Revisiting ultraproducts in fuzzy predicate logics

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In this paper we examine different possibilities of defining reduced products and ultraproducts in fuzzy predicate logics. We present analogues to the Łoś theorem for these notions and discuss the advantages and drawbacks of each definition introduced.

Key words: model theory, fuzzy predicate logics, ultraproducts, reduced products

1 INTRODUCTION

Ultraproducts are a powerful tool in classical model theory. Its applications range from an ultraproduct version of the compactness theorem to algebraic characizations of elementary classes. Its attraction comes from the fact that it preserves all properties expressible in first-order logic and also from its algebraic nature. The method originated with Skolem in the 1930's, and has been used extensively since the work of Łoś in the 1950's. For a general survey on the subject I refer the reader to [3] and [17].

Being one of the basic methods of constructing models in classical mathematical logic, it is a natural question to ask for the fuzzy predicate case. Do ultraproducts play also such a relevant role? What are the conditions for their existence? In this paper we present the first step to answer these questions. Here we examine different possibilities of defining reduced products

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and ultraproducts in fuzzy predicate logics. We prove analogues to the Łos Theorem for these notions and discuss the advantages and drawbacks of each definition introduced.

In fuzzy mathematical logic, ultraproducts have been used in different ways. In a classical sense, ultrapowers on the real unit interval give a characterization of MV-algebras [18]. In [13], the authors classify and axiomatize all universal classes generated by a certain infinite totally ordered MV-algebra by using ultrapowers of the additive group of integers. In [4], distinguished algebraic semantics for t-norm based fuzzy logics are studied, being the semantics of hyperreals another meaningful semantics close to the standard one. Strong non-standard completeness for fuzzy logics was introduced in [10] and more recently [19], the notion of hyperreal state have been defined.

Non-classical proposals are also present (for a reference see [9], [11], [21], [20], [12] and [1]). We have tried to encompass the most commonly used non-classical definitions of ultraproduct of the mathematical fuzzy logic literature. We extend, when available, their results to work in arbitrary fuzzy predicate logics.

This paper is an extended and full revised version of the contribution [8] of the author to the conference ISMVL'10. It is structured as follows: we start with some preliminaries on fuzzy predicate logics. In section 3, we study ultraproducts over a fixed MTL-algebra. In section 4 we introduce the notion of reduced product defined from pairs of filters. Finally, in section 5 we present some future research lines.

2 PRELIMINARIES

Our study of the model theory of fuzzy predicate logics is focused on the basic fuzzy predicate logic MTL$\forall$.

Now we introduce the syntax of MTL$\forall$. A *predicate language* Γ is a triple (**P**,**F**,**A**) where **P** is a non-empty set of predicate symbols, **F** is a set of function symbols and **A** is a mapping assigning to each predicate and function symbol a natural number called the *arity of the symbol*. The function symbols F for which $\mathbf{A}(F) = 0$ are called the *object constants*. The predicate symbols P for which $\mathbf{A}(P) = 0$ are called the *truth constants*.

Formulas of the predicate language Γ are built up from the symbols in (**P**,**F**,**A**), the connectives and truth constants of MTL, the logical symbols $\forall$ and $\exists$, variables and punctuation. From now on, the formulas of a predicate language Γ will be called Γ-*formulas*. A Γ-sentence is a Γ-formula without free variables.

Throughout the paper we consider the equality symbol as a binary predicate symbol not as a logical symbol, we work in equality-free fuzzy predicate logics. That is, the equality symbol is not necessarily present in all the languages and its interpretation is not fixed. We introduce now an axiomatic system for the predicate logic MTL$\forall$:

(P) the axioms resulting from the axioms of MTL by the substitution of the propositional variables by the Γ-formulas.

($\forall$1) $\forall x\phi(x) \rightarrow \phi(t)$, where t is substitutable for x in ϕ.

($\exists$1) $\phi(t) \rightarrow \exists x\phi(x)$, where t is substitutable for x in ϕ.

($\forall$2) $\forall x(\phi \rightarrow \varphi) \rightarrow (\phi \rightarrow \forall x\varphi)$, where x is not free in ϕ.

($\forall$3) $\forall x(\phi \vee \varphi) \rightarrow (\phi \vee \forall x\varphi)$, where x is not free in ϕ.

The deduction rules are those of MTL and *generalization*: from ϕ infer $\forall x\phi$. By $\Sigma \vdash_{\mathrm{MTL}\forall} \alpha$ we denote that the formula α follows from the set of formulas Σ in the axiomatic system of the fuzzy predicate logic MTL$\forall$. When it is clear by the context we omit the subscript MTL$\forall$.

