On the expressive power of Łukasiewicz's square operator

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Abstract

The aim of the paper is to analyze the expressive power of the square operator of Łukasiewicz logic: $*x = x \odot x$, where $\odot$ is the strong Łukasiewicz conjunction. In particular, we aim at understanding and characterizing those cases in which the square operator is enough to construct a finite MV-chain from a finite totally ordered set endowed with an involutive negation. The first of our main results shows that, indeed, the whole structure of MV-chain can be reconstructed from the involution and the Łukasiewicz square if and only if the obtained structure has only trivial subalgebras and, equivalently, if and only if the cardinality of the starting chain is of the form $n + 1$ where n belongs to a class of prime numbers that we fully characterize. Secondly, we axiomatize the algebraizable matrix logic whose semantics is given by the variety generated by a finite totally ordered set endowed with an involutive negation and Łukasiewicz's square operator. Finally, we propose an alternative way to account for Łukasiewicz square operator on involutive Gödel chains. In this setting, we show that such an operator can be captured by a rather intuitive set of equations.

1 Introduction

The framework of the so-called *mathematical fuzzy logic* (MFL) encompasses a number of deductive systems conceived for reasoning with vague (in the sense of gradual) information with a notion of comparative truth, and so formulas are usually interpreted in linearly ordered scales of truth values, which intend to represent gradual aspects of vagueness (or fuzziness). For a comprehensive and up-to-date account of MFL see the three volumes handbook [8]. Two interesting families of logics belonging to the family of MFL systems are given by the Łukasiewicz hierarchy of n-valued logics $Ł_n$ together with the infinite-valued version Ł, on the one hand, and the Gödel n-valued logics G_n, together with the infinite version G, on the other.

The semantics of MFL systems follows, in general, the paradigm of (full) *truth-preservation*, according to which a formula is a consequence of a set of premises if every algebraic valuation that interprets the premises as absolutely true (value 1) also interprets the conclusion as absolutely true (value 1). It was observed (see [20]) that the *degree-preservation* paradigm (see [22, 4]), according to which a formula follows from a set of premises if, for every evaluation, the truth degree of the conclusion is not lower than those of the premises, is more coherent with the many-valued approach to fuzzy logic. Indeed, within the degree-preserving consequence relations all the truth-values play an equally important role. As an intermediate alternative, it is possible to consider matrix logics in which the designated truth-values are given by (products of) order filters, see for instance [13] and [14] for the case of (products of) Łukasiewicz logics or Gödel's logics (possibly expanded with an involution) respectively.

Concerning Łukasiewicz logics, it is well-known that Ł is algebraizable in the sense of Blok-Pigozzi ([2]), having the variety $\mathbb{MV}$ of all MV-algebras as its equivalent quasivariety semantics, which is generated by the real interval $[0,1]$ equiped with suitable MV-operators. This MV-algebra will be denoted by $[0,1]_{MV}$. Algebraizability is preserved by finitary extensions, hence each finite-valued Łukasiewicz logic $Ł_{n+1}$ is also Blok-Pigozzi algebraizable by means of the subvariety $\mathbb{MV}_{n+1}$ of MV-algebras generated by the standard $(n+1)$-valued Łukasiewicz chain $\mathbf{Ł}_{n+1}$ with domain $\{0, 1/n, \ldots, (n-1)/n, 1\}$. By means of a general result concerning equivalences between logics, based on translations presented in [3], the logic L^i_n characterized by the logical matrix $\langle \mathbf{Ł}_{n+1}, F_{i/n}\rangle$ (where $F_{i/n}$ is the order filter generated by i/n) is also algebraizable by means of the variety $\mathbb{MV}_{n+1}$, see [13][1].

Hilbert calculi characterizing the logics $Ł_{n+1}$ are well known (see, for instance, [8]). By a general result on equivalence between logics introduced in [3], a sound and complete axiomatization can be obtained for each logic L^i_n by translating the axioms and rules of a Hilbert calculus for $Ł_{n+1} = \mathsf{L}^n_n$. However, the original signature of $Ł_{n+1}$ does not result to be very natural for axiomatizing L^i_n: in these logics the Łukasiewicz implication can be hardly considered as a proper implication since it does not satisfy modus ponens whenever $i < n$.

Because of this, in [13] we proposed an axiomatization of L^1_3 and L^2_3 in terms of another signature $\Sigma_0 = (\vee, \sim, \star)$, where $\vee$ denotes the supremum, $\sim$ represents the Łukasiewicz negation and $\star$ represents the square $*$ (w.r.t. the strong Łukasiewicz conjunction $\odot$) in $\mathbf{Ł}_{3+1}$, namely $*x := x \odot x$. It turns out that in this signature it is possible to define the 'classical' negation $-_{i/3}$ of the filter $F_{i/3}$ (for $i = 1, 2$): $-_{i/3}x = 0$ if $x \geq i/3$, and $-_{i/3}x = 1$ otherwise. In turn, this induces a 'classical' (deductive) implication $x \to_{i/3} y := -_{i/3}x \vee y$, obtaining in this way a suitable and very natural language for axiomatizing the logics L^i_3, for $i = 1, 2$.

Despite the success in axiomatizing L^1_3 and L^2_3 in the signature $\Sigma_0 = (\vee, \sim, \star)$, it was observed in [13] that the issue of obtaining a 'natural' axiomatization defined over such signature for every L^i_n with $n > 3$ is a problem which "appears to be much more complicated, and certainly it lies outside the scope of this paper" ([13, p. 150]). A crucial feature for the case $n = 3$ mentioned in [13] is that Łukasiewicz implication can be recovered from such signature. This feature does not hold for any n, not even for any prime number, as it is the case e.g. of $n = 17$, as we shall see in Section 3 of this paper. From this observation, a second question was posed in [13, p. 153]: the algebraic study of the fragment of $\mathbf{Ł}_{n+1}$ defined in the signature $\Sigma_0 = (\vee, \sim, \star)$. These two questions stated in [13], namely, the formal study – from the algebraic point of view – of the implication-less reduct of the $(n+1)$-valued Łukasiewicz chain $\mathbf{Ł}_{n+1}$ expanded with the square operator $*$ (which will be denoted here by $\mathbf{Ł}^*_{n+1}$), as well as the associated matrix logics $\Lambda^*_{n+1,i} = \langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$ for every filter $F_{i/n}$ of designated values, constitute the starting point of the present paper.[2] By convenience, the signature Σ_0 will be expanded, in this paper, to $\Sigma = (\vee, \sim, \star, \bot, \top)$. In this manner, the present study, already initiated in two preliminary extended abstracts [11] and [12], will encompass both questions and more.

Note that the square operator $*$ in the logics $Ł_{n+1}$, or in Ł, can be interpreted as a truth-stresser operator, in the sense of the class of truth-hedge operators axiomatically introduced by Hájek in [27] in the context of Hájek's Basic Fuzzy Logic BL to formalize the notion of 'very true'. In fact, $*$ is a model of Hájek's truth-stresser operators for both Łukasiewicz's and Gödel fuzzy logics, as well as of the operators considered in a more general logical in the setting of MFL studied in [18].

With respect to expressiveness, it is firstly proved in Section 3 that, for $n \neq 4$, $\mathbf{Ł}^*_{n+1}$ can define Łukasiewicz implication (in other words, $\mathbf{Ł}^*_{n+1}$ is term-equivalent to $\mathbf{Ł}_{n+1}$) iff it is stricly simple, that is, it has no non-trivial proper subalgebras. Surprisingly, and in contrast with the case of finite Łukasiewicz chains, it will be shown that this does not hold true for all n prime. Indeed, for any prime number $n \geq 3$, $\mathbf{Ł}^*_{n+1}$ is term-equivalent to $\mathbf{Ł}_{n+1}$ if and only if n satisfies a certain arithmetic property (see Theorem 3.20). For instance, $\mathbf{Ł}^*_{n+1}$ cannot define Łukasiewicz implication whenever $n > 5$ is a Fermat prime number (that is, n is a prime of the form $n = 2^{2^m} + 1$ for some $m > 1$) such as $n = 17$, $n = 257$ or $n = 65537$.[3] On the other hand, any $\mathbf{Ł}^*_{n+1}$ (n being prime or not) can always define the order implication ($x \Rightarrow_c y = 1$ if $x \leq y$, and $x \Rightarrow_c y = 0$ otherwise) and Gödel implication $\Rightarrow_G$. This is an important fact from the point of view of the algebraic study of these structures, as we shall see.

Concerning axiomatizations, it is proved in Section 4 that all the matrix logics $\Lambda^*_{n+1,i} = \langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$ are Blok-Pigozzi algebraizable with the same quasivariety over the signature $\Sigma = (\vee, \sim, \star, \bot, \top)$. Then, an uniform axiomatization for all of these logics is obtained. The definition of these Hilbert calculi, together with the results on (un)characterizability of $\mathbf{Ł}_{n+1}$ in terms of Σ,

[1]We warn the reader that the notation used in the present paper and that of [13] are not exactly the same. Indeed, while in [13] the MV-chain with $n+1$ elements is called *MV_n-chain*, here, as we already used above, that chain will be called *MV_{n+1}-chain*. The same variation applies when we will speak about varieties generated by chains with $n+1$ elements.

[2]To be precise, in [13] both questions were posed only with respect to n prime. This was motivated by the fact that L^i_n, for n prime and $i/n \leq 1/2$, constitute an interesting family of paraconsistent logics.

[3]As of 2021, these are the only known Fermat primes greater than 5.

constitute a complete solution of (an extended version of) the first problem posed in [13]. Concerning the algebraic study of these structures –the second question posed in [13]– it is also proved in Section 4 that the variety generated by $\mathbf{Ł}^*_{n+1}$ is constituted by $(n+1)$-valued Gödel algebras with involution expanded by an unary operator which, by simplicity, will be also denoted by $\star$, satisfying certain equations. This means that this class of algebras can be axiomatized by means of equations, thus being a variety.

Since not every subalgebra of $\mathbf{Ł}^*_{n+1}$ is isomorphic to $\mathbf{Ł}^*_{m+1}$ for some $m \leq n$, the question of studying the behaviour of the square operator in subalgebras of $\mathbf{Ł}^*_{n+1}$ is also tackled in the first part of Section 5. Let $[0,1]^*_{MV}$ be the algebra defined over the real unit interval by the Łukasiewicz operations $\vee, \neg, *$. Since every $\mathbf{Ł}^*_{n+1}$ is a (finite) subalgebra of $[0,1]^*_{MV}$, such study is performed by analyzing the finite subalgebras of this algebra.

As observed above, every $\mathbf{Ł}^*_{n+1}$ can define the Gödel implication $\Rightarrow_G$; however, this operator (as well as the Monteiro-Baaz Δ operator) is not definable in $[0,1]^*_{MV}$. This suggests the definition of a more comprehensive class of algebras, obtained by adding a unary $*$-like operator, denoted by $\star$, to Gödel chains with an involutive negation, as it is done in the second part of Section 5. Finally, the Gödel algebras with involutive negation and a $\star$ operation such that their implication-free reducts coincide with subalgebras of $\mathbf{Ł}^*_{n+1}$ are axiomatically characterized.

