Algebras and relational frames for Gödel modal logic and some of its extensions

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Abstract

Gödel modal logics can be seen as extenions of intutionistic modal logics with the prelinearity axiom. In this paper we focus on the algebraic and relational semantics for Gödel modal logics that leverages on the duality between finite Gödel algebras and finite forests, i.e. finite posets whose principal downsets are totally ordered. We consider different subvarieties of the basic variety $\mathbb{GAO}$ of Gödel algebras with two modal operators (GAOs for short) and their corresponding classes of forest frames, either with one or two accessibility relations. These relational structures can be considered as prelinear versions of the usual relational semantics of intuitionistic modal logic. More precisely we consider two main extensions of finite Gödel algebras with operators: the one obtained by adding Dunn axioms, typically studied in the fragment of positive classical (and intuitionistic) logic, and the one determined by adding Fischer Servi axioms. We present Jónsson-Tarski like representation theorems for the different types of finite GAOs considered in the paper.

1 Introduction

Extending modal logics to a non-classical propositional ground has been, and still is, a fruitful research line that encompasses several approaches, ideas and methods. In the last years, this topic has significantly impacted on the community of many-valued and mathematical fuzzy logic that have proposed ways to expand fuzzy logics (t-norm based fuzzy logics, in the terminology of Hájek [14]) by modal operators so as to capture modes of truth that can be faithfully described as "graded".

In this line, one of the fuzzy logics that has been an object of major interest without any doubt is the so called *Gödel logic*, i.e., the axiomatic extension of intuitionistic propositional calculus given by the *prelinearity axiom*: $(\varphi \to \psi) \vee (\psi \to \varphi)$. As first observed by Horn in [16], prelinearity implies completeness of Gödel logic with respect to totally ordered Heyting algebras, i.e., *Gödel chains*. Indeed, prelinear Heyting algebras form a proper subvariety of that of Heyting algebras, usually called the variety of Gödel algebras and denoted $\mathbb{G}$ whose subdirectly irreducible elements are totally ordered. Furthermore, in contrast with the intuitionistic case, $\mathbb{G}$ is locally finite, whence the finitely generated free Gödel algebras are finite.

Modal extensions of Gödel logic have been intensively discussed in the literature [7, 8, 18]. Following the usual methodological and philosophical approach to fuzzy logic, they have been mainly approached semantically by generalizing the classical definition of Kripke model $\langle W, R, e\rangle$ by allowing both the evaluation of (modal) formulas and the accessibility relation R to range over a Gödel algebra, rather than the classical two-valued set $\{0, 1\}$ (see [5] for a general approach). More precisely, a model of this kind, besides evaluating formulas in a more general structure than the classical two-element boolean algebra, regards the accessibility relation R as a function from the cartesian product $W \times W$ to a Godel algebra $\mathbf{A}$ so that, for all $w, w' \in W$, $R(w, w') = a \in A$ means that a is the *degree of accessibility* of w' from w.

In this chapter we will put forward a novel approach to Gödel modal logic that leverages on the duality between finite Gödel algebras and finite forests. This line, that was previously presented in [13], is deepened and extended in the present paper. In particular, we will focus on Gödel modal algebras and their dual structures, that is, the prime spectra of Gödel algebras ordered by reverse-inclusion. These ordered structures can be regarded as the prelinear version of posets and they are known in the literature as *forests*: posets whose principal downsets are totally ordered. The algebras we will consider form a variety denoted by $\mathbb{GAO}$ for *Gödel algebras with operators*. Hence, the algebras we are concerned with are those belonging to the finite slice of $\mathbb{GAO}$. The associated relational structures based on forests, as we briefly recalled above, might hence be regarded as the prelinear version of the usual relational semantics of intuitionistic modal logic. Accessibility relations $R_\Box$ and $R_\Diamond$ on finite forests are defined, in our frames, by ad hoc properties that we express in terms of (anti)monotonicity on the first argument of the relations themselves. These relational frames will be called *forest frames*.

Along the whole chapter, we will take care of comparing our approach to the ones that have been proposed for intuitionistic modal logic and, in particular, those developed by Palmigiano in [21] and Orłowska and Rewitzky in [20]. By analyzing the role that these different relational frames (namely, those presented by Palmigiano, Orłowska and Rewitzky and ours) have in proving a Jónsson-Tarski like representation theorem for Gödel algebras with modal operators, we realized that forest frames situate in a middle level of generality between those of Palmigiano and those of Orłowska and Rewitzky. The former being the less and the latter being the more general ones in a sense that will be made clear in Section 4.

More in detail, we will observe that, if we start from any Gödel algebra with operators $(\mathbf{A}, \Box, \Diamond)$, its associated forest frame $(\mathbf{F_A}, R_\Box, R_\Diamond)$ allows to construct another algebraic structure $(\mathbf{G}(\mathbf{F_A}), \beta_\Box, \delta_\Diamond)$ isomorphic to the starting one. Interestingly, the forest frame $(\mathbf{F_A}, R_\Box, R_\Diamond)$ is not the unique one that reconstructs $(\mathbf{A}, \Box, \Diamond)$ up to isomorphism. Indeed, as we will show in Section 4, for every Gödel algebra with operators $(\mathbf{A}, \Box, \Diamond)$, there are not isomorphic forest frames, Palmigiano-like and Orłowska and Rewitzky-like frames that determine the same original modal algebra $(\mathbf{A}, \Box, \Diamond)$ up to isomorphism.

In Section 3 we will start by considering the most general way to define the operators $\Box$ and $\Diamond$ on Gödel algebras while in Section 4 we investigate the relational structures corresponding to the resulting algebraic structures. In Section 5 we will focus on particular and well-known extensions. Precisely we will consider two main extensions of Gödel algebras with operators: (1) the first one is obtained by adding the Dunn axioms, typically studied in the fragment of positive classical (and intuitionistic) logic [11, 9]; (2) the second one is determined by adding the Fischer Servi axioms [12]. From the algebraic perspective, adding these identities to Gödel algebras with operators identifies two proper subvarieties of $\mathbb{GAO}$ that will be respectively denoted by $\mathbb{DGAO}$ and $\mathbb{FSGAO}$. Section 5 is hence complemented by some examples that showing that $\mathbb{DGAO}$ and $\mathbb{FSGAO}$ can be distinguished.

In contrast with the case of general Gödel algebras with operators discussed in Sections 3 and 4 whose relational structures need two independent relations to treat the modal operators, the structures belonging to $\mathbb{DGAO}$ and $\mathbb{FSGAO}$ only need, for their Jónnson-Tarski like representation, frames with only one accessibility relation. Forest frames with one relation are hence studied in Section 6 where, in addition to a comparison with the usual intuitionistic case, we will also study in detail the relational structures corresponding to two further subvarieties of $\mathbb{GAO}$. The first one is the variety $\mathbb{FSDGAO}$ obtained as the intersection $\mathbb{DGAO} \cap \mathbb{FSGAO}$. The algebras belonging to such variety have been called *bi-modal Gödel algebras* in [8]. The second subvariety that we will consider in Section 6.4 is another refinement of $\mathbb{DGAO}$ and it will be denoted by $\mathbb{WGAO}$. Algebras in this class are characterized by the requirement that $\Box a$ and $\Diamond a$ are Boolean for of each element a. A final proposition will make clear the inclusions between subvarieties of $\mathbb{GAO}$ studied in this paper.

Next section on preliminaries is devoted to introduce the basics of finite Gödel algebras and their dual structures of finite forests.

2 Preliminaries: Gödel algebras and forests

Gödel algebras, the algebraic semantics of infinite-valued Gödel logic [14], are idempotent, bounded, integral, commutative residuated lattices of the form $\mathbf{A} = (A, \wedge, \vee, \to, \bot, \top)$ satisfying the prelinearity equation: $(a \to b) \vee (b \to a) = \top$. In other words, Gödel algebras are *prelinear Heyting algebras*. All algebras we will consider in this paper are finite.

Let $\mathbf{A}$ be a Gödel algebra. A non-empty subset f of $\mathbf{A}$ is said to be a *filter* provided that: (1) $\top \in f$, (2) if $x, y \in f$, then $x \wedge y \in f$, (3) if $x \in f$ and $y \geq x$ then $y \in f$. A filter $f \neq A$ (that is a *proper* filter) is said to be *prime* if $x \vee y \in f$ implies that either $x \in f$ or $y \in f$. A filter f is *principal* (or *principally generated*) if there exists an element $x \in A$ such that $f = \uparrow x = \{y \in A \mid y \geq x\}$. By a standard result [,], in every finite Gödel algebra prime filters coincide with those filters principally generated by the join-irreducible elements of A.

A non-empty subset h of A is said to be a *co-filter*, if (1) if $x \in h$ and $y \geq x$ implies $y \in h$ and (2) $x \vee y \in f$ implies that either $x \in h$ or $y \in h$. Therefore a subset f of A is a prime filter iff it is both a filter and a co-filter.

A non-empty subset i of A is an *ideal* provided that: (1) i is downward closed and such that, if $x, y \in i$ then $x \vee y \in i$. It is easy to see that the set-theoretical complement of a proper co-filter is an ideal.

Let $\mathbf{A}$ be a finite Gödel algebra and denote by $F_{\mathbf{A}}$ the finite set of its prime filters. Unlike the case of boolean algebras, prime and maximal filters are not the same for Gödel algebras and indeed $F_{\mathbf{A}}$ can be ordered in a nontrivial way. In particular, if for $f_1, f_2 \in F_{\mathbf{A}}$ we define $f_1 \leq f_2$ iff (as prime filters) $f_1 \supseteq f_2$, $\mathbf{F_A} = (F_{\mathbf{A}}, \leq)$ turns out to be a finite *forest*, i.e., a poset such that the downset of each element is totally ordered.

Finite forests play a crucial role in the theory of finite Gödel algebras. Indeed, let $\mathbf{F} = (F, \leq)$ be a finite forest, $G(\mathbf{F})$ be the set of all downward closed subsets of F (i.e., the *subforests* of $\mathbf{F}$) and consider the following operations on $G(\mathbf{F})$: for all $x, y \in F$,

1. $x \wedge y = x \cap y$ (the set-theoretic intersection);
2. $x \vee y = x \cup y$ (the set-theoretic union);
3. $x \to y = (\uparrow(x \setminus y))^c = F \setminus \uparrow(x \setminus y)$, where $\setminus$ denotes the set-theoretical difference, for every $z \in F$, $\uparrow z = \{k \in F \mid k \geq z\}$ and ^{c} denotes the set-theoretical complement.[1]

The algebra $\mathbf{G}(\mathbf{F}) = (G(\mathbf{F}), \wedge, \vee, \to, \emptyset, F)$ is a Gödel algebra [2, §4.2] and the following Stone-like representation theorem holds.

Lemma 2.1 ([2, Theorem 4.2.1])**.** *Every finite Gödel algebra* $\mathbf{A}$ *is isomorphic to* $\mathbf{G}(\mathbf{F_A})$ *through the map* $r : \mathbf{A} \to \mathbf{G}(\mathbf{F_A})$

$$r : a \in A \mapsto \{f \in F_{\mathbf{A}} \mid a \in f\}.$$

Example 2.2. Let Free_1 be the 1-generated free Gödel algebra (Fig. 3). Its prime filters, which are all principally generated as upsets of its join-irreducible elements, are $f_1 = \{y \in \mathrm{Free}_1 \mid y \geq x\} = \{x, x \vee \neg x, \neg\neg x, \top\}$, $f_2 = \{y \in \mathrm{Free}_1 \mid y \geq \neg x\} = \{\neg x, x \vee \neg x, \top\}$, and $f_3 = \{y \in \mathrm{Free}_1 \mid y \geq \neg\neg x\} = \{\neg\neg x, \top\}$. The forest $\mathbf{F}_{\mathrm{Free}_1}$ is obtained by ordering $\{f_1, f_2, f_3\}$ by reverse inclusion.

[1] Without danger of confusion, and thanks to the following result, we will not distinguish the symbols of a Gödel algebra $\mathbf{A}$ from those of $\mathbf{G}(\mathbf{F})$

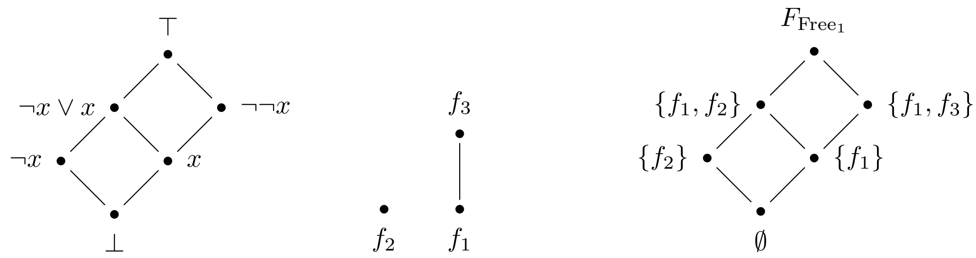

Figure 1: From left to right: The Hasse diagram of the free Gödel algebra over one generator Free_1, the forest $\mathbf{F}_{\text{Free}_1}$ of its prime filters, and the Hasse diagram of its isomorphic copy $\mathbf{G}(\mathbf{F}_{\text{Free}_1})$

Let us consider the set $G(\mathbf{F}_{\text{Free}_1})$ of subforests of $\mathbf{F}_{\text{Free}_1}$:

$$G(\mathbf{F}_{\text{Free}_1}) = \{\emptyset, F_{\text{Free}_1}, \{f_2\}, \{f_1\}, \{f_2, f_1\}, \{f_3, f_1\}\}$$

with operations $\wedge, \vee, \to$ as in (1-3) above. Lemma 2.1 shows that algebra $\mathbf{G}(\mathbf{F}_{\text{Free}_1})$ is a Gödel algebra which is isomorphic to Free_1.

3 Gödel algebras with operators

In this section we introduce the basic class of Gödel algebras with two modal operators and prove a representation theorem for them à la Jónsson-Tarski. Next definition is from [13, Definition 5].

Definition 3.1. A *Gödel algebra with operators* (*GAO* for short) is a triple $(\mathbf{A}, \Box, \Diamond)$ where $\mathbf{A}$ is a Gödel algebra, $\Box$ and $\Diamond$ are unary operators on A satisfying the following equations:

($\Box$1) $\Box\top = \top$;

($\Box$2) $\Box(x \wedge y) = \Box x \wedge \Box y$;

($\Diamond$1) $\Diamond\bot = \bot$;

($\Diamond$2) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$.