We introduce now the semantics for the fuzzy predicate logic MTL$\forall$:

Definition 1 *Given an* MTL*-algebra* **B***, a* **B***-structure for a predicate language* Γ *is a tuple*

$$\mathbf{M} = (M, (P_{\mathbf{M}})_{P\in\Gamma}, (F_{\mathbf{M}})_{F\in\Gamma})$$

where:

1. *M is a non-empty set.*

2. *For each n-ary predicate $P \in \Gamma$, if $n > 0$, $P_{\mathbf{M}}$ is a* **B***-fuzzy relation $P_{\mathbf{M}}: M^n \rightarrow B$. If $n = 0$, $P_{\mathbf{M}}$ is an element of B.*

3. *For each n-ary function symbol $F \in \Gamma$, if $n > 0$, $F_{\mathbf{M}}: M^n \rightarrow M$ is a crisp function. If $n = 0$, $F_{\mathbf{M}}$ is an element of M.*

Given an MTL-algebra **B** and a **B**-structure **M**, an **M**-*evaluation* of the object variables is a mapping v which assigns to each variable an element from M. By $\phi(x_1, \ldots, x_k)$ we mean that all the free variables of ϕ are among $x_1, \ldots, x_k$. Let v be an **M**-evaluation, x a variable, and $d \in M$, we denote by $v[x \rightarrow d]$ the **M**-evaluation such that $v[x \rightarrow d](x) = d$ and for each variable y different from x, $v[x \rightarrow d](y) = v(y)$.

Let $\mathbf{B}$ be an MTL-algebra, $\mathbf{M}$ be a $\mathbf{B}$-structure and v be an $\mathbf{M}$-evaluation, we define the values of the terms and *truth values* of the formulas as follows:

$$\|c\|_{\mathbf{M},v}^{\mathbf{B}} = c_{\mathbf{M}}, \|x\|_{\mathbf{M},v}^{\mathbf{B}} = v(x)$$

$$\|F(t_1,\ldots,t_n)\|_{\mathbf{M},v}^{\mathbf{B}} = F_{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}^{\mathbf{B}},\ldots,\|t_n\|_{\mathbf{M},v}^{\mathbf{B}})$$

for each variable x, each object constant $c \in \Gamma$, each n-ary function symbol $F \in \Gamma$ for $n > 0$ and Γ-terms $t_1,\ldots,t_n$, respectively.

$$\|P(t_1,\ldots,t_n)\|_{\mathbf{M},v}^{\mathbf{B}} = P_{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}^{\mathbf{B}},\ldots,\|t_n\|_{\mathbf{M},v}^{\mathbf{B}})$$

for each n-ary predicate $P \in \Gamma$,

$$\|\delta(\phi_1,\ldots,\phi_n)\|_{\mathbf{M},v}^{\mathbf{B}} = \delta_{\mathbf{B}}(\|\phi_1\|_{\mathbf{M},v}^{\mathbf{B}},\ldots,\|\phi_n\|_{\mathbf{M},v}^{\mathbf{B}})$$

for each n-ary connective δ of MTL and Γ-formulas $\phi_1,\ldots,\phi_n$. Finally, for the quantifiers,

$$\|\forall x\phi\|_{\mathbf{M},v}^{\mathbf{B}} = \inf\{\|\phi\|_{\mathbf{M},v[x\to d]}^{\mathbf{B}} : d \in M\}$$

$$\|\exists x\phi\|_{\mathbf{M},v}^{\mathbf{B}} = \sup\{\|\phi\|_{\mathbf{M},v[x\to d]}^{\mathbf{B}} : d \in M\}$$

Remark that, since the MTL-algebras we work with are not necessarily complete, the above suprema and infima could be not defined in some cases. It is said that a $\mathbf{B}$-structure is *safe* if such suprema and infima are defined for all the formulas and all evaluations. From now on we assume that all our structures are safe. If v is an evaluation such that for each $0 < i \le n$, $v(x_i) = d_i$, and λ is either a Γ-term or a Γ-formula, we abbreviate by $\|\lambda(d_1,\ldots,d_n)\|^{(\mathbf{B},\mathbf{M})}$ the expression $\|\lambda(x_1,\ldots,x_n)\|_{\mathbf{M},v}^{\mathbf{B}}$.

Definition 2 *Let ϕ be a Γ-sentence, given an* MTL*-algebra $\mathbf{B}$ and a $\mathbf{B}$-structure $\mathbf{M}$, it is said that $\mathbf{M}$ is a* model *of ϕ iff $\|\phi\|^{(\mathbf{B},\mathbf{M})} = 1$. And that $\mathbf{M}$ is a model of a set of Γ-sentences Σ iff for all $\phi \in \Sigma$, $\mathbf{M}$ is a model of ϕ.*

Definition 3 *Let $T \cup \{\phi\}$ be a set of Γ-sentences. We say that ϕ is a* semantical consequence *of T (denoted by $T \models \phi$) iff for every* MTL*-algebra $\mathbf{B}$ and every $\mathbf{B}$-structure $\mathbf{M}$, if $\mathbf{M}$ is a model of T, then $\mathbf{M}$ is also a model of ϕ.*

From now on, given an MTL-algebra $\mathbf{B}$, we say that $(\mathbf{B}, \mathbf{M})$ is a Γ-*structure* instead of saying that $\mathbf{M}$ is a $\mathbf{B}$-structure for a predicate language Γ.

