text{ if } w \models \bar{b} \end{cases} .$$

Now, since $\Pi(w|b) = 0$ whenever $w \models \bar{b}$, we have

$$\mathbb{E}(X_{(a|b)}|b) = [1 \odot \Pi(ab|b)] \vee [0 \odot \Pi(\bar{a}b|b)] \vee [\mathbb{E}(X_{(a|b)}|b) \odot 0] = \Pi(ab|b) = \Pi(a|b).$$

Therefore we get the following three-valued possibilistic representation of $(a|b)$:

$$X_{(a|b)}(w) = \begin{cases} 1, & \text{if } w \models ab, \\ 0, & \text{if } w \models \bar{a}b, \\ \Pi(a|b), & \text{if } w \models \bar{b}. \end{cases}$$

If $t = \neg(a|b)$, one gets an analogous expression for $X_{\neg(a|b)}$, just replacing above a by $\bar{a}$, and hence $\Pi(a|b)$ by $\Pi(\bar{a}|b)$ as well. Thus, one has $X_{\neg(a|b)} = X_{(\bar{a}|b)}$.

Fact 2 *From the above cases it follows that, for any $a \in A$ and $b \geqslant a$, the following equalities hold:*

- $X_{(a|b)} = X_{(a\wedge b|b)}$, $X_{\neg(a|b)} = X_{(\bar{a}|b)}$, *and* $X_{\neg\neg(a|b)} = X_{\neg(\bar{a}|b)} = X_{(a|b)}$
- $X_a = X_{(a|\top)}$, *and* $X_{\neg(a|\top)} = X_{(\bar{a}|\top)} = X_{\bar{a}} = 1 - X_a$
- $X_{(a|a)} = X_{(b|a)} = X_{(\top|\top)} = X_\top = \mathbf{1}$, *and* $X_{\neg(a|a)} = X_{(\bar{a}|a)} = X_{(\bot|\top)} = \mathbf{0}$

where $\mathbf{0}$ *and* $\mathbf{1}$ *denote the variables of constant value 0 and 1 respectively.*

In general, a possibilistic random quantity X_t can be specified in a more compact way: let $Red^0(t) = \{t^w|w \in \Omega\} = \{t_1, t_2, ..., t_k\}$ and let $E_1, E_2, ..., E_k$ be the corresponding interpretations leading to a same element of $Red^0(t)$, then

$$X_t(w) = \mathbb{E}(X_{t^w}|\mathbf{b}(t^w)) = \begin{cases} \mathbb{E}^c(X_{t_1}), \text{ if } w \models E_1 \\ \ldots, \quad \ldots \\ \mathbb{E}^c(X_{t_k}), \text{ if } w \models E_k \\ \hdashline \mathbb{E}^c(X_t), \quad \text{if } w \models \neg(E_1 \vee \ldots \vee E_k) \end{cases}$$

where $\mathbb{E}^c(X_t)$ stands for $\mathbb{E}(X_t|\mathbf{b}(t))$, and the dashed line separates the cases where w satisfies $\mathbf{b}(t)$ from those which do not. It follows that X_t can be expressed as a max-$\odot$ combination of the indicator functions X_{E_i}'s :

$$X_t = \max(\mathbb{E}^c(X_{t_1}) \odot X_{E_1}, \ldots, \mathbb{E}^c(X_{t_k}) \odot X_{E_k}, \mathbb{E}^c(X_t) \odot X_{E_{k+1}}),$$

where $E_{k+1} = \bar{E}_1 \ldots \bar{E}_k = \overline{\mathbf{b}}(t)$, and hence, the possibilistic expectation of X_t is given by:

$$\mathbb{E}^c(X_t) = \max(\mathbb{E}^c(X_{t_1}) \odot \Pi(E_1|\mathbf{b}(t)), \ldots, \mathbb{E}^c(X_{t_k}) \odot \Pi(E_k|\mathbf{b}(t))).$$

Next result shows two interesting properties of the possibilistic prevision of X_t, that are similar to the probabilistic case. In particular it shows that the prevision $\mathbb{E}(X_t)$ coincides with its conditional previsions given both $\mathbf{b}(t)$ and $\overline{\mathbf{b}}(t)$.

