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From Fuzzy Sets to Mathematical Fuzzy Logic

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In this paper our aim is to provide a short survey of main historical developments of systems of fuzzy logic in narrow sense, today under the umbrella of the discipline called Mathematical Fuzzy Logic, arising from the birth of Zadeh's fuzzy sets in 1965. Particular attention is devoted to show how the tools of mathematical logic have allowed to define logical systems which form the core of Mathematical Fuzzy Logic and allow for a formalization of some topics included in Zadeh's agenda spanning from fuzzy sets to approximate reasoning and probability theory of fuzzy events.

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1 Introduction

During the 20th century, three main variations of classical logic have been proposed in order to accommodate the formalization of properties and concepts which, otherwise, would not be feasible in the classical setting. Those are modal, intuitionistic and many-valued logics. The first one is an expansion of classical logic by means of logical operators, called modalities, intended to represent *modes of truth* such as "the formula φ is necessary/possibly true". Intuitionistic logic is a logical system which arises from constructivism; the main idea behind it is that proofs obtained by the *reductio ad absurdum* argument should not be acceptable. By doing so, intuitionistic logic gives a different meaning, for instance, to the negation connective (which is no longer involutive) and to the existential quantifier: $\exists(x)P(x)$ is provable if it is possible to find a witness a such that $P(a)$ is provable. Finally, in many-valued logics propositions can take intermediate truth-values between the classical "true" (1) and "false" (0) and thus, they do not satisfy the classical *tertium non datur* law.

Although Aristotle already considered to adopt not bivalent logics to treat future contingent propositions, it is common to fix the birth of many-valued logics around 1920, when the first three-valued logics of Łukasiewicz, Kleene, Bochvar were firstly introduced. Remarkably, these logics are distinguishable by the intended semantics given to the third-value, that is the new intermediate value between 0 and 1. These three-valued logics and their further generalizations to larger truth-values sets share the property of truth-functionality, that is, the truth-value of a compound formula is determined by the values of its subformulas. Truth-functionality for the many-valued logics marks a clear difference from other non-classical graded logics which aim at formalizing uncertain reasoning and modeling intentional notions, such as probability logics, which are well-known to be not truth-functional.

Among many-valued logics, the family of *fuzzy* logics usually denotes those whose truth-values set is linearly ordered. In most of the cases the whole real unit interval $[0, 1]$, or a finite subset of it, is taken as set of truth-values. This allows to model a notion of *comparative truth* underlying the interpretation of gradual properties.

This paper focuses on many-valued systems within the scope of fuzzy logics and, in particular, on the following topics. In the next section we will present an historical overview on the born of fuzzy logics and their relation to Zadeh's fuzzy set theory and, in Subsection 3.1 in particular we will recall how the discipline that nowadays goes under the name of Mathematical Fuzzy Logic was initiated in the 1990's. Section 3 is devoted to present the main propositional logics based on continuous (BL) and left-continuous t-norms (MTL) while in Section 4, we will recall their main schematic extensions and expansions. As for the latter, we will show how the already powerful expressive power of the main t-norm based fuzzy logics can be further enriched by, respectively, truth-constants, truth-stressing and truth-depressing hedges and an involutive negation. Section 5 deals with first-order t-norm based fuzzy logics and the complexity issues for both propositional and predicate logics. Finally, in Section 6 we will discuss on further topics related to formal fuzzy logics and in particular on their probability theory and fuzzy modal logics. We conclude with some final comments in Section 7.

A significant part of the material contained in this manuscript has been taken from the handbook chapter [38] and the survey paper [64].

2 From fuzzy sets to fuzzy logics

Consider a non-empty set X of objects. A *fuzzy set* A in X was described by Zadeh as a function μ_A, called the *membership function of* A, which associates to each element x of X a real number $\mu_A(a)$ in $[0, 1]$ and which represents the *grade* (or *degree*) *of membership* of x to A. Fuzzy sets hence generalize (classical) set: if μ_A only takes values $\{0, 1\}$ then, upon identifying A with its characteristic function, μ_A describes a (classical) subset of X.

Besides defining the notion of fuzzy set, in [131] Zadeh also discussed on how to extend the usual operations of intersection, union and complementation to this setting. The natural choice he made was the following: the intersection of two fuzzy sets μ_A and μ_B is modeled by the minimum function, i.e. for each $x \in X$, $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$, their union by the maximum function, i.e. $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$, and the complement of μ_A was computed as $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$.

In the beginning 1980s it became common use in the mathematical fuzzy community to consider t-norms as suitable candidates for connectives upon which generalized intersection operations for fuzzy sets should be based, see [5, 37, 120, 88, 130]. These t-norms, a shorthand for "triangular norms", first became important in discussions of the triangle inequality within probabilistic metric spaces, see the monographs [125] and later [89, 4]. They are binary operations in the real unit interval that endow it with an structure of an ordered commutative monoid with 1 as unit element. The three most prominent and well-known t-norms are the so-called Łukasiewicz, Product and Gödel t-norms, denoted respectively T_L, T_P and T_G, and defined as $T_L(x, y) = \max\{x+y-1, 0\}$, $T_P(x, y) = x \cdot y$, and $T_G(x, y) = \min\{x, y\}$, for every $x, y \in [0, 1]$.

In an analogous way, the so-called t-conorms appeared as suitable operations for the union operations of fuzzy sets as dual counterparts of t-norms. Indeed, as t-norms, t-conorms are binary operations in the real unit interval providing $[0, 1]$ with an structure of ordered commutative monoid, but now replacing 1 by 0 as unit element. More precisely, if $n : [0, 1] \to [0, 1]$ is a bijective involutive order reversing mapping such that $n(0) = 1$ (and hence $n(1) = 0$), then given a t-norm T, the operation S defined as $S(x, y) = n(T(n(x), n(y))$ is a t-conorm, and viceversa, T can be defined from S and n in an analogous way. The following are the corresponding t-conorms of T_L, T_P and T_G t-norms for $n(x) = 1 - x$: $S_L(x, y) = \min\{1, x + y\}$, $T_P(x, y) = x + y - x \cdot y$, and $T_G(x, y) = \max\{x, y\}$, for every $x, y \in [0, 1]$.

The general understanding in the context of fuzzy connectives is that t-norms form a suitable class of generalized conjunction operators.

Although today it seems completely natural to relate fuzzy set theory and formal systems for many-valued logics, that was not the case when fuzzy sets were introduced by Zadeh in the '60s of the last century [131]. In fact, Zadeh himself, although presenting fuzzy sets as a tool to model vague notions —and by doing so he surely recognized the many-valuedness of his own approach— he did not relate fuzzy sets to many-valued logics at the beginning. Not surprisingly, also the overwhelming majority of fuzzy set papers that followed [131] treated fuzzy sets in the standard mathematical context, i.e. with an implicit reference to a naive understanding of classical logic as argumentation structure.

Goguen was the first among Zadeh's immediate followers who emphasized an intimate relationship between fuzzy sets and many-valued logics. In his 1969 paper [65], he considers membership degrees as generalized truth values, i.e. as truth degrees. Additionally he sketches a "solution" of the sorites paradox, i.e. the heap paradox, using – but only implicitly – the ordinary product in $[0, 1]$ as a generalized conjunction operation. Based on these ideas, he proposes, as suitable structures for the membership degrees of fuzzy sets, completely distributive lattice ordered monoids $(A, \leq, *, 0, 1)$ enriched, whenever definable, with an operation $\Rightarrow$ which is the (right) residuum of the monoidal operation $*$, and hence characterized by the well known adjointness condition

$$a * b \leq c \quad \text{iff} \quad b \leq a \Rightarrow c\,, \tag{1}$$

and with the "implies falsum"-negation $\neg$, i.e. defined as $\neg a = a \Rightarrow 0$ In other words, completely distributive commutative, bounded and integral residuated lattices.

Gottwald in [66] noticed that a monoid over the unit real interval $[0,1]$ defined by a left continuous t-norm $*$ always has residuum $\Rightarrow$, defined as

$$x \Rightarrow y = \max\{z \in [0,1] \mid x * z \leq y\}.$$

Structures of this kind were found to be relevant examples over which one could define a propositional language (with connectives $\wedge, \vee$ for the truth degree functions $\min, \max$, and connectives $\&, \to$ for a left-continuous t-norm and its residuum) to develop fuzzy set theory within – at least as long as the complementation of fuzzy sets remains out of scope.

Indeed, each (left continuous) t-norm $*$ uniquely determines a semantical (propositional) calculus over formulas defined in the usual way from a countable set of propositional variables, connectives $\wedge$, $\&$ and $\to$ and truth-constant $\overline{0}$ [69]. Further connectives are defined as follows:

$$\begin{aligned} \varphi \vee \psi &\text{ is } ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\ \neg\varphi &\text{ is } \varphi \to \bar{0}, \\ \varphi \equiv \psi &\text{ is } (\varphi \to \psi)\&(\psi \to \varphi). \end{aligned}$$

Evaluations of propositional variables are mappings e assigning each propositional variable p a truth-value $e(p) \in [0,1]$, which extend univocally to compound formulas as follows:

$$\begin{aligned} e(\overline{0}) &= 0 \\ e(\varphi \wedge \psi) &= \min(e(\varphi), e(\psi)) \\ e(\varphi \& \psi) &= e(\varphi) * e(\psi) \\ e(\varphi \to \psi) &= e(\varphi) \Rightarrow e(\psi) \end{aligned}$$

Note that, from the above defintions, $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$, $\neg\varphi = e(\varphi) \Rightarrow 0$ and $e(\varphi \equiv \psi) = e(\varphi \to \psi) * e(\psi \to \varphi)$.