Definition 4 *Let* $(\mathbf{B}, \mathbf{M})$ *be a* Γ-*structure, by* $Alg(\mathbf{B}, \mathbf{M})$ *we denote the subalgebra of* $\mathbf{B}$ *whose domain is the set*

$$\{\|\phi(d_1, \ldots, d_n)\|^{(\mathbf{B}, \mathbf{M})} : d_1, \ldots, d_n \in M \text{ and } \phi(x_1, \ldots, x_n) \text{ is a } \Gamma\text{-formula}\}$$

Then, it is said that $(\mathbf{B}, \mathbf{M})$ *is* exhaustive *iff* $Alg(\mathbf{B}, \mathbf{M}) = \mathbf{B}$.

Given two Γ-structures $(\mathbf{B}_1, \mathbf{M}_1)$ and $(\mathbf{B}_2, \mathbf{M}_2)$, we denote by $(\mathbf{B}_1, \mathbf{M}_1) \equiv (\mathbf{B}_2, \mathbf{M}_2)$ the fact that $(\mathbf{B}_1, \mathbf{M}_1)$ and $(\mathbf{B}_2, \mathbf{M}_2)$ are *elementarily equivalent*, that is, that they are models of exactly the same Γ-sentences. In this section we have presented only a few definitions and notation, a detailed introduction to the syntax and semantics of fuzzy predicate logics can be found in [14] and [5].

2.1 Mappings, Homomorphisms and Congruences

Different definitions have been used so far for basic model-theoretic operations on structures. For instance, the notion of *elementary submodel, morphism* [9], *elementary embeddings and submodels* [15], *fuzzy submodel, elementary fuzzy submodel* and *isomorphism* of structures of first-order fuzzy logic with graded syntax [20], *complete morphism* in languages with a similarity predicate [1] and the notion of σ-embedding [4]. Being our starting point all these works, in this section we recall the notions of mapping, of homomorphism and of congruence of a model, as introduced in [6]. In the aforementioned paper we tried both to encompass the most commonly used definitions in the literature and to extend the corresponding notions of classical predicate logics.

Definition 5 *Let* $(\mathbf{B}_1, \mathbf{M}_1)$ *be a* Γ_1-*structure and* $(\mathbf{B}_2, \mathbf{M}_2)$ *be a* Γ_2-*structure, with* $\Gamma_1 \subseteq \Gamma_2$. *We say that the pair* (g, f) *is a* mapping *iff*

1. *$g \colon \mathbf{B}_1 \to \mathbf{B}_2$ is an* MTL-*algebra homomorphism of* $\mathbf{B}_1$ *into* $\mathbf{B}_2$.

2. *$f \colon M_1 \to M_2$ is a mapping of* M_1 *into* M_2.

3. *For each atomic* Γ_1-*formula* $\phi(x_1, \ldots, x_n)$ *and elements* $d_1, \ldots, d_n \in M_1$, $g(\|\phi(d_1, \ldots, d_n)\|^{(\mathbf{B}_1, \mathbf{M}_1)}) = \|\phi(f(d_1), \ldots, f(d_n))\|^{(\mathbf{B}_2, \mathbf{M}_2)}$

Moreover, if in addition:

- Condition 3. *above holds for every Γ_1-formula, (g, f) is said to be an* elementary mapping.

- *g preserves the existing infima and suprema, (g, f) is said to be a σ-mapping.*

- *For each n-ary function symbol $F \in \Gamma_1$ and elements $d_1, \ldots, d_n \in M_1$,*

$$f(F_{\mathbf{M}_1}(d_1, \ldots, d_n)) = F_{\mathbf{M}_2}(f(d_1), \ldots, f(d_n))$$

(g, f) *is said to be a* homomorphism

It is easy to check by induction on the complexity of the formulas, that for every mapping (g, f) the following holds: For each quantifier-free Γ_1-formula $\phi(x_1, \ldots, x_n)$ and elements $d_1, \ldots, d_n \in M_1$, $g(\|\phi(d_1, \ldots, d_n)\|^{(\mathbf{B}_1, \mathbf{M}_1)}) = \|\phi(f(d_1), \ldots, f(d_n))\|^{(\mathbf{B}_2, \mathbf{M}_2)}$.