The structure of the paper is completed with some needed preliminaries gathered in the next section and with Section 6 containing some conclusions and open problems. Finally, in the Appendix we will present an alternative proof for the algebraizability of the logics we will study in Subsection 4.2 that uses Abstract Algebraic Logic methods and that has been suggested by one of the anonymous referees.

2 Preliminaries

Along this paper we will be mainly concerned with the classes of finite chains belonging to the varieties $\mathbb{MV}$ of MV-algebras and $\mathbb{G}$ of Gödel algebras. One of the most relevant class of algebras that contains both MV and Gödel-algebras is the variety $\mathbb{BL}$ of Hájek's BL-algebras [26]. Let us start recalling that a BL-algebra is a bounded, integral and commutative residuated lattice $\mathbf{A} = (A, \wedge, \vee, \odot, \Rightarrow, 0, 1)$ that further satisfies the following equations:

- $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ (*prelinearity*)
- $x \wedge y = x \odot (x \Rightarrow y)$ (*divisibility*)

In every BL-algebra $\mathbf{A}$ one can define further operations. In particular, for all $a \in A$, the *residual negation* (or simply the *negation*) of a is denoted by $\neg a$ and stands for $a \Rightarrow 0$; also, for all $a, b \in A$, $a \Leftrightarrow b$ is an abbreviation for $(a \Rightarrow b) \wedge (b \Rightarrow a)$.

Further, a partial order relation $\leq$ can be defined: for all $a, b \in A$

$$a \leq b \text{ iff } a \Rightarrow b = 1 \text{ holds.}$$

The partial order $\leq$ coincides with the lattice order of $\mathbf{A}$. The BL-algebra $\mathbf{A}$ is said to be a *BL-chain* if $\leq$ is linear.

Definition 2.1. A BL-algebra $\mathbf{A}$ is said to be

- An *MV-algebra* if the equation $\neg\neg x = x$ holds in $\mathbf{A}$;
- A *Gödel-algebra* (or simply a G-algebra) if $x \odot y = x \wedge y$ holds in $\mathbf{A}$.

A BL-algebra, MV-algebra or G-algebra, is said to be *finite* if its universe is a finite set.

It is worth to point out that finite MV and Gödel chains are the "building blocks" of finite BL-chains. Indeed [7, Corollary 3.7] shows that finite BL-chains can only be ordinal sums of MV-chains and G-chains. One of the basic properties that distinguishes finite MV-chains from finite G-chains lies in the fact that, while MV-operations allow to describe the arithmetic sum between real numbers, in Gödel chains is only possible to describe the order of their elements.

In the rest of this paper, in order to ease the reading, we will distinguish MV-operations from Gödel operations adopting subscripts: the implication operator of MV-algebras (also called Łukasiewicz implication) will be denoted by $\Rightarrow_{\text{Ł}}$, while Gödel implication will be written $\Rightarrow_G$. The negation operators are defined as usual: MV-negation (or Łukasiewicz negation) $\neg_{\text{Ł}} x = x \Rightarrow_{\text{Ł}} 0$ and Gödel negation $\neg_G x = x \Rightarrow_G 0$.

The main differences between MV-algebras and Gödel algebras can be easily grasped by recalling how their operations behave in the standard algebras of the relative varieties. Recall in fact that both the variety $\mathbb{MV}$ and $\mathbb{G}$ can be generated by structures based on the real unit interval $[0,1]$. Those algebras, called respectively the *standard* MV-algebra (written $[0,1]_{MV}$) and the *standard* Gödel algebra (denoted by $[0,1]_G$) interpret operations as follows: for all $x, y \in [0,1]$,

- $x \odot y = \max\{0, x+y-1\}$; $x \wedge y = \min\{x, y\}$;
- $x \Rightarrow_{\text{Ł}} y = \min\{1, 1-x+y\}$; $x \Rightarrow_G y = 1$ if $x \leq y$ and $x \Rightarrow_G y = y$ otherwise;
- $\neg_{\text{Ł}} x = 1-x$; $\neg_G x = 1$ if $x = 0$ and $\neg_G x = 0$ otherwise.

In addition to the ones recalled above, in every MV-algebra, one can define further arithmetic operations like the bounded sum $x \oplus y = \neg_{\text{Ł}} x \Rightarrow_{\text{Ł}} y$ whose semantics in $[0,1]_{MV}$ is $x \oplus y = \min\{1, x+y\}$ and the square operator $*x = x \odot x$ that will play a main role in this paper and whose behavior in $[0,1]_{MV}$ is $*x = \max\{0, 2x-1\}$.

Finite MV-chains are easily characterized. Indeed, for each natural number n, the set $\text{Ł}_{n+1} = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$ is the domain of the $(n+1)$-valued MV-chain. Such algebra will be henceforth denoted by $\mathbf{Ł}_{n+1}$. The Gödel chain with $n+1$ elements will be denoted by $\mathbf{G}_{n+1}$.

Every finite MV-chain $\mathbf{Ł}_{n+1}$ (resp. every finite Gödel chain $\mathbf{G}_{n+1}$) generates a proper subvariety of $\mathbb{MV}$ (resp. of $\mathbb{G}$). Equations describing these subvarieties, within $\mathbb{MV}$ and $\mathbb{G}$, can be found e.g. in [25] for the case of MV and in [24] for the Gödel case.

Notice that, by definition, Łukasiewicz negation is involutive and thanks to this, all operations of any MV-algebra can be defined starting only from the signature $\{\Rightarrow_{\text{Ł}}, 0\}$. In fact, we will use that reduced signature when we will deal with MV-algebras in the remaining of the present paper.

Gödel negation $\neg_G$ does not satisfy the involution equation $\neg\neg x = x$. For this reason, an expansion of Gödel algebras by an involution has been studied in [17] (see also [19]). The corresponding algebraic structures are defined as follows.

Definition 2.2. A *Gödel algebra with involution* (IG-algebra for short) is a pair $(\mathbf{A}, \sim)$ where $\mathbf{A}$ is a Gödel algebra and $\sim : A \to A$ is a unary operator satisfying the following equations:

1. $\sim\sim x = x$
2. $\neg_G x \leq \sim x$
3. $\Delta(x \Rightarrow_G y) = \Delta(\sim y \Rightarrow_G \sim x)$
4. $\Delta x \vee \neg_G \Delta x = 1$
5. $\Delta(x \vee y) \leq \Delta x \vee \Delta y$
6. $\Delta(x \Rightarrow_G y) \leq \Delta x \Rightarrow_G \Delta y$

where $\Delta x = \neg_G \sim x$.

The class of IG-algebras form a variety that will be denoted by $\mathbb{IG}$. As it is proved in [17, Theorem 7], $\mathbb{IG}$ is generated by the IG-algebra $([0,1]_G, \sim)$ where $[0,1]_G$ is the standard G-algebra and $\sim x = 1-x$. The variety generated by the IG-chain with $n+1$ elements will be henceforth denoted by $\mathbb{IG}_{n+1}$.

It is worth noticing that the operator Δ appearing in Definition 2.2, and that is definable in IG-algebras by combining the two negations $\neg_G$ and $\sim$, is the Baaz-Monteiro operator [1]. In every totally ordered algebra, Δ behaves as follows: $\Delta(x) = 1$ if $x = 1$ and $\Delta(x) = 0$ otherwise. Such an operator is indeed also definable in every finite MV-chain $\mathbf{Ł}_{n+1}$ by the term $\Delta(x) = x^n = x \odot \ldots \odot x$ (n-times). Indeed, for every $0 \leq k < n$, $(k/n)^n = 0$ while $1^n = 1$. However, Δ is not definable in infinite MV-chains, whence in particular, it is not definable in $[0,1]_{MV}$.

3 Analyzing the square operator: first steps

In this section we start the study on the expressive power of Łukasiewicz's square operator $*$ by means of $(n+1)$-valued algebraic structures denoted by $\mathbf{Ł}^*_{n+1}$. After defining a fundamental algorithmic tool which allows to compute the subalgebras of $\mathbf{Ł}^*_{n+1}$, we will analyse the relationship between primality of n and term-equivalence between $\mathbf{Ł}^*_{n+1}$ and $\mathbf{Ł}_{n+1}$.

The next definition introduces the algebraic structures that will play a key role in this paper. Let $\mathbf{Ł}_{n+1} = (\mathrm{Ł}_{n+1}, \Rightarrow_{\mathrm{Ł}}, \neg_{\mathrm{Ł}}, 0, 1)$ be the MV-chain with $n+1$ elements on the domain $\mathrm{Ł}_{n+1} = \{0, 1/n, \ldots, (n-1)/n, 1\}$ of $\mathbf{Ł}_{n+1}$, where the strong conjunction $\odot$ is definable as usual, i.e. $x \odot y = \neg_{\mathrm{Ł}}(x \Rightarrow_{\mathrm{Ł}} \neg_{\mathrm{Ł}} y)$.

Definition 3.1. The algebra $\mathbf{Ł}^*_{n+1} = (\mathrm{Ł}_{n+1}, \vee, \neg_{\mathrm{Ł}}, *, 0, 1)$ is the structure obtained by adding the unary square operator $* : x \mapsto x \odot x$ and the join $\vee$ to the $\{\Rightarrow_{\mathrm{Ł}}\}$-free reduct of $\mathbf{Ł}_{n+1}$.

Therefore, for every n, $\mathbf{Ł}^*_{n+1}$ is the linearly-ordered algebra on the domain $\{0, 1/n, \ldots, (n-1)/n, 1\}$ endowed with the operations $x \vee y = \max\{x, y\}$, $\neg_{\mathrm{Ł}} x = 1 - x$ and

$$*x = \max\{0, 2x - 1\},$$

besides the constants 0 and 1. In every $\mathbf{Ł}^*_{n+1}$-chain we can define the operation $+$ that is the dual operation of $*$ w.r.t. the negation $\neg$:

$$+x = \neg_{\mathrm{Ł}} * \neg_{\mathrm{Ł}} x = \min\{1, 2x\}.$$

Furthermore, for every natural number $k \geq 1$, we will denote by $*^k$ the k-times iteration of $*$, that is $*^k x = *x$ if $k = 1$ and $*^k x = *^{k-1}(*x)$ for $k > 1$. This gives $*^k x = \max\{0, 2^k x - (2^k - 1)\}$. Similarly, we define $+^k x = +x$ if $k = 1$ and $+^k x = +^{k-1}(+x)$ otherwise, yielding $+^k x = \min\{1, 2^k x\}$.

Recall that an element $x \in \mathrm{Ł}_{n+1}$ is called *positive* if $x > \neg_{\mathrm{Ł}} x$, i.e. $x > 1/2$, otherwise it is called *negative*.

3.1 On the subalgebras of $\mathbf{Ł}^*_{n+1}$

For every subset X of $\mathrm{Ł}_{n+1}$ we will denote by $\langle X \rangle^*$ the subalgebra of $\mathbf{Ł}^*_{n+1}$ generated by X. In case $X = \{x\}$ we will write $\langle x \rangle^*$ instead of $\langle \{x\} \rangle^*$.

In the rest of this section we will only deal with Łukasiewicz operations. Thus, in order to ease the reading, we will omit the subscript Ł from operations without danger of confusion.

Let us present now an algorithmic tool that will be central in the rest of this paper.