A worthwhile comment at this point is that, due to the fact that the residuation property (1) holds for the pair $(*, \Rightarrow)$, these calculi admit a *graded form of modus ponens.* Indeed, for any evaluation e and $\alpha, \beta \in [0,1]$, from lower bounds on the value of an implication and on its premise, we can derive a lower bound for the value of the conclusion according to the following expression:

$$\frac{e(\varphi) \geq \alpha, \quad e(\varphi \to \psi) \geq \beta}{e(\psi) \geq \alpha * \beta}.$$

Two prominent many-valued logics that fall in this class of calculi, namely Łukasiewicz and Gödel infinitely-valued logics, [92, 63], denoted Ł and G respectively, were defined much before fuzzy logic was born. They indeed correspond to the calculi defined by Łukasiewicz and min t-norms T_L and T_G respectively. Łukasiewicz logic Ł has received much attention from the fifties, when completeness results were proved by Rose and Rosser [121], and by algebraic means by Chang [16, 17], who developed the theory of MV-algebras largely studied in the literature. Many results about Łukasiewicz logic and MV-algebras can be found in the books [29, 109]. On the other hand, a completeness theorem for Gödel logic was already given in the fifties by Dummett [39]. Note that the algebraic structures related to Gödel logic are linear Heyting algebras (known as Gödel algebras in the context of fuzzy logics), that have been studied in the setting of intermediate or superintuitionistic logics, i.e. logics between intuitionistic and classical logic.

In fact, the type of logical setting based on left-continuous t-norms for fuzzy set theory was considered in the 1980s and beginning of the 1990s, although in a naive way: these early approaches were mainly semantically oriented calculi and what was in general missing, with the exception of infinite-valued Łukasiewicz logic Ł and Gödel logic G,[1] was a systematic investigation on suitable syntactic calculi providing adequate axiomatizations of them.

The first proposal to fill in this gap was made by Ulrich Höhle [84–86] who offered in 1994 his *monoidal logic* ML. This common generalization of Łukasiewicz logic Ł, intuitionistic logic and the additive fragment of Girard's integral, commutative linear logic aMALL [62], was determined by an algebraic semantics given by the class of all M-algebras, namely the variety of commutative, integral and bounded residuated lattices. At this point, it is interesting to notice that Höhle's monoidal logic belongs to the family of substructural logics, namely M-algebras are nothing but the algebras of the logic FL_{ew}, i.e. Full Lambek calculus with exchange and weakening. Adequate axiomatizations for the propositional as well as for the first-order version of this logic were given in [84, 86].

3 The logics of continuous t-norms and the beginning of "Mathematical Fuzzy Logic"

Monoidal logic intended to grasp the relationship between fuzzy set theory and the t-norm based setting of their set-algebraic operations. But it was not strongly enough tied with this background. In contrast to Höhle's general approach, Hájek's proposal was to restrict the analysis and he devoted himself to the task of axiomatising the *common tautologies* of all *continuous t-norm* based calculi. In short, to define the logic of all continuous t-norms [68, 69], called *Basic fuzzy logic* and usually denoted BL.

There are two crucial properties which pave the way to the original algebraic semantics for BL and that marks a difference w.r.t. the M-algebras in general.

– The first one is that for any t-norm $*$ and their residuation operation $\Rightarrow$ one has
$$(a \Rightarrow b) \vee (b \Rightarrow a) = 1\,, \tag{2}$$
with $\vee$ denoting the lattice join here, i.e. the max-operation for a linearly ordered carrier. This equation is known as the *prelinearity* condition, and it is not in general satisfied in M-algebras. For instance, if this condition is imposed upon the Heyting algebras, which form an adequate algebraic

[1] In 1996 the product logic [77] was added to this list.

semantics for intuitionistic logic, the resulting class of prelinear Heyting algebras is an adequate algebraic semantics for the infinite-valued Gödel logic.
– The second observation is that the continuity condition of a t-norm can be given in algebraic terms: for any left-continous t-norm $*$ and its residuum $\Rightarrow$ one has that the *divisibility* condition

$$a * (a \Rightarrow b) = a \wedge b \tag{3}$$

is satisfied if and only if $*$ is a continuous t-norm, see [85]. In the condition above $*$ denotes the semigroup operation and $\wedge$ the lattice meet (i.e. the min operation on $[0, 1]$).

BL-algebras were then defined just as M-algebras which satisfy the prelinearity and the divisibility conditions. In particular all BL-algebras over the real unit interval $[0, 1]$ are those defined by a continuous t-norm and its residuum, and they are known as *standard* BL-algebras.

In his highly influential monograph [69], Hájek characterized his basic (propositional) fuzzy logic BL as the logic whose algebraic semantics is the class of BL-algebras, and gave an axiomatization via the following set of axioms:

($\mathsf{Ax_{BL}}1$) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
($\mathsf{Ax_{BL}}2$) $\varphi \,\&\, \psi \to \varphi$,
($\mathsf{Ax_{BL}}3$) $\varphi \,\&\, \psi \to \psi \,\&\, \varphi$,
($\mathsf{Ax_{BL}}4$) $\varphi \,\&\, (\varphi \to \psi) \to \psi \,\&\, (\psi \to \varphi)$,
($\mathsf{Ax_{BL}}5a$) $(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$,
($\mathsf{Ax_{BL}}5b$) $(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$,
($\mathsf{Ax_{BL}}6$) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,
($\mathsf{Ax_{BL}}7$) $\overline{0} \to \varphi$,

and with *modus ponens* as its only inference rule.

Routine calculations show that the axioms $\mathsf{Ax_{BL}}5a$ and $\mathsf{Ax_{BL}}5b$ express the adjointness condition (1). Also by elementary calculations one can show that $\mathsf{Ax_{BL}}6$ formulates the prelinearity condition (2). This was one of the interesting reformulations Hájek gave to the standard algebraic properties. Another one was that he recognized that the weak disjunction, i.e. the connective which corresponds to the lattice join operation in the truth degree structures, could be defined as

$$\varphi \vee \psi \;=_{\mathrm{def}}\; ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \,. \tag{4}$$

Here $\wedge$ is the weak conjunction with the lattice meet as truth degree function which can, according to the divisibility condition, be defined as

$$\varphi \wedge \psi \;=_{\mathrm{def}}\; \varphi \,\&\, (\varphi \to \psi) \,. \tag{5}$$

Hájek proved that BL logic was complete w.r.t. the whole class of BL-algebras, but he also conjectured that the subclass of all standard BL-algebras would be enough for completeness. Although he was very close to prove the conjecture, it was finally Cignoli et al. who proved the standard completeness of BL in [31].

Yet another fundamental property related to the BL logic was proved in [46] by algebraic methods: for any continuous t-norm, its corresponding calculus can be axiomatized as a finitary extension of BL. This result was later improved by Haniková in [81] where she shows that the equational theory of an arbitrary class of standard BL-algebras is finitely based as well.

3.1 The beginning of "Mathematical Fuzzy Logic"

Hájek's book [69] contains not only the introduction of BL but also many logical results related to it and some others concerning the formalization of some of the tasks which formed Zadeh's agenda, the so-called Fuzzy Logic in wide sense. In fact Hájek quotes Zadeh's preface of the book [134] where he made a very clear distinction between the two main meanings of the term fuzzy logic. Indeed, he writes:

> The term "fuzzy logic" has two different meanings: wide and narrow. In a narrow sense it is a logical system which aims a formalization of approximate reasoning. In this sense it is an extension of multi-valued logic. However the agenda of fuzzy logic is quite different from that of traditional many-valued logic. Such key concepts in FL as the concept of linguistic variable, fuzzy if-then rule, fuzzy quantification and defuzzification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not addressed in traditional systems. In its wide sense, FL, is fuzzily synonymous with the fuzzy set theory of classes of unsharp boundaries.

Hájek, in the introduction of his monograph [69] makes the following comment to Zadeh's quotation:

> Even if I agree with Zadeh's distinction (. . .) I consider formal calculi of many-valued logic to be the kernel of fuzzy logic in the narrow sense and the task of explaining things Zadeh mentions by means of this calculi to be a very promising task.

Also, Novák *et al.*, in the introduction of their monograph [115], write:

> Fuzzy logic in narrow sense is a special many-valued logic which aims at providing formal background for the graded approach to vagueness.

Actually, the book by Hájek [69] has been considered as a sort of official start of the new discipline called Mathematical Fuzzy Logic, with the goal of providing rigorous, logical foundations to fuzzy logic in narrow sense. This branch of mathematical logic has had a great development over the past twenty years and from many points of view (logical, algebraic, proof-theoretical, functional representation, complexity analysis, etc.), as witnessed by a number of important monographs that have appeared in the literature at the beginning of this century see [69, 67, 115, 98], and culminated more recently by the series of three handbooks [32–34].