If (g, f) is both a σ-mapping and a homomorphism we would say that (g, f) is a *σ-homomorphism*. Moreover, we would say that (g, f) is an *embedding* when (g, f) is a σ-homomorphism with both g and f one-to-one, and we denote by $(\mathbf{B}, \mathbf{M}) \cong (\mathbf{A}, \mathbf{N})$ when these two structures are *isomorphic* (that is, there is an embedding (g, f) from $(\mathbf{B}, \mathbf{M})$ into $(\mathbf{A}, \mathbf{N})$ with g and f onto). Homomorphisms that in addition are elementary mappings will be called *elementary homomorphisms*. Remark that, unlike [1], homomorphisms are crisp when restricted to the algebraic reducts of the models. Observe also that, working with predicate languages without function symbols, the notions of mapping and homomorphism coincide, but it is not the case for arbitrary structures. Finally,

Note that, by definition, homomorphisms are not always σ-homomorphisms, as are in [9] or [1] (mappings are not always σ-mappings respectively). The following proposition is a reformulation of Propositions 6.1 and 6.2 in [9] :

Proposition 6 *Let (g, f) be a mapping of the Γ-structure $(\mathbf{B}_1, \mathbf{M}_1)$ into the Γ-structure $(\mathbf{B}_2, \mathbf{M}_2)$. If (g, f) is a σ-mapping with f onto, then (g, f) is an elementary mapping and $(\mathbf{B}_1, \mathbf{M}_1) \equiv (\mathbf{B}_2, \mathbf{M}_2)$.*

Finally, we introduce now the notion of congruence.

Definition 7 *A* congruence *on a Γ-structure $(\mathbf{B}, \mathbf{M})$ is a pair (θ, E) where:*

1. *θ is an MTL-congruence on the algebra $\mathbf{B}$.*

2. *E is an equivalence relation $E \subseteq M \times M$ such that:*

- *For each n-ary function symbol $F \in \Gamma$ and elements $d_1, \ldots, d_n, e_1, \ldots, e_n \in M$, if for each $0 < i \leq n$, $(d_i, e_i) \in E$, then*
$$(F_{\mathbf{M}}(d_1, \ldots, d_n), F_{\mathbf{M}}(e_1, \ldots, e_n)) \in E$$

- *For each n-ary predicate $P \in \Gamma$ and elements $d_1, \ldots, d_n, e_1, \ldots, e_n \in M$, if for each $0 < i \leq n$, $(d_i, e_i) \in E$, then*
$$(P_{\mathbf{M}}(d_1, \ldots, d_n), P_{\mathbf{M}}(e_1, \ldots, e_n)) \in \theta$$

Now, given a congruence (θ, E) on $(\mathbf{B}, \mathbf{M})$ we define the *quotient structure* $(\mathbf{B}/\theta, \mathbf{M}/E)$ by:

- For each n-ary function symbol $F \in \Gamma$ and elements $d_1, \ldots, d_n \in M$,

$$F_{\mathbf{M}/E}([d_1]_E, \ldots, [d_n]_E) = [F_{\mathbf{M}}(d_1, \ldots, d_n)]_E$$

- For each n-ary predicate $P \in \Gamma$ and elements $d_1, \ldots, d_n \in M$,

$$P_{\mathbf{M}/E}([d_1]_E, \ldots, [d_n]_E) = [P_{\mathbf{M}}(d_1, \ldots, d_n)]_\theta$$

where, given an element $d \in M$ and $b \in B$, $[d]_E$ and $[b]_\theta$ denote respectively the equivalence classes of d modulo E and of b modulo θ. Observe that, like the classical case, given a homomorphism we obtain a congruence and conversely, given a congruence we obtain the so-called *canonical mapping*. We will say that (θ, E) is an *elementary* congruence (σ-congruence, respectively) iff its canonical mapping (g_θ, f_E) is an elementary homomorphism (σ-homomorphism, respectively).

3   ULTRAPRODUCTS OVER AN MTL-ALGEBRA

The first ultraproducts we study are defined over a fixed MTL-algebra. See for instance [21] for Rational Pavelka's logic (RPL) or [20] in the case of first-order fuzzy logic with graded syntax. Here we work over a fixed MTL-algebra not necessarily complete, and for arbitrary fuzzy predicate languages, using κ-complete ultrafilters.

Definition 8 *Let I be a non-empty set and κ an infinite cardinal. A filter H over I is said to be κ-complete iff the intersection of any non-empty set of fewer than κ elements of H belongs to H.*

Given a set X we denote by $|X|$ the cardinality of X.