Note that GAOs are in fact the axiomatic extensions of the so-called HK-algebras in [20] with the prelinearity axiom.

Clearly the class of Gödel algebras with operators forms a variety (i.e., an equational class) that we will henceforth denote by $\mathbb{GAO}$.

Let us start showing some easy properties which will turn out to be useful for the rest of this section.

Proposition 3.2. *For every GAO* $(\mathbf{A}, \Box, \Diamond)$ *and for every filter* f *of* $\mathbf{A}$ *the following facts hold.*

1. $\Box^{-1}(f)$ *is a filter;*
2. *if* f *is prime, then* $\Diamond^{-1}(f)$ *is a co-filter.*

Proof. (1) Since f is a filter, $\top \in \Box^{-1}(f)$ because of ($\Box$1); $\Box^{-1}(f)$ is $\wedge$-closed because of ($\Box$2); $\Box^{-1}(f)$ is upward closed because $\Box$ is monotone.

(2) Since f is prime, f is a co-filter. By ($\Diamond$1), $\Diamond\bot = \bot$ and hence $\Diamond\bot \notin f$ because f is proper. Therefore the claim follows from [20, Lemma 3.2 (a)]. ◻

Let $(\mathbf{A}, \Box, \Diamond)$ be a GAO and let $\mathbf{F}_{\mathbf{A}}$ the forest of its prime filters. Define $R_\Box$ and $R_\Diamond$ on $F_{\mathbf{A}} \times F_{\mathbf{A}}$ as follows: for each $f_1, f_2 \in F_{\mathbf{A}}$,

$$f_1 R_\Box f_2 \text{ iff } \Box^{-1}(f_1) \subseteq f_2 \tag{1}$$

and

$$f_1 R_\Diamond f_2 \text{ iff } \Diamond(f_2) \subseteq f_1 \text{ iff } f_2 \subseteq \Diamond^{-1}(f_1). \tag{2}$$

Lemma 3.3. *For every GAO $(\mathbf{A}, \Box, \Diamond)$, the relations $R_\Box$ and $R_\Diamond$ respectively satisfy*

(M) for all $f, g, h \in F_{\mathbf{A}}$, if $f \leq g$ and $f R_\Box g$, then $g R_\Box h$;

(A) for all $f, g, h \in F_{\mathbf{A}}$, if $g \leq f$ and $f R_\Diamond h$, then $g R_\Diamond h$.

Proof. Let $f, g, h \in F_{\mathbf{A}}$. If $f \leq g$ in the order of $\mathbf{F_A}$, then $f \supseteq g$ as prime filters, whence if $\Box^{-1}(f) \subseteq h$ then $\Box^{-1}(g) \subseteq h$. Therefore, if $f R_\Box h$, then $g R_\Box h$ that shows that $R_\Box$ satisfies (M).

As for the second claim, let $f, g, h \in F_{\mathbf{A}}$ and assume $f R_\Diamond h$ (i.e., $\Diamond(h) \subseteq f$ as prime filters) and $f \geq g$, meaning that, as prime filters, $f \subseteq g$. Then, $\Diamond(h) \subseteq f \subseteq g$ and hence $g R_\Diamond h$. □

Now, let $(\mathbf{F}, R_\Box, R_\Diamond)$ be such that $\mathbf{F}$ is a forest and $R_\Box, R_\Diamond \subseteq F \times F$ respectively satisfy (M) and (A) of Lemma 3.3. Let $\mathbf{G}(\mathbf{F})$ be the Gödel algebra of downsets of $\mathbf{F}$ defined as in the previous section and consider the maps $\beta, \delta : G(\mathbf{F}) \to G(\mathbf{F})$ such that, for every $a \in G(\mathbf{F})$

$$\beta(a) = \{y \in F \mid \forall z \in F,\ (y R_\Box z \Rightarrow z \in a)\}, \tag{3}$$

and

$$\delta(a) = \{y \in F \mid \exists z \in a,\ y R_\Diamond z\}. \tag{4}$$

Remark 3.4. (1) For all $a \in G(\mathbf{F})$, $\beta(a)$ is a subforest of $\mathbf{F}$. Indeed, if $x \in \beta(a)$ then $\forall z \in F$, $(x R_\Box z \Rightarrow z \in a)$. Let $y \leq x$. For all z, if $y R_\Box z$, then $x R_\Box z$ as well, because of (M), and hence $z \in a$. Thus $y \in \beta(a)$.

(2) For all $a \in G(\mathbf{F})$, $\delta(a) \in G(\mathbf{F})$, i.e., $\delta(a)$ is a subforest of $\mathbf{F}$. Indeed if $x \in \delta(a)$ then there exists $z \in a$ such that $x R_\Diamond z$. Let $y \leq x$ in $\mathbf{F}$. Then (A) of Lemma 3.3 implies $y R_\Diamond z$ as well, that is $y \in \delta(a)$ and hence $\delta(a)$ is downward closed.

Moreover, the following properties hold.

Proposition 3.5. *Let $\mathbf{F}$ be a finite forest and let $R_\Box, R_\Diamond \subseteq F \times F$ such that $R_\Box$ satisfies (M) and $R_\Diamond$ satisfies (A). Let $\beta, \delta : G(\mathbf{F}) \to G(\mathbf{F})$ be defined as in (7) and (8) respectively. Then:*

1. *$\beta(\top) = \top$;*
2. *For all $a, b \in G(\mathbf{F})$, $\beta(a \wedge b) = \beta(a) \cap \beta(b)$.*
3. *$\delta(\bot) = \bot$;*
4. *For all $a, b \in G(\mathbf{F})$, $\delta(a \vee b) = \delta(a) \cup \delta(b)$.*

Proof. (1) Recall from Section 2 that the top element of $\mathbf{G}(\mathbf{F})$ is F. Thus, $\beta(\top) = \beta(F) = \{y \in F \mid \forall z \in F,\ (y R_\Box z \Rightarrow z \in F)\}$. Obviously, the condition $(y R_\Box z \Rightarrow z \in F)$ is true for all $z \in F$ and hence $\beta(F) = F$.

(2) For all $a, b \in G(\mathbf{F})$, we have,

$$\begin{aligned}
\beta(a \wedge b) &= \{y \in F \mid \forall z \in F,\ y R_\Box z \Rightarrow z \in a \wedge b\} \\
&= \{y \in F \mid \forall z \in F,\ y R_\Box z \Rightarrow z \in a \cap b\} \\
&= \{y \in F \mid \forall z \in F,\ y R_\Box z \Rightarrow z \in a\} \cap \\
&\ \{y \in F \mid \forall z \in F,\ y R_\Box z \Rightarrow z \in b\} \\
&= \beta(a) \cap \beta(b).
\end{aligned}$$

(3) The bottom element of $\mathbf{G}(\mathbf{F})$ is the empty forest, whence $\emptyset = \{y \in F \mid \exists z \in \emptyset,\ y R_\Diamond z\} = \delta(\bot)$.

(4) $\delta(a \vee b) = \{y \in F \mid \exists z \in a \vee b,\ y R_\Diamond z\} = \{y \in F \mid \exists z \in a \cup b,\ y R_\Diamond z\} = \{y \in F \mid \exists z \in a,\ y R_\Diamond z\} \cup \{y \in F \mid \exists z \in b,\ y R_\Diamond z\} = \delta(a) \cup \delta(b)$. □

Now we can prove the following representation theorem.

Theorem 3.6. *For every finite GAO* $(\mathbf{A}, \Box, \Diamond)$*, the function* $r : (\mathbf{A}, \Box, \Diamond) \to (\mathbf{G}(\mathbf{F_A}), \beta, \delta)$ *mapping every* $a \in A$ *to* $\{f \in F_{\mathbf{A}} \mid z \in f\}$ *is a isomorphism. In particular, for all* $a \in A$*,*

$$r(\Box(a)) = \beta(r(a)) \text{ and } r(\Diamond(a)) = \delta(r(a)). \tag{5}$$

Proof. We proved in Lemma 2.1 that $\mathbf{G}(\mathbf{F_A})$ is a Gödel algebra and the map $r : \mathbf{A} \to \mathbf{G}(\mathbf{F_A})$ is a Gödel isomorphism. Thus, it remains to show that (5) holds.

(1) $r(\Box(a)) = \beta(r(a))$. Let us start proving that for all $a \in A$, $\beta(r(a)) \subseteq r(\Box(a))$. By definition,

$$\begin{aligned} \beta(r(a)) &= \{f \in F_{\mathbf{A}} \mid \forall g \in F_{\mathbf{A}}\ (fR_{\Box}g \Rightarrow g \in r(a))\} \\ &= \{f \in F_{\mathbf{A}} \mid \forall g \in F_{\mathbf{A}}\ (\Box^{-1}(f) \subseteq g \Rightarrow a \in g)\}. \end{aligned}$$

Let $f \in \beta(r(a))$ and assume, by way of contradiction, that $f \notin r(\Box(a))$, that is to say, $a \notin \Box^{-1}(f)$. Notice that this assumption forces $a \neq \top$. By Proposition 3.2 (1), $\Box^{-1}(f)$ is a filter. Thus, if $a \notin \Box^{-1}(f)$ and since $a \neq \top$, by [14, Lemma 2.3.15], there exists a prime filter g of $\mathbf{A}$ such that $g \supseteq \Box^{-1}(f)$ and $a \notin g$. On the other hand, $fR_{\Box}g$ because g extends $\Box^{-1}(f)$ and $a \notin g$. Thus, $f \notin \beta(r(a))$ and a contradiction has been reached.

For the other inclusion, we have to prove that if $\Box(a) \in f$, then for all $g \in F_{\mathbf{A}}$, $fR_{\Box}g \Rightarrow a \in g$. If $\Box(a) \in f$, then $a \in \Box^{-1}(f)$. Therefore, for all $g \in F_{\mathbf{A}}$, if $fR_{\Box}g$, then $\Box^{-1}(f) \subseteq g$ and hence $a \in g$ which settles the claim.

(2) $r(\Diamond(a)) = \delta(r(a))$. First of all notice that it is sufficient to prove it for the case of a being a join-irreducible element of $\mathbf{A}$. Indeed, assume that the right-hand-side of (5) holds for join irreducible elements and let b be not join irreducible. Then b can be displayed as $b = a_1 \vee \ldots \vee a_k$, where the a_i's are join irreducible. By ($\Diamond$2), $\Diamond(b) = \Diamond(a_1) \vee \ldots \vee \Diamond(a_k)$. Therefore, since r is a Gödel algebra isomorphism,

$$r(\Diamond(b)) = r(\Diamond(a_1)) \vee \ldots \vee r(\Diamond(a_k)).$$

By assumption, $r(\Diamond a_i) = \delta(r(a_i))$ for all $i = 1, \ldots, k$. Thus, $r(\Diamond(b)) = \delta(a_1) \vee \ldots \vee \delta(a_k)$ which equals $\delta(b)$ by Proposition 3.5(2).

Let hence a be join irreducible and let us prove that $r(\Diamond(a)) \supseteq \delta(r(a))$ and $r(\Diamond(a)) \subseteq \delta(r(a))$. As for the first inclusion, notice that for all $a \in A$ (being a join irreducible or not), by Lemma 2.1,

$$\begin{aligned} \delta(r(a)) &= \{f \in F_{\mathbf{A}} \mid \exists g \in r(a),\ fR_{\Diamond}g\} \\ &= \{f \in F_{\mathbf{A}} \mid \exists g \in F_{\mathbf{A}},\ (a \in g \ \&\ fR_{\Diamond}g\} \\ &= \{f \in F_{\mathbf{A}} \mid \exists g \in F_{\mathbf{A}},\ (a \in g \ \&\ \Diamond(g) \subseteq f)\} \end{aligned}$$

Therefore, if $f \in \delta(r(a))$, $\Diamond(a) \in f$ and hence $f \in r(\Diamond(a))$ and hence $r(\Diamond(a)) \supseteq \delta(r(a))$.

To prove the other inclusion we have to show that if $f' \in r(\Diamond(a))$, there exists an $f \in F_{\mathbf{A}}$ such that $a \in f$ and $\Diamond(f) \subseteq f'$. Since a is join irreducible, the filter $f_a = \{b \in A \mid b \geq a\}$ is prime. Let us prove that $\Diamond(f_a) \subseteq f'$.

Claim 1. $\Diamond(f_a) \subseteq f_{\Diamond(a)} = \{x \in A \mid x \geq \Diamond(a)\}$.

As a matter of fact, if $z \in \Diamond(f_a)$, then there exists $b \geq a$ such that $z = \Diamond(b)$. Since $\Diamond$ is monotone, $\Diamond(b) \geq \Diamond(a)$, whence $z = \Diamond(b) \in f_{\Diamond(a)}$.

Claim 2. *For all* $f' \in r(\Diamond(a))$, $f_{\Diamond(a)} \subseteq f'$.

Indeed, if $x \in f_{\Diamond(a)}$, then $x \geq \Diamond(a)$ and hence $x \in f'$ because $\Diamond(a) \in f'$ and f' is upward closed.

By the above claims, for all $f' \in r(\Diamond(a))$, $\Diamond(f_a) \subseteq f'$, whence

$$r(\Diamond(a)) \subseteq \delta(r(a)).$$

Thus, for all a, $r(\Diamond(a)) = \delta(r(a))$ which settles the claim. □

Theorem 3.6 above shows that every finite GAO can be isomorphically represented as the algebra of subforests of the forests of its prime filters. This kind of representation will be henceforth call *forest-based representation.* In these latter algebras, the modal operators are obtained by two binary relations $R_\Box$ and $R_\Diamond$ satisfying (M) and (A) respectively. Let us further notice that, although these two relations $R_\Box$ and $R_\Diamond$ are independent in general, there are significant cases in which they are not, or they can even coincide. An example of the latter is the case of classical Kripke frames, the dual semantics of Boolean algebras with operators, a proper subvariety of $\mathbb{GAO}$.

We will deepen the investigation on such relational models in the next section, but it is worth to point out that, indeed, the theorem above is sufficiently general to be rephrased in every subclass $\mathbb{K}$ of $\mathbb{GAO}$ that is closed under isomorphic images. i.e., such that $\mathbb{K} = \mathbf{I}(\mathbb{K})$. Thus, in particular, it applies to all subvarieties of $\mathbb{GAO}$. The following easy consequence of Theorem 3.6 makes this fact clear.