3.2 The logic of all left-continuous t-norms

Only a short time after Hájek's axiomatization of BL (the logic of continuous t-norms), Esteva and Godo in [42] introduced the logic MTL (for *Monoidal t-norm based Logic*) inspired by the following basic property that we already recalled in Section 2:

- A t-norm has residuum if and only if it is left-continuous.

Let us observe that, in general, left-continuous t-norms do not satisfy the divisibility property (3) and hence, in MTL, the weak conjunction connective $\wedge$ cannot be defined in terms of the strong conjunction & (interpreted by a left-continuous t-norm $*$) and the implication $\rightarrow$ (modeled by the residuum of $*$), whence it has to be introduced as primitive connective. Indeed, the axioms of MTL are obtained from those of BL by replacing axiom ($\mathsf{Ax_{BL}}4$) by the three following ones:

($\mathsf{Ax_{MTL}}4a$) $\varphi \wedge \psi \rightarrow \varphi$
($\mathsf{Ax_{MTL}}4b$) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$
($\mathsf{Ax_{MTL}}4c$) $\varphi \& (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$

The algebraic semantics of the logic MTL is the variety of MTL-algebras, which can either be obtained by removing the divisibility equation (3) from the equations of Hájek BL-algebras, or as the subvariety of those Höhle's M-algebras which satisfy the prelinearity condition (2).

As Hájek did for BL, also Esteva and Godo conjectured that MTL was the logic of all left-continuous t-norms, that is, *standard* MTL-algebras form a complete semantics for the monoidal t-norm based logic. That claim was confirmed a short later by Jenei and Montagna who, in the paper [87], actually proved that MTL is complete with respect to the class of standard MTL-algebras over $[0, 1]$.

The semantic calculi defined by each left-continuous t-norm and its residuum correspond to different extensions of MTL. Some of them have been studied and axiomatized (e.g. the case of nilpotent minimum t-norm) but there are no general results, as in the case of continuous t-norms, about the axiomatization of these calculi.

Remark 1. All t-norm based fuzzy logics, except Gödel logic, belong to the framework of substructural logics as they fail to satisfy the structural rule of contraction, equivalent in this framework to the contraction axiom $\varphi \rightarrow \varphi \& \varphi$.

4 Distinguished extensions and expansions

In this section we introduce a summary of the main extensions and expansions of the basic systems of t-norms based fuzzy logics. The logics so defined are intended to cope with the main notions of the agenda proposed by Zadeh.

4.1 Main axiomatic extensions of t-norm based fuzzy logics

As we recalled in Section 3.2, the logic MTL can be regarded as the weakest t-norm based logic, in the sense of Hájek. This means, among other things, that the other logics, such as BL, or Gödel, product and Łukasiewicz logic can be obtained as axiomatic extension from MTL. Basic references for this topic are [69] where Łukasiewicz, product and Gödel logic are presented as schematic extension of BL; [42] which, in addition to MTL, also introduces the logics IMTL (*involutive* MTL), NM (*nilpotent minimum*), WNM (*weak* nilpotent minimum) and SBL (*strict* BL). The logic ΠMTL (*product* MTL) was defined in [71], while SMTL (*strict* MTL) was introduced and studied in [40]. An exhaustive and systematic treatment of MTL and its schematic extension can be found in [110].

The following table collects some of the axioms schema which allow to obtain, starting from MTL, some of its most relevant extensions.

Name	Formula
Involution (Inv)	$\neg\neg\varphi \to \varphi$
Pseudocomplementation (PC)	$\varphi \wedge \neg\varphi \to \bot$
Cancellation (Π1)	$\neg\neg\chi \to ((\varphi \& \chi \to \psi \& \chi) \to (\varphi \to \psi))$
Weak Nilpotent Minimum (WNM)	$(\varphi \& \psi \to \bot) \vee (\varphi \wedge \psi \to \varphi \& \psi)$
Divisibility (Div)	$\varphi \wedge \psi \to \varphi \& (\varphi \to \psi)$
Contraction (Con)	$\varphi \to \varphi \& \varphi$

Before recalling how the main extensions of MTL can be obtained (they will be collected in the following Table 1), let us briefly comment on the above axioms.

The first one, involution, forces the negation of a logic to be involutive, as the classical negation. Also the second axiom schema, pseudocomplementation, forces a logics to have a classical behavior. Indeed, if a logic satisfies (PC) it then enjoys the contradiction law with respect to the lattice operation of meet. Divisibility, the peculiar axiom of BL, allows a logic to define the lattice operations, and the $\wedge$ in particular, in terms of the t-norm and its residuum. Contraction can be understood as *idempotency*. It can be shown, in fact, that if a logic satisfies the contraction schema, then $\varphi \& \varphi \leftrightarrow \varphi \wedge \varphi$ and hence $\varphi \& \varphi \leftrightarrow \varphi \wedge \varphi \leftrightarrow \varphi$ since $\wedge$ is idempotent. As for (Π1), (WNM), let us explain them on the standard semantics based on $[0, 1]$. The latter defines t-norms which behave similarly to the usual product between real numbers. Indeed, for a cancellative t-norm, either an element c is 0 (this fact is expressed by the first part of the formula: $\neg\neg\chi$) or $a * c = b * c$ gives that $a = b$ (second part: $(\varphi \& \chi \to \psi \& \chi) \to (\varphi \to \psi)$). Thus, if c is not 0, then one can cancel it on the right and on the left of $a * c = b * c$. Finally, the axiom (WNM) indeed captures the behavior of a (weak) nilpotent minumum t-norm: either $a * b = 0$, or $a * b$ coincides with $\min\{a, b\}$.

Figure 2 presents the main extensions of the logic MTL within the framework of substructural logics where the t-norm based fuzzy logics, are those logics in the blue oval.

MTL extension	Additional axioms schema
IMTL	(Inv)
SMTL	(PC)
*Π*MTL	(PC) and (*Π*1)
WNM	(WNM)
NM	(Inv) and (WNM)
BL	(Div)
SBL	(Div) and (PC)
Ł	(Div) and (Inv)
G	(Con)
P	(Div), (PC) and (*Π*1)

Fig. 1. The main schematic extensions of MTL (names on the left) and the axioms schema which characterizes each of them.

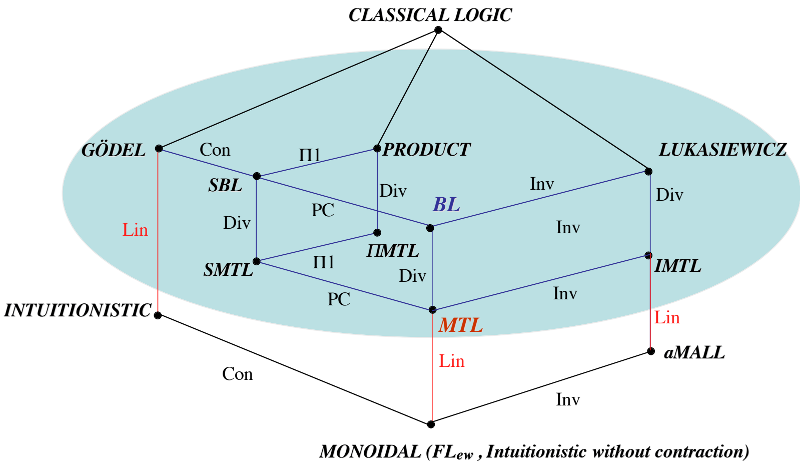

Fig. 2. Main t-norm based fuzzy logics as extensions of MTL and Höhle monoidal logic. In particular, the main extension of MTL are those logics in the blue oval. Further notice that *Lin* stands for the prelinearity axiom that we discussed in Section 3.

4.2 Expansions of t-norm based fuzzy logics

By an *expansion* of a logic L, we intend a logical system whose language is endowed with additional symbols intended to model operations or constants that would not be otherwise definable in the initial language of the logic L.

The first expansions of t-norm based fuzzy logics we consider are by truth constants, that is symbols which ideally corresponds to real, or rational, truth-values for formulas. The logics arising in this way hence allow to handle partial truth directly at the syntactical level. It is worth to recall that the first work in this direction is the one put forward by Pavelka in [119].

Second we expand basic systems with unary connectives that model linguistic qualifiers called *truth stressing* and *truth depressing hedges*. In particular the maximal stressing hedge is the so called Baaz-Monteiro projection Δ that semantically correspond to the crisp connective $\Delta(1) = 1$ and $\Delta(x) = 0$ for every $x \neq 1$ firstly introduced in [7].

Finally, we present a way to expand t-norm based fuzzy logics with an involutive negation. Recall in fact that the negation connective defined by $\neg\varphi = \varphi \to \overline{0}$ usually fails to be involutive. Indeed, it is involutive if and only if the logic is an extension of IMTL as for instance NM and Łukasiewicz logics.