Definition 3.2 (The procedure $\mathtt{P}$)**.** Let us consider a procedure, that we will henceforth denote by $\mathtt{P}$, defined as follows: given n and an element $a \in \mathrm{Ł}_{n+1} \setminus \{0, 1\}$, $\mathtt{P}(n, a)$ iteratively computes a sequence $[a_1, \ldots, a_m, \ldots]$ of elements of $\mathrm{Ł}_{n+1}$ such that $a_1 = a$ and for all $i \geq 1$,

$$a_{i+1} = \begin{cases} *a_i & \text{if } a_i > 1/2, \\ \neg a_i & \text{otherwise.} \end{cases}$$

We say that $\mathtt{P}(n, a)$ *stops at* k (or $\mathtt{P}(n, a)$ *stops at* a_k) if k is the first i such that $a_{i+1} = a_j$ for some $j < i$. Since $\mathrm{Ł}_{n+1}$ is finite then, for every $a \in \mathrm{Ł}_{n+1} \setminus \{0, 1\}$, there exists $k \geq 1$ such that $\mathtt{P}(n, a)$ stops at k. If $\mathtt{P}(n, a)$ stops at k, the *sequence generated by* $\mathtt{P}(n, a)$ is denoted also by $\mathtt{P}(n, a) = [a_1, \ldots, a_k]$, while the *image* and the *negated image* of $\mathtt{P}(n, a)$ are $\mathtt{I}(n, a) = \{a_1, \ldots, a_k\}$ and $\mathtt{NI}(n, a) = \{\neg a_1, \ldots, \neg a_k\}$, respectively. The *range* of $\mathtt{P}(n, a)$ is $\mathtt{R}(n, a) = \mathtt{I}(n, a) \cup \mathtt{NI}(n, a)$.

In order to exemplify the procedure $\mathtt{P}$ defined above, let us present two concrete numerical examples that will also turn out to be useful for what follows.

Example 3.3. (1) Let us fix $n = 9$ so that $\mathrm{Ł}_{9+1} = \mathrm{Ł}_{10}$ is the MV-chain of 10 elements on the domain $\{0, 1/9, \ldots, 8/9, 1\}$. Take $a = 8/9$, its coatom. Then, $\mathtt{P}(9, 8/9)$ produces a sequence $[a_1, a_2, a_3, a_4]$ in the following way:

- $a_1 = a = 8/9$.
- Since $a_1 = 8/9 > 1/2$, $a_2 = *a_1 = 2a_1 - 1 = 7/9$. Again $7/9 > 1/2$ and thus $a_3 = *(7/9) = 5/9 > 1/2$. Therefore, $a_4 = *(5/9) = 1/9$.

- Now, $a_4 = 1/9 < 1/2$, and hence $a_5 = \neg a_4 = 8/9$. Since $a_5 = a_1$, then $\mathtt{P}$ stops and outputs the string $\mathtt{P}(9, 8/9) = [8/9, 7/9, 5/9, 1/9]$.

Therefore, the image of $\mathtt{P}(9, 8/9)$ is $\mathtt{I}(9, 8/9) = \{8/9, 7/9, 5/9, 1/9\}$, its negated image is $\mathtt{NI}(9, 8/9) = \{1/9, 2/9, 4/9, 8/9\}$ while its range is $\mathtt{R}(9, 8/9) = \mathtt{I}(9, 8/9) \cup \mathtt{NI}(9, 8/9) = \{8/9, 7/9, 5/9, 4/9, 2/9, 1/9\}$.

(2) Now, let us fix $n = 17$ and hence the MV-chain $Ł_{18}$ and let $a = 1/17$, its atom. Then, $\mathtt{P}(17, 1/17)$ starts by $a_1 = a < 1/2$ and hence $a_2 = \neg a_1 = 16/17$ and, proceeding as above, it meets the elements $a_3 = *(16/17) = 15/17$, $a_4 = *(15/17) = 13/17$, $a_5 = *(13/17) = 9/17$, $a_6 = *(9/17) = 1/17$ and it stops since $a_6 = a_1$.

Hence, the image of $\mathtt{P}(17, 1/17)$ is $\mathtt{I}(17, 1/17) = \{1/17, 16/17, 15/17, 13/17, 9/17\}$, the negated image of $\mathtt{P}(17, 1/17)$ is $\mathtt{NI}(17, 1/17) = \{16/17, 1/17, 2/17, 4/17, 8/17\}$ and its range is $\mathtt{R}(17, 1/17) = \{1/17, 2/17, 4/17, 8/17, 9/17, 13/17, 15/17, 16/17\}$.

Observe that, for every $a \in Ł^*_{n+1} \setminus \{0, 1\}$, the set of positive elements of $\mathtt{R}(n, a)$ coincides with the set of positive elements of $\mathtt{I}(n, a)$, i.e. the set $\mathtt{NI}(n, a)$ does not introduce new positive elements, i.e. all positive elements of $\mathtt{R}(n, a)$ belong to the sequence $\mathtt{P}(n, a)$ obtained by the procedure P starting at a.

As a first application of the procedure $\mathtt{P}$, it will be shown that it allows us to compute the subalgebras of $\mathbf{Ł}^*_{n+1}$ of the form $\langle a \rangle^*$.

Proposition 3.4. *Let $a \in Ł^*_{n+1} \setminus \{0, 1\}$. Then, the domain of the subalgebra $\langle a \rangle^*$ of $\mathbf{Ł}^*_{n+1}$ is $\mathtt{R}(n, a) \cup \{0, 1\}$.*

Proof. By definition of the procedure $\mathtt{P}$ and the above observation it is easy to prove that $\mathtt{R}(n, a) \cup \{0, 1\}$ is closed under $*$ and $\neg$ and so it is the domain of a subalgebra containing a, i.e. $\langle a \rangle^* \subseteq \mathtt{R}(n, a) \cup \{0, 1\}$. Moreover every element of $\mathtt{R}(n, a)$ is obtained from a using only the operations $*$ and $\neg$, hence $\mathtt{R}(n, a) \cup \{0, 1\} \subseteq \langle a \rangle^*$. Therefore $\mathtt{R}(n, a) \cup \{0, 1\} = \langle a \rangle^*$. □

Notice that if a and b are positive elements, and b is reached by the procedure $\mathtt{P}(n, a)$, i.e. $b \in \mathtt{I}(n, a)$, then the sequence generated by $\mathtt{P}(n, b)$ is in fact a subsequence of the one generated by $\mathtt{P}(n, a)$, and hence, $\langle b \rangle^* \subseteq \langle a \rangle^*$. On the other hand, it is clear that for each $a \in Ł^*_{n+1} \setminus \{0, 1\}$, $\langle a \rangle^* = \langle \neg a \rangle^*$. Therefore from now on we will consider only subalgebras generated by positive elements since they cover all the one-generated subalgebras.

<u>*Notation*</u>: In what follows, for every $\mathbf{Ł}^*_{n+1}$-algebra, we will denote by $\mathbf{c}$ its coatom $(n-1)/n$.

Lemma 3.5. *For every $\mathbf{Ł}^*_{n+1}$-algebra, if $\langle \mathbf{c} \rangle^* = \mathbf{Ł}^*_{n+1}$, then every positive element a of $Ł^*_{n+1} \setminus \{1\}$ is reached by the procedure $\mathtt{P}$ starting at $\mathbf{c}$, that is, $a \in \mathtt{I}(n, \mathbf{c})$.*

Proof. It follows from the fact that the set of positive elements of $\mathtt{R}(n, \mathbf{c})$ (which, by hypothesis and by Proposition 3.4 is the set of positive elements of $\mathbf{Ł}^*_{n+1} \setminus \{1\}$) is included in $\mathtt{I}(n, \mathbf{c})$. □

In the rest of this paper we will make often use of the notion of *strictly simple* algebra whose definition is recalled below adapting to our case the general definition that can be found in [23].

Definition 3.6. An algebra $\mathbf{Ł}^*_{n+1}$ is said to be *strictly simple* if its unique proper subalgebra is the two-element chain $\{0, 1\}$.[4]

Then we can prove the following.

Lemma 3.7. *For every $\mathbf{Ł}^*_{n+1}$-algebra, if $\langle \mathbf{c} \rangle^* = \mathbf{Ł}^*_{n+1}$ and $\mathtt{P}(n, \mathbf{c}) = [a_1, \ldots, a_k]$ with $a_k = 1/n$, then $\mathbf{Ł}^*_{n+1}$ is strictly simple.*

Proof. Let $\langle \mathbf{c} \rangle^* = \mathbf{Ł}^*_{n+1}$ and let b be a positive element of $\mathbf{Ł}^*_{n+1}$. By Lemma 3.5, $b \in \mathtt{P}(n, \mathbf{c})$ and so the initial segment of $\mathtt{P}(n, b)$ is a subsequence of $\mathtt{P}(n, \mathbf{c})$. In particular, the last element $1/n$ of $\mathtt{P}(n, \mathbf{c})$ belongs to $\mathtt{I}(n, b)$ and so $c \in \mathtt{R}(n, b)$. In other words $\langle b \rangle^* = \langle \mathbf{c} \rangle^* = \mathbf{Ł}^*_{n+1}$, and the latter does not contain proper subalgebras and hence it is strictly simple. □

Finally, we have the following characterization for strictly simple $\mathbf{Ł}^*_{n+1}$-algebras.

[4]The definition of *strictly simple* algebra usually requires an algebra $\mathbf{A}$ to have no non-trivial proper subalgebras, in other words, $\mathbf{A}$ is strictly simple if the trivial, one element algebra, is its unique subalgebra. However, in our context we always assume $0 \neq 1$, so that algebras have at least two elements, and thus the trivial algebra is the (Boolean) two-element algebra $\{0, 1\}$.

Theorem 3.8. *For all $n > 1$ and $n \neq 4$, $\mathbf{Ł}^*_{n+1}$ is strictly simple iff $\langle \mathbf{c} \rangle^* = \mathbf{Ł}^*_{n+1}$.*

Proof. The left-to-right direction is obvious.

Let us hence assume that $\langle \mathbf{c} \rangle^* = \mathbf{Ł}^*_{n+1}$. We distinguish the following cases:

- n is even: The case $n = 2$ clearly fulfils the claim. Then notice that for any even number $n > 2$, $\langle 1/2 \rangle^*$ is a proper subalgebra of $\mathbf{Ł}^*_{n+1}$, hence $\mathbf{Ł}^*_{n+1}$ is not strictly simple. Thus the case $n = 4$ does not fulfill the claim since $\langle 3/4 \rangle^* = \mathbf{Ł}^*_5$. Now suppose $n > 4$. In order to get the claim, we have hence to prove that $\langle \mathbf{c} \rangle^* \neq \mathbf{Ł}^*_{n+1}$. Since n is even, it is easy to see that every application of either $*$ or $\neg$ to a rational number with even denominator, will output a rational with the same denominator and even numerator. In other words $\langle \mathbf{c} \rangle^* \setminus \{\mathbf{c}, \neg\mathbf{c}\}$ only contains rationals with even numerators, hence $\langle \mathbf{c} \rangle^*$ is a proper subalgebra of $\mathbf{Ł}^*_{n+1}$.
- $n > 1$ and odd: Let $a = ((n+1)/2)/n$ be the least positive element of $\mathbf{Ł}^*_{n+1}$ and let $\mathtt{P}(n, \mathbf{c}) = [a_1, \ldots, a_k]$. By Lemma 3.5, $a = a_t$ for some $a_t \in \mathtt{I}(n, \mathbf{c})$. A direct computation shows that $a_{t+1} = *a = 1/n$, hence it must be $a_{t+1} = a_k$. Thus, by Lemma 3.7, $\mathbf{Ł}^*_{n+1}$ is strictly simple.