Corollary 3.7. *Let* $\mathbb{K}$ *be any subset of* $\mathbb{GAO}$ *that is closed under isomorphic images. Then, every algebra in* $\mathbb{K}$ *has an isomorphic forest-based representation in* $\mathbb{K}$*.*

4 Forest frames with two relations

This section is dedicated to investigate the relational structures used in the previous results and that are made, for a GAO $(\mathbf{A}, \Box, \Diamond)$, of the forest $\mathbf{F_A}$ of its prime filters and two binary relations $R_\Box$ and $R_\Diamond$ respectively satisfying the properties (M): *monotonicity* in the first argument, and (A): *antimonotonicity* in the fist argument.

The idea of defining relational structures on the prime spectrum of the modal algebra is not new and indeed the below definition of forest frame is strongly inspired by the usual way relational structures are defined for intuitionistic modal logic and, in particular, for the logic denoted **IntK** in [23] in which the two modalities $\Box$ and $\Diamond$ have no axioms in common and hence they are treated, on the relational side, by two (independent) accessibility relations.

After defining forests frames, we will compare these structures with their analogous defined by Orłowska and Rewitzky in [20] and by Palmigiano in [21].

Definition 4.1. A *forest frame* is a triple $(\mathbf{F}, R_\Box, R_\Diamond)$ where $\mathbf{F} = (F, \leq)$ is a finite forest and $R_\Box, R_\Diamond \subseteq F \times F$ respectively satisfy the following conditions:

(M) for all $x, y, z \in F$, if $x \leq y$ and $xR_\Box z$, then $yR_\Box z$;

(A) for all $x, y, z \in F$, if $y \leq x$ and $xR_\Diamond z$, then $yR_\Diamond z$.

We have seen in the previous section that for every forest frame $(\mathbf{F}, R_\Box, R_\Diamond)$ we have an associated GAO $(\mathbf{G}(\mathbf{F}), \beta, \delta)$ and for every GAO $(\mathbf{A}, \Box, \Diamond)$ we have an associated forest frame $(\mathbf{F_A}, R_\Box, R_\Diamond)$ where the relations $R_\Box, R_\Diamond \subseteq F_\mathbf{A} \times F_\mathbf{A}$ are the ones defined in the previous section.

Remark 4.2. Notice that conditions (M) and (A) in the definition above can be equivalently expressed as follows:

(M) $(\geq \circ R_\Box) \subseteq R_\Box$

(A) $(\leq \circ R_\Diamond) \subseteq R_\Diamond$

where $\circ$ denotes the composition of relations. Since the converse inclusions always hold, these conditions can be equivalently expressed as identities as follows:

(M) $(\geq \circ R_\Box) = R_\Box$

(A) $(\leq \circ R_\Diamond) = R_\Diamond$.

Now, let $(\mathbf{F}, R_\Box, R_\Diamond)$ be any forest frame and let us define the following two binary relations on F:

$$R'_\Box = R_\Box \circ \geq \text{ and } R'_\Diamond = R_\Diamond \circ \leq . \tag{6}$$

In other words, for all $x, y \in F$, $xR'_\Box y$ iff there exists $z \in F$ such that $xR_\Box z$ and $z \geq y$. Analogously, $xR'_\Diamond y$ iff there exists $z \in F$ such that $xR_\Diamond z$ and $z \leq y$. Then, the following holds.

Proposition 4.3. *For every forests frame* $(\mathbf{F}, R_\Box, R_\Diamond)$ *the following conditions hold:*

1. $(\mathbf{F}, R'_\Box, R'_\Diamond)$ *is a forest frame;*

2. $R'_\Box(x) = \downarrow R_\Box(x)$ *and* $R'_\Diamond(x) = \uparrow R_\Diamond(x)$*;*

3. $(\geq \circ R'_\Box \circ \geq) = R'_\Box$ *and* $(\leq \circ R'_\Diamond \circ \leq) = R'_\Diamond$*.*

Proof. (1) Let $x, y, z \in F$ such that $y \leq x$ and $xR'_\Diamond z$. Then, there exists $w \in F$ such that $xR_\Diamond w$ and $w \leq z$. Since $R_\Diamond$ satisfies (A), $yR_\Diamond w$. From, $yR_\Diamond w$ and $w \leq z$, we get $yR'_\Diamond z$.

Let $x, y, z \in F$ such that $x \leq y$ and $xR'_\Box z$. Then, there exists $w \in F$ such that $xR_\Box w$ and $z \leq w$. Since $R_\Box$ satisfies (M), $yR_\Box w$. From, $yR_\Box w$ and $z \leq w$, we get $yR'_\Box z$.

(2) It follows from the definition of $R'_\Diamond$ and $R'_\Box$.

(3) The inclusion $R'_\Box \subseteq (\geq \circ R'_\Box \circ \geq)$ is immediate. Let $x, y, z, w \in F$ such that $x \geq y$, $yR'_\Box z$ and $z \geq w$. We will prove that $xR'_\Box w$. Since $R'_\Box$ satisfies (M), $xR'_\Box z$. Since $R'_\Box(x)$ is an downset of $\mathbf{F}$ we get that $xR'_\Box w$.

The inclusion $R'_\Diamond \subseteq (\leq \circ R'_\Diamond \circ \leq)$ is immediate. Let $x, y, z, w \in F$ such that $x \leq y$, $yR'_\Diamond z$ and $z \leq w$. We will prove that $xR'_\Diamond w$. Since $R'_\Diamond$ satisfies (A), $xR'_\Diamond z$. Since $R'_\Diamond(x)$ is an upset of $\mathbf{F}$ we get that $xR'_\Diamond w$. □

For every forest frame $(\mathbf{F}, R_\Box, R_\Diamond)$, let $\mathbf{G}(\mathbf{F})$ be the Gödel algebra of downsets of $\mathbf{F}$ and let the maps $\beta, \delta : G(\mathbf{F}) \to G(\mathbf{F})$ be as in the previous section: for every $a \in G(\mathbf{F})$

$$\beta(a) = \{y \in F \mid \forall z \in F,\ (yR_\Box z \Rightarrow z \in a)\}. \tag{7}$$

$$\delta(a) = \{y \in F \mid \exists z \in a,\ yR_\Diamond z\}. \tag{8}$$

Notation 1. Along this section, we will make use of subscripts to distinguish a binary relation R from those that we will write R', R'', etc. More precisely, we will use symbols $R_\Box$, $R_\Diamond$ as usual for the binary relations of a forest frame and $R'_\Box$, $R'_\Diamond$, $R''_\Box$, $R''_\Diamond$ for derived binary relations on the same forest. In addition, we will denote by β', δ' and β'', δ'' the operations on $\mathbf{G}(\mathbf{F})$ defined through (7) and (8) and by the relations $R'_\Box$, $R'_\Diamond$ and $R''_\Box$, $R''_\Diamond$ respectively.

Lemma 4.4. *Let* $(\mathbf{F}, R_\Box, R_\Diamond)$ *be a forest frame, then*

1. $\delta(a) = \delta'(a)$ *for all* $a \in G(\mathbf{F})$*.*

2. $\beta(a) = \beta'(a)$ *for all* $a \in G(\mathbf{F})$*.*

Proof. (1) We will prove $\delta'(a) \subseteq \delta(a)$. The other inclusion follows immediately. Let $y \in \delta'(a)$. Then, there exists $z \in a$ such that $yR'_\Diamond z$. From definition of $R'_\Diamond$ there exists $w \in F$ such that $yR_\Diamond w$ and $w \leq z$. Since $a \in G(\mathbf{F})$, a is a downward closed subset of $\mathbf{F}$ and we get that $w \in a$. Therefore $y \in \delta(a)$.

(2) We will prove $\beta(a) \subseteq \beta'(a)$. The other inclusion follows immediately. Let $y \in \beta(a)$. Let $z \in F$ such that $yR'_\Box z$, we will prove that $z \in a$. From definition of $R'_\Box$ there exists $w \in F$ such that $yR_\Box w$ and $z \leq w$. Then, $w \in a$ and since $a \in G(\mathbf{F})$, a is a downward closed subset of $\mathbf{F}$ and we get that $z \in a$. Therefore $y \in \beta'(a)$. □

From the previous Lemma we get that the forest frames $(\mathbf{F}, R_\Box, R_\Diamond)$ and $(\mathbf{F}, R'_\Box, R'_\Diamond)$ induce the same Gödel algebra with operators, i.e., $(\mathbf{G}(\mathbf{F}), \beta, \delta) = (\mathbf{G}(\mathbf{F}), \beta', \delta')$.

In [20] Orłowska and Rewitzky defined a class of relational frames, based on posets, being a dual semantics for Heyting algebras with operators. Our interest now is to compare forest frames with them. For this, we will define Orłowska and Rewitzky frames on forests as follows.

Definition 4.5. An *Orłowska-Rewitzky frame* (or *OR-frame* for short) is a triple $(\mathbf{F}, R_\Box, R_\Diamond)$ where $\mathbf{F}$ is a forest and $R_\Box, R_\Diamond \subseteq F \times F$ satisfy the following conditions:

(OR1) $(\geq \circ R_\Box \circ \geq) \subseteq R_\Box$,

(OR2) $(\leq \circ R_\Diamond \circ \leq) \subseteq R_\Diamond$.

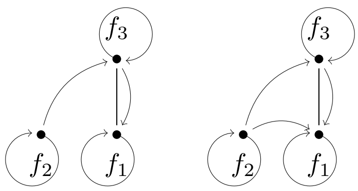

Figure 2: From left to right: The frame $(\{f_1, f_2, f_3\}, \leq, R_\Box)$; the frame $(\{f_1, f_2, f_3\}, \leq, R'_\Box)$.

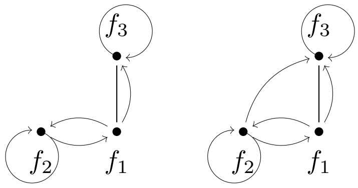

Figure 3: From left to right: The frame $(\{f_1, f_2, f_3\}, \leq, R_\Diamond)$; the frame $(\{f_1, f_2, f_3\}, \leq, R'_\Diamond)$.

The next result is hence a direct consequence of Proposition 4.3 and Lemma 4.4.

Corollary 4.6. *Let* $(\mathbf{F}, R_\Box, R_\Diamond)$ *be a forest frame. Then,* $(\mathbf{F}, R'_\Box, R'_\Diamond)$ *is a OR-frame. Moreover,* $(\mathbf{G}(\mathbf{F}), \beta, \delta) = (\mathbf{G}(\mathbf{F}), \beta', \delta')$.

Note that every OR-frame is a forest frame, but the converse is not always the case. In the following example we will show a forest frame that is not a OR-frame.

Example 4.7. Consider the following forest frame $(\{f_1, f_2, f_3\}, \leq, R_\Box, R_\Diamond)$ where

$$R_\Box = \{(f_1, f_1), (f_2, f_3), (f_2, f_2), (f_3, f_1), (f_3, f_3)\}$$

as we see in Figure 2 and

$$R_\Diamond = \{(f_1, f_2), (f_1, f_3), (f_2, f_2), (f_2, f_1), (f_3, f_3)\}$$

as we see in Figure 3. If we compute $R'_\Box$ and $R'_\Diamond$ as in (6) we get that:

$$R'_\Box = \{(f_1, f_1), (f_2, f_3), (f_2, f_2), (f_2, f_1), (f_3, f_1), (f_3, f_3)\}.$$

and

$$R'_\Diamond = \{(f_1, f_2), (f_1, f_3), (f_2, f_2), (f_2, f_1), (f_2, f_3), (f_3, f_3)\}.$$

Summing up, what we presented so far, shows that OR-frames form a class of relational frames strictly contained in that of forest frames. However, for every forest frame $(\mathbf{F}, R_\Box, R_\Diamond)$, it is always possible to define an OR-frame $(\mathbf{F}, R'_\Box, R'_\Diamond)$ based on the same forest $\mathbf{F}$ such that they define the same GAO $(\mathbf{G}(\mathbf{F}), \beta, \delta)$.

Now, we turn our attention on the relational frames introduced by Palmigiano in [21]. Again, we will consider the particular case of relational structures based on forests, rather than the more general case studied in [21].

Definition 4.8. A *Palmigiano frame* (or *P-frame* for short), is a triple $(\mathbf{F}, R_\Box, R_\Diamond)$ where $\mathbf{F}$ is a forest and $R_\Box, R_\Diamond \subseteq F \times F$ satisfy the following conditions:

(P1) $(\geq \circ R_\Box) \subseteq (R_\Box \circ \geq)$;

(P2) $(\leq \circ R_\Diamond) \subseteq (R_\Diamond \circ \leq)$.

Our next result shows that P-frames include forest frames.

Proposition 4.9. *Every forest frame* $(\mathbf{F}, R_\Box, R_\Diamond)$*is a P-frame.*

Proof. Let $(\mathbf{F}, R_\Box, R_\Diamond)$ be a forest frame. It is easy to see that $(\geq \circ R_\Box) \subseteq R_\Box \subseteq (R_\Box \circ \geq)$ and $(\leq \circ R_\Diamond) \subseteq R_\Diamond \subseteq (R_\Diamond \circ \leq)$ and the result follows. □

So, in particular we have that every OR-frame is a forest frame and every forest frame is a P-frame over the same ordered set. So we have an inclusion of frame classes.

Now, given a P-frame $(\mathbf{F}, R_\Box, R_\Diamond)$, consider the following relations $R''_\Box, R''_\Diamond \subseteq F \times F$ defined by:

$$R''_\Box = (\geq \circ R_\Box) \text{ and } R''_\Diamond = (\leq \circ R_\Diamond). \tag{9}$$

Proposition 4.10. *Let* $(\mathbf{F}, R_\Box, R_\Diamond)$ *be a P-frame. Then,* $(\mathbf{F}, R''_\Box, R''_\Diamond)$ *is a forest frame such that*

1. $\delta(a) = \delta''(a)$ *for all* $a \in G(\mathbf{F})$.
2. $\beta(a) = \beta''(a)$ *for all* $a \in G(\mathbf{F})$.