Expansions with truth constants T-norm based fuzzy logics are basically logics of *comparative truth*. In fact, as already mentioned, the residuum $\Rightarrow$ of a (left-continuous) t-norm satisfies for all $x, y \in [0, 1]$ the condition

$$x \Rightarrow y = 1 \text{ if, and only if, } x \leq y.$$

This means that a formula $\varphi \to \psi$ is a logical consequence of a theory if the truth degree of φ is at most as high as the truth degree of ψ in any interpretation which is a model of the theory. In fact, the logic of continuous t-norms as it is presented in Hájek's seminal book [69] only deals with valid formulae and deductions taking 1 as the only truth value to be preserved by inference (in the sense of yielding true consequences from true premises for each interpretation). This line is followed by the majority of logical papers written from then in the setting of Mathematical Fuzzy Logic. However, in general, these truth-preserving logics do not exploit in depth neither the idea of comparative truth nor the potentiality of dealing with explicit partial truth that a many-valued logic setting offers.

In some situations one might be also interested to explicitly represent and reason with intermediate degrees of truth. A way to do so, while keeping the truth preserving framework, is to introduce truth-constants into the language. This approach actually goes back to Pavelka [119] who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz logic obtained by adding into the language a truth-constant $\overline{r}$ for each real $r \in [0, 1]$, together with some additional axioms. Pavelka proved that his logic is strongly complete in a non-finitary sense (known as Pavelka-style completeness), heavily relying on the continuity of Łukasiewicz truth-functions.

Hájek [69] showed that Pavelka's logic PL could be significantly simplified while keeping the completeness results. Indeed, Rational Pavelka Logic (RPL) is defined as the expansion of Łukasiewicz logic Ł by adding a truth constant $\overline{r}$ for each rational $r \in [0, 1]$ together with the following two book-keeping axioms for truth constants:

(RPL1) $\overline{r} \& \overline{s} \leftrightarrow \overline{r *_{Ł} s}$
(RPL2) $\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow_{Ł} s}$

where $*_{Ł}$ and $\Rightarrow_{Ł}$ are Łukasiewicz t-norm and implication respectively. An evaluation e of propositional variables by reals from $[0, 1]$ extends to an evaluation of all formulas as in Łukasiewicz logic over the standard MV-algebra $[0, 1]_{*_{Ł}}$ provided that $e(\overline{r}) = r$ for each rational r.

Notice that a formula of the form $\overline{r} \to \varphi$ gets value 1 by an evaluation e whenever φ gets a value by e greater or equal than r. Therefore, the RPL-formula $\overline{r} \to \varphi$ expresses that the truth-value of φ is at least r. Similarly, $\varphi \to \overline{r}$ expresses that the truth-value of φ is at most r.

A *theory* T over RPL is just a set of formulas. The notion of proof denoted $\vdash_{\mathrm{RPL}}$ is defined as usual from the axioms of RPL and modus ponens.

Given a theory T, the *truth degree* of a formula φ in T is defined as

$$||\varphi||_T = \inf\{e(\varphi) \mid e \text{ is a model of } T, \text{i.e. } e(\psi) = 1 \text{ for all } \psi \in T\},$$

and the *provability degree* of φ over T as

$$|\varphi|_T = \sup\{r \text{ rational of } [0, 1] \mid T \vdash_{\mathrm{RPL}} \overline{r} \to \varphi\}.$$

Note that the provability degree is a supremum, which is not necessarily attained as a maximum; for an infinite T, $|\varphi|_T = 1$ does not always imply $T \vdash \varphi$. (For finite T it does, see e.g. [69, 3.3.14]).

Pavelka style completeness was proved for RPL (see [119, 69]), namely: for any theory T and formula φ of RPL,

$$||\varphi||_T = |\varphi|_T.$$

The proof is strongly related to the fact that in Łukasiewicz logic truth-functions are continuous. Similar expansions with truth-constants for other propositional t-norm based fuzzy logics can be analogously defined but Pavelka-style completeness cannot be obtained since, in contrast to the Łukasiewicz case, not all truth-functions are continuous.

A more general approach has been developed in a series of papers, see e.g. [47, 41, 123, 48], where rather than Pavelka-style completeness the authors have focused on the usual notion of completeness of a logic. It is interesting to note that in this approach: (1) the logic to be expanded with truth-constants has to be the logic of a given left-continuous t-norm; (2) the expanded logic is still a truth-preserving logic, but its richer language admits formulae of type $\overline{r} \to \varphi$ saying that, when evaluated at 1, the truth degree of φ is greater or equal than r; and (3) the expanded logic is still algebraizable in the sense of Blok and Pigozzi. A summary of all results in this topic can be found in the book chapter [44].

In all these works, special attention has been paid to formulas of the kind $\overline{r} \to \varphi$, where $\overline{r}$ denotes the truth-constant r and φ is a formula without additional truth-constants. Actually, this kind of formulas have been extensively considered in other frameworks for reasoning with partial degrees of truth, like in Novák's evaluated syntax formalism based on Łukasiewicz Logic (see e.g. [115]) or in fuzzy logic programming (see e.g. [128]). In particular, these formulas can be seen as a special kind of Novák's *evaluated* formulas, which are expressions a/A

where a is a truth value (from a given algebra) and A is a formula that may contain truth-constants again, and whose interpretation is that the truth-value of A is at least a. Hence the formula $\overline{r} \to \varphi$ would be expressed as r/φ in Novák's evaluated syntax. We invite the reader to consult Subsection 6.3 for further details.

Finally, truth-degrees in the syntax also appear in the Gerla's framework of abstract fuzzy logics [61] which is based on the notion of fuzzy consequence operators over fuzzy sets of formulas, where the membership degree of formulas are again interpreted as lower bounds of their truth degrees.

Expansions with truth-stressing and truth-depressing hedges Typical examples of fuzzy truth-values in the sense of Zadeh (see [133]) are "very true", "quite true", "more or less true", "slightly true", etc. They are represented in fuzzy logic in narrow sense as fuzzy subsets on the set of truth values, typically the real unit interval. In order to cope with these fuzzy truth values in the setting of mathematical fuzzy logic, Hájek proposes in [70] to understand them as truth functions of new unary connectives called either *truth-stressing* or *truth depressing hedges* (depending on whether they reinforce or weaken a truth value). The intuitive interpretation of a truth-stressing (resp. depressing) hedge like *very true* (resp. *slightly true*) on a chain of truth-values is a subdiagonal (resp. superdiagonal) non-decreasing function preserving 0 and 1, called *hedge functions* from now on. Notice that the projection operator Δ (introduced independently first by Monteiro in the context of intuitionistic logic [103] and posteriorly by Baaz in the context of Gödel logics [7]) is a limit case of a truth stresser since, over a chain, it sends 1 to 1 and all the other elements to 0, and the intuitive interpretation would be *definitely true*.

Hájek [70] and Vychodil [129] propose an axiomatization of truth-stressing and depressing hedges respectively as expansions of BL (and of some of their prominent extensions, like Łukasiewicz, Product or Gödel logics) by new unary connectives vt, for *very true*, and st, for *slightly true*, respectively. The logics they define are shown to be algebraizable and to enjoy completeness with respect to the classes of chains of their corresponding varieties. However the axiomatics that Hájek proposes (also used by Vychodil) is quite restrictive since not any BL-chain expanded with a hedge function is a model of the proposed logic, as one would expect from the traditional use of hedges in fuzzy logic in wide sense. Moreover, the defined logics are not proved to enjoy general standard completeness, except for the case of logics over Gödel logic. One of the main reasons for both problems is the presence in the axiomatizations of the axiom K for the vt connective i.e. $vt(\varphi \to \psi) \to (vt(\varphi) \to vt(\psi))$, which puts quite a lot of constraints on the hedges to be models of these logics without a natural algebraic interpretation.

Quite simple and general axiomatizations with very intuitive properties and nice completeness results are given in [50]. Indeed, let L be t-norm based fuzzy logic, and consider L_S the expansion of L with a new unary connective s (for *stresser*) defined by the following additional axioms:

(VTL1) $s\varphi \to \varphi$,
(VTL2) $s\bar{1}$,

and the following additional inference rule:

(MON) from $(\varphi \to \psi) \vee \chi$ infer $(s\varphi \to s\psi) \vee \chi$.

Observe that monotonicity rule (MON) has a rather complicate expression (with the addition of the $\vee\chi$). In fact it is only a technical addition to assure linearity. In [50] it is proved that L_S is complete w.r.t. the chains of the corresponding quasi-variety. Moreover it is also standard complete but, in general only for finite set of premises. In the same paper an axiomatization of truth depresser hedges is presented in a similar way than the one for truth stressers. Nevertheless the problem is not dual from truth stressers since, in general, the negation is not involutive.

Expansions with an involutive negation In the logico-algebraic framework, the problem of extending a relevant class of algebras with an involutive negation probably goes back as early as Moisil in 1942 [99], who considered the expansion of Heyting algebras with an involution. These algebras were extensively investigated by Monteiro under the name of *symmetric Heyting algebras* [103]. They were also considered by Sankappanavar [122], and more recently in [30] in the more general framework of residuated lattices.