Proposition 2. *The following properties hold for any conditional term $t \in \mathbb{T}(A)$ and event $a \in A$:*

(i) $\mathbb{E}(X_t \wedge X_a) = \mathbb{E}(X_t \odot X_a) = \mathbb{E}(X_t|a) \odot \Pi(a)$
(ii) $\mathbb{E}(X_t|\bar{\mathbf{b}}(t)) = \mathbb{E}(X_t|\mathbf{b}(t)) = \mathbb{E}(X_t)$

Proof. (i) Since $a \in A$, $X_a(w) \in \{0,1\}$, whence for every term t, $X_t \wedge X_a = X_t \odot X_a$. Now, $\mathbb{E}(X_t \wedge X_a) = \mathbb{E}(X_t \odot X_a) = \max_w\{X_t(w) \odot X_a(w) \odot \Pi(w)\}$. Now, observe that $X_a(w) \odot \Pi(w)) = \Pi(w \wedge a)$ and, by $(C\Pi3)$, $\Pi(w \wedge a) = \Pi(w \wedge a|\top) = \Pi(w|a) \odot \Pi(a)$ and hence the previous expression equals $\max_w\{X_t(w) \odot \Pi(w|a) \odot \Pi(a)\} = \Pi(a) \odot \max_w\{X_t(w) \odot \Pi(w|a)\} = \Pi(a) \odot \mathbb{E}(X_t|a)$.

(ii-a) By definition, $\mathbb{E}(X_t|\bar{\mathbf{b}}(t)) = \max_{w \models \bar{\mathbf{b}}(t)} X_t(w) \odot \Pi(w|\bar{\mathbf{b}}(t))$ and this latter equals $\max_{w \models \bar{\mathbf{b}}(t)} \mathbb{E}(X_{t^w}|\bar{\mathbf{b}}(t^w)) \odot \Pi(w|\bar{\mathbf{b}}(t))$. By Fact 1 (1) if $w \models \bar{\mathbf{b}}(t)$, $t^w = t$ and hence $\mathbb{E}(X_t|\bar{\mathbf{b}}(t)) = \max_{w \models \bar{\mathbf{b}}(t)} \mathbb{E}(X_t|\mathbf{b}(t)) \odot \Pi(w|\bar{\mathbf{b}}(t)) = \mathbb{E}(X_t|\mathbf{b}(t)) \odot \max_{w \models \bar{\mathbf{b}}(t)} \Pi(w|\bar{\mathbf{b}}(t)) = \mathbb{E}(X_t|\mathbf{b}(t)) \odot \Pi(\bar{\mathbf{b}}(t)|\bar{\mathbf{b}}(t)) = \mathbb{E}(X_t|\mathbf{b}(t))$.

(ii-b) Since b is an event, X_b only takes value 0 or 1, and thus $X_t = (X_t \odot X_b) \vee (X_t \odot X_{\bar{b}})$. Now, from (i) and (ii-a) above, the following equalities hold: $\mathbb{E}(X_t) = \mathbb{E}(X_t \odot X_{\mathbf{b}(t)}) \vee \mathbb{E}(X_t \odot X_{\bar{\mathbf{b}}(t)}) = \max(\mathbb{E}(X_t|\mathbf{b}(t)) \odot \Pi(\mathbf{b}(t)), \mathbb{E}(X_t|\bar{\mathbf{b}}(t)) \odot \Pi(\bar{\mathbf{b}}(t))) = \max(\mathbb{E}(X_t|\mathbf{b}(t)) \odot \Pi(\mathbf{b}(t)), \mathbb{E}(X_t|\mathbf{b}(t)) \odot \Pi(\bar{\mathbf{b}}(t))) = \mathbb{E}(X_t|\mathbf{b}(t)) \odot \max(\Pi(\mathbf{b}(t)), \Pi(\bar{\mathbf{b}}(t))) = \mathbb{E}(X_t|\mathbf{b}(t))$. □

We end this section with two further instantiations of the definition of X_t, namely for the cases of a conjunction and a disjunction of basic conditionals.