Proof. It is immediate to see that $(\mathbf{F}, R''_\Box, R''_\Diamond)$ is a forest frame.

1. Let $a \in G(\mathbf{F})$. Since $R_\Diamond \subseteq R''_\Diamond$, $\delta(a) \subseteq \delta''(a)$. Let $x \in \delta''(a)$. Then, $R''_\Diamond(x) \cap a \neq \emptyset$. Let $y \in R''_\Diamond(x)$ such that $y \in a$. Since $(\mathbf{F}, R_\Box, R_\Diamond)$ is a P-frame we have that $R''_\Diamond \subseteq (R_\Diamond \circ \leq)$. So, there exists $z \in F$ such that $xR_\Diamond z$ and $z \leq y$. Thus, since a is a downset, $z \in a$. Therefore $R_\Diamond(x) \cap a \neq \emptyset$ and $\delta''(a) \subseteq \delta(a)$.

2. Let $a \in G(\mathbf{F})$. Since $R_\Box \subseteq R''_\Box$, $\beta''(a) \subseteq \beta(a)$. Let $x \in \beta(a)$. Then, $R_\Box(x) \subseteq a$. Let $y \in R''_\Box(x)$. Since $(\mathbf{F}, R_\Box, R_\Diamond)$ is a P-frame, we have that $R''_\Box \subseteq (R_\Box \circ \geq)$. Therefore, there exists $z \in F$ such that $xR_\Box z$ and $y \leq z$. Thus, $z \in a$ and since a is a downset, $y \in a$. Therefore $R''_\Box(x) \subseteq a$ and $\beta(a) \subseteq \beta''(a)$. □

The next example shows that forest frames form a proper subclass of P-frames. Thus, together with the above results and Example 4.7, we have that OR-frames are strictly contained in the class of forest frames that, in turn, are strictly contained in P-frames.

Example 4.11. Consider the forest $\mathbf{F}$ (tree) depicted as in Figure 4. Consider the

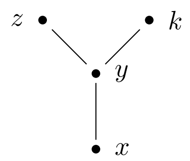

Figure 4: A tree with 4 points used in Example 4.11.

following relations $R_\Box$ and $R_\Diamond$ on F:

$$R_\Box = \{(x, y), (y, z), (z, z), (k, z)\}$$

and

$$R_\Diamond = \{(x, x), (y, y)\}$$

Now, we compute the following composed relations:

$$(\geq \circ R_\Box) = \{(x, y), (y, y), (y, z), (z, y), (z, z), (k, y), (k, z)\}$$

and

$$(R_\Box \circ \geq) = \{(x, x), (x, y), (y, x), (y, y), (y, z), (z, x), (z, y), (z, z), (k, x), (k, y), (k, z)\}.$$

Moreover, we have:

$$(\leq \circ R_\Diamond) = \{(x, x), (x, y), (y, y)\}$$

and

$$(R_\Diamond \circ \leq) = \{(x, x), (x, y), (x, z), (x, k), (y, y), (y, z), (y, k)\}.$$

This shows that $(\mathbf{F}, R_\Box, R_\Diamond)$ is a P-frame but it is not a forest frame.

Now, we end this section with a (graphical) comparison between the characterizing properties for $R_\Box$ and $R_\Diamond$ of OR-frames, forest-frames and P-frames.

Remark 4.12. So far, we have considered three kind of relational structures based on forests, namely forest frames, OR-frames, and P-frames. For the next comparison, let us recall what properties are asked for the binary relations $R_\Box$ an $R_\Diamond$ in each of the aforementioned models.

As for forest frames, we have the following two properties to be satisfied by $R_\Box$ and $R_\Diamond$ respectively.

(M) for all $x, y, z \in F$, if $x \leq y$ and $xR_\Box z$, then $yR_\Box z$;

(A) for all $x, y, z \in F$, if $y \leq x$ and $xR_\Diamond z$, then $yR_\Diamond z$.

As for OR-frames and P-frames, let us express (OR1), (OR2), (P1) and (P2) by first-order formulas as follows:

(OR1) for all $x, y \in F$, if there exist $z, w \in F$ such that $x \geq z$, $zR_\Box w$ and $w \geq y$, then $xR_\Box y$.

(OR2) for all $x, y \in F$, if there exist $z, w \in F$ such that $x \leq z$, $zR_\Diamond w$ and $w \leq y$, then $xR_\Diamond y$.

(P1) for all $x, y \in F$, if there exists $z \in F$ such that $x \geq z$ and $zR_\Box y$, then there exists $w \in F$ such that $xR_\Box w$ and $w \geq y$.

(P2) for all $x, y \in F$, if there exists $z \in F$ such that $x \leq z$ and $zR_\Diamond y$, then there exists $w \in F$ such that $xR_\Diamond w$ and $w \leq y$.

Figures 5 and 6 below present a graphical representation for the properties recalled above. The dashed arrows will represent the final relation corresponding to the right-hand side of the above quasi-equations. Also, for a point a, we will label it by $\exists a$ (instead of simply a) to highlight that the existence of a is ensured by the left-hand side of the quasi-equations above. In particular, this is the case of (P1) and (P2).

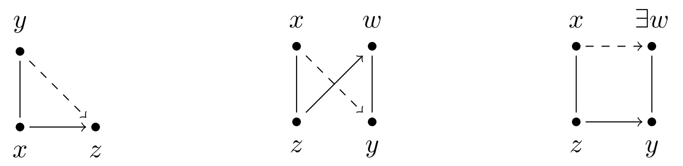

Figure 5: A graphical representation of the properties (M) (left-hand side), (OR1) (center) and (P1) (right-hand side).

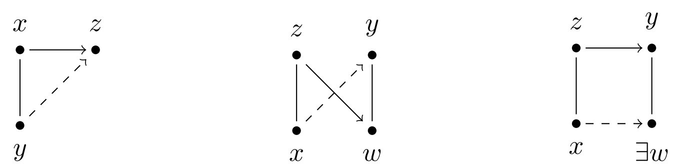

Figure 6: A graphical representation of the properties (A) (left-hand side), (OR2) (center) and (P2) (right-hand side).

5 Adding structure to Gödel algebras with operators

In this section we will be concerned with two extensions of Gödel algebras with operators and their forest frame semantics. The first one is the subvariety $\mathbb{DGAO}$ of $\mathbb{GAO}$ obtained by the equations (D1) and (D2) below:

$(D1)$ $\Box(a \vee b) \leq \Box a \vee \Diamond b$;

$(D2)$ $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.

The above axioms have been firstly studied by Dunn in [11] in the logical setting of the positive fragment of classical (or intuitionistic) logic. The algebras in $\mathbb{DGAO}$ will be henceforth called *Dunn* GAOs.

Before moving to the second variety, let us show an interesting property of Dunn GAOs, the fact that the modal operators are closed on the set of Boolean elements algebras in $\mathbb{DGAO}$. Let $(\mathbf{A}, \Box, \Diamond)$ be a GAO and let us consider the set of Boolean elements of $\mathbf{A}$:

$$B(\mathbf{A}) = \{x \in A : x \vee \neg x = \top\}.$$

Then the following holds.

Proposition 5.1. *Let $(\mathbf{A}, \Box, \Diamond)$ be a Dunn GAO, then $\Box a, \Diamond a \in B(\mathbf{A})$ for all $a \in B(\mathbf{A})$.*

Proof. Let $a \in B(\mathbf{A})$. Then, $\top = \Box(a \vee \neg a) \leq \Box a \vee \Diamond(\neg a)$. Also, $\Box a \wedge \Diamond(\neg a) \leq \Diamond(a \wedge \neg a) = \bot$. So, we get that $\Diamond(\neg a) \leq \neg\Box a$. Therefore, $\top = \Box a \vee \Diamond(\neg a) \leq \Box a \vee \neg\Box a$. And thus $\Box a \in B(\mathbf{A})$.

On the other hand, let $a \in B(\mathbf{A})$. Then, $\top = \Box(\neg a \vee a) \leq \Box(\neg a) \vee \Diamond a$. Also, $\Box(\neg a) \wedge \Diamond a \leq \Diamond(a \wedge \neg a) = \bot$. So, we get that $\Box(\neg a) \leq \neg\Diamond a$. Therefore, $\top = \Box(\neg a) \vee \Diamond a \leq \neg\Diamond a \vee \Diamond a$. And thus $\Diamond a \in B(\mathbf{A})$. □

The second is the variety $\mathbb{FSGAO}$ given by the well-known Fischer Servi axioms (FS1) and (FS2):

$(FS1)$ $\Diamond(a \to b) \leq (\Box a \to \Diamond b)$;

$(FS2)$ $(\Diamond a \to \Box b) \leq \Box(a \to b)$.

Algebras in $\mathbb{FSGAO}$ will be called *Fischer Servi* GAOs.

In [17] it has been proved that (D2) and (FS1) are equivalent. For the sake of completeness, we provide another proof of this fact in the result below.

Proposition 5.2. *Every GAO $(\mathbf{A}, \Box, \Diamond)$ satisfies (D2) iff it satisfies (FS1).*

Proof. Let us start assuming that $(\mathbf{A}, \Box, \Diamond)$ satisfies (D2), that is, for all $a, b \in A$, $\Box a \wedge \Diamond b \leq \Box(a \wedge b)$. Let us show that $(\mathbf{A}, \Box, \Diamond)$ thus satisfies (FS2): $\Diamond(a \to b) \leq \Box a \to \Diamond b$. The latter, by residuation, is equivalent to $\Diamond(a \to b) \wedge \Box a \leq \Diamond b$. Now, by (D2) $\Box a \wedge \Diamond(a \to b) \leq \Diamond(a \wedge (a \to b))$ and the latter equals $\Diamond(a \wedge b) \leq \Diamond b$.

Conversely, let us assume that $(\mathbf{A}, \Box, \Diamond)$ satisfies (FS1), that is, for all $a, b \in A$, $\Diamond(a \to b) \leq \Box a \to \Diamond b$. By residuation, $\Diamond b \leq \Diamond(a \to (a \wedge b))$ and by (FS1), $\Diamond(a \to (a \wedge b)) \leq \Box a \to \Diamond(a \wedge b)$. Thus, we conclude that $\Diamond b \wedge \Box a \leq \Diamond(a \wedge b)$. □

A full comparison between Dunn's and Fischer Servi's axioms, at best of our knowledge, has not be presented. This section is hence dedicated to a comparison between these two axiom schema.

By Corollary 3.7, it is clear that each GAO in either $\mathbb{DGAO}$ or $\mathbb{FSGAO}$ has an isomorphic representation, within the same classes $\mathbb{DGAO}$ and $\mathbb{FSGAO}$ respectively, in the sense of Theorem 3.6. However, in these particular cases, it is possible to consider special forest frames with only one accessibility relation that, equivalently to the forest frames studied in Subsection 4, allows to recover, up to isomorphism, the Gödel algebra with operators we started with. Let us hence introduce the following.

Definition 5.3. A *basic-frame* is a pair $(\mathbf{F}, R)$ where $\mathbf{F}$ is a finite forest, $R \subseteq F \times F$ and there exist $R_\Box, R_\Diamond \subseteq F \times F$ such that:

(1) $R_\Box$ satisfies (M) and $R_\Diamond$ satisfies (A);

(2) $R = R_\Box \cap R_\Diamond$.

Clearly, every forest frame defines a basic-frame by taking $R = R_\Box \cap R_\Diamond$ and, vice-versa, is $(\mathbf{F}, R)$ is a basic-frame, then $(\mathbf{F}, R_\Box, R_\Diamond)$ is a forest frame.

In order for our next result to be clear, let us introduce the following notation: let $(\mathbf{A}, \Box, \Diamond)$ be a GAO. According with the notation used in the previous section, let us denote by $\mathbf{F}_\mathbf{A}$ the forest of its prime filters and by $R_\Box$ and $R_\Diamond$ the binary relations on $F_\mathbf{A}$ defined as in (1) and (2) respectively. Let $R_\Box$ and $R_\Diamond$ be as in Section 4, and put $R = R_\Box \cap R_\Diamond$. Then, by Proposition 4.3 (1), it follows that $(\mathbf{F}_\mathbf{A}, R)$ is a basic frame.

Then, let us call β and δ as in Subsection 4, while β_R and δ_R will denote the unary maps $\mathbf{G}(\mathbf{F}_\mathbf{A}) \to \mathbf{G}(\mathbf{F}_\mathbf{A})$ defined as in (7) and (8) where R replaces $R_\Box$ and $R_\Diamond$ respectively.

As the following result shows, basic and forest frames are *equivalent* in the sense that they define the same Dunn GAO in the isomorphic representation theorem.

Theorem 5.4. *Let $(\mathbf{A}, \Box, \Diamond)$ be any Dunn GAO. Then*

$$(\mathbf{A}, \Box, \Diamond) \cong (\mathbf{G}(\mathbf{F}_\mathbf{A}), \beta, \delta) \cong (\mathbf{G}(\mathbf{F}_\mathbf{A}), \beta_R, \delta_R)$$

via the isomorphism r. Thus, in particular, for all $a \in A$,

$$r(\Box a) = \beta(r(a)) = \beta_R(r(a))$$

and

$$r(\Diamond a) = \delta(r(a)) = \delta_R(r(a)).$$

Proof. For $\mathbf{A}$ being any Gödel algebra, let us denote by $\mathbf{A}^-$ its $\{\to, \neg\}$-free reduct. Then, if $(\mathbf{A}, \Diamond, \Box)$ satisfies $(D1)$ and $(D2)$, $(\mathbf{A}^-, \Diamond, \Box)$ is a *positive modal algebra* in the sense of [11, 9]. Since the set of prime filters of $\mathbf{A}$ and that of $\mathbf{A}^-$ coincide, $\mathbf{F}_{\mathbf{A}^-} = \mathbf{F}_\mathbf{A}$ and, following [9], let us define $R_\mathbf{A} \subseteq F_{\mathbf{A}^-} \times F_{\mathbf{A}^-}$ as follows: for all $f_1, f_2 \in F_\mathbf{A}$,

$$R_\mathbf{A}(f_1, f_2) \text{ iff } \Box^{-1}(f_1) \subseteq f_2 \subseteq \Diamond^{-1}(f_1).$$

Since $f_2 \subseteq \Diamond^{-1}(f_1)$ iff $\Diamond(f_2) \subseteq f_1$, by [9, Lemma 2.1(1)], we have that $R_\mathbf{A} = R_\Box \cap R_\Diamond$, where $R_\Box$ and $R_\Diamond$ are defined as usual.