In the framework of t-norm based fuzzy logics, the negation connective $\neg$ is defined from the implication $\to$ and the truth constant $\bar{0}$, namely $\neg\varphi$ is $\varphi \to \bar{0}$. However, this negation may have a very differently behaviour in different varieties of algebras. Indeed, for instance, the associated negation function is involutive in any IMTL chain (in particular in algebras associated to Łukasiewicz logic) but it is not any longer involutive outside the variety of IMTL-algebras. The most paradigmatic case are the chains of the variety of SMTL-algebras, where $\neg$ is interpreted by the so-called Gödel's negation n_G, defined by:

$$n_G(x) = x \Rightarrow 0 = \begin{cases} 1, \text{ if } x = 0 \\ 0, \text{ otherwise} \end{cases}$$

In what follows we will summarize results on the expansion of any t-norm based fuzzy logic with an independent involutive negation $\sim$. Particularly interesting are the cases of $\mathrm{SMTL}_\sim$ and their axiomatic extensions $\mathrm{G}_\sim$ and $\Pi_\sim$, where Δ is definable as a composition of the two negations (residuated and involutive) defined there.

Notice that having an involutive negation in the logic enriches, in a non trivial way, the representational power of the logical language. For instance, in the enriched language we can define:

- a strong disjunction $\varphi \veebar \psi$ is definable now by duality, i.e. as $\sim(\sim\varphi \mathbin{\&} \sim\psi)$, with truth function in real algebras $[0,1]_*$ defined by the *dual t-conorm* $\oplus$ defined as $x \oplus y = n(n(x) * n(y))$

– a contrapositive implication $\varphi \hookrightarrow \psi$ is definable as $\sim\varphi \veebar \psi$, with truth function the *strong implication* function $\overset{c}{\Rightarrow}$ defined as $x \overset{c}{\Rightarrow} y = \sim x \oplus y$.

The first paper studying this topic in the context of Mathematical Fuzzy logic is [43] where the authors defined the expansion of the logic SBL (that has a negation $\neg$) with an involutive engation $\sim$. Recall that with both negations the projection connective Δ is definable since $\Delta\varphi$ is $\neg\sim\varphi$.

In the paper the authors gives an axiomatization of what is called $\mathrm{SBL}_\sim$ by the axioms of SBL plus

- $\sim\sim\varphi \equiv \varphi$
- $\neg\varphi \to \sim\varphi$
- $\Delta(\varphi \to \psi) \to \Delta(\sim\psi \to \sim\varphi)$
- $\Delta(\varphi) \vee \neg\Delta(\varphi)$
- $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \delta\psi)$
- $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \delta\psi)$

and modus ponens and necessitation for Δ as inference rules (where Δ is the abreviation for $\neg\sim$).

In this setting it is possible to prove completeness with the chains of the corresponding variety. Only in the case of Gödel logic it is also possible to prove standard completenes with respect to the real unit interval with Zadeh connective $\max, \min, 1 - Id$ and the corresponding residuated implication. In the paper the authors give an example that such standard completeness is not true for Product Logic with an involutive negation. Flaminio and Marchioni in [57] studied the more general case of adding an involution to MTL_Δ and their axiomatic extensions. On the other hand, Cintula et al. investigate in [25, 26] the lattice of subvarieties generated by $\mathrm{SBL}_\sim$-chains and $\Pi_\sim$-chains, and Haniková and Savický [82] go further in the study of subvarieties generated by $\mathrm{SBL}_\sim$-chains studying isomorphisms between pairs formed by a (SBL) t-norm and different involutive negations.

5 Predicate fuzzy logics

The extensions of propositional t-norm based logics to first-order ones follows the standard lines of approach: one has to start from a first-order language $\mathcal{L}$ with the two standard quantifiers $\forall, \exists$ and a suitable commutative, residuated, integral lattice ordered monoid $\mathbf{A}$ over a bounded lattice,[2] and has to define $\mathbf{A}$-interpretations $\mathbf{M}$ by fixing a nonempty domain $M = |\mathbf{M}|$ and by assigning to each predicate symbol of $\mathcal{L}$ an $\mathbf{A}$-valued relation in M (of suitable arity) and to each constant an element from (the carrier of) $\mathbf{A}$.

The satisfaction relation is also defined in the standard way. The quantifiers $\forall$ and $\exists$ are interpreted as taking the infimum or supremum, respectively, of all the values of the relevant instances.

[2] Integrality means that the monoidal unit coincides with the upper bound of the lattice.

Unfortunately, infima and suprema do not always exist in lattices. So one could suppose to consider only complete lattices. A less restrictive assumption is to assume that, for any formula, all the infima and suprema do exist which have to be considered for any evaluation this formula. Interpretations which satisfy this last condition are called *safe* by Hájek [69].

The first-order version BL$\forall$ of Hájek's logic BL is obtained by adding the following axioms and rules to the propositional system (see [69])

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ,
($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ,
($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ,
($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, where x is not free in χ,
($\forall$3) $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, where x is not free in χ,

and the *rule of generalization.*

Here, substitutability and the rule of generalization have the same meaning as in classical first-order logic.

Then he was able to prove the following general *chain completeness theorem*: A first-order formula φ is a theorem of BL$\forall$ iff it is valid in all safe interpretations over BL-chains.

That result can be extended to elementary theories as well as to a lot of other first-order fuzzy logics, e.g. to MTL$\forall$. For this logic MTL$\forall$ one has also a strong *standard completeness* result [102]: a formula φ is MTL$\forall$-provable from a set T of formulas iff φ holds true in all safe $\mathbf{A}$-interpretations $\mathbf{M}$ which are models of T and are based a standard MTL-algebra.

We will not discuss further completeness results here but rather we refer to the extended survey [10]. Anyway it should be mentioned that, as suprema are not always maxima and infima not always minima, the truth degree of an existentially/universally quantified formula may not be the maximum/minimum of the truth degrees of the instances. It is, however, interesting to have conditions which characterize models in which the truth degrees of each existentially/universally quantified formula is witnessed as the truth degree of an instance. This problem was first considered by Hájek in the framework of fuzzy description logics [72], later studied by Cintula and Hájek in [23], and surveyed in [10] as well.

5.1 The Complexity issue

A key problem for a logic is to determine the complexity of the set of its theorems. Usually, the way to perform that analysis is semantic. Indeed, if a logic L is complete with respect a class of structures $\mathbb{K}$ (or models in the first order case) deciding if a formula φ is a theorem of L is the same as deciding if φ is a tautology of $\mathbb{K}$. It is conveniente to define the following sets, for a given class $\mathbb{K}$ of MTL-algebra

- $TAUT^1(\mathbb{K})$ is the set of $\mathbb{K}$-*tautology.* A formula φ is a $\mathbb{K}$-tautology if for every $\mathbf{A} \in \mathbb{K}$ and for every $\mathbf{A}$-valuation v, $v(\varphi) = 1$;

- $TAUT^{>0}(\mathbb{K})$ is the set of a *positive* $\mathbb{K}$-*tautology.* A formula φ is a positive $\mathbb{K}$-tautology if for every $\mathbf{A} \in \mathbb{K}$ and for every $\mathbf{A}$-valuation v, $v(\varphi) > 0$;
- $SAT^1(\mathbb{K})$ is the set of $\mathbb{K}$-*satisfiable* formulas, that is, the set of formulas φ for which there exists $\mathbf{A} \in \mathbb{K}$ and a $\mathbf{A}$-valuation v such that $v(\varphi) = 1$;
- $SAT^{>0}(\mathbb{K})$ is the set of *positively* $\mathbb{K}$-*satisfiable*, that is, the set of formulas φ for which there exists $\mathbf{A} \in \mathbb{K}$ and a $\mathbf{A}$-valuation v such that $v(\varphi) > 0$;

Clearly $TAUT^1(\{0,1\}) = TAUT^{>0}(\{0,1\})$ and $SAT^1(\{0,1\}) = SAT^{>0}(\{0,1\})$. Moreover, in classical propositional and first-order logic, a formula φ is a $\{0,1\}$-tautology iff φ is a $\{0,1\}$-positive tautology iff $\neg\varphi$ is not $\{0,1\}$-satisfiable iff φ is not positively $\{0,1\}$-satisfiable. By Cook's theorem, [35], deciding if a propositional formula φ is $\{0,1\}$-satisfiable is an NP-complete problem and hence deciding if φ is a $\{0,1\}$-tautology is coNP-complete, while, for the first-order case, Church-Turing theorem shows that $TAUT^1(\{0,1\})$ is undecidable.

As for t-norm based logics, however, $TAUT^1(\mathbb{K}) \neq TAUT^{>0}(\mathbb{K})$, $SAT^1(\mathbb{K}) \neq SAT^{>0}(\mathbb{K})$ and $\mathbb{K}$-satisfiable formulas and $\mathbb{K}$-tautologies do not form dual sets in the above sense. Indeed, the formula $x \vee \neg x$ in Łukasiewicz logic neither is a $[0,1]_{MV}$-tautology as witnessed by the valuation $x \mapsto 1/2$, nor its negation $\neg x \wedge x$ is $[0,1]_{MV}$-satisfiable since for all $\alpha \in [0,1]$ one has $\min\{\alpha, 1-\alpha\} \leq 1/2$.