Example 2. Let $t = (a|b) \wedge (c|d)$. Here we have $\mathbf{b}(t) = b \vee d$, and

$$X_t(w) = \begin{cases} 1, & \text{if } w \models abcd \\ 0, & \text{if } w \models (\bar{a}b) \vee (\bar{c}d) \\ \mathbb{E}^c(X_{a|b}) = \Pi(a|b), & \text{if } w \models \bar{b}cd \\ \mathbb{E}^c(X_{c|d}) = \Pi(c|d), & \text{if } w \models ab\bar{d} \\ \hdashline \mathbb{E}^c(X_{(a|b) \wedge (c|d)}), & \text{if } w \models \bar{b}\bar{d} \end{cases}$$

Then, by definition we get:

$$\mathbb{E}^c(X_{(a|b) \wedge (c|d)}) = \\ \max(\Pi(abcd|b \vee d), \Pi(a|b) \odot \Pi(\bar{b}cd|b \vee d), \Pi(c|d) \odot \Pi(ab\bar{d}|b \vee d))^2$$

In the particular case when $a \leqslant b = c \leqslant d$ everything simplifies, indeed it is not difficult to check that $\mathbb{E}^c(X_{(a|b) \wedge (c|d)}) = \Pi(a|d)$ and $X_{(a|b) \wedge (c|d)} = X_{a|d}$.

Now, consider $t = (a|b) \vee (c|d)$. Again here $\mathbf{b}(t) = b \vee d$, and X_t is defined as:

$$X_t(w) = \begin{cases} 1, & \text{if } w \models ab \vee cd \\ 0, & \text{if } w \models \bar{a}b\bar{c}d \\ \mathbb{E}^c(X_{a|b}) = \Pi(a|b), & \text{if } w \models \bar{b}\bar{c}d \\ \mathbb{E}^c(X_{c|d}) = \Pi(c|d), & \text{if } w \models \bar{a}b\bar{d} \\ \hdashline \mathbb{E}^c(X_{(a|b) \vee (c|d)}), & \text{if } w \models \bar{b}\bar{d} \end{cases}$$

[2] This is a possibilistic counterpart of the formula given in [30] for the probability of the conjunction of two conditionals.

where, by definition we have: $\mathbb{E}^c(X_{(a|b)\vee(c|d)}) = \mathbb{E}(X_{(a|b)\vee(c|d)}|b\vee d) = \max(\Pi(ab\vee cd|b\vee d), \Pi(a|b)\odot\Pi(\bar{b}\bar{c}d|b\vee d), \Pi(c|d)\odot\Pi(\bar{a}b\bar{d}|b\vee d))$. One can show that the last expression is equal to $\max(\Pi(a|b), \Pi(c|d))$ (we omit the proof for the lack of space). Therefore we have

$$\mathbb{E}^c(X_{(a|b)\vee(c|d)}) = \max(\Pi(a|b), \Pi(c|d)). \qquad \square$$

From the above example, the following equalities among variables readily follow by simple inspection:

$X_{(a|b)\wedge(c|d)} = X_{(c|d)\wedge(a|b)}$ and $X_{(a|b)\vee(c|d)} = X_{(c|d)\vee(a|b)}$,
$X_{(a|b)\wedge(c|b)} = X_{(a\wedge c|b)}$ and $X_{(a|b)\vee(c|b)} = X_{(a\vee c|b)}$,
$X_{(a|b)\wedge(a|b)} = X_{(a|b)\vee(a|b)} = X_{(a|b)}$,
$X_{(a|b)\wedge(\bar{a}|b)} = X_{(a|b)\wedge\neg(a|b)} = X_{\perp} = \mathbf{0}$,
$X_{(a|b)\vee(\bar{a}|b)} = X_{(a|b)\vee\neg(a|b)} = X_{\top} = \mathbf{1}$.