Now, let $\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}}$ be the Gödel algebra of subforests of $\mathbf{F}_{\mathbf{A}^-}$ and define $\delta_{R_\mathbf{A}}$ and $\beta_{R_\mathbf{A}}$ on $S_{\mathbf{F}_{\mathbf{A}^-}}$ by (8) and (7) respectively. Then, [9, Theorem 2.2] (see also [11, Theorem 8.1]), shows that $(\mathbf{A}^-, \Diamond, \Box)$ and the positive algebra $((\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}})^-, \delta_{R_\mathbf{A}}, \beta_{R_\mathbf{A}})$ are isomorphic (as positive modal algebras).

Since $\mathbf{F}_\mathbf{A} = \mathbf{F}_{\mathbf{A}^-}$, one has that $\mathbf{S}_{\mathbf{F}_{\mathbf{A}^-}} = \mathbf{G}(\mathbf{F}_\mathbf{A})$. Now, it is not difficult to extend the above result to Dunn GAOs by expanding the positive modal algebra $((\mathbf{S}_{\mathbf{F}_\mathbf{A}})^-, \delta_{R_\mathbf{A}}, \beta_{R_\mathbf{A}})$ by the operator $\to$ defined as in Section 2: for all $x, y \in S_{\mathbf{F}_\mathbf{A}}$,

$$x \to y = (\uparrow(x \setminus y))^c = F_\mathbf{A} \setminus \uparrow(x \setminus y).$$

Then, $(\mathbf{S}_{\mathbf{F}_\mathbf{A}})^-$ plus $\to$ and $\neg$ (defined as usual by $\neg x = x \to \emptyset$) is a Gödel algebra isomorphic to $\mathbf{G}(\mathbf{F}_\mathbf{A})$. $\Box$

The case of Fischer Servi GAOs is similar and indeed the same basic frames are enough to construct the isomorphic copy of any algebra belonging to $\mathbb{FSGAO}$. The unique necessary modification consists in considering, in place of β_R, the map $\beta_{(\geq \circ R)} : \mathbf{G}(\mathbf{F}_\mathbf{A}) \to \mathbf{G}(\mathbf{F}_\mathbf{A})$ as in (7), but adopting the composed relation $\geq \circ R$ instead of $R_\Box$.

In order to prove that basic frames allows to show a forest-based representation result for Fischer Servi GAOs, let us recall [20, Theorem 4.1] that has been proved in the more general setting of Heyting algebras with operators that satisfy Fischer Servi equations. In the aforementioned paper, these latter algebras have been denoted by *HK1-algebras.*

Theorem 5.5. *Every HK1-algebra $\mathbf{W}$ is embeddable into the complex algebra of its canonical frame $C(X(W))$ through the mapping $h : W \to C(X(W))$ defined as $h(w) = \{F \in X(W) : w \in F\}$.*

Now, as we recalled above, HK1-algebras are the Heyting analogues of our Fisher-Servi GAOs, so every $(\mathbf{A}, \Box, \Diamond) \in \mathbb{FSGAO}$ is a HK1-algebra in particular. Moreover, the canonical frame of $(\mathbf{A}, \Box, \Diamond)$ is exactly the basic frame $(\mathbf{F_A}, R_\mathbf{A})$, where $\mathbf{F_A}$ is the forest of its prime filters and $R_\mathbf{A}$ is the binary relation on $F_\mathbf{A}$ defined as usual. Finally, still following [20], it is immediate to see that the complex algebra of $(\mathbf{F_A}, R_\mathbf{A})$ is exactly the Fischer Servi GAO $(\mathbf{G}(\mathbf{F_A}), \beta_{\geq \circ \mathbf{R_A}}, \delta_{\mathbf{R_A}})$ and the mapping h is the same as the mapping r.

Therefore, according to the above theorem, every finite Fischer Servi GAO $(\mathbf{A}, \Box, \Diamond)$ can be embedded (as HK1-algebra) into $(\mathbf{G}(\mathbf{F_A}), \beta_{\geq \circ R_\mathbf{A}}, \delta_{R_\mathbf{A}})$ by means of r. Now, since $(\mathbf{A}, \Box, \Diamond)$ is indeed a Fischer Servi GAO, then $(\mathbf{G}(\mathbf{F_A}), \beta_{\geq \circ R_A}, \delta_{R_A})$ must be a GAO as well, and since $\mathbf{A}$ is finite, according to Lemma 2.1, r is an isomorphism. Therefore we have the following theorem.

Theorem 5.6. *Let* $(\mathbf{A}, \Box, \Diamond)$ *be any Fischer Servi GAO. Then*

$$(\mathbf{A}, \Box, \Diamond) \cong (\mathbf{G}(\mathbf{F_A}), \beta_{(\geq \circ R_A)}, \delta_{R_A})$$

via the isomorphism r.

The next result is meant to show what is the effect on the relations of the associated frames, of Dunn and Fischer Servi equations once added to Gödel algebras with operators.

Theorem 5.7. *Let* $(\mathbf{A}, \Box, \Diamond)$ *be a GAO, let* $(\mathbf{F_A}, R_\Box, R_\Diamond)$ *be the dual frame, and let* $R_A = R_\Box \cap R_\Diamond$. *Then the following conditions hold:*

1. $(\mathbf{A}, \Box, \Diamond)$ *satisfies (D1) iff* $R_\Box = R_A \circ \geq$.
2. $(\mathbf{A}, \Box, \Diamond)$ *satisfies (D2) iff* $R_\Diamond = R_A \circ \leq$.
3. $(\mathbf{A}, \Box, \Diamond)$ *satisfies (FS2) iff* $R_\Box = \geq \circ R_A$.

Proof. The claims concerning left-to-right implications in (2) and (3) have been proved in [21, Lemma 5.4]. In particular, part of claim (2) is proved by [21, Lemma 5.4 (2)] together with Proposition 5.2 above concerning the equivalence between (D2) and (FS1).

Concerning (1), the claim has been stated in [9] without proof. We are hence going to show it here.

(Left-to-right). Let us start observing that the inclusion $R_\Box \supseteq R_A \circ \geq$ is straightforward. Let us hence prove that $R_\Box \subseteq R_A \circ \geq$.

Let $f_1 R_\Box f_2$ and let us prove that there exists $f_3 \in F_\mathbf{A}$ such that

$$\Box^{-1}(f_1) \subseteq f_3 \subseteq \Diamond^{-1}(f_1) \text{ and } f_2 \leq f_3.$$

Let $i = Id((f_2)^c \cup (\Diamond^{-1}(f_1))^c)$ be the ideal generated by $(f_2)^c \cup (\Diamond^{-1}(f_1))^c$ and let us prove that $\Box^{-1}(f_1) \cap i = \emptyset$. By way of contradiction, assume that $a \in \Box^{-1}(f_1) \cap i$. Thus, in particular, $a \in \Box^{-1}(f_1)$, that is to say, $\Box a \in f_1$. Moreover, $a \in i$, whence there exists $c \in (f_2)^c$ and $b \in (\Diamond^{-1}(f_1))^c$ such that $a \leq c \vee b$. Therefore, by the monotonicity of $\Box$ and (D1), $\Box a \leq \Box(c \vee b) \leq \Box c \vee \Diamond b$. Now, since $\Box a \in f_1$ and f_1 is a filter, $\Box c \vee \Diamond b \in f_1$ as well. Notice that $\Box c \notin f_1$. Indeed, since $c \notin f_2$ and $f_2 \supseteq \Box^{-1}(f_1)$, $\Box c \notin f_1$. However, f_1 is prime and then $\Diamond b \in f_1$ that is absurd. Thus, $\Box^{-1}(f_1) \cap i = \emptyset$.

By Birkhoff prime filter theorem, there exists a prime filter g such that $\Box^{-1}(f_1) \subseteq g$ and $g \cap i = \emptyset$. Therefore, $g \subseteq f_2$, meaning that in $\mathbf{F_A}$, $f_2 \leq g$, and $g \subseteq \Diamond^{-1}(f_1)$. Thus the claim is completed by taking $g = f_3$.

(Right-to-left). Let us assume, by way of contradiction, that $R_\Box = R_\mathbf{A} \circ \geq$ and let $a, b \in A$ such that $\Box(a \vee b) \not\leq \Box a \vee \Diamond b$. Then, there exists a prime filter f such that $\Box(a \vee b) \in f$ and $\Box a \vee \Diamond b \notin f$. Then, $\Box(a \vee b) \in f$ implies that $a \vee b \in \Box^{-1}(f)$, while $\Box a \vee \Diamond b \notin f$ entails that, in particular,

$$a \notin \Box^{-1}(f) \text{ and } b \notin \Diamond^{-1}(f). \tag{10}$$

Since $\Box^{-1}(f)$ is a filter and $a \notin \Box^{-1}(f)$, there is a prime filter g such that $g \supseteq \Box^{-1}(f)$ and $a \notin g$. In particular, $g \supseteq \Box^{-1}(f)$ implies that $fR_\Box g$. By hypothesis $R_\Box = R_{\mathbf{A}} \circ \geq$, and since $fR_\Box g$, one has that there exists a prime filter h such that $\Box^{-1}(f) \subseteq h \subseteq \Diamond^{-1}(f)$ and $h \subseteq g$. Since $a \notin g$, and $h \subseteq g$, $a \notin h$. However, $a \vee b \in \Box^{-1}(f) \subseteq h$, whence $b \in h \subseteq \Diamond^{-1}(f)$ contradicting (10).

Now, in order to prove the right-to-left claim (2), let us assume that $R_\Diamond = R_A \circ \leq$ and let $a, b \in A$. We want to prove that $\Diamond(a \to b) \leq \Box a \to \Diamond b$ or equivalently $\Diamond(a \to b) \wedge \Box a \leq \Diamond b$. So, let us consider the filter $f = \uparrow(\Diamond(a \to b) \wedge \Box a)$, and we will prove $\Diamond b \in f$. Suppose $\Diamond b \notin f$. Then, there exists a prime filter f_1 such that $f \subseteq f_1$ and $\Diamond b \notin f_1$. Then, $a \to b \in \Diamond^{-1}(f_1)$ and $a \in \Box^{-1}(f_1)$. By Lemmas [21, 3.5 and (a) of 3.2], there exists a prime filter f_2 such that $a \to b \in f_2 \subseteq \Diamond^{-1}(f_1)$. So, $f_1 R_\Diamond f_2$ and, by assumption, there exists a prime filter f_3 such that $\Box^{-1}(f_1) \subseteq f_3 \subseteq \Diamond^{-1}(f_1)$ and $f_2 \subseteq f_3$. Since $a \to b \in f_2 \subseteq f_3$ and $a \in \Box^{-1}(f_1) \subseteq f_3$, we get that $a, a \to b \in f_3$ and therefore $b \in f_3$. From $b \in f_3 \subseteq \Diamond^{-1}(f_1)$ we obtain $\Diamond b \in f_1$ which is a contradiction. Therefore, $\Diamond b \in f$.

To prove the right-to-left claim (3), let us assume that $R_\Box = \geq \circ R_A$ and let $a, b \in A$. We will prove that $\Diamond a \to \Box b \leq \Box(a \to b)$. So, let us consider a prime filter f_1 such that $\Diamond a \to \Box b \in f_1$ and suppose that $\Box(a \to b) \notin f_1$. Since $\Box^{-1}(f_1)$ is a filter, $b \notin Fi(\Box^{-1}(f_1) \cup \{a\})$, where $Fi(\Box^{-1}(f_1) \cup \{a\})$ is the filter generated by $\Box^{-1}(f_1) \cup \{a\}$. Then, there exists a prime filter f_2 such that $\Box^{-1}(f_1) \subseteq f_2$, $a \in f_2$ and $b \notin f_2$. By assumption, since $f_1 R_\Box f_2$, there exists a prime filter f_3 such that $f_1 \subseteq f_3$ and $\Box^{-1}(f_3) \subseteq f_2 \subseteq \Diamond^{-1}(f_3)$. Thus, from $\Diamond a \to \Box b \in f_1 \subseteq f_3$ and $a \in f_2 \subset \Diamond^{-1}(f_3)$ we get $\Diamond a \to \Box b, \Diamond a \in f_3$. We can imply that $\Box b \in f_3$ and hence $b \in \Box^{-1}(f_3) \subseteq f_2$ which is a contradiction because $b \notin f_2$. Therefore $b \in Fi(\Box^{-1}(f_1) \cup \{a\})$, and we get $a \to b \in \Box^{-1}(f_1)$. ☐

Let $F = (\mathbf{X}, R^\Box, R^\Diamond)$ be a forest frame, where $\mathbf{X} = (X, \leq)$ and let $(\mathbf{G}(\mathbf{X}), \beta, \delta)$ be its associated GAO.

Lemma 5.8. *Let $(X, \leq)$ be a finite forest and let $F_{G(X)}$ be the set of prime filters of $\mathbf{G}(\mathbf{X})$. Define the mapping $k : X \to F_{G(X)}$ as follows: for any $x \in X$, by $k(x) = \{f \in G(X) \mid x \in f\}$. Then k is a bijective mapping such that $x \leq y$ iff $k(x) \supseteq k(y)$. Therefore, the forests $\mathbf{X} = (X, \leq)$ and $\mathbf{F_{G(X)}} = (F_{G(X)}, \supseteq)$ are isomorphic through the mapping k.*

Moreover, the mapping k preserves forest frames in the following sense.

Lemma 5.9. *Let $F = (\mathbf{X}, R^\Box, R^\Diamond)$ be a forest frame. Then:*

(1) if $xR^\Box y$ then $k(x) R^\Box_{G(X)} k(y)$.

(2) if $R^\Box \circ \geq = R^\Box$, then if $k(x) R^\Box_{G(X)} k(y)$ then $xR^\Box y$.

(3) if $xR^\Diamond y$ then $k(x) R^\Diamond_{G(X)} k(y)$.

(4) if $R^\Diamond \circ \leq = R^\Diamond$, then if $k(x) R^\Diamond_{G(X)} k(y)$ then $xR^\Diamond y$.