The propositional case As for the propositional t-norm based logics whose decision problem has been settled (this is the case of continuous t-norm based logics such as BL, product, Gödel, Łukasiewicz logics) the situation is not dissimilar to the classical case. Indeed, with respect to the class of standard algebras, we have that both SAT^1 and $SAT^{>0}$ are NP-complete while $TAUT^1$ and $TAUT^{>0}$ are coNP-complete. However, complexity results for left-continuous t-norm based logics are much more fragmented: the SAT^1 and $SAT^{>0}$ are NP-complete for both the NM and the WNM logics, while for MTL, SMTL, ΠMTL and IMTL, we only know that their the above sets are decidable, but no complexity bounds have been established so far. The following table collects known complexities for the main extension of MTL with respect to the class of their standard algebras. Complexity results for the propositional logics mentioned below have been established in a series of paper by multiple authors, [8, 24, 79, 94, 104]. We also invite the interest reader to consult [80] for a concise and exhaustive treatment.

	$TAUT^1$	$TAUT^{>0}$	SAT^1	$SAT^{>0}$
MTL	decidable	decidable	decidable	decidable
IMTL	decidable	decidable	decidable	decidable
SMTL	decidable	decidable	decidable	decidable
ΠMTL	decidable	decidable	decidable	decidable
NM	co-NP-complete	coNP-complete	NP-complete	NP-complete
WNM	co-NP-complete	co-NP-complete	NP-complete	NP-complete
BL	co-NP-complete	co-NP-complete	NP-complete	NP-complete
Ł	co-NP-complete	co-NP-complete	NP-complete	NP-complete
G	co-NP-complete	co-NP-complete	NP-complete	NP-complete
P	co-NP-complete	co-NP-complete	NP-complete	NP-complete

The first-order case Not surprisingly, first order t-norm based logics, besides few exceptions, turn out to be undecidable and hence the main problem is to determine, for each logic, its undecidability degree. The known results in this predicate setting much depends on the chosen semantics we may fix for each logic.

The chapter [78] provides an exhaustive analysis on this topic considering several semantics but focusing in particular on the *general semantics*, given by the class of all chains for the logic, and the *standard semantics*, given by those chains whose lattice reduct is a sublattice of the real unit interval $[0, 1]$. The distinction between the general and the standard semantics is quite important. Indeed, while for the former the undecidability degree for each logic is relatively low (the SAT^1 and $\mathrm{SAT}^{>0}$ are in Σ_1 while TAUT^1 and $\mathrm{TAUT}^{>0}$ are in Π_1), the standard semantics offers a much more variegate range of situations: for MTL, SMTL, ΠMTL, IMTL and G the satisfiability and the tautology problems behave as for the general semantics, for Łukasiewicz first order logic the undecidability degrees of SAT^1, $\mathrm{SAT}^{>0}$ and TAUT^1, $\mathrm{TAUT}^{>0}$ are higher than Σ_1 and Π_1 respectively, but still arithmetical, and for product logic and BL, all the above sets fall outside the arithmetical hierarchy.

	$TAUT^1$	$TAUT^{>0}$	SAT^1	$SAT^{>0}$
MTL∀	Σ_1-complete	Σ_1-complete	Π_1-complete	Π_1-complete
IMTL∀	Σ_1-complete	Σ_1-complete	Π_1-complete	Π_1-complete
SMTL∀	Σ_1-complete	Σ_1-complete	Π_1-complete	Π_1-complete
ΠMTL∀	Σ_1-hard	Σ_1-hard	Π_1-hard	Π_1-hard
NM∀	Σ_1-complete	Σ_1-complete	Π_1-complete	Π_1-complete
WNM∀	Σ_1-complete	Σ_1-complete	Π_1-complete	Π_1-complete
BL∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic
Ł∀	Π_2-complete	Σ_1-complete	Π_1-complete	Σ_2-complete
G∀	Σ_1-complete	Σ_1-complete	Π_1-complete	Π_1-complete
P∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic

The first paper exhibiting a result of computational complexity for first-order fuzzy logics is Scarpellini's [124], published in 1962, showing the non-axiomatizability of Łukasiewicz predicate logic. The undecidability of MTL∀, as well as any other predicate logic between MTL∀ and classical predicate logic, is proved by Montagna and Ono in [102]. A general approach to complexity problems extending the scope predicate fuzzy logics to arbitrary semantics can be found in [101]. We also invite the reader to consult the above mentioned chapter [78], the references therein and also the survey paper [74].

6 Further topics and recent research lines

In this final section we will briefly review on some topics which fall under the scope of Mathematical Fuzzy Logic and which, nowadays, constitute active research areas. In specific terms we will present basic notions, result and ongoing research directions on probability theory on fuzzy events and fuzzy modal logics.

6.1 Probability of fuzzy events

In 1968, Zadeh published a fundamental paper where he approached probability theory for fuzzy events by addressing the following key questions: *what are fuzzy events?* and *how to measure their probability?* Zadeh's answers are grounded on the observation that in a probability space of the form $\langle \mathbb{R}^n, \mathcal{B}(n), P\rangle$ where $\mathcal{B}(n)$ stands for the σ-field of Borel subsets of $\mathbb{R}^n$, a classical event E is nothing else than a Borel subset of $\mathbb{R}^n$ and its probability can be represented as

$$P(E) = \int_{\mathbb{R}^n} \chi_E \, \mathrm{d}P, \tag{6}$$

where χ_E denotes the characteristic function of E.

In the light of this observation, it is natural to propose the following.

Definition 1 ([132]). *Let $\langle \mathbb{R}^n, \mathcal{B}(n), P\rangle$ be a probability space. Then a* fuzzy event *in $\mathbb{R}^n$ is a fuzzy set F whose membership function $\mu_F : \mathbb{R}^n \to [0,1]$ is Borel measurable. The* probability *of a fuzzy event F is hence defined by the Lebesgue integral*

$$P(F) = \int_{\mathbb{R}^n} \mu_F \, \mathrm{d}P. \tag{7}$$

Some remarks are in order. First of all, we must observe the deep analogy between the above equations (6) and (7). It is indeed evident that they differ for the definition of *event*, while *probabilities* (of events) remain essentially unchanged. Second, Zadeh's definition of probability (of fuzzy events) is not axiomatic *à la Kolmogorov* and, although the basic properties of *normalization* ($P(\mathbb{R}^n) = 1$), *monotonicity* (if $\mu_A \leq \mu_B$, then $P(A) \leq P(B)$) and *finite additivity* ($P(A \cup B) = P(A) + P(B) - P(A \cap B)^3$) hold for a probability function on fuzzy events, in [132] it is not proved that they indeed characterize these measures. Finally it must be observed that the definition of fuzzy event as given in Definition 1 is

[3] Where $\mu_{A\cup B} = \max(\mu_A, \mu_B)$ and $\mu_{A\cap B} = \min(\mu_A, \mu_B)$.

essentially measure-theoretical and it does not rely on a specific formal logic fixed in advance to represent fuzzy sets and fuzzy events by its formulas.

The development of formal systems in the realm of Mathematical Fuzzy Logic provided a logical and algebraic setting upon which one can address the question on how to axiomatically (*à la* Kolmogorov) capture Zadeh's probability. More precisely, once a fuzzy logic L is fixed, we may ask if its formulas can be regarded as fuzzy events in the sense of the above Definition 1 and which are the axioms that a $[0,1]$-valued function f defined on the formulas of L (univocally) characterize f as a probability according to the above Equation (7).

Although one of the first references for the axiomatic treatment of the probability of fuzzy events can be traced back to the paper [18] by Chovanec, it is Mundici's definition of *state of an MV-algebra*, [106], which paved the way, within the community of fuzzy logicians, for a discipline that nowadays goes under the name of *state theory* whose aim is, among other things, to answer the questions above (see also [56, 109]).

The choice of MV-algebras, and hence of Łukasiewicz logic, for Mundici's analysis is justified by the fact that formulas of Łukasiewicz logic can be regarded as $[0,1]$-valued continuous, and hence Borel-measurable functions. Thus, every formula in this setting is a fuzzy event in the above sense. A state of Łukasiewicz formulas, is hence a $[0,1]$-valued function s which satisfies the following conditions:

1. If $\vdash_{Ł} \varphi$, then $s(\varphi) = 1$,
2. If $\vdash_{Ł} \neg(\varphi \& \psi)$, then $s(\varphi \underline{\vee} \psi) = s(\varphi) + s(\psi)$,

where & and $\underline{\vee}$ here denote, respectively, the strong conjunction and strong disjunction connectives of Łukasiewicz logic. This first axiom clearly corresponds to the *normalization axiom* for probability functions on formulas of classical logic, and it says that the state of every Łukasiewicz theorem is 1. The second one forces states to be *additive*. Indeed it says that, if it is provable in Łukasiewicz logic that φ and ψ are disjoint with respect to Łukasiewicz strong conjunction, then the state of the strong disjunction of φ and ψ coincides with the sum of their states. It is worth noticing that the above definition of state properly generalizes the one of classical probability on classical logic formulas. Indeed, if φ and ψ are classical formulas, then the axiom (2) implies, in algebraic terms, that if $\varphi \wedge \psi = \bot$, then $s(\varphi \vee \psi) = s(\varphi) + s(\psi)$ and classical Kolmogorov axioms are so recovered.