□

Let us end this subsection with a comparison between $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$-algebras concerning subalgebras and strictly simple algebras. In particular, recall that a finite MV-chain $\mathbf{Ł}_{n+1}$ is strictly simple iff n is prime [25].

Proposition 3.9. *The following holds for every $n, m \geq 2$:*

- *If $\mathbf{Ł}_{n+1}$ is subalgebra of $\mathbf{Ł}_{m+1}$, then $\mathbf{Ł}^*_{n+1}$ is subalgebra of $\mathbf{Ł}^*_{m+1}$.*
- *If $\mathbf{Ł}^*_{n+1}$ is strictly simple, then n is prime.*

Proof. The first item is a consequence of the fact that the operation $*$ of $\mathbf{Ł}^*_{n+1}$ is definable in $\mathbf{Ł}_{n+1}$. The second item is a consequence of the first item plus the already recalled fact from [25] stating that $\mathbf{Ł}_{n+1}$ is strictly simple if and only if n is prime. □

Notice that for every $\mathbf{Ł}_{n+1}$, the subalgebras are algebras of type $\mathbf{Ł}_{m+1}$ with m being a divisor of n and $\mathbf{Ł}_{n+1}$ is strictly simple if n is a prime number. Both statements are not true for $\mathbf{Ł}^*_{n+1}$. There exists subalgebras of $\mathbf{Ł}^*_{n+1}$ that are not of type $\mathbf{Ł}^*_{m+1}$ and there exists prime numbers n such that $\mathbf{Ł}^*_{n+1}$ is not strictly simple as the following examples show.

Example 3.10. This is a follow-up of Example 3.3.

(1) Let $n = 9$. Then, $\langle 8/9 \rangle^* = \{0, 1/9, 2/9, 4/9, 5/9, 7/9, 8/9, 1\}$. This subalgebra is a chain of 8 elements which is not isomorphic to $\mathbf{Ł}^*_{7+1}$. Indeed, in $\mathbf{Ł}^*_{7+1}$, we have $*(5/7) = 3/7$, and the correspondent (w.r.t. the order) element of $5/7$ in $\langle 8/9 \rangle^*$ is $7/9$. But in this algebra $*(7/9) = 5/9$, that corresponds to $4/7$, instead of $3/7$, in $\mathbf{Ł}^*_{7+1}$. This shows that, although both $\langle 8/9 \rangle^*$ and $\mathbf{Ł}^*_{7+1}$ are 8-element algebras, they are not isomorphic.

(2) Let $n = 17$ (that is prime) and consider the subalgebra of $\mathbf{Ł}^*_{18}$ generated by its co-atom $16/17$. A direct computation shows that

$$\left\langle \frac{16}{17} \right\rangle^* = \left\{0, \frac{1}{17}, \frac{2}{17}, \frac{4}{17}, \frac{8}{17}, \frac{9}{17}, \frac{13}{17}, \frac{15}{17}, \frac{16}{17}, 1\right\},$$

which is in fact a proper non-trivial subalgebra of $\mathbf{Ł}^*_{18}$, showing that the latter is not strictly simple.

A more detailed study on the subalgebras of $\mathbf{Ł}^*_{n+1}$-algebras will be the object of Subsection 5.1.

3.2 Term-equivalence between $\mathbf{Ł}^*_{n+1}$ and $\mathbf{Ł}_{n+1}$

In this section we will characterize those algebras $\mathbf{Ł}^*_{n+1}$ that allow to define Łukasiewicz implication $\Rightarrow_{\mathrm{L}}$ and hence, that are term-equivalent to the original finite MV-chain $\mathbf{Ł}_{n+1}$.

Let us start proving that in every $\mathbf{Ł}^*_{n+1}$ we can define terms characterizing the principal order filter $F_a = \{b \in \mathrm{Ł}_{n+1} \mid b \geq a\}$ generated by a.

Proposition 3.11. *For each $a \in L_{n+1}$, the unary operation Δ_a defined as*

$$\Delta_a(x) = \begin{cases} 1 & \textit{if } x \in F_a \\ 0 & \textit{otherwise.} \end{cases}$$

is definable in $\mathbf{Ł}^*_{n+1}$. As a consequence, for every $a \in Ł_{n+1}$, the operation χ_a that corresponds to the characteristic function of a (i.e. $\chi_a(x) = 1$ if $x = a$ and $\chi_a(x) = 0$ otherwise) is definable as well.

Proof. The case $a = 1$ corresponds to the Monteiro-Baaz Δ operator and, as is well-known, it can be defined as $\Delta_1(x) = *^n x$. For the case $a = 0$, define $\Delta_0(x) = \Delta_1(x) \vee \neg\Delta_1(x)$; this gives $\Delta_0(x) = 1$ for every x.

In order to define $\Delta_a(x)$ for $0 < a < 1$, consider the following notions.

Given $a, b \in Ł_{n+1}$ such that $a > b$ we say that (a, b) is *separated* if either: (1) $a > 1/2 \geq b$, or (2) $b = 0$, or (3) $a = 1$. Clearly, if (a, b) is not separated then either $1 > a > b > 1/2$ or $1/2 \geq a > b > 0$.

From now on we will consider terms $t(x)$ on a variable x formed by combining applications of $*$ and $+$. Such terms are monotonic, i.e., if $a \geq b$ then $t(a) \geq t(b)$. Observe that, for any $0 < a < 1$ there exists m and k such that $+^m a = 1$ and $*^k a = 0$.

Fact 1. *If (a, b) is separated then there exists a term $t(x)$ as above such that $t(a) = 1$ and $t(b) = 0$.*

Indeed, if (1) holds then $*a > *b = 0$. Let $t(x) = +^k * x$ such that $k = \min\{m \mid +^m * a = 1\}$. If (2) holds let $t(x) = +^k x$ such that $k = \min\{m \mid +^m a = 1\}$. If (3) holds let $t(x) = *^k x$ such that $k = \min\{m \mid *^m b = 0\}$. In any case, t is as required, by observing that $*1 = 1$ and $+0 = 0$.

Now, given $0 < a < 1$, let a^- be its immediate predecessor in the chain, i.e. $a^- = a - 1/n$. If (a, a^-) is separated then, by **Fact 1**, $\Delta_a(x) = t(x)$ is as required, since t is monotonic. If (a, a^-) is not separated, a sequence of pairs (x_i, y_i) of elements in $Ł_{n+1}$ such that $x_i > y_i$ will be defined by taking $x_0 = a$, $y_0 = a^-$ and by considering, for every $i \geq 0$, the following two cases:

Case A: Let (x_i, y_i) such that $1 > x_i > y_i > 1/2$. Let $k_i = \max\{m \mid *^m x_i > *^m y_i\}$. Let $x_{i+1} = t_{i+1}(x_i)$ and $y_{i+1} = t_{i+1}(y_i)$, where $t_{i+1}(x)$ is the term on a variable x given by $t_{i+1}(x) = *^{k_i} x$. Note that $x_{i+1} > y_{i+1}$. If (x_{i+1}, y_{i+1}) is separated the procedure stops. Otherwise, note that $1/2 \geq x_{i+1} > y_{i+1} > 0$. Go to **Case B** with input (x_{i+1}, y_{i+1}).

Case B: Let (x_i, y_i) such that $1/2 \geq x_i > y_i > 0$. If $x_i = 1/2$ let $x_{i+1} = +x_i$ and $y_{i+1} = +y_i$. Then, (x_{i+1}, y_{i+1}) is separated and the procedure stops. Otherwise, if $1/2 > x_i$, let $k_i = \max\{m \mid +^m x_i > +^m y_i\}$. Let $x_{i+1} = t_{i+1}(x_i)$ and $y_{i+1} = t_{i+1}(y_i)$, where $t_{i+1}(x)$ is the term given by $t_{i+1}(x) = +^{k_i} x$. Note that $x_{i+1} > y_{i+1}$. If (x_{i+1}, y_{i+1}) is separated the procedure stops. Otherwise, note that $1 > x_{i+1} > y_{i+1} > 1/2$. Go to **Case A** with input (x_{i+1}, y_{i+1}).

By definition, if the procedure defined above stops then the output (x_{i+1}, y_{i+1}) is separated. Note that $x_{i+1} = \bar{t}(a)$ while $y_{i+1} = \bar{t}(a^-)$ for some term $\bar{t}(x)$. In such a case, $\Delta_a(x)$ can be defined by the term $t(\bar{t}(x))$, where $t(x)$ is a term as specified in **Fact 1**. Indeed, $\Delta_a(a) = t(x_{i+1}) = 1$ and $\Delta_a(a^-) = t(y_{i+1}) = 0$ and so $\Delta_a(x)$ is as required, and being a term constructed by combining applications of $*$ and $+$, it is monotonic. Thus, it remains to prove that the procedure above always stops. But this is easy to see from the following observation: if (a, b) is not separated then either $(\#a, \#b)$ is separated or $\#a - \#b = 2(a - b)$, for $\# \in \{+, *\}$. This means that, either (x_{i+1}, y_{i+1}) is separated, or the distance $x_{i+1} - y_{i+1}$ between x_{i+1} and y_{i+1} is strictly greater than the distance $x_i - y_i$ between x_i and y_i. Given that the distance between two elements a and b of $Ł_{n+1}$ is itself an element of $Ł_{n+1}$ (which is a finite set) and it is defined as $\neg(a \Rightarrow b) \vee \neg(b \Rightarrow a)$, a separated (x_{i+1}, y_{i+1}) must be found at some step $i + 1$.

From the previous constructions we have shown that $\Delta_a(x)$ can always be constructed by means of a term which combines applications of $*$ and $+$. Finally, as for the operations χ_a, define $\chi_1 = \Delta_1$; $\chi_0(x) = \neg\Delta_{1/n}(x)$, and if $0 < a < 1$, then define $\chi_a(x) = \Delta_a(x) \wedge \neg\Delta_{a^+}(x)$, where $a^+ = a + 1/n$ is the immediate successor of a. □

Next we show an example of the above procedure to find the unary operations Δ_a.