Proof. First of all, note that, by definition:

$$\begin{aligned} k(x) R^\Box_{G(X)} k(y) &\text{ iff } \beta^{-1}(k(x)) \subseteq k(y) \\ &\text{ iff } \forall f \in G(X),\ x \in \beta(f) \text{ implies } y \in f \\ &\text{ iff } \forall f \in G(X),\ (\forall z)(xR^\Box z \text{ implies } z \in f) \text{ implies } y \in f \end{aligned}$$

$$\begin{aligned} k(x) R^\Diamond_{G(X)} k(y) &\text{ iff } k(y) \subseteq \delta^{-1}(k(x)) \\ &\text{ iff } \forall f \in G(X),\ y \in f \text{ implies } x \in \delta(f) \\ &\text{ iff } \forall f \in G(X),\ y \in f \text{ implies } (\exists z)(xR^\Diamond z \text{ and } z \in f) \end{aligned}$$

(1) Assume $xR^\Box y$. Let $f \in G(X)$ such that $(\forall z)(xR^\Box z$ implies $z \in f)$. Hence, taking $z = y$, we get $y \in f$, and thus $k(x) R^\Box_{G(X)} k(y)$.
(2) Assume $k(x) R^\Box_{G(X)} k(y)$. Let $f = R^\Box(x)$. Since $R^\Box \circ \geq = R^\Box$, $R^\Box(x)$ is a down-subset of X, i.e. $R^\Box(x) \in G(X)$. Therefore, $y \in R^\Box(x)$, i.e. $xR^\Box y$.

(3) Assume $xR^{\Diamond}y$. Let $f \in G(X)$ such that $y \in f$. Then taking $z = y$, the condition $(\exists z)(xR^{\Diamond}z$ and $z \in f)$ is trivially satisfied. Thus $k(x)R^{\Diamond}_{G(X)}k(y)$.

(4) Assume $k(x)R^{\Diamond}_{G(X)}k(y)$. Let $f = \downarrow y = \{z \in X \mid z \leq y\}$. Clearly, $f \in G(X)$ and $y \in f$. Then $(\exists z)(xR^{\Diamond}z$ and $z \leq y)$. Since $R^{\Diamond} \circ \leq = R^{\Diamond}$, it follows that $xR^{\Diamond}y$. □

Corollary 5.10. *Let $F = (\mathbf{X}, R^{\Box}, R^{\Diamond})$ be a forest frame such that $R^{\Box} \circ \geq = R^{\Box}$ and $R^{\Diamond} \circ \leq = R^{\Diamond}$. Then the frame $F = (\mathbf{X}, R^{\Box}, R^{\Diamond})$ is isomorphic to the frame $(\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R^{\Box}_{G(X)}, R^{\Diamond}_{G(X)})$.*

Theorem 5.11. *Let $(\mathbf{X}, R_{\Box}, R_{\Diamond})$ be a forest frame. Let $R'_{\Box} = R_{\Box} \circ \geq$, $R'_{\Diamond} = R_{\Diamond} \circ \leq$ and $R' = R'_{\Box} \cap R'_{\Diamond}$. Then*

(i) $(\mathbf{G}(\mathbf{X}), \beta, \delta)$ satisfies (D1) iff $R'_{\Box} = R' \circ \geq$

(ii) $(\mathbf{G}(\mathbf{X}), \beta, \delta)$ satisfies (D2) iff $R'_{\Diamond} = R' \circ \leq$

(iii) $(\mathbf{G}(\mathbf{X}), \beta, \delta)$ satisfies (FS2) iff $R'_{\Box} = \geq \circ R'$

Proof. First of all, due to Lemma 4.4, notice that the GAO generated by a forest frame $(\mathbf{X}, R_{\Box}, R_{\Diamond})$, $(\mathbf{G}(\mathbf{X}), \beta, \delta)$, is the same a the one generated by the frame $(\mathbf{X}, R'_{\Box}, R'_{\Diamond})$, and hence we can apply Corollary 5.10 and get that $(\mathbf{X}, R'_{\Box}, R'_{\Diamond})$ is isomorphic to $(\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R^{\Box}_{G(X)}, R^{\Diamond}_{G(X)})$. Then, letting $R_{G(X)} = R^{\Box}_{G(X)} \cap R^{\Diamond}_{G(X)}$, we have:

(i) By Theorem 5.7, $(\mathbf{G}(\mathbf{X}), \beta, \delta)$ satisfies (D1) iff $R^{\Box}_{G(X)} = R_{G(X)} \circ \geq$, and this holds, by Corollary 5.10, iff $R'_{\Box} = R' \circ \geq$.

(ii) By Theorem 5.7, $(\mathbf{G}(\mathbf{X}), \beta, \delta)$ satisfies (D2) iff $R^{\Diamond}_{G(X)} = R_{G(X)} \circ \leq$, and this holds, by Corollary 5.10, iff $R'_{\Diamond} = R' \circ \leq$.

(iii) By Theorem 5.7, $(\mathbf{G}(\mathbf{X}), \beta, \delta)$ satisfies (FS2) iff $R^{\Box}_{G(X)} = \geq \circ R_{G(X)}$ and this holds, by Corollary 5.10, iff $R'_{\Box} = \geq \circ R'$. □

As we recalled at the beginning if this section, Dunn axioms have been mainly studied on the positive fragment of classical (intuitionistic) logic. Axiomatic extension of full intuitionistic modal logic by (D1) and (D2) have also been considered in [17], while [18] studies the extension of bi-modal Gödel logic by (D1) and (D2). On the other hand, Fischer Servi axioms, for they own formulation that in fact needs the implication connective, have been quite deeply studied in the realm of intuitionistic modal logic, see [22, 23].

We end this section with two examples showing that, indeed, $\mathbb{DGAO} \not\subseteq \mathbb{FSGAO}$ and $\mathbb{FSGAO} \not\subseteq \mathbb{DGAO}$ and hence showing that, in particular, (D1) and (FS2) are independent when considered in Gödel modal logic and hence also on the modal intuitionistic base, a fortiori.

The fist example is a GAO based on the free 1-generated Gödel algebra that is a DGAO but it fails to prove (FS2).

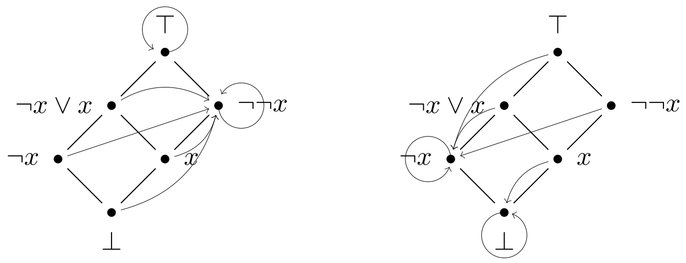

Figure 7: A finite Gödel algebra with a □ (left-hand-side) and a ◊ (right-hand-side) that satisfies Dunn's axioms but it does not satisfy (FS2).

Define, on Free(1), the following operators as in Figure 7:

$$\Box\top = \top;\ \Box y = \neg\neg x \text{ for all } y \neq \top;\ \Diamond\bot = \Diamond x = \bot;$$
$$\Diamond\neg x = \Diamond(x \vee \neg x) = \Diamond\neg\neg x = \Diamond\top = \neg x.$$

Then, one has, $\Diamond x \to \Box\bot = \bot \to \neg\neg x = \top \not\leq \Box(x \to \bot) = \neg\neg x$ and hence (FS2) fails.

However, (D1) and (D2) holds. In fact, as for (D1), notice that, for all a, b such that $a \vee b \neq \top$, $\Box(a \vee b) = \neg\neg x$ and in these cases $\Box a = \Box b = \neg\neg x$. Thus, $\Box(a \vee b) = \Box a \leq \Box a \vee \Diamond b$. Now, if $a \vee b = \top$ and avoiding the trivial case in which either a or b equals $\top$, one has that either a or b are $\neg\neg x$. Assume $a = \neg\neg x$. Then, if $b = \neg x$, $\Box(\neg\neg x \vee \neg x) = \top = \Box\neg\neg x \vee \Diamond\neg x = \neg\neg x \vee \neg x$. Conversely, if $b = \neg x \vee x$, again $\Box(\neg\neg x \vee (\neg x \vee x)) = \top = \Box\neg\neg x \vee \Diamond(\neg x \vee x) = \neg\neg x \vee \neg x$.

That (D2) also holds can be proved in a similar manner and the proof is omitted.

As for the second example, consider the (directly indecomposable) Gödel algebra $\mathbf{A}$ of Figure 8 below where $\Box$ and $\Diamond$ are so defined:

$$\Box\top = \Box d = \top;\ \Box c = \Box b = a;\ \Box a = \Box\bot = \bot;\ \Diamond\top = \Diamond d = \Diamond c = \Diamond b = \Diamond a = c;\ \Diamond\bot = \bot.$$

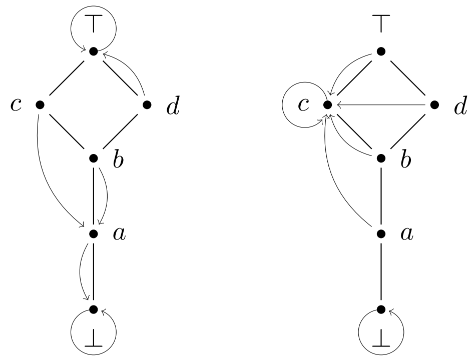

Figure 8: A finite Gödel algebra with a $\Box$ (left-hand-side) and a $\Diamond$ (right-hand-side) that satisfies Fischer Servi's axioms but it does not satisfy (D1).

In that GAO, one has: $\Box(a \vee d) = \Box d = \top \not\leq \Box a \vee \Diamond d = \bot \vee c = c$ and hence (D1) fails. On the other hand, it satisfies (FS1) and (FS2). In order to prove that, consider the forest (tree) $\mathbf{F_A}$ of the prime filters of $\mathbf{A}$ as in Figure 9 where x denotes the

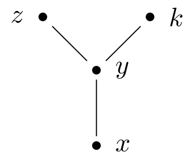

Figure 9: The tree of prime filters of the Gödel algebra $\mathbf{A}$ of Figure 8.

principal filter generated by a; y the principal filter generated by b; z is the principal filter generated by c and k is the principal filter generated by d. Then, compute the relations $R_\Box$ and $R_\Diamond$:

$$R_\Box = \{(x, x), (x, y), (y, x), (y, y), (y, z), (z, x), (z, y), (z, z), (k, x), (k, y), (k, z)\}$$

and

$$R_\Diamond = \{(x, x), (x, y), (x, z), (x, k), (y, x), (y, y), (y, z), (y, k), (k, x), (k, y), (k, z), (k, k)\}$$

so that

$$R = R_\Box \cap R_\Diamond = \{(x, x), (x, y), (y, x), (y, y), (y, z), (k, x), (k, y), (k, z)\}$$

It is not difficult to prove that $(\mathbf{F_A}, R)$ is an *IK-frame* in the sense of [21] and therefore the algebra $(\mathbf{G}(\mathbf{F_A}), \beta_R, \delta_R)$ is isomorphic to $(\mathbf{A}, \Box, \Diamond)$ and it satisfies (FS1) and (FS2).

6 Forest frames with a single relation

In this section we review results in the literature about relational frames for Heyting and positive modal algebras with a single binary relation, and adapt them to our setting to finite GAOs. In particular, we will consider Celani and Jansana's results on duality for positive modal logic [9], as well a results by Palmigiano [21] and Orlowska and Rewitzky [20] on dualities for Intuitionistic modal logics. In the second part of the section we consider special forest frames in which the binary relation satisfies both the properties we considered above, namely, monotonicity and antimonotonicity in the first argument.

For the remining of this section, it is useful to recall the above Lemma 5.8 showing that every finite forest is isomorphic to the forest of prime filters of its associated Gödel algebra.

Furthermore, recall from Section 4 that, for each GAO, $(\mathbf{A}, \Box, \Diamond)$, we can consider the associated relational frame $(\mathbf{F_A}, R_A)$, where R_A is the binary relation on F_A defined as follows: for every $f, g \in F_A$,

$$f R_A g \quad \text{iff} \quad \Box^{-1}(f) \subseteq g \subseteq \Diamond^{-1}(f).$$

Equivalently, one could define R_A as the intersection of the two relations $R_A^{\Box}$ and $R_A^{\Diamond}$, where $f R_A^{\Box} g$ if $\Box^{-1}(f) \subseteq g$, and $f R_A^{\Diamond} g$ if $\Diamond(g) \subseteq f$.

6.1 Forest frames for GAOs satisfying Dunn axioms

In [9] Celani and Jansana study the duality theory for Dunn's positive modal logic [11]. Positive modal algebras can always be expanded with a implication operation such that the resulting structure is a Dunn-GAO. Many results can be easily extended to our finitary setting of Gödel algebras with operators.

Definition 6.1. A relational frame $F = (\mathbf{X}, R)$ is a *CJ-forest frame* provided that $\mathbf{X}$ is a forest and the binary relation $R \subseteq X \times X$ satisfies the following two conditions:

(CJ1): $(\leq \circ R) \subseteq (R \circ \leq)$
(CJ2): $(\geq \circ R) \subseteq (R \circ \geq)$.

Lemma 6.2. *Conditions (CJ1) and (CJ22) are respectively equivalent to:*

(CJ1') $(\leq \circ R \circ \leq) = (R \circ \leq)$
(CJ2') $(\geq \circ R \circ \geq) = (R \circ \geq)$.

Proof. First of all, note that (CJ1') and (CJ2') are respectively equivalent to

(CJ1") $(\leq \circ R \circ \leq) \subseteq (R \circ \leq)$
(CJ2") $(\geq \circ R \circ \geq) \subseteq (R \circ \geq)$.

since the reverse inclusions always hold. Now let us prove that (CJ2) is equivalent to (CJ2"):

- Assume $(\geq \circ R) \subseteq (R \circ \geq)$ holds. Then, by monotonicity (wrt set inclusion) of the composition, $(\geq \circ R \circ \geq) \subseteq (R \circ \geq \circ \geq)$, but $\geq \circ \geq \, = \, \geq$, and thus, $(\geq \circ R \circ \geq) \subseteq (R \circ \geq)$.

- Conversely, assume $(\geq \circ R \circ \geq) \subseteq (R \circ \geq)$. But, trivially, $(\geq \circ R) \subseteq (\geq \circ R \circ \geq)$, and hence, by transitivity, $(\geq \circ R) \subseteq (R \circ \geq)$.