One could now wonder whether the above definition also gives, in the special case of MV-algebras and Łukasiewicz logic, also a notion of probability of fuzzy events as in Definition 1. The answer is affirmative. Indeed, the following result, which has been independently proved by Kroupa [90] and Panti [118], shows that for every state s of Łukasiewicz formulas there exists a unique regular Borel, and hence σ-additive, probability measure P such that s is the Lebesgue integral with respect to P.

Theorem 1. *For every state s of Łukasiewicz formulas (on m variables) there exists a unique regular, Borel probability measure P on $[0,1]^m$ such that, for* *every Łukasiewicz formula* φ*,*

$$s(\varphi) = \int_{[0,1]^m} f_\varphi \, \mathrm{d}P.^4$$

This theorem, in addition to show the correctness of Zadeh's intuition, inspired a line of research which aims at providing similar characterizations for other t-norm based fuzzy logics.

Recent results in this direction shows that similar results to Theorem 1 can be obtained for states defined on the two other fundamental logics based on a continuous t-norms, i.e. Gödel and Product fuzzy logics. We want to recall the following articles in which axioms and an integral representation for states have been quite intensively studied:

- States on formulas of Gödel logic have been introduced by Aguzzoli, Gerla and Marra in [3]. In the same paper the authors also extend the foundational theorem of de Finetti to the framework of Gödel events.[5]
- States for Product logic were studied by Flaminio, Godo and Ugolini in [55]. There, also a modal logic for reasoning about the uncertainty of product events has been introduced and completeness result has been established.
- For the case of the t-norm based logic of nilpotent minimum, states have been studied by Aguzzoli and Gerla in [1]. This logical setting is particularly interesting since nilpotent minimum is an example of a non-continuous but left-continuous t-norm, and with an involutive negation.

Entering in a detailed discussion on the axiomatization of states in the above recalled logical realms is out of the scope of the present paper. However, it is worth to point out the following things: (1) In contrast to the case of Łukasiewicz logic in which all formulas correspond to continuous functions, formulas of Gödel, product and nilpotent minumum logics do not. However they are representable by Borel measurable, and hence Lebesgue integrable, functions; (2) the different kinds of fuzzy events which each of the above logical system is able to represent, inevitably forces different axiomatizations of states. Each of these axiomatizations, however, collapses to Kolmogorov's axioms of classical probability functions as a special case.

The main definitions and results for state theory on MV-algebras can be found in the book [109] and the book chapter [56]. In particular, the former also contains a concise treatment on foundational approaches to the probability of fuzzy (Łukasiewicz) events, and the latter also recalls how conditional probability [91, 100, 108] and internal states [58] can be approached in the realm of MV-algebras and Łukasiewicz logic.

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[4] Here, f_φ denotes the continuous, and hence Borel-measurable function defined on a certain cube $[0,1]^m$ which represent the formula φ in the Lindenbaum-Tarski algebra of formulas of Łukasiewicz logic, see [107] for more details.

[5] De Finetti's theorem for Łukasiewicz events was generalized by Mundici in [107].

6.2 Fuzzy modal logics

Fuzzy modal logic is an active and rapidly growing area of research that aims at generalizing classical modal logic to a many-valued, or fuzzy, framework. The first attempts to generalize modal logic to a many-valued setting can be traced back to the papers by Ostermann [116, 117] and Fitting [59, 60]. In particular, the latter papers start by generalizing the relational semantics of classical modal logics to the many-valued setting, and defines two families of many-valued modal logics: the first one is characterized by Kripke models with a classical (two-valued) accessibility relations in which at each possible world formulas are evaluated by a finite many-valued logic; the second one allows also the accessibility relation to be many-valued.

These works on many-valued modal logics paved the way to the birth of fuzzy modal logic as a discipline and inspired several other researches who, following these ideas, further generalized classical modal logic to the ground of infinite-valued fuzzy logics and in the formal setting of Mathematical Fuzzy Logic. For instance, Hájek's book [69] already contains a chapter dedicated to present a generalization of the modal logic S5 to a fuzzy framework. This approach was then further generalized by Hájek himself in [73].

The language of a fuzzy modal logic is not particularly different form that of classical modal logics and it can be defined as the expansion of the basic language of the logic MTL by two unary modalities which, following the tradition, are denoted by $\Box$ and $\Diamond$. Modal formulas are hence defined as usual. However, the relational semantics of fuzzy modal logic is significantly more general than the classical one and it consists of a MTL-algebra $\mathbf{A} = (A, \wedge, \vee, \odot, \to, 0, 1)$ upon which one can define an $\mathbf{A}$-*frame* as a triple $(W, \mathbf{A}, R)$ where W is a non-empty set of *worlds* and $R : W \times W \to A$ is a $\mathbf{A}$-*valued accessibility relation* on W. If for all $w, w' \in W$, $R(w, w') \in \{0, 1\}$, then R is said to be *crisp*.

$\mathbf{A}$-frames $(W, \mathbf{A}, R)$ extend to $\mathbf{A}$-*Kripke models* by adding a valuation map V from propositional formulas to the fixed MTL-algebra $\mathbf{A}$. Thus, a modal formula ϕ is evaluated in a $\mathbf{A}$-model $(W, \mathbf{A}, R, V)$, at a given world $w \in W$ (and we will write $\|\phi\|_w$ to denote its truth-value), by the following inductive conditions:

- If p is a propositional variable, $\|p\|_w = V(p)$;
- If $\phi = \psi \& \gamma$, then $\|\phi\|_w = \|\psi\|_w \odot \|\gamma\|_w$;
- If $\phi = \psi \to \gamma$, then $\|\phi\|_w = \|\psi\|_w \Rightarrow \|\gamma\|_w$;
- If $\phi = \Box\psi$, then $\|\phi\|_w = \bigwedge_{w' \in W}\{R(w, w') \Rightarrow \|\psi\|_{w'}\}$;
- If $\phi = \Diamond\psi$, then $\|\phi\|_w = \bigvee_{w' \in W}\{R(w, w') \odot \|\psi\|_{w'}\}$.

It is worth noticing that, if the MTL-algebra $\mathbf{A}$ is the Boolean chain $\{0, 1\}$, then $\mathbf{A}$-frames are classical Kripke frames and the evaluation of modal formulas in a model are as in the classical modal setting.

The very general definition of relational semantics for fuzzy modal logics, can be specified by choosing a precise class of algebras such as BL, MV, Gödel algebras and so forth. By doing so, the general problems related to axiomatizability and proof-theoretic questions have been addressed. In these respects, it is worth to remember the general approaches of Bou, Esteva, Godo and Rodriguez

[12] in which the authors axiomatize the minimal modal logic of **A**-models in which **A** is, in particular, a finite MTL-chain. Further works which follow Fitting's ideas of generalizing Kripke models to the fuzzy environment and focus on modal expansions of specific propositional logics are also worth to recall. In particular that by Caicedo and Rodriguez [15] that investigate bi-modal expansion of Gödel logic; the paper [83] by Hansoul and Teheux that instead consider modal expansions of propositional Łukasiewicz logic, and that by Vidal, Esteva and Godo [127] where the propositional base is fixed in product fuzzy logic.

Concerning computability and decidability, in [13, 14] it is proved that the minimal (local) modal Gödel logics with both and $\Box$ and $\Diamond$ modal operators are decidable (both over models with a crisp accessibility relation and with a $[0,1]$-valued one). In the same papers, Caicedo, Metcalfe, Rodriguez and Rogger also showed that the S5 extension of the previous logic with crisp accessibility (equivalent to the one-variable fragment of predicate Gödel logic) is decidable too. As for the case of modal Łukasiewicz and product logics, the decidability problems are addressed in the forthcoming paper by Vidal [126].

Finally, it is interesting to recall that a weaker modal expansion of Łukasiewicz propositional logic can be used to model probabilistic reasoning on both classical and fuzzy events. The first modal logic of this kind was axiomatized in 1995 by Hájek, Esteva and Godo in [76] and then it has been further generalized and extended (see [53] for an overview). The basic idea which lies at the ground of a fuzzy-modal approach to uncertainty (and probability in particular) is to consider modal formulas of the form $U(\varphi)$ to be read "the formula φ is uncertain" and provide axioms for U in such a way that the truth-degree of the formula $U(\varphi)$ becomes the uncertain degree of φ. Specific axioms for the modality U can be provided to model the specific measure we are interested in. Thus, for instance, U can be axiomatized to behave as a probability function by imposing normalization and finite additivity [69, 76], but alternative axiomatizations for more general uncertainty measures on both classical or fuzzy events such as possibility and necessity measures [52], belief functions [54], lower and upper probabilities [93] are also feasible.

6.3 A brief recap of additional developments

Finally, we briefly overview some further topics and research lines, still under the umbrella of Mathematical Fuzzy Logic, that have also been developed and contributed to extend this field in several directions.