Example 3.12. Let us consider the $\mathbf{Ł}^*_{n+1}$-chain for $n = 11$, and let $a = 8/11$. We show how we can find the operation $\Delta_{8/11}$ according to the procedure described in the proof of the previous proposition. In this case a and $a^- = 7/11$ are both positive, so it fits with Case A above. Hence, the procedure above produces the following sequence of pairs (by simplicity, the denominator 11

x	0	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{3}{11}$	$\frac{4}{11}$	$\frac{5}{11}$	$\frac{6}{11}$	$\frac{7}{11}$	$\mathbf{\frac{8}{11}}$	$\mathbf{\frac{9}{11}}$	$\frac{10}{11}$	1
$*x$	0	0	0	0	0	0	$\frac{1}{11}$	$\frac{3}{11}$	$\frac{5}{11}$	$\frac{7}{11}$	$\frac{9}{11}$	1
$+x$	0	$\frac{2}{11}$	$\frac{4}{11}$	$\frac{6}{11}$	$\frac{8}{11}$	$\frac{10}{11}$	1	1	1	1	1	1
$+*x$	0	0	0	0	0	0	$\frac{2}{11}$	$\frac{6}{11}$	$\frac{10}{11}$	1	1	1
$*^2+*x$	0	0	0	0	0	0	0	0	$\frac{7}{11}$	1	1	1
$+*^2+*x$	0	0	0	0	0	0	0	0	1	1	1	1
$*^2x$	0	0	0	0	0	0	0	0	0	$\frac{3}{11}$	1	1
$+^2*^2x$	0	0	0	0	0	0	0	0	0	1	1	1

Table 1: Some definable operations in $\mathbf{Ł}^*_{12}$

will be omitted): $(8,7) \overset{*}{\mapsto} (5,3) \overset{+}{\mapsto} (10,6) \overset{*^2}{\mapsto} (7,0)$. Since $+(7/11) = 1$, we obtain that $\Delta_{8/11}(x) = t(t_3(t_2(t_1(x)))) = +*^2+*x$, by using the notation of the proof of Proposition 3.11. Similarly, one can check that $\Delta_{9/11}(x) = t(t_1(x))$, where $t_1(x) = *^2x$ and $t(x) = +^2x$, i.e. $\Delta_{9/11}(x) = +^2*^2x$. In this case, the pairs produced by the procedure described in the proof above are $(9,8) \overset{*^2}{\mapsto} (3,0)$. Therefore, $\chi_{8/11}(x) = \Delta_{8/11}(x) \wedge \neg\Delta_{9/11}(x) = \min(+*^2+*x, 1 - +^2*^2x)$.

Table 1 shows, besides the operations $*$ and $+$, the different steps to obtain $\Delta_{8/11}$ and $\Delta_{9/11}$. The reader can easily obtain the other operators Δ_a from such table by applying the given procedure.

Actually, Proposition 3.11 can be straightforwardly generalized to any subalgebra of a $\mathbf{Ł}^*_{n+1}$.

Corollary 3.13. *Let* $\mathbf{A}$ *be a subalgebra of* $\mathbf{Ł}^*_{n+1}$. *Then, we have the following:*

(i) for any element $a \in \mathbf{A}$, *the operations* Δ_a *and* χ_a *are also definable in* $\mathbf{A}$.

(ii) $\mathbf{A}$ *is simple.*

Proof. (i) Indeed, the same procedure defined in the proof of Proposition 3.11 to find the terms for Δ_a and χ_a in $\mathbf{Ł}^*_{n+1}$ works in $\mathbf{A}$ as well, as the operations $*$ and $\neg$ are obviously closed in $\mathbf{A}$. The argument given in that proof to show that the procedure always stops remains the same.

As for (ii), it comes as a corollary of the fact that in $\mathbf{Ł}^*_{n+1}$ and in any of its subalgebras $\mathbf{A}$, the operator Δ_1 is definable, and hence any congruence θ of these algebras has to be closed under Δ_1. This implies that θ is either the trivial congruence or the identity. □

It is now almost immediate to check that the crisp (or order) implication as well as the Gödel implication are definable in every $\mathbf{Ł}^*_{n+1}$.

Proposition 3.14. *The order implication and Gödel implication,*

$$x \Rightarrow_c y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases} \qquad x \Rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

are both definable in $\mathbf{Ł}^*_{n+1}$.

Proof. Indeed, $\Rightarrow_c$ can be defined as

$$x \Rightarrow_c y = \bigvee_{0 \leq i \leq n} (\chi_{i/n}(x) \wedge \Delta_{i/n}(y)). \tag{1}$$

In turn, Gödel implication is given by $x \Rightarrow_G y = (x \Rightarrow_c y) \vee y$. □

Now we are ready to prove the main result of this section, that is a characterization of those algebras $\mathbf{Ł}^*_{n+1}$ that define Łukasiewicz implication $\Rightarrow_{Ł}$ or, equivalently, of those algebras $\mathbf{Ł}^*_{n+1}$ that are term-equivalent to $\mathbf{Ł}_{n+1}$. For the next result, recall how strictly simple algebras are introduced in Definition 3.6.

Theorem 3.15. *For all* $n \neq 4$, *the finite MV-chain* $\mathbf{Ł}_{n+1}$ *is term equivalent to* $\mathbf{Ł}^*_{n+1}$ *iff* $\mathbf{Ł}^*_{n+1}$ *is strictly simple.*

Proof. Left-to-right: If $\mathbf{Ł}_{n+1}$ is term-equivalent to $\mathbf{Ł}^*_{n+1}$ then Łukasiewicz product $\odot$ is definable in $\mathbf{Ł}^*_{n+1}$, and hence $\langle (n-1)/n\rangle^* = \mathbf{Ł}^*_{n+1}$. Indeed, we can obtain $(n-i-1)/n = ((n-1)/n)\odot((n-i)/n)$ for $i = 1,\dots,n-1$, and $1 = \neg 0$. By Theorem 3.8 it follows that $\mathbf{Ł}^*_{n+1}$ is strictly simple.

Right-to-left: since $\mathbf{Ł}^*_{n+1}$ is strictly simple then, for each $a,b \in \mathrm{Ł}_{n+1}$ where $a \notin \{0,1\}$ there is a definable term $t_{a,b}(x)$ such that $t_{a,b}(a) = b$. Otherwise, if for some $a \notin \{0,1\}$ and $b \in \mathrm{Ł}_{n+1}$ there is no such term then $\mathbf{A} = \langle a\rangle^*$ would be a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ (since $b \notin \mathbf{A}$) different from $\{0,1\}$, a contradiction. Now, for $0 \le j < i \le n$ consider terms $\mathbf{t}_{i,j}(x,y)$ such that $\mathbf{t}_{i,j}(i/n, j/n) = (i/n) \Rightarrow_{\mathrm{L}} (j/n)$. Such terms can be defined as follows: if $n > i > j \ge 0$ then $\mathbf{t}_{i,j}(x,y) = t_{i/n,a_{ij}}(x)$, where $a_{ij} = 1 - i/n + j/n$; and $\mathbf{t}_{n,j}(x,y) = y$ for $0 \le j < n$. Since by Proposition 3.11 the operations $\chi_a(x)$ are definable for each $a \in \mathrm{Ł}_{n+1}$, then in $\mathbf{Ł}^*_{n+1}$ we can define the Łukasiewicz implication $\Rightarrow_{\mathrm{L}}$ as follows:

$$x \Rightarrow_{\mathrm{L}} y = (x \Rightarrow_c y) \vee \left(\bigvee_{n \ge i > j \ge 0} \chi_{i/n}(x) \wedge \chi_{j/n}(y) \wedge \mathbf{t}_{i,j}(x,y) \right)$$

where $x \Rightarrow_c y$ is defined as in Proposition 3.14. □

Remark 3.16. The case $n = 4$ is a singular one: it is the only counterexample for Theorems 3.8 and 3.15. First, $\mathbf{Ł}^*_5$ is generated by its coatom: $\mathbf{Ł}^*_5 = \langle 3/4\rangle^*$. In addition, it is term-equivalent to $\mathbf{Ł}_5$. Indeed, Łukasiewicz implication $\Rightarrow_{\mathrm{L}}$ can be defined in $\mathbf{Ł}^*_5$ as in the proof of Theorem 3.15, with suitable adaptations. For $0 \le j < i \le 4$ consider terms $\mathbf{t}_{i,j}(x,y)$ such that $\mathbf{t}_{i,j}(i/4, j/4) = (i/4) \Rightarrow_{\mathrm{L}} (j/4)$. Such terms can be defined as follows (observing that $1/2 = 2/4$ in $\mathbf{Ł}^*_5$): $\mathbf{t}_{4,j}(x,y) = y$ for $0 \le j < 4$; $\mathbf{t}_{i,0}(x,y) = \neg x$ for $0 < i \le 4$; $\mathbf{t}_{3,2}(x,y) = x$; $\mathbf{t}_{3,1}(x,y) = *x$; and $\mathbf{t}_{2,1}(x,y) = \neg y$. Then, $x \Rightarrow_{\mathrm{L}} y$ is given by the term

$$x \Rightarrow_{\mathrm{L}} y = (x \Rightarrow_c y) \vee \left(\bigvee_{4 \ge i > j \ge 0} \chi_{i/4}(x) \wedge \chi_{j/4}(y) \wedge \mathbf{t}_{i,j}(x,y) \right).$$

However, $\mathbf{Ł}^*_5$ is not strictly simple, since it has the non-trivial subalgebra with domain $\{0, 1/2, 1\}$.

3.3 Strictly simple $\mathbf{Ł}^*_{n+1}$-chains and prime numbers

As we have shown in Proposition 3.9, n being prime is a necessary condition for $\mathbf{Ł}^*_{n+1}$ to be strictly simple which, in turn, is equivalent to the term-equivalence between $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ (if $n \neq 4$) by Theorem 3.15. However, the primality of n is not a sufficient condition for $\mathbf{Ł}^*_{n+1}$ to be strictly simple. In fact, as the following result shows, there are prime numbers n for which $\mathbf{Ł}^*_{n+1}$ contains non-trivial subalgebras. This fact was already observed in Example 3.10 (2).

Lemma 3.17. *If $n > 5$ is of the form $n = 2^m + 1$ then $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are not term-equivalent.*

Proof. Let n be of the form $n = 2^m + 1$ for some $m > 2$. If $\mathbf{c} = (n-1)/n$ then $*^m\mathbf{c} = 1/n$. By Proposition 3.4, the algebra $\langle \mathbf{c}\rangle^*$ has domain $\mathtt{I}(n,\mathbf{c}) \cup \mathtt{NI}(n,\mathbf{c}) \cup \{0,1\}$ (recall Definition 3.2). Since $\mathtt{I}(n,\mathbf{c})$ has $m+1$ elements then $\mathtt{NI}(n,\mathbf{c}) \setminus \mathtt{I}(n,\mathbf{c})$ has at most $m-1$ elements (since $\mathbf{c} = \neg(1/n)$ and $1/n = \neg\mathbf{c}$ belong to $\mathtt{NI}(n,\mathbf{c}) \cap \mathtt{I}(n,\mathbf{c})$). Hence, the algebra $\langle \mathbf{c}\rangle^*$ has at most $2+2(m-1)+2 = 2m+2$ elements. Since $2m+2 < 2^m+1 = n$ as $m > 2$, $\langle \mathbf{c}\rangle^*$ is properly contained in $\mathbf{Ł}_{n+1}$, and it is different from $\{0,1\}$. Therefore $\langle \mathbf{c}\rangle^*$ is a proper non trivial subalgebra of $\mathbf{Ł}^*_{n+1}$, and the result follows from Theorem 3.15. □

It is well-known that if $n = 2^m + 1$ is prime then m is of the form 2^k; in such case, $n = 2^{(2^k)} + 1$ is said to be a *Fermat prime*. As mentioned in the introduction, up to 2021 the only known Fermat primes are 3, 5, 17, 257, and 65537, and it is an open problem to determine whether there are infinitely many such prime numbers. Therefore, for any prime $n > 5$ of the form $n = 2^m + 1$ (i.e., for any Fermat prime > 5), $\mathbf{Ł}^*_{n+1}$ contains non-trivial subalgebras, and hence it is not strictly simple.