Moreover, by iterating or combining the above expressions for the conjunction and disjunction of basic conditionals, the following further equalities also hold:

$X_{(a|b)\wedge((c|d)\wedge(e|f))} = X_{((a|b)\wedge(c|d))\wedge(e|f)}$ and $X_{(a|b)\vee((c|d)\vee(e|f))} = X_{((a|b)\vee(c|d))\vee(e|f)}$,
$X_{(a|b)\wedge((c|d)\vee(e|f))} = X_{((a|b)\wedge(c|d))\vee((a|b)\wedge(e|f))}$,
$X_{(a|b)\vee((c|d)\wedge(e|f))} = X_{((a|b)\vee(c|d))\wedge((a|b)\vee(e|f))}$,
$X_{\neg((a|b)\wedge(c|d))} = X_{\neg(a|b)\vee\neg(c|d)}$, $X_{\neg((a|b)\vee(c|d))} = X_{\neg(a|b)\wedge\neg(c|d)}$.

5 A Boolean algebraic structure on the set of compound conditionals

The aim of this section is to show that $\mathbb{T}(A)$ can be endowed with a Boolean algebraic structure. To prove this, we start showing some elementary properties whose proof can be shown by induction on the structure of the terms and whose base cases only involve basic conditionals and are listed at the end of Section 4.

Proposition 3. *For every $t, s, r \in \mathbb{T}(A)$ the following conditions hold:*

1. $X_t = X_{t\wedge t}$ 2. $X_{t\wedge s} = X_{s\wedge t}$ 3. $X_{t\wedge(s\wedge r)} = X_{(t\wedge s)\wedge r}$
4. $X_{t\wedge\neg t} = 0$ 5. $X_{\neg(t\wedge s)} = X_{\neg t\vee\neg s}$ 6. $X_{t\wedge(s\vee r)} = X_{(t\wedge s)\vee(t\wedge r)}$
7. $X_{\neg\neg t} = X_t$ 8. $X_{t\vee s} = \max(X_t, X_s)$ 9. *If* $a \leqslant b$, $X_{(a|b)\wedge(a|b\vee c)} = X_{(a|b\vee c)}$.

The next step consists in partitioning $\mathbb{T}(A)$ in equivalence classes, each of which contains compound conditionals giving the same possibilistic quantity in any conditional possibility space over $\mathbf{A}$.

Definition 5. *For all $t, s \in \mathbb{T}(A)$, t is equivalent to s, written $t \equiv s$ whenever $X_t = X_s$ under any conditional possibility Π on $\mathbf{A} \times \mathbf{A}'$.*

It is clear that $\equiv$ is an equivalence relation, and hence we can consider the quotient $\mathbb{T}(A)/\equiv$. Letting $[t]$ being the equivalence class of a generic term $t \in \mathbb{T}(A)$ under $\equiv$, define $\wedge^*, \vee^*, \neg^*$ on $\mathbb{T}(A)$ as follows: for all $[t], [s] \in \mathbb{T}(A)$, $[t]\wedge^*[s] = [s\wedge t]$, $[t]\vee^*[s] = [s\vee t]$, $\neg^*[t] = [\neg t]$, $0 = [(\perp|\top)]$, $1 = [(\top|\top)]$. By the properties of X_t, the operations are well defined (we skip details due to lack of space) and, by Proposition 3, they endow $\mathbb{T}(A)/\equiv$ with a Boolean structure.

Theorem 3. *$\mathcal{T}(\mathbf{A}) = (\mathbb{T}(A)/\equiv, \wedge^*, \vee^*, \neg^*, 0, 1)$ is a Boolean algebra.*

Next shows natural properties of conditionals that hold in the current setting.

Proposition 4. *The following properties hold in $\mathcal{T}(\mathbf{A})$:*
(i) $[(a|a)] = 1$, *(ii)* $[(a|b) \wedge (c|b)] = [(a \wedge c|b)]$,
(iii) $[\neg(a|b)] = [(\bar{a}|b)]$, *(iv)* $[(a \wedge b|b)] = [(a|b)]$,
(v) $[(a|b) \wedge (b|c)] = [(a|c)]$, *if* $a \leqslant b \leqslant c$.

Proof. For each one of the equalities above, of the form $[t] = [s]$, we proved in previous examples that $X_t = X_s$. □

Properties (*i*)-(*v*) turn out to be to conditions (C1)-(C5) in [18] required in the construction of a finite Boolean algebra $\mathcal{C}(\mathbf{A})$ of conditional objects starting from a finite algebra of events $\mathbf{A}$. In particular (C5) stands for a qualitative counterpart of the Bayes rule for conditional probabilities ($P(a \wedge b|c) = P(a|c) \cdot P(b|a \wedge c)$) and for condition ($C\Pi 3$) of Definition 2 for $\odot$-conditional possibilities, equivalently expressed when $a \leqslant b \leqslant c$. These properties are enough to prove that the sets of atoms of both $\mathcal{T}(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$ are in bijective correspondence and hence the following holds.