The case of (CJ1) and (CJ1") can be proved analogously. □

Moreover, if $F = (\mathbf{X}, R)$ is a CJ-forest frame, then the operations β_R and δ_R on subsets of X are in fact closed on the set $G(X)$ of downsets of $(X, \leq)$. Then, based on [9], one can check that the modal algebra

$$\mathbf{G}(F) = (\mathbf{G}(\mathbf{X}), \beta_R, \delta_R)$$

is a Dunn-GAO. Hence, by Proposition 5.1, the operators β_R and δ_R are closed on the set of Boolean elements of $\mathbf{G}(\mathbf{X})$.

Now, starting from the algebra $\mathbf{G}(F)$, one can consider its associated relational frame $(\mathbf{F_{G(X)}}, R_{G(F)})$ as defined above. Next proposition shows that we basically recover the initial frame F.

Proposition 6.3. *For every CJ-forest frame* $(\mathbf{X}, R)$*,* $(\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G(F)})$ *is a CJ-forest frame and, moreover,* $(\mathbf{X}, R) \cong (\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G(F)})$*.*

Proof. That $(\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G(F)})$ is a CJ-forest frame directly follows from [9, Lemma 2.1]. By Lemma 5.8, the mapping $k : X \to F_{G(X)}$, defined as $k(x) = \{f \in G(X) \mid x \in f\}$ for any $x \in X$, is a bijection. Finally, a very slight adaptation of [20, Lemma 4.5] shows that, for all $x, y \in X$, xRy iff $k(x)R_{G(F)}k(y)$. □

It is also very interesting to observe that any CJ-forest frame is equivalent to a forest frame with two (different) relations in the sense of generating the same algebra. Indeed, given a CJ-frame $F = (\mathbf{X}, R)$, let us consider the following two relations: $R_\Box = R \circ \geq$ and $R_\Diamond = R \circ \leq$. Then in [9] the authors prove that

$$\beta_R = \beta_{R_\Box}, \quad \delta_R = \delta_{R_\Diamond}.$$

Now, consider the intersection of these two relations $R' = R_\Box \cap R_\Diamond = (R \circ \geq) \cap (R \circ \leq)$. Clearly, $R \subseteq R'$, and it is not hard to prove the following further properties.

Proposition 6.4. *For any CJ-forest frame* $F = (\mathbf{X}, R)$*, define a new relation* $R' = (R \circ \geq) \cap (R \circ \leq)$*. Then:*

(i) R' *satisfies (CJ1) and (CJ2)*

(ii) $(R' \circ \geq) = (R \circ \geq)$*,* $(R' \circ \leq) = (R \circ \leq)$*, i.e.* $R'_\Box = R_\Box$ *and* $R'_\Diamond = R_\Diamond$

(iii) $R' = (R' \circ \geq) \cap (R' \circ \leq)$*, i.e.* $R' = R'_\Box \cap R'_\Diamond$

(iv) $\beta_{R'} = \beta_R$*,* $\delta_{R'} = \delta_R$

Proof. (i) As for (CJ1), we have $(\leq \circ R') = (\leq \circ ((R \circ \geq) \cap (R \circ \leq))) \subseteq (\leq \circ R \circ \leq) = (R \circ \leq) \subseteq (R' \circ \leq)$. And as for (CJ2) we have $(\geq \circ R') = (\geq \circ ((R \circ \geq) \cap (R \circ \leq))) \subseteq (\geq \circ R \circ \geq) = (R \circ \geq) \subseteq (R' \circ \geq)$.

(ii) The inclusions $\supseteq$'s are direct, let us prove the inclusions $\subseteq$'s. We have: $(R' \circ \geq) = (((R \circ \geq) \cap (R \circ \leq)) \circ \geq) \subseteq ((R \circ \geq) \circ \geq) = (R \circ \geq)$, and similarly $(R' \circ \leq) = (((R \circ \geq) \cap (R \circ \leq)) \circ \leq) \subseteq ((R \circ \leq) \circ \leq) = (R \circ \leq)$.

(iii) It directly follows from the definition of R' and (ii).

(iv) It directly follows from (ii) and the fact that $\beta_R = \beta_{R_\Box}$ and $\delta_R = \delta_{R_\Diamond}$.

□

The following result, in which we will adopt the notation introduced above, is a direct consequence of the properties proved in Proposition 6.3 and Proposition 6.4 above.

Corollary 6.5. *For every CJ-forest frame* $F = (\mathbf{X}, R)$*,* $F' = (\mathbf{X}, R')$ *is a CJ-forest frame that is equivalent to* $F = (\mathbf{X}, R)$*, i.e.* $\mathbf{G}(F) = \mathbf{G}(F')$*.*

6.2 Forest frames for GAOs satisfying Fisher-Servi axioms

Palmigiano in [21] and Orłowska and Rewitzky in [20] study duality theory for some intuitionistic modal logics. It is worth noticing that the notions of relational frames with a single relation in these papers that are relevant for our setting, namely the so-called IK-frames in [21] and HK1-frames in [20], coincide. They are shown to capture Heyting modal algebras satisfying the Fischer Servi axioms.

As already recalled, Gödel logic is the axiomatic extension of Intuitionistic logic with the pre-linearity axiom, and hence the variety of Gödel algebras are the subvariety of Heyting algebras generated by the linearly-ordered ones. Thus, again, many results in [21] and [20] also extend to our finitary setting of Gödel algebras with operators, the basic difference being at the level of relational frames is that we consider here frames on forests rather than on preordered sets.

Definition 6.6. A relational frame $F = (\mathbf{X}, R)$ is a *FS-forest frame* provided that $\mathbf{X}$ is a forest and the binary relation $R \subseteq X \times X$ satisfies the following two conditions:
(FS1) $(\leq \circ R) \subseteq (R \circ \leq)$
(FS2) $(R \circ \geq) \subseteq (\geq \circ R)$.

Note that condition (FS1) is in fact the same as (CJ1), and conditions (FS1) and (FS2), that appear in the definition of IK-frames in [21], are respectively equivalent to the conditions that appear in the definition of HK1-frames in [20].

The proof of the following lemma is very similar to that of Lemma 6.2 and it is omitted.

Lemma 6.7. *The conditions (FS1) and (FS2) are respectively equivalent to:*
(FS1') $(\leq \circ R \circ \leq) = (R \circ \leq)$
(FS2') $(\geq \circ R \circ \geq) = (\geq \circ R)$.

Then, if $F = (\mathbf{X}, R)$ is a FS-forest frame, then the operations $\beta'_R = \beta_{\geq \circ R}$ and δ_R on subsets of X are closed on the set $G(X)$ of downsets of $(X, \leq)$ and adapting the results in [21, 20], we have that the modal algebra

$$\mathbf{G}'(F) = (\mathbf{G}(\mathbf{X}), \beta'_R, \delta_R)$$

is a FS-GAO.

Similarly to the case of CJ-frames, $\mathbf{G}'(F)$ induces its associated relational frame $(\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G'(F)})$. Also here, next proposition shows that we recover the initial frame F up to a isomorphism.

Proposition 6.8. *For every FS-forest frame $F = (\mathbf{X}, R)$, $(\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G'(F)})$ is a FS-forest frame and, moreover, $(\mathbf{X}, R) \cong (\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G'(F)})$.*

Proof. That $(\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G'(F)})$ is a FS-forest frame directly follows from (3) of [20, Corollary 5.7]. Let the mapping $k : X \to F_{G(X)}$ be defined, for any $x \in X$, by $k(x) = \{f \in G(X) \mid x \in f\}$. Then Lemma 5.8 shows that k is bijective and order preserving, while [20, Lemma 4.5] shows that, for all $x, y \in X$, xRy iff $k(x)R_{G'(F)}k(y)$. □

Proposition 6.9. *For any FS-frame $F = (\mathbf{X}, R)$, let $R' = (\geq \circ R) \cap (R \circ \leq)$. Then:*

(i) R' *satisfies (FS1) and (FS2)*

(ii) $(\geq \circ R') = (\geq \circ R)$, $(R' \circ \leq) = (R \circ \leq)$

(iii) $R' = (\geq \circ R') \cap (R' \circ \leq)$

(iv) $\beta_{\geq \circ R'} = \beta_{\geq \circ R}$, $\delta_{R'} = \delta_R$

Proof. (i) As for (FS1), the proof is practically the same than for (CJ1) in (i) of Prop. 6.4. And as for (FS2) we have $(R' \circ \geq) = (((\geq \circ R) \cap (R \circ \leq)) \circ \geq) \subseteq (\geq \circ R \circ \geq) = (\geq \circ R) \subseteq (\geq \circ R')$.

(ii) The inclusions $\supseteq$'s are direct, let us prove the inclusions $\subseteq$'s. We have: $(R' \circ \geq) = (((\geq \circ R) \cap (R \circ \leq)) \circ \geq) \subseteq ((\geq \circ R) \circ \geq) = (\geq \circ R)$, and similarly $(R' \circ \leq) = (((\geq \circ R) \cap (R \circ \leq)) \circ \leq) \subseteq ((R \circ \leq) \circ \leq) = (R \circ \leq)$.

(iii) It directly follows from the definition of R' and (ii).

(iv) It directly follows from (ii) and the fact that $\delta_R = \delta_{R \circ \leq}$.

□

As in the previous subsection, we now present a final result on FS-forest frames. We will adopt the notation introduced above and the next is a direct consequence of the properties proved in Proposition 6.8 and Proposition 6.9 above.

Corollary 6.10. *For every FS-forest frame $F = (\mathbf{X}, R)$, let $R' = (\geq \circ R) \cap (R \circ \leq)$. Then $F' = (\mathbf{X}, R')$ is a FS-forest frame that is equivalent to $F = (\mathbf{X}, R)$, i.e. $\mathbf{G}'(F') = \mathbf{G}(F)$.*

6.3 Forest frames for GAOs satisfying both Dunn and Fischer Servi axioms

In this section we consider the class of frames that are both CJ- and FS-forest frames.

Definition 6.11. A relational frame $F = (\mathbf{X}, R)$ is a *FSD-forest frame* provided that $\mathbf{X}$ is a forest and the binary relation $R \subseteq X \times X$ satisfies the following two conditions:
(FS1) $(\leq \circ R) \subseteq (R \circ \leq)$
(FS2) $(R \circ \geq) \subseteq (\geq \circ R)$
(CJ2) $(\geq \circ R) \subseteq (R \circ \geq)$.

By definition, it is clear that $F = (\mathbf{X}, R)$ is a FSD-forest frame iff F is both a CJ-forest frame and a FS-forest frame.

It is easy to check that requiring the above three conditions is equivalent to require the following two conditions:

(FS1') $(R \circ \leq) = (\leq \circ R \circ \leq)$

(FSCJ2) $(R \circ \geq) = (\geq \circ R)$

It is clear that (FS1') is a simple reformulation of (FS1), which is commonly satisfied by both CJ- and FS-forest frames, while (FSCJ2) is obtained by combining (FS2) and (CJ2).

In this case, notice that if $F = (\mathbf{X}, R)$ is a FSD-forest frame, then $\beta_R = \beta_{\geq \circ R}$, and thus the Gödel modal algebra

$$\mathbf{G}(F) = (\mathbf{G}(\mathbf{X}), \beta_R, \delta_R)$$

is both a FS-GAO and a Dunn-GAO. Note that, as in the case of CJ-forest frames, the operators β_R and δ_R in $\mathbf{G}(F)$ keep being closed on the set of Boolean elements of $\mathbf{G}(\mathbf{X})$.

Similarly to the previous cases, now we have the following two propositions that can be easily proved by combining respectively Props 6.3 and 6.4 on the one hand and Props. 6.8 and 6.9 on the other.

Proposition 6.12. *Let $F = (\mathbf{X}, R)$ be a FSD-forest frame and let $(\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G(F)})$ be the frame such that $\mathbf{F}_{\mathbf{G}(\mathbf{X})}$ is the forest of prime filters of $\mathbf{G}(F)$ and $R_{G(F)} = R^{\Box}_{G(F)} \cap R^{\Diamond}_{G(F)}$. Then $(\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G(F)})$ is a FSD-forest frame and, moreover, $F = (\mathbf{X}, R) \cong (\mathbf{F}_{\mathbf{G}(\mathbf{X})}, R_{G(F)})$.*

Proposition 6.13. *Let $F = (\mathbf{X}, R)$ be a FSD-forest frame, and define a new relation $R' = (\geq \circ R) \cap (R \circ \leq) = (R \circ \geq) \cap (R \circ \leq)$. Then:*

(i) R' satisfies (FS1), (FS2) and (CJ2)

(ii) $(\geq \circ R') = (\geq \circ R)$, $(R' \circ \leq) = (R \circ \leq)$

(iii) $R' = (\geq \circ R') \cap (R' \circ \leq) = (R' \circ \geq) \cap (R' \circ \leq)$

(iv) $\beta_{\geq \circ R'} = \beta_{\geq \circ R} = \beta_R$, $\delta_{R'} = \delta_R$

Proof. The proof of this proposition follows straightforwardly from Props. 6.4 and 6.9 by noticing that the relations R' defined in those propositions coincide in a FSD-forest frame, i.e. if R satisfies (FSJ2) then $(\geq \circ R) = (R \circ \geq)$ and hence, $(\geq \circ R) \cap (R \circ \leq) = (R \circ \geq) \cap (R \circ \leq)$ and all the properties in Props. 6.4 and 6.9 are valid in a a FSD-forest frame. □

Corollary 6.14. *For every FSD-forest frame $F = (\mathbf{X}, R)$, let $R' = (\geq \circ R) \cap (R \circ \leq)$. Then $F' = (\mathbf{X}, R')$ is a FSD-forest frame that is equivalent to $F = (\mathbf{X}, R)$, i.e. $\mathbf{G}'(F') = \mathbf{G}(F)$.*

6.4 Forest frames with one relation satisfying (A) and (M)

In this section we finally consider forest frames with a single relation satisfying both monotonicity properties (A) and (M) on the first variable. As we will show, this class of frames determine a proper subvariety of $\mathbb{DGAO}$.