Notice that, as we showed in Example 3.10 (2), the subalgebra of $\mathbf{Ł}^*_{18}$ generated by its coatom is a proper subalgebra of $\mathbf{Ł}^*_{18}$ and indeed, 17 is the first Fermat prime number greater than 5.

We have seen that, in contrast with the case of the $\mathbf{Ł}_{n+1}$-algebras, which are strictly simple iff n is prime, there are prime numbers for which $\mathbf{Ł}^*_{n+1}$ contains proper non-trivial subalgebras. It is however possible to characterize those prime numbers which ensure the term-equivalence between $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$. Let us start by the following definition.

Definition 3.18. Let Π be the set of odd primes n such that 2^m is not congruent with ± 1 mod n for all m such that $0 < m < (n-1)/2$.[5]

By Fermat's little theorem, 2^{n-1} is congruent with 1 mod n, for every odd prime n. Since $2^{n-1} = b^2$ for $b = 2^m$ and $m = (n-1)/2$, it follows that b^2 is congruent with 1 mod n. But then, using that n is prime, we conclude that $b = 2^m$ is congruent with ± 1 mod n, for $m = (n-1)/2$. From this, n is in Π iff n is an odd prime such that $(n-1)/2$ is the least $m > 0$ such that 2^m is congruent with ± 1 mod n.

As a matter of example, the first prime numbers (below 200) in the set Π are: 3, 5, 7, 11, 13, 19, 23, 29, 37, 47, 53, 59, 61, 67, 71, 79, 83, 101, 103, 107, 131, 139, 149, 163, 167, 173, 179, 181, 191, 197 and 199.

The following Theorem 3.20 is the main result of this subsection and it characterizes the class of prime numbers for which the Łukasiewicz implication is definable in $\mathbf{Ł}^*_{n+1}$ (besides $n = 2$). Before proving it, we need to show the following lemma.

Lemma 3.19. *For each odd number n, the procedure* $\mathtt{P}$ *starting at $\mathbf{c} = (n-1)/n$ stops after reaching $1/n$, that is, if $\mathtt{P}(n, \mathbf{c}) = [a_1, a_2, \ldots, a_t]$ then $a_t = 1/n$.*

Proof. We already observed that $\mathtt{P}$ always stops since $Ł^*_{n+1}$ is finite. Thus assume, by way of contradiction, that $\mathtt{P}$ stops at $a_t = k/n$ with $k > 1$.

Fact 2. *Let q and n be positive integers such that $q < n$ and n is odd. Then: (1) if $q > n/2$, $*(q/n)$ has always an odd numerator; (2) if q is odd, then $\neg(q/n)$ has even numerator.*

By Fact 2, k cannot be even. Indeed, if k were even, then a_t would be obtained from a_{t-1} by negating it, i.e. $a_t = \neg a_{t-1}$, and then $a_{t+1} = *(a_t)$ should coincide with a previous element a_i in the list $[a_1, a_2, \ldots, a_t]$ such that $a_i < a_1 = (n-1)/n$. But then $a_{t+1} = a_i$ should have an odd numerator, and hence $a_i = *(a_{i-1})$, that is, we would have $a_{t+1} = *(a_t) = *(a_{i-1})$, hence $a_t = a_{i-1}$, and the procedure should have stopped at a_{t-1}, contradiction. Therefore k must be odd.

Since $\mathtt{P}$ stops at $a_m = k/n$, there exists an $a_j < a_1$ already met by the procedure such that $a_{t+1} = a_j$. If $a_{t+1} = *(a_t)$, then we reason as above and get $a_{j-1} = a_t$, contradiction. So let us assume $a_{t+1} = \neg(a_t) = n - (k/n) = (n-k)/n = a_j$. Notice that the numerator $n - k$ of a_j is even. Thus, a_j must have been obtained as $\neg(a_{j-1})$ in a previous step, that is, $a_j = \neg a_{j-1} = \neg a_t = a_j$, and hence it must be the case that $a_{j-1} = a_t$. In other words, the procedure should have stopped earlier at a_{t-1}, contradiction. Therefore, necessarily $k = 1$, that is to say, $a_t = 1/n$. $\square$

Theorem 3.20. *Let $n \geq 3$ be an odd number. Then: $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term-equivalent iff n is a prime number belonging to the set Π.*

Proof. Let $\mathbf{c} = (n-1)/n$ and let $\mathtt{P}(n, \mathbf{c}) = [a_1, \ldots, a_l]$ be the sequence generated by the procedure $\mathtt{P}(n, \mathbf{c})$. This sequence can be regarded as the concatenation of a number r of subsequences

$$[a^1_1, \ldots, a^1_{l_1}],\ [a^2_1, \ldots, a^2_{l_2}],\ \ldots,\ [a^r_1, \ldots, a^r_{l_r}],$$

with $a^1_1 = a_1$ and $a^r_{l_r} = a_l$, where for each subsequence $1 \leq j \leq r$, only the last element $a^j_{l_j}$ is below $1/2$, while the rest of elements are above $1/2$.

By the very definition of $*$, it follows that the last elements $a^j_{l_j}$ of every subsequence are of the form

$$a^j_{l_j} = \begin{cases} \frac{k_j n - 2^{m_j}}{n}, & \text{if } j \text{ is odd} \\ \frac{2^{m_j} - k_j n}{n}, & \text{otherwise, i.e. if } j \text{ is even} \end{cases}$$

for some $m_j, k_j > 0$, where in particular m_j is the number of strictly positive elements of $Ł_{n+1}$ which are obtained by the procedure before getting $a^j_{l_j}$.

By Lemma 3.19, since n is odd, $\mathtt{P}(n, \mathbf{c})$ stops at $1/n$, i.e. $a_l = a^r_{l_r} = 1/n$. Thus, writing m and k instead of m_r and k_r:

[5]These prime numbers are known in the literature as those *odd primes with one coach*. Properties satisfied by such a set of prime numbers can be found in the following webpage of the Online Encyclopedia of Integer Sequences (OEIS): https://oeis.org/A216371. Further interesting properties on the class Π can be found in [28]

$$\begin{cases} kn - 2^m = 1, \text{ if } r \text{ is odd (i.e., } 2^m \equiv -1 \pmod{n} \text{ if } r \text{ is odd)} \\ 2^m - kn = 1, \text{ otherwise (i.e., } 2^m \equiv 1 \pmod{n} \text{ if } r \text{ is even)} \end{cases}$$

where m is now the number of strictly positive elements in the list $\mathtt{P}(n, \mathbf{c})$, i.e. that are reached by the procedure before stopping. Therefore, 2^m is congruent with ± 1 mod n.

Suppose now that $n \geq 3$ is an odd number such that $\mathbf{Ł}_{n+1}$ is term equivalent to $\mathbf{Ł}^*_{n+1}$. Then, $\mathbf{Ł}^*_{n+1}$ is strictly simple and by Proposition 3.9, n is prime. This being so, the integer m defined above must be exactly $(n-1)/2$, the number of strictly positive elements of $Ł_{n+1}$ (different from 1). Otherwise, $\langle \mathbf{c} \rangle^*$ would be a proper subalgebra of it, which is absurd. Moreover, for no $m' < m$ one has that $2^{m'}$ is congruent with ± 1 mod n because, in this case, the algorithm would stop producing, again, a proper subalgebra of $\mathbf{Ł}^*_{n+1}$. This shows that $n \in \Pi$, i.e., the left-to-right direction of our claim.

In order to show the other direction, assume that $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are not term-equivalent. By Theorem 3.15, this implies that $\mathbf{Ł}^*_{n+1}$ is not strictly simple. Thus, by Theorem 3.8, $\langle a_1 \rangle^*$ is a proper subalgebra of $\mathbf{Ł}^*_{n+1}$ and hence the algorithm above stops at $1/n$, after reaching $m < (n-1)/2$ strictly positive elements of $Ł_{n+1}$. Thus, 2^m is congruent with ± 1 mod n (depending on whether r is even or odd, where r is the number of subsequences in the sequence $\mathtt{P}(n, \mathbf{c})$ as described above), showing that $n \notin \Pi$. $\square$

Observe that 3 and 5 are the only Fermat primes belonging to Π. Indeed, by Lemma 3.17, if n is a Fermat prime such that $n > 5$ then $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are not term-equivalent. By Theorem 3.20, n does not belong to the set Π.

4 The matrix logics of the $\mathbf{Ł}^*_{n+1}$-chains

Given the algebra $\mathbf{Ł}^*_{n+1}$, it is possible to consider, for every $1 \leq i \leq n$, the matrix logic $\Lambda^*_{n+1,i} = \langle \mathbf{Ł}^*_{n+1}, F_{i/n} \rangle$, where $F_{i/n} = \{a \in Ł_{n+1} \mid a \geq i/n\}$. Observe that the logic $\Lambda^*_{n+1,i}$, regarded as a consequence relation over a propositional language $\mathcal{L}$ with signature $\Sigma = (\vee, \sim, \star, \bot, \top)$ of type $(2, 1, 1, 0, 0)$, is defined as follows: for every subset of formulas $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$,

$$\Gamma \models_{\Lambda^*_{n+1,i}} \varphi \quad \text{if} \quad \begin{array}{l} \text{for every } \mathbf{Ł}^*_{n+1}\text{-evaluation } e, \\ e(\psi) \geq i/n \text{ for every } \psi \in \Gamma \text{ implies } e(\varphi) \geq i/n \end{array}$$

where an $\mathbf{Ł}^*_{n+1}$-evaluation is a homomorphism $e : \mathcal{L} \to \mathbf{Ł}^*_{n+1}$ of algebras over Σ, namely: $e(\varphi \vee \psi) = \max\{e(\varphi), e(\psi)\}$, $e(\sim\varphi) = \neg_Ł e(\varphi)$, $e(\star\varphi) = *e(\varphi)$, $e(\bot) = 0$ and $e(\top) = 1$.

In the following subsections we will first show that the logics $\Lambda^*_{n+1,i}$ are algebraizable, then we will describe their equivalent algebraic semantics, and finally we will provide an axiomatization.

4.1 Algebraizability of the logics $\Lambda^*_{n+1,i}$

In this section we show that all the logics $\Lambda^*_{n+1,i}$ are algebraizable in the sense of Blok-Pigozzi [2], and that, for every i, j, the quasivarieties associated to $\Lambda^*_{n+1,i}$ and $\Lambda^*_{n+1,j}$ are the same.