Theorem 4. *The algebras $\mathcal{T}(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$ are isomorphic.*

By the above and [18] we hence know that each atom of $\mathcal{T}(\mathbf{A})$ can be regarded as terms $(\alpha_{i_1}|\top) \wedge (\alpha_{i_2}|\bar{\alpha}_{i_1}) \wedge ... \wedge (\alpha_{i_{n-1}}|\bar{\alpha}_{i_1}...\bar{\alpha}_{i_{n-2}})$ where $\mathrm{at}(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$ and $\{i_1, ..., i_{n-1}\}$ are $n-1$ pairwise different indices from $\{1, ..., n\}$.

6 Possibility Measures on $\mathcal{T}(\mathbf{A})$ and canonical extensions

Since $\mathcal{T}(\mathbf{A})$ is a Boolean algebra, we can define possibility measures on it. Actually, we can show that the possibilistic expectations $\mathbb{E}(X_t)$'s of the variables X_t's determine in fact an (unconditional) possibility on $\mathcal{T}(\mathbf{A})$.

Definition 6. *Given a $\odot$-conditional possibility $\Pi : A \times A' \to [0, 1]$, we define the mapping $\Pi^* : \mathcal{T}(\mathbf{A}) \to [0, 1]$ as follows: for every $[t] \in \mathcal{T}(\mathbf{A})$,*

$$\Pi^*([t]) =_{def} \mathbb{E}(X_t) = \max_w \mathbb{E}^c(X_{t^w}) \odot \Pi(w|\mathbf{b}(t)).$$

Again, this is well defined, as if t and t' are terms such $t \equiv t'$, it is immediate to check that $\Pi^*([t]) = \Pi^*([t'])$. Moreover, Π^* is a possibility measure in $\mathcal{T}(\mathbf{A})$:

- $\Pi^*(\bot) = \mathbb{E}(X_\bot) = 0$, $\Pi^*(\top) = \mathbb{E}(X_\top) = 1$, and
- $\Pi^*(t \vee s) = \mathbb{E}(X_{t \vee s}) = \mathbb{E}(X_t \vee X_s) = \max(\mathbb{E}(X_t), \mathbb{E}(X_s)) = \max(\Pi^*(t), \Pi^*(s))$

Notice that, given a conditional possibility Π on $A \times A'$, Π^* is a (unconditional) possibility measure in $\mathcal{T}(\mathbf{A})$ such that, for every basic conditional $(a|b)$,

$$\Pi^*([(a|b)]) = \mathbb{E}(X_{(a|b)}) = \Pi(a|b),$$

as we checked after Definition 4. In other words, Π^* satisfies the possibilistic counterpart of *Stalnaker's hypothesis* for the probabilistic case. Moreover, Definition 6 provides a recursive procedure to compute the possibility measure $\Pi^*([t])$ of any compound conditional t, in terms of conditional possibilities of basic conditionals. For instance, based on Example 2, we get the following expression for the possibility measure of the conjunction of two conditionals:

$$\Pi^*([(a|b)\wedge(c|d)]) = \Pi(abcd|b\vee d)\vee[\Pi(a|b)\odot\Pi(\bar{b}cd|b\vee d)]\vee[\Pi(c|d)\odot\Pi(ab\bar{d}|b\vee d)].$$

It turns out that Π^* is not an arbitrary possibility measure on the algebra $\mathcal{T}(\mathbf{A})$ of (equivalence classes of) possibilistic variables, but a very special one. As a matter of fact, next theorem shows that Π^* can be seen as the *canonical extension* of the conditional possibility Π on $A \times A'$ to $\mathcal{T}(\mathbf{A})$.