Definition 6.15. A relational frame $F = (\mathbf{X}, R)$ is a W-forest frame provided that $\mathbf{X}$ is a forest and the binary relation $R \subseteq X \times X$ satisfies the following two conditions:
(W1): $(\leq \circ R) \subseteq R$
(W2): $(\geq \circ R) \subseteq R$

It is easy to check that conditions (W1) and (W2) are equivalent to the compound condition

(W): $(\leq \circ R) = R = (\geq \circ R)$

Moreover, in a W-forest frame we further have all the following relations:

$$R = (\leq \circ R) = (\geq \circ R) \begin{cases} \subseteq (R \circ \leq) = (\leq \circ R \circ \leq) = (\geq \circ R \circ \leq) \\ \subseteq (R \circ \geq) = (\leq \circ R \circ \geq) = (\geq \circ R \circ \geq) \end{cases}$$

Also note the following.

Remark 6.16. (i) W-forest frames are CJ-forest frames, since (W1) and (W2) imply (CJ1) and (CJ2) respectively, but the converse is not true.

(ii) W-forest frames are not FS-forest frames in general, but if $F = (\mathbf{X}, R)$ is a W-forest frame, then $F' = (\mathbf{X}, R')$, where $R' = R \circ \geq$, is both a W-forest frame and a FS-forest frame.

Indeed, we have:

- (W1) $\leq \circ R' = \leq \circ R \circ \geq = R \circ \geq = R'$,
- (W2) $\geq \circ R' = \geq \circ R \circ \geq = R \circ \geq = R'$,
- (FS2) $R' \circ \geq = R \circ \geq \circ \geq = R \circ \geq = R' \subseteq \geq \circ R'$.

(iii) From (i) and (ii) it follows that if $F = (\mathbf{X}, R)$ is a W-forest frame, since $\geq \circ R' = R'$, then $\beta_{R'}$ and $\delta_{R'}$ are closed on $Down(X)$ and

$$\mathbf{G}(F') = (\mathbf{G}(\mathbf{X}), \beta_{R'}, \delta_{R'})$$

is both a Dunn-GAO and a FS-GAO. Notice that, in general $\mathbf{G}(F')$ is not isomorphic to the GAO $\mathbf{G}(F)$, for otherwise, every algebra defined by a W-forest frame would belong to $\mathbb{FSGAO}$ and this is not the case because of next Proposition 6.19 (iv).

Next, we axiomatise the subvariety of Dunn-GAOs whose associated frames are W-forest frames.

Definition 6.17. An algebra $(\mathbf{A}, \Box, \Diamond)$ is a W-GAO if it is a D-GAO that satisfies the following two equations:

(BB) $\Box(x) \vee \neg\Box(x) = 1$,

(DB) $\Diamond(x) \vee \neg\Diamond(x) = 1$.

Theorem 6.18. *(1) Let $F = (\mathbf{X}, R)$ be a W-forest frame. Then $\mathbf{G}(F) = (\mathbf{G}(\mathbf{X}), \beta_R, \delta_R)$ is a W-GAO.*

(2) Let $(\mathbf{A}, \Box, \Diamond)$ be a W-GAO. Then $(F_{\mathbf{A}}, R_{\mathbf{A}})$ is a W-forest frame.

Proof. (1) Let $F = (X, R)$ a W-forest frame, that is, R is such that $(\leq \circ R) = R = (\geq \circ R)$. This means that if xRy and either $z \leq x$ or $z \geq x$, then zRy as well. This implies that, for any $a \in G(X)$, if $x \in \beta_R(a)$ (resp. $x \in \delta_R(a)$) then $y \in \beta_R(a)$ (resp. $y \in \delta_R(a)$) for any $y \in X$ such that $y \leq x$ or $y \geq x$. In other words, for any a, $\beta_R(a)$ and $\delta_R(a)$ are both a downset and an upset. Since any subset of a forest that is both downwards and upwards closed must be a union of a collection of maximal trees of the forest. But maximal trees correspond to joint-irreducible Boolean elements in the algebra $\mathbf{G}(\mathbf{F})$, and therefore, for any $a \in G(X)$, $\beta_R(a)$ and $\delta_R(a)$ must be Boolean elements of $\mathbf{G}(\mathbf{F})$.

(2) Let $f, g \in F_A$ such that fR_Ag, that is, such that $\Box^{-1}(f) \subseteq g$ and $g \subseteq \Diamond^{-1}(f)$. We have to show that if $f' \in F_A$ is such that $f' \subseteq f$ or $f \subseteq f'$, then $f'R_Ag$ as well.

Suppose $f \subseteq f'$. Then clearly, $g \subseteq \Diamond^{-1}(f')$, so let us how that $\Box^{-1}(f') \subseteq g$ as well. By definition, $\Box^{-1}(f') = \{y \in G(X) \mid \Box(y) \in f'\}$. But by assumption every such $\Box(y)$ is a Boolean element of the prime filter f', and hence $\Box(y) \in f$ as well. Indeed, by contradiction, suppose $\Box(y) \notin f$. Then, since f is prime, $\neg\Box(y) \in f$, and thus $\neg\Box(y) \in f'$ as well, that is a contradiction with the fact that $\Box(y) \in f'$. Therefore, $\Box^{-1}(f') = \Box^{-1}(f) \subseteq G$ and thus $f'R_Ag$.

The case $f' \subseteq f'$ can be proved in a similar way. $\Box$

6.5 A final comparison

It is now convenient to summarize how the relational frames and their associated classes of Gödel algebras with operators relate each other.

First of all, notice that the two subvarieties $\mathbb{DGAO}$ and $\mathbb{FSGAO}$ have a non empty intersection as, for instance, the variety $\mathbb{BAO}$ of Boolean algebras with operators is a subvariety of their intersection $\mathbb{FSDGAO} = \mathbb{DGAO} \cap \mathbb{FSGAO}$. Moreover, $\mathbb{DGAO}$ and $\mathbb{FSGAO}$ can be distinguished as we showed in Section 5.

As for the variety $\mathbb{WGAO}$ that we introduced in the above Subsection 6.4, the next result is going to make clear how it relates with the aforementioned varieties as depicted in Figure 10.

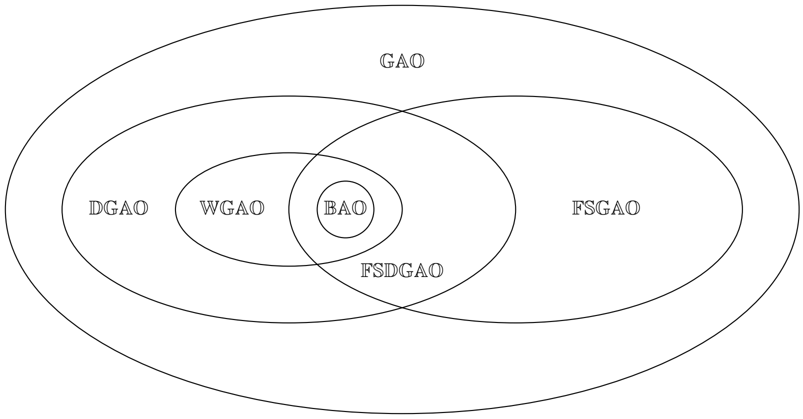

Figure 10: A diagram explaining the inclusions among the subvarieties of $\mathbb{GAO}$ that we have studied in the present paper.

Proposition 6.19. *The following properties hold:*

(i) $\mathbb{WGAO} \subsetneq \mathbb{DGAO}$*,*

(ii) $\mathbb{WGAO} \cap \mathbb{FSDGAO} \neq \emptyset$*,*

(iii) $\mathbb{WGAO} \cap \mathbb{FSDGAO} \subsetneq \mathbb{FSDGAO}$*,*

(iv) $\mathbb{WGAO} \setminus \mathbb{FSGAO} \neq \emptyset$*,*

(v) $\mathbb{BAO} \subsetneq (\mathbb{WGAO} \cap \mathbb{FSGAO})$*.*

Proof. (i) By Remark 6.16 (i), every W-forest frame satisfies the conditions that characterize Dunn GAOs. Therefore, $\mathbb{WGAO} \subseteq \mathbb{DGAO}$. So as to prove that the inclusion is proper, consider the tree $\mathbf{F}$ as in Figure 9 with a relation $R = \{(x, y), (y, y), (z, k), (k, z)\}$. Then $(\mathbf{F}, R)$ is a CJ-forest frame. Indeed, $(\geq \circ R) \subseteq (R\circ \geq)$ as $(\geq \circ R) = R \cup$

$\{(z,y),(k,y)\}$ and $(R\circ \geq) = R \cup \{(y,x),(z,y),(z,x),(k,y),(k,x),(x,x)\}$. Similarly, $(\leq \circ R) \subseteq (R\circ \leq)$, because $(\leq \circ R) = R \cup \{(y,k),(x,k),(y,z),(x,z)\}$ and $(R\circ \leq) = R \cup \{(x,k),(x,z),(y,k),(y,z)\}$. On the other hand, in the corresponding GAO $(\mathbf{G}(\mathbf{F}),\beta,\delta)$, whose Gödel algebra is that one as in Figure 8, one has that $\beta(\{x,y\}) = \{x,y\}$ that is not Boolean and hence $(\mathbf{G}(\mathbf{F}),\beta,\delta) \notin \mathbb{WGAO}$.

(ii) follows because Boolean algebra with operators are, at the same time, a proper subvariety of both $\mathbb{WGAO}$ and $\mathbb{FSDGAO}$.

As for (iii), consider the finite forest $\mathbf{F}$ being the tree as in Figure 9 and whose associated Gödel algebra is as in Figure 8. Let R be the following relation on F: $\{(x,x),(x,y),(y,x),(y,y),(y,z),(k,x),(k,y),(k,z),(z,x),(z,y),(z,z)\}$. Then one can check that $(\leq \circ R) \subseteq (R\circ \leq)$ and $(R\circ \geq) = (\geq \circ R)$. In other words $(\mathbf{F},R)$ is a FSD-forest frame and hence its associated GAO $(\mathbf{G}(\mathbf{F}),\beta,\delta)$ belongs to $\mathbb{FSDGAO}$ (recall Subsection 6.3). However, $\beta(\{x,y\}) = \{x\}$ in $G(\mathbf{F})$ and $\{x\}$ is not Boolean. Therefore, $(\mathbf{G}(\mathbf{F}),\beta,\delta) \notin \mathbb{WGAO}$ by definition.

In order to prove that (iv) holds, consider the algebra of Figure 7 and recall that it does not belong to $\mathbb{FSGAO}$. Furthermore, notice that it belongs to $\mathbb{WGAO}$ as it satisfies (BB) and (DB) of Definition 6.17.

Finally, in order to prove (v), i.e., that $\mathbb{BAO}$ is strictly contained in both $\mathbb{WGAO}$ and $\mathbb{FSGAO}$, let us consider the three element Gödel chain on domain $A = \{\bot, a, \top\}$ together with the following modal operators: $\Box\top = \Diamond\top = \Diamond a = \top$, $\Box\bot = \Box a = \Diamond\bot = \bot$. Obviously $(\mathbf{A},\Box,\Diamond)$ is not a BAO and it is a WGAO. Let us hence see that $(\mathbf{A},\Box,\Diamond)$ satisfies (FS2). For all $x, y \in A$ such that $x \leq y$, $x \to y = \top$ and $\Box\top = \top$. Thus, in these cases (FS2) holds. The remaining three cases are the following: (1) $\Box(\top \to a) = \Box a = \bot$ and, in this case, $\Diamond\top \to \Box a = \top \to \bot = \bot$, whence (FS2) holds; (2) $\Box(a \to \top) = \bot$; (3) $\Box(\top \to \bot) = \bot$. As for (2), notice that $\Diamond a = \top$ and $\Box\bot = \bot$. Thus $\Diamond a \to \Box\bot = \bot$. In case (3), $\Diamond\top = \top$ and hence $\Diamond\top \to \Box\bot = \bot$. Thus, (FS2) holds in $(\mathbf{A},\Box,\Diamond)$. □

7 Conclusions and future work

In this paper we have been concerned with finite Gödel modal algebras from several varieties and their corresponding classes of forest frames, which are their dual relational structures. In particular, we have first considered the more basic class of Gödel algebras with operators (GAOs) where there is no interaction between the operators and proved a representation theorem in terms of algebras defined on the set of downsets of their prime spectra. The dual relational structures based on forests, called forest frames, are defined by two independent binary relations, one per each modal operator, that satisfy (anti)monotonicity properties on the first argument of them. Then we have considered two subvarieties of Gödel modal algebras where the operators are not independent any longer, namely those satisfying the so-called Dunn's axioms of positive modal logic and Fischer Servi's axioms for intuitionistic modal logic. For these algebras, the associated dual forest frames are basically specified by a single relation that accounts for the relationship among the operators. In this aspect, we have essentially adapted available results in the general case of Heyting algebras with operators [9, 21, 20] to our case and have further investigated on the relations between the additional axioms and properties of the relations in the forest frames. Finally we have considered two further subvarieties of Gödel modal algebras, the one whose algebras satisfy both Dunn and Fischer Servi axioms and the one whose modal operators always yield Boolean elements, and their corresponding forest frames have a single binary relation satisfying both monotonicity and antimonotonicity properties in the first argument.

As for future work, there are at least a couple of interesting issues that deserve further research. A first issue is the relationship between the forest-based relational semantics for Gödel modal logics, considered in this paper along the line of intuitionistic modal logics, and the $[0,1]$-valued Kripke models that have been used in other venues like e.g. in [8, 5, 18] following the strand of fuzzy modal logics. It is not clear whether there exists a direct relationship between them. On the one hand, the minimal modal logic complete with respect to $[0,1]$-valued Kripke models, i.e. the bimodal Gödel logic studied in [8], already satisfies the Fischer Servi axioms. So it seems that $[0,1]$-valued Kripke models only account for Gödel modal logics satisfying both Fischer Servi axioms. Thus, for instance $[0,1]$-valued Kripke models seems not to be a semantics for the logic of our GAOs algebras nor those extended with axioms (D1), (BB) and (DB).

A second issue is to extend our research on Gödel modal algebras over related algebraic structures with nice duality theories. A clear candidate is the variety of Nilpotent Minimum algebras (NM-algebras for short) that, similarly to Gödel algebras, is dual to a forest-based category (see [6]). Another class of algebras that is dual to forests is that of IUML-algebras, studied in [1]. The latter are algebras based on uninorms rather than t-norms, but similar methods to those developed in the present chapter might be used to define and study modal operators on them.

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