Observe that, as shown in Proposition 3.14, the Gödel implication $\Rightarrow_G$ of the $(n+1)$-valued Gödel logic is definable within the chain $\mathbf{Ł}^*_{n+1} = (Ł_{n+1}, \vee, \neg_Ł, *, 0, 1)$. Thus, the logic $\Lambda^*_{n+1} := \Lambda^*_{n+1,n} = \langle \mathbf{Ł}^*_{n+1}, \{1\} \rangle$ is a (Rasiowa) *implicative* logic, since it satisfies the following characteristic properties (see, for instance, [21, Definition 2.3]):

(R1) $\models_{\Lambda^*_{n+1}} \varphi \Rightarrow_G \varphi$

(R2) $\varphi \Rightarrow_G \psi, \psi \Rightarrow_G \chi \models_{\Lambda^*_{n+1}} \varphi \Rightarrow_G \chi$

(R3) $\varphi \Rightarrow_G \psi, \psi \Rightarrow_G \varphi \models_{\Lambda^*_{n+1}} \#\varphi \Rightarrow_G \#\psi$ for $\# \in \{\star, \sim\}$

$\varphi_1 \Rightarrow_G \psi_1, \psi_1 \Rightarrow_G \varphi_1, \varphi_2 \Rightarrow_G \psi_2, \psi_2 \Rightarrow_G \varphi_2 \models_{\Lambda^*_{n+1}} (\varphi_1 \vee \varphi_2) \Rightarrow_G (\psi_1 \vee \psi_2)$

(R4) $\varphi, \varphi \Rightarrow_G \psi \models_{\Lambda^*_{n+1}} \psi$

(R5) $\varphi \models_{\Lambda^*_{n+1}} \psi \Rightarrow_G \varphi$

And it is well-known that implicative logics are algebraizable (see e.g. [21, Proposition 3.15]). This lead us to the following:

Lemma 4.1. *For every n, the logic Λ^*_{n+1} is implicative and so it is also algebraizable.*

Blok and Pigozzi introduce in [3] the following notion of equivalent deductive systems. Two propositional deductive systems S_1 and S_2 in the same language are *equivalent* if there are translations $\tau_i : S_i \to S_j$ for $i \neq j$ such that: $\Gamma \vdash_{S_i} \varphi$ iff $\tau_i(\Gamma) \vdash_{S_j} \tau_i(\varphi)$, and $\varphi \dashv\vdash_{S_i} \tau_j(\tau_i(\varphi))$. From very general results in [3], it follows that two equivalent logic systems are indistinguishable from the algebraic point of view, namely: if one of the systems is algebraizable then the other will be also algebraizable w.r.t. the same quasivariety. This can be applied to $\Lambda^*_{n+1,i}$.

Lemma 4.2. *For every n and every $1 \leq i \leq n-1$, the logics Λ^*_{n+1} and $\Lambda^*_{n+1,i}$ are equivalent.*

Proof. Indeed, it is enough to consider the translation mappings $\tau_1 : \Lambda^*_{n+1} \to \Lambda^*_{n+1,i}$, $\tau_1(\varphi) = \Delta_1(\varphi)$, and $\tau_{i,2} : \Lambda^*_{n+1,i} \to \Lambda^*_{n+1}$, $\tau_{i,2}(\varphi) = \Delta_{i/n}(\varphi)$. □

Therefore, as a direct consequence of Lemma 4.1, Lemma 4.2 and the above observations, the algebraizability of $\Lambda^*_{n+1,i}$ easily follows.

Theorem 4.3. *For every n and every $1 \leq i \leq n$, the logic $\Lambda^*_{n+1,i}$ is algebraizable.*

Therefore, for each logic $\Lambda^*_{n+1,i}$ there is a quasivariety $\mathbb{A}(i, n+1)$ which is its equivalent algebraic semantics. The question of describing $\mathbb{A}(i, n+1)$ is dealt with in the next section, where it is shown that it is in fact the variety generated by $\mathbf{Ł}^*_{n+1}$.

Remarks 4.4. The last three results shown above deserve some comments.[6]

(1) Although, as stated in Lemma 4.1, the logic Λ^*_{n+1} is implicative (where $\Rightarrow_G$ is an implication for it), the same is not true for the logics the logic $\Lambda^*_{n+1,i}$ when $i \neq n$ (despite all of them are algebraizable). More precisely, the translated implication $\varphi \Rightarrow_i \psi := \Delta_1(\varphi \Rightarrow_G \psi)$ does not satisfy condition (R5) in $\Lambda^*_{n+1,i}$, as it can be checked. Hence, $\Rightarrow_i$ is not an implication in $\Lambda^*_{n+1,i}$ in the sense of Rasiowa considered above. Moreover, $\Rightarrow_i$ does not satisfy the weaker condition $\varphi, \psi \models_{\Lambda^*_{n+1,i}} \varphi \Rightarrow_i \psi$, hence the logic $\Lambda^*_{n+1,i}$ is not even *regularly algebraizable* w.r.t. $\varphi \Leftrightarrow_i \psi := \{\varphi \Rightarrow_i \psi, \psi \Rightarrow_i \varphi\}$ for $i \neq n$ (for the notion of regularly algebraizable logics see, for instance, [21] pp. 100-101 and 140-141).

(2) In addition to the above point (1), it is easy to see that $\Lambda^*_{n+1,i} \neq \Lambda^*_{n+1,j}$ whenever $i \neq j$. Indeed, assume, w.l.o.g., that $i < j$, and let p be a propositional variable. Then $\Delta_{i/n}\, p \models_{\Lambda^*_{n+1,i}} p$ while $\Delta_{i/n}\, p \not\models_{\Lambda^*_{n+1,j}} p$. On the other hand, $p \models_{\Lambda^*_{n+1,j}} \Delta_{j/n}\, p$ but $p \not\models_{\Lambda^*_{n+1,i}} \Delta_{j/n}\, p$. This shows that $\Lambda^*_{n+1,i} \neq \Lambda^*_{n+1,j}$ and that, moreover, they are incomparable.

(3) As one would expect, there are several differences between the logics $\Lambda^*_{n+1,i}$ and $\Lambda^*_{n+1,j}$ for $i \neq j$, besides the ones observed in points (1) and (2). Let p and q be two different propositional variables, and let φ be an arbitrary formula. Then, $p, \sim p \not\models_{\Lambda^*_{n+1,i}} q$ if $i/n \leq 1/2$ while $p, \sim p \models_{\Lambda^*_{n+1,j}} \varphi$ if $j/n > 1/2$. That is, in $\Lambda^*_{n+1,i}$ the negation $\sim$ is paraconsistent if $i/n \leq 1/2$, and explosive otherwise.[7] Moreover, $\models_{\Lambda^*_{n+1,i}} p \vee \sim p$ if $i/n \leq 1/2$, but $\not\models_{\Lambda^*_{n+1,j}} p \vee \sim p$ if $j/n > 1/2$. That is, $\Lambda^*_{n+1,i}$ satisfies the excluded-middle principle if $i/n \leq 1/2$, while $\Lambda^*_{n+1,j}$ is paracomplete if $j/n > 1/2$.

4.2 The equivalent algebraic semantics of $\Lambda^*_{n+1,i}$

Due to Lemma 4.2, all logics $\Lambda^*_{n+1,i}$'s are equivalent to Λ^*_{n+1} and so they have the same equivalent algebraic semantics, i.e. $\mathbb{A}(i, n+1) = \mathbb{A}(j, n+1)$, for every $1 \leq i, j \leq n$. Hence, we will simplify the notation and refer to $\mathbb{A}(n+1)$ for this common quasivariety. In order to characterize it, in the following we consider, without loss of generality, the case $i = n$, i.e. the algebras corresponding to the matrix logic $\Lambda^*_{n+1} = \langle \mathbf{Ł}^*_{n+1}, F_1 \rangle$ defined by the filter $F_1 = \{1\}$. From this point forward, throughout the paper we will write $\neg$ instead of $\neg_{Ł}$ to ease the reading (as it was already done in Subsection 3.1).

We start by observing that from the chain $\mathbf{Ł}^*_{n+1} = (Ł_{n+1}, \vee, \neg, *, 0, 1)$ we can obtain the algebra

$$\mathbf{IG}_{n+1} = (Ł_{n+1}, \wedge, \vee, \Rightarrow_G, \neg, 0, 1)$$

where $\Rightarrow_G$ is Gödel implication, which is definable in $\mathbf{Ł}^*_{n+1}$ as shown in Proposition 3.14. Hence, $\mathbf{IG}_{n+1}$ is in fact the standard $(n+1)$-valued Gödel algebra expanded with the involution $\neg x = 1 - x$

[6] We thank one of the anonymous referees by pointing out these facts to us.

[7] This situation also occurs in the logic $\mathsf{L}^i_n = \langle \mathbf{Ł}_{n+1}, F_{i/n} \rangle$, as observed in [13].

[17, 19]. Conversely, $\mathbf{Ł}^*_{n+1}$ can be seen as the expansion of $\mathbf{IG}_{n+1}$ with the $*$ operation. Recalling also from Proposition 3.11 the definition, for a given n and for every $a \in Ł_{n+1}$, of the terms Δ_a (as suitable sequences of the $\neg$ and $*$ operations), the above motivates the following definition.

Definition 4.5. An $\Lambda^\star_{n+1}$-*algebra* is a triple $(\mathbf{A}, \sim, \star)$, where

- $\mathbf{A} = (A, \wedge, \vee, \Rightarrow, 0, 1)$ is a $(n+1)$-valued Gödel algebra (a G_{n+1}-algebra for short),
- $(\mathbf{A}, \sim)$ is a $(n+1)$-valued Gödel algebra with involution (a IG_{n+1}-algebra for short), and
- $\star$ is a unary operation on A such that the following equations hold, where for every $a \in \{0, 1/n, \ldots, (n-1)/n, 1\}$, the operation Δ_a is defined as a sequence of $\sim$'s and $\star$'s obtained from its definition in Proposition 3.11 by replacing the occurrences of $\neg$ and $*$ by $\sim$ and $\star$ respectively:

(Eq1) $\Delta_1(x) = \Delta(x)$, $\Delta_0(x) = 1$

(Eq2) $\Delta_a \Delta_b x = \Delta_b x$

(Eq3) $\Delta_a x \vee {\sim}\Delta_a x = 1$

(Eq4) $\Delta_{a^+} x \Rightarrow \Delta_a x = 1, \quad \text{if} \quad a < 1$

(Eq5) $\Delta_a(x \vee y) = (\Delta_a x \vee \Delta_a y)$

(Eq6) $\Delta_{\neg a} {\sim} x = {\sim}\Delta_{a^+} x, \quad \text{if} \quad a < 1$

(Eq7) $\Delta(x \Rightarrow y) \Rightarrow (\star x \Rightarrow \star y) = 1$

(Eq8) $\Delta_a x \Rightarrow \Delta_{*a} \star x = 1$

(Eq9) $\Delta_{(*a)^+} \star x \Rightarrow \Delta_{a^+} x = 1$

where $\Delta(x) = {\sim}x \Rightarrow 0$ is the Baaz-Monteiro operator, and $a^+ = a + 1/n$.

Observe that $\Lambda^\star_{n+1}$-algebras are defined over the signature $\Sigma_+ = (\wedge, \vee, \Rightarrow, \sim, \star, \bot, \top)$ (by simplicity, we will use the same symbols for the connectives $\wedge$, $\vee$, $\Rightarrow$ and $\star$ and for the respective operators in $\Lambda^\star_{n+1}$-algebras). Since the class of IG_{n+1}-algebras is a variety (it is a subvariety of the class of Gödel algebras with an involution), from the above definition it is clear that the quasivariety $\mathbb{\Lambda}(n+1)$ coincides, up to language, with the variety $\mathbb{\Lambda}^\star_{n+1}$ of $\Lambda^\star_{n+1}$-algebras, hence it is in fact a variety.