Theorem 5. *Let* $\mathbf{A}$ *be a Boolean algebra with* $\mathrm{at}(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$ *and let* Π *be a conditional possibility on* $A \times A'$. *Then, for each sequence* $\langle\beta_1, \ldots, \beta_m\rangle$ *of* m *pairwise incompatible events from* $\mathbf{A}$, *with* $m \leqslant n$, *it holds that:*

(1) $\Pi^*((\beta_1|\top) \wedge (\beta_2|\overline{\beta_1}) \wedge ... \wedge (\beta_m|\overline{\beta_1} \wedge \ldots \wedge \overline{\beta_{m-1}})) =$
$= \Pi(\beta_1) \odot \Pi(\beta_2|\overline{\beta_1}) \odot \ldots \odot \Pi(\beta_m|\overline{\beta_1} \wedge \ldots \wedge \overline{\beta_{m-1}})$, *and in particular*

(2) $\Pi^*((\alpha_1|\top) \wedge (\alpha_2|\overline{\alpha_1}) \wedge ... \wedge (\alpha_{n-1}|\overline{\alpha_1} \wedge \ldots \wedge \overline{\alpha}_{n-2})) =$
$= \Pi(\alpha_1) \odot \Pi(\alpha_2|\overline{\alpha_1}) \odot \ldots \odot \Pi(\alpha_{n-1}|\overline{\alpha_1} \wedge \ldots \wedge \overline{\alpha_{n-2}})$.

Proof. We prove (1) and first show by induction that $X_{(\beta_1|\top)\wedge\ldots\wedge(\beta_m|\overline{\beta_1}\wedge\ldots\wedge\overline{\beta_{m-1}})} = \Pi(\beta_m|\bar{\beta}_1...\bar{\beta}_{m-1})\odot\Pi(\beta_{m-1}|\bar{\beta}_1...\bar{\beta}_{m-2})\odot...\odot\Pi(\beta_2|\bar{\beta}_1)\odot X_{\beta_1}$. For $k \in \{1, ..., m-1\}$, let $t_k = (\beta_k|\bar{\beta}_1...\bar{\beta}_{k-1}) \wedge ... \wedge (\beta_m|\bar{\beta}_1...\bar{\beta}_{m-1})$, where $\mathbf{b}(t_k) = \bar{\beta}_1...\bar{\beta}_{k-1}$. Then:
($\bullet$) Let $k = 1$. Hence $t_1 = \beta_1 \wedge (\beta_2|\bar{\beta}_1) \wedge ... \wedge (\beta_m|\bar{\beta}_1...\bar{\beta}_{m-1})$, and $\mathbf{b}(t_1) = \top$. Then:

$$X_{t_1}(w) = \begin{cases} 1, & \text{if } w \models \bot \\ 0, & \text{if } w \models \bar{\beta}_1 \\ \mathbb{E}^c(X_{(\beta_2|\bar{\beta}_1)\wedge\ldots\wedge(\beta_m|\bar{\beta}_1\ldots\bar{\beta}_{m-1})}), & \text{if } w \models \beta_1 \end{cases}$$

Thus, $X_{t_1} = \mathbb{E}^c(X_{t_2})\odot X_{\beta_1}$, and $\mathbb{E}(X_{t_1}) = \mathbb{E}^c(X_{t_2})\odot\mathbb{E}(X_{\beta_1}) = \mathbb{E}^c(X_{t_2})\odot\Pi(\beta_1)$.
($\bullet$) Let $k \leqslant m-2$ and assume, by inductive hypothesis, that the following hold:
- $\mathbb{E}^c(X_{t_k}) = \mathbb{E}^c(X_{t_{k+1}}) \odot \Pi(\beta_k|\bar{\beta}_1...\bar{\beta}_{k-1})$,
- $X_{t_1} = \mathbb{E}^c(X_{t_{k+1}}) \odot \Pi(\beta_k|\bar{\beta}_1...\bar{\beta}_{k-1}) \odot ... \odot \Pi(\beta_2|\bar{\beta}_1) \odot X_{\beta_1}$.