Definition 9 *Let I be a non-empty set, κ an infinite cardinal and for each $i \in I$, let $(\mathbf{B}, \mathbf{M}_i)$ be a Γ-structure. Assume that U is a κ-complete ultrafilter over I such that $|B| < \kappa$. The* ultraproduct $(\mathbf{B}, \prod \mathbf{M_i}/U)$ *of the structures $\{(\mathbf{B}, \mathbf{M}_i) : i \in I\}$ has as algebraic part of $\prod \mathbf{M_i}/U$ (regarded as a classical first-order structure) the usual ultraproduct construction, that is, the quotient of the direct product $\prod M_i$ modulo the congruence θ_U defined as usual: for every $\overline{d}, \overline{e} \in \prod M_i$, $(\overline{d}, \overline{e}) \in \theta_U$ iff $\{i \in I : \overline{d}(i) = \overline{e}(i)\} \in U$. And for each n-ary predicate $P \in \Gamma$, and every $\overline{d_1}, \ldots, \overline{d_n} \in \prod M_i$,*

$$\|P([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{(\mathbf{B}, \prod \mathbf{M_i}/U)} = b \text{ iff}$$
$$\{i \in I : \|P(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b\} \in U$$

Observe that there is a unique structure with the above properties, the ultraproduct is well-defined because, by Lemma 4.2.3. of [3], a proper ultrafilter U over a nonempty set I is κ-complete iff for every partition of I into fewer that κ parts, exactly one of the parts belongs to U. Then we consider the partition $(X_b : b \in B)$ where for each $b \in B$, $X_b = \{i \in I : \|P(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b\}$. Since $|B| < \kappa$, by Lemma 4.2.3. of [3], for some $b \in B$, $X_b \in U$. Here we encounter a first limitation of this definition of ultraproduct, the assumption of κ-completeness of the ultrafilters.

Remark that, so defined, ultraproducts of classical first-order structures are two-valued and thus, this definition is an extension of the classical notion of ultraproduct. In case that, for every $i \in I$, $(\mathbf{B}, \mathbf{M}_i) = (\mathbf{B}, \mathbf{M})$ for the same Γ-structure, the ultraproduct is called the *ultrapower* of $(\mathbf{B}, \mathbf{M})$ and will be denoted by $(\mathbf{B}, \mathbf{M})^U$. The *natural embedding* is the pair $(id_{\mathbf{B}}, d)$, where $id_{\mathbf{B}}$ is the identity on the algebra $\mathbf{B}$ and $d \colon M \to M^I/U$ is the mapping such that for every $a \in M$, $d(a)$ is the U-equivalence class of the constant function with value a. By definition, d is clearly one-to-one. Moreover, by using Theorem 10, it is easy to check that $(id_{\mathbf{B}}, d)$ is an elementary embedding just as in the classical case.

Now we present an analogue to the classical Łoś Theorem for Ultraproducts (cf. [3] and [17]) in fuzzy predicate logics. Intuitively speaking, this theorem will show us how the elementary properties of ultraproducts are related to those of their constituent factors.

Theorem 10 *Let I be a non-empty set, κ an infinite cardinal and for each $i \in I$, let $(\mathbf{B}, \mathbf{M}_i)$ be a Γ-structure. Assume that U is a κ-complete ultrafilter*

over I such that $|B| < \kappa$. Then for every Γ-formula $\phi(x_1, \ldots, x_n)$, elements
$\overline{d_1}, \ldots, \overline{d_n} \in \prod M_i$ *and* $b \in B$,

$$\|\phi([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{(\mathbf{B}, \prod \mathbf{M_i}/U)} = b \text{ iff}$$
$$\{i \in I : \|\phi(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b\} \in U$$

Proof: By induction on the complexity of ϕ. For ϕ atomic it is clear, because U is an ultrafilter and by definition of the ultraproduct. Assume that for the Γ-formulas $\phi_1, \ldots, \phi_k$ the property holds. Let δ be a k-ary MTL-connective and for every $0 < j \leq k$, $\|\phi_j([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{(\mathbf{B}, \prod \mathbf{M_i}/U)} = a_j$.

($\Rightarrow$) Assume that $\|\delta(\phi_1, \ldots, \phi_k)([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{(\mathbf{B}, \prod \mathbf{M_i}/U)} = b$, then $\delta_{\mathbf{B}}(a_1, \ldots, a_k) = b$. Now for every $0 < j \leq k$, let Ψ_j be the following set:
$$\Psi_j = \{i \in I : \|\phi_j(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = a_j\}.$$
By inductive hypothesis, $\Psi_1 \cap \ldots \cap \Psi_k \in U$ and since $\Psi_1 \cap \ldots \cap \Psi_k$ is included in $\{i \in I : \|\delta(\phi_1, \ldots, \phi_k)(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b\}$ and U is an ultrafilter we have

$$\{i \in I : \|\delta(\phi_1, \ldots, \phi_k)(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b\} \in U.$$

($\Leftarrow$) Now we use the result we have just obtained to see that if

$$\{i \in I : \|\delta(\phi_1, \ldots, \phi_k)(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b\} \in U,$$

since U is an ultrafilter, necessarily we have $\delta_{\mathbf{B}}(a_1, \ldots, a_k) = b$. Therefore $\|\delta(\phi_1, \ldots, \phi_k)([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{(\mathbf{B}, \prod \mathbf{M_i}/U)} = b$.