The logics $\text{Ł}\Pi$, $\text{Ł}\Pi\frac{1}{2}$. The aim at defining the logics $\text{Ł}\Pi$ and $\text{Ł}\Pi\frac{1}{2}$ was, roughly speaking, putting together Łukasiewicz and Product logics, that is, to define a single logic containing both Łukasiewicz logic connectives (related to the arithmetic operations of addition and subtraction) and Product logic connectives (related to the arithmetic operations of product and division). Moreover, Gödel logic connectives are obtained for free as well. In this way, the logics obtained have a very high expressive power.[6] The logics were formally introduced by Esteva, Godo and Montagna in [45] and further developed by Cintula in [19–21]. See also the handbook chapter [44] for a comprehensive survey.

Just to give a flavour, the language of the ŁΠ logic is built in the usual way from a countable set of propositional variables, three binary connectives $\to_L$ (Łukasiewicz implication), $\odot$ (Product conjunction) and $\to_\Pi$ (Product implication), and the truth constant $\bar{0}$. Two negations are definable $\neg_\Pi\varphi := \varphi \to_\Pi \bar{0}$, $\neg_Ł\varphi := \varphi \to_Ł \bar{0}$, Baaz-Monteiro operator is $\Delta\varphi := \neg_\Pi\neg_Ł\varphi$, and the $\ominus$ connective is defined as $\varphi \ominus \psi := \neg_Ł(\varphi \to_Ł \psi)$. Then ŁΠ is axiomatized by the adding to the axioms of set of Łukasiewicz and Product logics only these three axioms and rules:

$(\neg)\ \neg_\Pi\varphi \to_Ł \neg_Ł\varphi$

$(\Delta)\ \Delta(\varphi \to_Ł \psi) \equiv_Ł \Delta(\varphi \to_\Pi \psi)$

$(Ł\Pi5)\ \varphi \odot (\psi \ominus \chi) \equiv_Ł (\varphi \odot \psi) \ominus (\varphi \odot \chi)$

and the rule of necessitation for Δ: from φ infer $\Delta\varphi$. While the logic ŁΠ$\frac{1}{2}$ is then obtained from ŁΠ by adding a truth constant $\overline{\frac{1}{2}}$ in the language together with the axiom: (ŁΠ$\frac{1}{2}$) $\overline{\frac{1}{2}} \equiv \neg_L \overline{\frac{1}{2}}$.

Weakly implicative and semilinear logics. In the last years there has been tendency on considering more and more general systems of fuzzy logic, going beyond MTL. For instance Cintula [22] introduced the framework of weakly implicative fuzzy logics. The main idea behind this class of logics is to capture the notion of comparative truth common to all fuzzy logics, that is to say, logics that are complete with respect to a semantics based on linearly ordered algebras. It actually corresponds to the main thesis of [9] that defends the claim that fuzzy logics are the logics of chains. Along this line, with the aim of dealing in a uniform way the variety of fuzzy logics studied in the literature, Cintula and Noguera provide in [28, 27] a new framework (the hierarchy of the so-called *implicational logics*) where one can develop in a natural way a technical notion corresponding to the intuition of fuzzy logics as the logics of linearly ordered algebras of truth-values. Indeed, they introduce the notion of implicational *semilinear* logic as a property related to the implication, namely a logic L is an implicational semilinear logic iff it has an implication such that L is complete w.r.t. the class of logical matrices where the implication induces a linear order on the set of truth-values. The above mentioned hierarchy, when restricted to the semilinear case, provides a classification of implicational semilinear logics that encompasses almost all the known examples of fuzzy logics.

Proof theory. Besides the development of fuzzy logics has been mainly motivated by the goal of representing and reasoning about truth-dergrees, they also deal with a notion of proof. Although the usual axiomatic presentation for these systems is a Hilbert-style calculus, most of fuzzy logics investigated in the literature have a natural proof-theoretical, i.e. *Gentzen-style* formulation. How-

[6] Indeed, all rationals in $[0, 1]$ are definable in ŁΠ$\frac{1}{2}$, and Marchioni and Montagna showed [95] that the universal theory of real closed fields is definable in the equatinal theory of ŁΠ$\frac{1}{2}$-algebras.

ever typical sequent-calculi are not enough to present fuzzy logics and an extra step in complexity is required to *hypersequents*: multisets of sequents interpreted as disjunctions. Moving from sequents to hypersequents naturally improve in complexity, but at the same time allows to gain in expressive power. Indeed, hypersequent-like proof systems for the majority of fuzzy logics can be obtained by transferring sequent calculi to the hypersequent level and by adding specific extra rules such as Avron's *communication rule* (see [6]) which corresponds, in the Hilbert-style presentation, to the prelinearity axiom. The book chapter [97] presents a wide overview on this subject and we urge the interest reader to consult it and the references therein.

Functional representation. In the same way formulas of classical propositional logic can be represented, in Lindenbaum-Tarski algebras, as functions from $\{0,1\}^n$ to $\{0,1\}$ (where n denotes the number of propositional variables in the language we are dealing with), also formulas of t-norm based fuzzy logics have a functional representation. The most famous result in this sense is McNaughton theorem [96, 105] which allows to represent formulas of Łukasiewicz logic as continuous, piecewise linear functions with integer coefficients from $[0,1]^n$ to $[0,1]$. For other t-norm based logics, similar results are known, whilst for other the problem of their functional representation is open. The interest reader may consult [2], and references therein, for an exhaustive treatment of this topic and the functional representation for the logics BL, Gödel, product, NM and WNM.

Fuzzy logic with evaluated syntax. In parallel to the development of t-norm based fuzzy logic as initiated by Hájek, *fuzzy logic with evaluated syntax*, $\mathrm{Ev}_{\text{Ł}}$, was developed by Novák in [111] (see also [115]) to generalize classical two-valued logics in both its syntax and semantics. Indeed, in $\mathrm{Ev}_{\text{Ł}}$ the axioms of formal theories can be partially true and hence form a fuzzy set, rather than a classical set and in the standard development of Mathematical Fuzzy Logic. To do so, the language of $\mathrm{Ev}_{\text{Ł}}$, besides containing constants from a truth-value set L (usually an MV-algebra), also allows *evaluated formulas* of the form a/A, where A is a formula and $a \in L$, denoting that A takes a value greater or equal than a. As a consequence of working this generalized context, not only axioms but also inference rules includes evaluated formulas and a n-ary inference rule r has the form of a function over evaluated formulas:

$$r : \frac{a_1/A_1, \ldots, a_n/A_n}{r^{evl}(a_1, \ldots, a_n)/r^{syn}(A_1, \ldots, A_n)}$$

where r^{syn} is a partial n-ary operation on (propositional) formulas, and the evaluation operation r^{evl} is an n-ary operation on the algebra of truth-values which is lower semicontinuous (i.e., it preserves arbitrary suprema in each variable).

The book chapter [114] presents a detailed discussion on this topic.

Fuzzy description logics. Description Logics (DLs) are a family of well established knowledge representation formalisms whose languages are based on *concepts*, *roles* and *individuals*. According to a well-defined Tarski style semantics, concepts, roles and individuals are respecively interpreted as sets (or unary predicates), binary relations (or binary predicates), and domain elements. Fuzzy Description Logics (FDLs) were born in the nineties as a generalization of DLs to the fuzzy framework. The first generalizations consisted in having the same structure as DLs, but interpreting concepts and roles as fuzzy sets and fuzzy relations respectively. Although these first approaches had interesting applications in several fields, they have no clear logical counterpart. The born of Mathematical Fuzzy Logic at the end of nineties, paved the way to a logical development of FDLs as proposed by Hájek in [72]. The already rich hierarchy in which the several languages of DLs are organized, further improved in the fuzzy case and hence the main task for FDLs was to investigate the balance between the expressive power from one side and the decidability of the resulting calculus from the other. The chapter [11] presents the topic of FDLs paying particular attention to disclose their expressive powers, the interpretation of FDLs in fragments of first-order t-norm based fuzzy logics and issues related with their decidability.

7 Concluding Remarks

In this paper our aim has been to provide a short survey of main historical developments of systems of fuzzy logic in narrow sense, today under the umbrella of the discipline called Mathematical Fuzzy Logic, arising from the birth of Zadeh's fuzzy sets in 1965. Particular attention has been devoted to show how the tools of mathematical logic have allowed to define logical systems which form the core of Mathematical Fuzzy Logic and allow for a formalization of some elements in Zadeh's agenda of fuzzy logic, which indeed spans from fuzzy sets to approximate reasoning and probability theory. Besides this fundamental core, we have also presented some further topics that have allowed, and are still allowing, this discipline to develop and grow.

Regarding the topics that we have selected to emphasize in Subsection 6.3, it must be remarked that they indeed represent only a part of a plethora of other interesting lines of research have been explored but, unfortunately, have been left out of the scope of this overview. For instance, we can mention topics like Fuzzy Type Theory [112], fuzzy/intermediate quantifiers [113], game semantics for fuzzy logics [51] or Fuzzy/Graded Model Theory (see for instance [36, 75]) among many others.

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