Now consider the variable $X_{t_{k+1}}$, where $\mathbf{b}(t_{k+1}) = \bar{\beta}_1...\bar{\beta}_k$. Then:

$$X_{t_{k+1}}(w) = \begin{cases} 1, & \text{if } w \models \bot \\ 0, & \text{if } w \models \bar{\beta}_1...\bar{\beta}_k\bar{\beta}_{k+1} \\ \mathbb{E}^c(X_{(\beta_{k+2}|\bar{\beta}_1\ldots\bar{\beta}_k)\wedge\ldots\wedge(\beta_m|\bar{\beta}_1\ldots\bar{\beta}_{m-1})}), & \text{if } w \models \bar{\beta}_1...\bar{\beta}_k\beta_{k+1} \end{cases}$$

Hence $X_{t_{k+1}} = \mathbb{E}^c(X_{t_{k+2}}) \odot X_{\bar{\beta}_1\ldots\bar{\beta}_k\beta_{k+1}}$, and thus we have:
- $\mathbb{E}^c(X_{t_{k+1}}) = \mathbb{E}^c(X_{t_{k+2}})\odot\Pi(\bar{\beta}_1...\bar{\beta}_k\beta_{k+1}|\bar{\beta}_1...\bar{\beta}_k) = \mathbb{E}^c(X_{t_{k+2}})\odot\Pi(\beta_{k+1}|\bar{\beta}_1...\bar{\beta}_k)$,
- $X_{t_1} = \mathbb{E}^c(X_{t_{k+1}}) \odot \Pi(\beta_k|\bar{\beta}_1...\bar{\beta}_{k-1}) \odot ... \odot \Pi(\beta_2|\bar{\beta}_1) \odot X_{\beta_1}$
$= \mathbb{E}^c(X_{t_{k+2}})\odot\Pi(\beta_{k+1}|\bar{\beta}_1...\bar{\beta}_k)\odot\Pi(\beta_k|\bar{\beta}_1...\bar{\beta}_{k-1})\odot...\odot\Pi(\beta_2|\bar{\beta}_1)\odot X_{\beta_1}$.

(•) In particular, taking $k = m - 2$, we have

$$\mathbb{E}^c(X_{t_{k+2}}) = \mathbb{E}^c(X_{t_n}) = \mathbb{E}^c(X_{(\beta_n|\bar{\beta}_1...\bar{\beta}_{m-1})}) = \Pi(\beta_n|\bar{\beta}_1...\bar{\beta}_{m-1})$$

and thus,

$$X_{t_1} = \Pi(\beta_n|\bar{\beta}_1...\bar{\beta}_{m-1}) \odot \Pi(\beta_{n-1}|\bar{\beta}_1...\bar{\beta}_{m-2}) \odot ... \odot \Pi(\beta_2|\bar{\beta}_1) \odot X_{\beta_1}.$$

Finally, taking expectations we have:

$$\Pi^*(X_{t_1}) = \mathbb{E}(X_{t_1}) = \Pi(\beta_n|\bar{\beta}_1...\bar{\beta}_{m-1})\odot\Pi(\beta_{n-1}|\bar{\beta}_1...\bar{\beta}_{m-2})\odot...\odot\Pi(\beta_2|\bar{\beta}_1)\odot\Pi(\beta_1),$$

that proves (1). Claim (2) follows from (1) when taking the set of atoms as the set of pair-wise incompatible events and noticing that $\overline{\alpha_1} \wedge \ldots \wedge \overline{\alpha_{n-1}} = \alpha_n$. □

Expression (2) in the above theorem tells us is that Π^* is nothing but the *canonical extension* of the original conditional possibility Π to the algebra $\mathcal{T}(\mathbf{A})$ (or $\mathcal{C}(\mathbf{A})$ if you prefer) in the sense of [20], where the original conditional probabilistic setting from [16] has been adapted to the possibilistic case.

7 Conclusions

In this paper we have proposed a possibilistic counterpart of the random quantity-based approach to (compound) conditionals, and have shown that it preserves all their main properties as well as the underlying Boolean algebraic structure of compound conditionals that arises from them, and thus appearing as an essential feature independent from the particular probabilistic or possibilistic uncertainty quantification model used.

As for future work, since possibility measures are a particular class of upper probabilities, we plan to explore the feasibility of using in the definition of the variables X_t the corresponding lower previsions. This might lead to an alternative model of conditionals.

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