Finally we prove the universal quantifier step. Assume inductively that the property holds for the Γ-formula $\phi(y, x_1 \ldots x_n)$.

($\Rightarrow$) If $\|\forall x \phi([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{(\mathbf{B}, \prod \mathbf{M_i}/U)} = b$ we define

$$\Theta = \{\|\phi([\overline{e}]_{\theta_U}, [\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{(\mathbf{B}, \prod \mathbf{M_i}/U)} : \overline{e} \in \prod M_i\}.$$

We have that $b = inf\Theta$. Now we choose, for every $a \in \Theta$, $\overline{e}_a \in \prod M_i$ such that $a = \|\phi([\overline{e}_a]_{\theta_U}, [\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{(\mathbf{B}, \prod \mathbf{M_i}/U)}$. Let us denote by X_a the set $\{i \in I : \|\phi(\overline{e}_a(i), \overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = a\}$. By inductive hypothesis, for every $a \in \Theta$, $X_a \in U$ and since U is a κ-complete ultrafilter over I such that $|B| < \kappa$, $\bigcap_{a \in \Theta} X_a \in U$. Now let $b_0 \in B$ be such that

$$\{i \in I : \|\forall x \phi(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b_0\} \in U.$$

Such a b_0 exists because the collection

$$(\{i \in I : \|\forall x \phi(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b'\} : b' \in B)$$

is a partition of I into fewer than κ parts. Thus exactly one of the parts belongs to U, by κ-completeness. Observe that this implies that $b_0 \leq b$, because $\bigcap_{a \in \Theta} X_a \in U$. We show now that $b_0 = b$. Let us assume the contrary, that $b_0 < b$. Then for each element

$$j \in \{i \in I : \|\forall x \phi(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b_0\}$$

we could choose $k_j \in M_j$ such that

$$\|\phi(k_j, \overline{d_1}(j), \ldots, \overline{d_n}(j))\|^{\mathbf{B}}_{\mathbf{M_j}} < b.$$

Now we define an element of the product $\overline{k} \in \prod M_i$ in the following way: for every $i \in I$,

$$\overline{k}(i) \;=\; \begin{cases} k_j, & \text{if } i = j \\ \text{arbitrary}, & \text{otherwise.} \end{cases}$$

And then we set $b_1 = \|\phi([\overline{k}]_{\theta_U}, [\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{(\mathbf{B}, \prod \mathbf{M_i}/U)}$. Hence we have that $b_1 \in \Theta$ and thus $b \leq b_1$, which is a contradiction, because this would imply that

$$\{i \in I : \|\phi(\overline{k}(i), \overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b_1\} \in U$$

by inductive hypothesis. And at the same time,

$$\{i \in I : \|\phi(\overline{k}(i), \overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} < b\} \in U$$

by definition of $\overline{k}$. Therefore we conclude that $b_0 = b$ and consequently,

$$\{i \in I : \|\forall x \phi(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B}, \mathbf{M_i})} = b\} \in U.$$

The ($\Leftarrow$) direction follows from the same kind of argument than the ($\Leftarrow$) direction in the quantifier-free step. For the existential quantifier the proof is analogous. $\qquad\square$

3.1 Some Drawbacks

Now we would like to mention some drawbacks of this definition of ultra-product. It is a well-known fact (see Proposition 4.2.1. of [3]) that a filter H over a nonempty set I is κ-complete for every cardinal κ iff H is principal. Nevertheless, as the following proposition shows, we are not interested in principal ultrafilters:

Proposition 11 *Let I be a non-empty set and for each $i \in I$, let $(\mathbf{B}, \mathbf{M}_i)$ be a Γ-structure. Assume that U is a principal ultrafilter over I. Then there is $i \in I$ such that $(\mathbf{B}, \prod \mathbf{M_i}/U) \cong (\mathbf{B}, \mathbf{M}_i)$.*

Proof: By definition of principal ultrafilter, since U is principal, there is $i \in I$ such that $U = \{X \subseteq I : i \in X\}$. Then, it is easy to see that the mapping $(id_{\mathbf{B}}, f) : (\mathbf{B}, \prod \mathbf{M_i}/U) \to (\mathbf{B}, \mathbf{M}_i)$, defined by: for every $\bar{e} \in \prod M_i$, $f([\bar{e}]_{\theta_U}) = \bar{e}(i)$ is an isomorphism. $\qquad\square$

Since every filter over a finite set is principal, as a corollary of the previous proposition we have that if I is a finite set, then there is $i \in I$ such that $(\mathbf{B}, \prod \mathbf{M_i}/U) \cong (\mathbf{B}, \mathbf{M}_i)$.

From the previous results we realize that we are clearly interested in non-principal ultrafilters, but what do we know about their existence? If κ is a singular cardinal, there are no nonprincipal κ-complete filters over κ (for a reference see [16]). Measurable cardinals κ are those for which there exists a nonprincipal κ-complete ultrafilter over κ. Clearly, ω is measurable. However κ-complete nonprincipal ultrafilters are much harder to come by for uncountable cardinals: the existence of uncountable measurable regular cardinals is a big cardinal axiom independent of ZFC. This is a serious drawback that makes that, apart form set-theoretical considerations, it is precisely the case in which $\mathbf{B}$ is a finite algebra that makes this definition of ultraproduct interesting.

Continuity of the connectives of the algebra gives us a better notion of ultraproduct for logics such as RPL. In [2] it was shown that continuity guarantees that the limit with respect to an ultrafilter always exists. In this case, the assumption of κ-complete ultrafilters is not needed. The proof of the corresponding Fundamental Ultrafilter Theorem for RPL can be found in [21].

4 D-FILTERS AND REDUCED PRODUCTS

The second definition we will examine is taken from [11]. The use of this notion would help us to overcome difficulties coming from the assumption of

κ-completeness of the ultrafilters.

First we recall some basic definitions and facts on direct products. Di Nola and Gerla introduced in [9] the notions of valuation structure and fuzzy model of a given first-order language in a categorial setting. There they show that certain operations such as direct products preserve first-order properties of this kind of models. In Proposition 2.1 of [9], they prove that the category of the valuation structures of a given type has direct products. Direct products for fuzzy structures were also studied by Bělohlávek in [1], but restricted to structures over complete residuated lattices and languages with an equality symbol interpreted as a similarity. The notion of *direct product of fuzzy algebras* is introduced in [5] and a kind of Birkhoff variety theorem for fuzzy algebras is presented (unpublished result of Hájek).

Definition 12 *Let I be a non-empty set and for each $i \in I$, $(\mathbf{B}_i, \mathbf{M}_i)$ be a Γ-structure. The* direct product $(\prod \mathbf{B_i}, \prod \mathbf{M_i})$ *of the structures* $\{(\mathbf{B}_i, \mathbf{M}_i) : i \in I\}$ *is defined as follows:*

- *The domain is the cartesian product $\prod M_i$.*

- *$\prod \mathbf{B_i}$ is the direct product of the MTL-algebras $\{\mathbf{B}_i : i \in I\}$.*

- *For each n-ary function symbol $F \in \Gamma$, and every $\overline{d_1}, \dots, \overline{d_n} \in \prod M_i$,*

$$F_{\prod \mathbf{M_i}}(\overline{d_1}, \dots, \overline{d_n}) = (F_{\mathbf{M}_i}(\overline{d_1}(i), \dots, \overline{d_n}(i)) : i \in I)$$

- *For each n-ary predicate $P \in \Gamma$, and every $\overline{d_1}, \dots, \overline{d_n} \in \prod M_i$,*

$$P_{\prod \mathbf{M_i}}(\overline{d_1}, \dots, \overline{d_n}) = (P_{\mathbf{M}_i}(\overline{d_1}(i), \dots, \overline{d_n}(i)) : i \in I)$$

Remark that the direct product is well-defined because the class of MTL-algebras is a variety and thus is closed under direct products.

Gerla introduced in [11] the notions of d-filter, of reduced product and of ultraproduct of a family of fuzzy models with definable quantifiers. That is, models such that for each quantifier there is a formula of the classical first-order language with equality with a unique monadic predicate A that defines it (for a reference see Definition 8.1 of [11]). He proved that these operations preserve first-order properties of fuzzy models with definable quantifiers. Our definition is based in that of Gerla, we use also pairs of filters (d-filters in Gerla's terms).

Definition 13 *Let I be a non-empty set and for each $i \in I$, $(\mathbf{B}_i, \mathbf{M}_i)$ be a Γ-structure. Let G and H be proper filters over I with $G \subseteq H$. The* reduced

product $(\prod \mathbf{B_i}/H, \prod \mathbf{M_i}/G)$ *of the structures* $\{(\mathbf{B}_i, \mathbf{M}_i) : i \in I\}$ *is the quotient structure of* $(\prod \mathbf{B_i}, \prod \mathbf{M_i})$ *modulo the congruence* (θ_H, θ_G)*, where* θ_G, θ_H *are defined as follows:*

$$(\overline{d}, \overline{e}) \in \theta_G \text{ iff } \{i \in I : \overline{d}(i) = \overline{e}(i)\} \in G$$

$$(\overline{a}, \overline{b}) \in \theta_H \text{ iff } \{i \in I : \overline{a}(i) = \overline{b}(i)\} \in H$$

for every $\overline{d}, \overline{e} \in \prod M_i$ *and* $\overline{a}, \overline{b} \in \prod B_i$.

Observe that the reduced product is well-defined because the class of MTL-algebras is a variety and thus is closed under reduced products. Special cases of reduced products are the *direct products*, when $H = \{I\}$; *ultra-products*, when G is an ultrafilter; and *reduced powers*, when for each $i \in I$, $(\mathbf{B}_i, \mathbf{M}_i) = (\mathbf{B}, \mathbf{M})$ for the same structure. Remark that, when G is an ultrafilter, then H is also an ultrafilter, because $G \subseteq H$. Now we present an analogue to the Łoś Theorem for reduced products in fuzzy predicate logics:

Theorem 14 *Let I be a non-empty set and for each $i \in I$, let $(\mathbf{B}_i, \mathbf{M}_i)$ be a Γ-structure and let G and H be proper filters over I with $G \subseteq H$ such that (θ_H, θ_G) is a σ-congruence. Then for every Γ-formula $\phi(x_1, \ldots, x_n)$ and elements $\overline{d_1}, \ldots, \overline{d_n} \in \prod M_i$, $\overline{b} \in \prod B_i$,*

$$\|\phi([\overline{d_1}]_{\theta_G}, \ldots, [\overline{d_n}]_{\theta_G})\|^{(\prod \mathbf{B_i}/H, \prod \mathbf{M_i}/G} = [\overline{b}]_{\theta_H} \text{ iff}$$
$$\{i \in I : \|\phi(\overline{d_1}, \ldots, \overline{d_n})\|^{(\prod \mathbf{B_i}, \prod \mathbf{M_i})}(i) = \overline{b}(i)\} \in H \text{ iff}$$
$$\{i \in I : \|\phi(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B_i}, \mathbf{M_i})} = \overline{b}(i)\} \in H$$

Proof: Let $(\mathbf{B}_i : i \in I)$ be a sequence of MTL-algebras and $(\overline{b_j} : j \in J)$ a sequence of elements of its direct product $\prod \mathbf{B_i}$. Then we have that, for every $i \in I$: if $sup_{j \in J}\overline{b_j}(i)$ exists then $[sup_{j \in J}\overline{b_j}](i) = sup_{j \in J}\overline{b_j}(i)$ (analogously for the infimum of a sequence of elements). As a direct consequence we obtain that the i-projection (f_i, g_i) defined in the natural way, is a σ-homomorphism with f_i and g_i onto. Thus, for every Γ-formula $\phi(x_1, \ldots, x_n)$ and every $\overline{d_1}, \ldots, \overline{d_n} \in \prod M_i$,

$$\|\phi(\overline{d_1}, \ldots, \overline{d_n})\|^{(\prod \mathbf{B_i}, \prod \mathbf{M_i})} = (\|\phi(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{(\mathbf{B_i}, \mathbf{M_i})} : i \in I)$$

Using this result and the fact that σ-congruences are elementary congruences (by Proposition 6) we obtain that

$$\|\phi([\overline{d_1}]_{\theta_G}, \ldots, [\overline{d_n}]_{\theta_G})\|^{(\prod \mathbf{B_i}/H, \prod \mathbf{M_i}/G} = [\|\phi(\overline{d_1}, \ldots, \overline{d_n})\|^{(\prod \mathbf{B_i}, \prod \mathbf{M_i})}]_{\theta_H}$$

and consequently, the desired result. $\square$

The advantage of working with reduced products in general (instead that working only with ultraproducts) is that we can guarantee the existence of such σ-congruences. Therefore we can use Theorem 14 to obtain the desired structures preserving first-order properties. σ-congruences can be obtained using κ-complete filters for κ big enough (depending of the chosen $(\mathbf{B}_i, \mathbf{M}_i)$ structures and the cardinality of the language). Unless κ-complete ultrafilters, the existence of κ-complete filters does not depend on the assumption of large cardinal axioms. Examples of κ-complete filters are the following:

- If κ is a regular cardinal, then the set of all $X \subseteq \kappa$ for which the cardinality of its complement $(\kappa - X)$ is smaller than κ, is a κ-complete nonprincipal filter over κ.

- Let $I = [0, 1]$ be the real unit interval and μ the Lebesgue measure. Then the set $H = \{X \subseteq [0, 1] : \mu(X) = 1\}$ is a countably complete filter.

It is easy to check that one of the important applications of ultraproducts holds for this definition of reduced products: the reduced power of one structure is an elementary extension of this structure. On the side of the drawbacks we have that, in general, reduced products or ultraproducts of classical first-order structures are not necessarily two-valued.

5   FUTURE WORK

Work in progress includes applications of the fundamental theorem to the characterization of elementary classes and of elementarily equivalent models. Using Theorems 10 and 14 it can be proved that the different ultraproduct constructions are elementarily equivalent. Future work will be devoted to a deeper study of the relationship between these two notions. Following [7], we plan also to extend the results of [8] to show that the constructions introduced are adequate for working in a reduced semantics.

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