Moreover, by defining $x \Leftrightarrow y := (x \Rightarrow y) \wedge (y \Rightarrow x)$, the following congruence law holds for $\star$:

$$\text{if } x \Leftrightarrow y = 1 \text{ then } \star x \Leftrightarrow \star y = 1 \tag{Cong}$$

If we look at a $\Lambda^\star_{n+1}$-algebra as an axiomatic expansion of its underlying (prelinear) IG_{n+1}-algebra with the additional $\star$ operation, (Cong) is in fact the necessary condition to be satisfied by $\star$ to keep the prelinearity property in the expanded algebra, see e.g. [10, in Vol. 1 of [8]]. Therefore, the variety $\mathbb{\Lambda}(n+1)$ is semilinear and the following subdirect representation holds.

Proposition 4.6. *Every $\Lambda^\star_{n+1}$-algebra is a subdirect product of linearly ordered $\Lambda^\star_{n+1}$-algebras.*

Since the operator Δ is definable in any $\Lambda^\star_{n+1}$-algebra, the same arguments of (ii) of Lemma 3.13 show that any linearly ordered $\Lambda^\star_{n+1}$-algebra is simple. Now, since any subdirectly irreducible $\Lambda^\star_{n+1}$-algebra is linearly ordered, we have the following corollary.

Corollary 4.7. *The variety of $\Lambda^\star_{n+1}$-algebras is semisimple.*

Looking at the above axioms, we observe that (Eq7) requires $\star$ to be a non-decreasing operation, while (Eq1) declares that the n-iteration of $\star$ results in the well-known Baaz-Monteiro's Δ operator. These two properties allows us to prove the following three further basic properties of the $\star$ operation.

Lemma 4.8. *The following identities hold in any $\Lambda^\star_{n+1}$-algebra:*

(i) $\star x \Rightarrow x = 1$

(ii) $\star 1 = 1$

(iii) $\star 0 = 0$

Proof.

(i) By the above representation theorem, it is enough to prove it for linearly-ordered $\Lambda^\star_{n+1}$-algebras. Let $\mathbf{A}$ be a $\Lambda^\star_{n+1}$-chain, and by way of contradiction, let $x \in A$ such that $x < \star x$. By (Eq7) and (Eq1), we have the following chain of inequalities: $x < \star x \leq \star\star x \leq \ldots \leq (\star)^n x = \Delta_1(x) = \Delta(x)$. But if $x < \star x$ it means that $x < 1$ and hence $\Delta x = 0$. It then follows that $\star x = 0$, in contradiction with the hypothesis $x < \star x$.

(ii) By (Eq8), $1 = \Delta_1 1 \Rightarrow \Delta_{*1}\star 1 = \Delta_1 1 \Rightarrow \Delta_1 \star 1$, but by (Eq1), $\Delta = \Delta_1$ and we know that $\Delta 1 = 1$, thus $\Delta\star 1 = 1$, and hence $\star 1 = 1$ as well.

(iii) It directly follows from (i) by taking $x = 0$. □

Recall from Proposition 3.11 that the operations χ_a's definable from the Δ_a's as $\chi_a(x) = \Delta_a(x) \wedge {\sim}\Delta_{a^+}(x)$ for $a < 1$ and $\chi_1(x) = \Delta_1(x)$. Next lemma shows some properties of these operations.

Lemma 4.9. *The following equations hold in the variety of $\Lambda^\star_{n+1}$-algebras:*

(i) $\bigvee_{a \in L_{n+1}} \chi_a x = 1$

(ii) $\chi_a x \wedge \chi_b x = 0$, *hence* ${\sim}(\chi_a x \wedge \chi_b x) = 1$, *for* $a \neq b$

(iii) $\chi_a x = \chi_{\neg a}{\sim}x$

(iv) $\chi_a x \Rightarrow \chi_{*a}\star x = 1$

Moreover, in any $\Lambda^\star_{n+1}$-chain, the following monotonicity condition holds:

(v) If $x \leq y$ and $\chi_a(x) = \chi_b(y) = 1$ then $a \leq b$.

Proof. (i) By definition of the operators χ_a, it is easy to check that $\bigvee_{0\leq a\leq 1} \chi_a x = \Delta_1 x \vee \Delta_{(n-1)/n} x \vee \ldots \vee \Delta_{1/n} x \vee {\sim}\Delta_{1/n} x$, but $\Delta_{1/n} x \vee {\sim}\Delta_{1/n} x = 1$, hence $\bigvee_{0\leq a\leq 1} \chi_a x = 1$ as well.

(ii) W.l.o.g., suppose $a > b$. By definition, $\chi_a x \wedge \chi_b x = (\Delta_a x \wedge {\sim}\Delta_{a^+} x) \wedge (\Delta_b x \wedge {\sim}\Delta_{b^+} x)$. Since $a > b$ then $a \geq b^+$ and so $\Delta_a x \leq \Delta_{b^+} x$ by (Eq4). Hence, $\chi_a x \wedge \chi_b x \leq \Delta_{b^+}\alpha \wedge {\sim}\Delta_{b^+}\alpha = 0$.

(iii) If $a = 0$ the result follows by (Eq6), namely: $\chi_0 x = {\sim}\Delta_{1/n} x = \Delta_1{\sim}x = \chi_1{\sim}x$. If $a = 1$ then $\chi_1 x = \chi_1{\sim}{\sim}x = \chi_0{\sim}x$. Now, suppose that $0 < a < 1$. Then, $\chi_a x = \Delta_a x \wedge {\sim}\Delta_{a^+} x$, and since $x = {\sim}{\sim}x$, $\Delta_a x = {\sim}\Delta_{(\neg a)^+}{\sim}x$. By (Eq6) again, ${\sim}\Delta_{a^+} x = \Delta_{\neg a}{\sim}x$. Therefore, $\chi_a x = {\sim}\Delta_{(\neg a)^+}{\sim}x \wedge \Delta_{\neg a}{\sim}x = \chi_{\neg a}{\sim}x$.

(iv) Note first that $\Delta_{(*a)^+}\star x \leq \Delta_{a^+} x$ iff ${\sim}\Delta_{a^+} x \leq {\sim}\Delta_{(*a)^+}\star x$. Then, from (Eq8), (Eq9) we get $\Delta_a x \wedge {\sim}\Delta_{a^+} x \leq \Delta_{*a}\star x \wedge {\sim}\Delta_{(*a)^+}\star x$, that is, $\chi_a x \leq \chi_{*a}\star x$.

(v) In a given $\Lambda^\star_{n+1}$-chain $\mathbf{A}$, the condition is equivalent to the following one: for all $x, y \in A$, if $\chi_a(x) = \chi_b(x \vee y) = 1$ then $a \leq b$; and in turn this equivalent to: if $\chi_a(x) = 1$ and $a > b$ then $\chi_b(x \vee y) = 0$. Now, by definition if $\chi_a(x) = 1$ we have $\Delta_a x = 1$ and, by Equation (Eq5), $\Delta_a(x \vee y) = 1$ as well. Then, since $b^+ \leq a$, by Equation (Eq4), we have $\Delta_{b^+}(x \vee y) = 1$, i.e., ${\sim}\Delta_{b^+}(x \vee y) = 0$, and again by definition of χ_b, we finally have $\chi_b(x \vee y) = 0$.

□

By the considerations made at the beginning of this section, each $\mathbf{L}^*_{n+1}$ can be regarded as an algebra over the expanded signature Σ_+ of $\Lambda^\star_{n+1}$-algebras introduced after Definition 4.5. This fact will be used in the sequel and, depending on the context, $\mathbf{L}^*_{n+1}$ will be considered indistinctly as a Σ-algebra and as a Σ_+-algebra. Thus, we have the following lemma.

Lemma 4.10. *Every $\Lambda^\star_{n+1}$-chain $(\mathbf{A}, \sim, \star)$ is isomorphic to a subalgebra of $\mathbf{L}^*_{n+1}$.*

Proof. Let $\mathbf{A}$ be a $\Lambda^\star_{n+1}$-chain. Since in particular the G-reduct of $\mathbf{A}$ is a G_{n+1}-chain, $\mathbf{A}$ is finite, and let $|A| = m + 1 \leq n + 1$ and $A = \{0 < a_1 < \ldots < a_{m-1} < 1\}$. Note that, by the symmetry induced by the involutive negation, we have ${\sim}a_j = a_{m-j}$. We will show that $\mathbf{A}$ embeds into the standard algebra $\mathbf{L}^*_{n+1}$.

By (i) and (ii) of Lemma 4.9, for each $a_j \in A$, there is a unique $i_j \in \{0, 1, \ldots, n\}$ such that $\chi_{i_j/n}(a_j) = 1$. Let us check that $\bar{A} = \{0, i_1/n, \ldots, i_{m-1}/n, 1\}$ is the domain of a subalgebra of cardinality $m + 1$ of $\mathbf{L}^*_{n+1}$. It is clear that $\bar{A}$ is closed under the Gödel operations $\wedge, \vee, \Rightarrow$, thus we only have to check that $\neg(i_j/n), *(i_j/n) \in \bar{A}$, for each $i_j/n \in \bar{A}$:

(i) by (iii) of Lemma 4.9, if $\sim a_j = a_k$, then $1 = \chi_{i_j/n}(a_j) = \chi_{i_k/n}(a_k) = \chi_{\neg(i_j/n)}(a_k)$, hence by (i) and (ii) of Lemma 4.9, $\neg(i_j/n) = i_k/n \in \bar{A}$.

(ii) by (iv) of Lemma 4.9, if $\star a_j = a_k$, then $1 = \chi_{i_j/n}(a_j) = \chi_{i_k/n}(a_k) = \chi_{*(i_j/n)}(a_k)$, hence by (i) and (ii) of Lemma 4.9, $*(i_j/n) = i_k/n \in \bar{A}$.

Note that, by the symmetry induced by the involutive negation, we have $n - i_j = i_{m-j}$ for every $j \in \{1, \ldots, m\}$. Then, we define a mapping $h : A \to Ł^*_{n+1}$ by stipulating $h(0) = 0, h(1) = 1$ and $h(a_j) = i_j/n$ for all $j = 1, \ldots, m-1$. It is clear that h is one-to-one and is order preserving (by (v) of Lemma 4.9), and hence a morphism w.r.t. Gödel operations. Moreover, h is a morphism w.r.t. to the $\sim$ and $\star$ operations as well:

- $h(\sim a_j) = h(a_{m-j}) = i_{(m-j)/n} = 1 - i_j/n = \neg h(a_j)$
- since $*(i_j/n) \in \bar{A}$, then let $i_k/n = *(i_j/n)$ and hence $\star a_j = a_k$. Then $h(\star a_j) = h(a_k) = i_k/n = *(i_j/n) = *h(a_j)$

Therefore, $\mathbf{A}$ is isomorphic to the subalgebra of $\mathbf{Ł}^*_{n+1}$ over the domain $\bar{A} = \{0, i_1/n, \ldots, i_{m-1}/n, 1\}$. □

As a consequence we have the following result.

Theorem 4.11. *The variety $\mathbb{\Lambda}^\star_{n+1}$ of $\Lambda^\star_{n+1}$-algebras is generated by the algebra $\mathbf{Ł}^*_{n+1}$ over Σ_+.*

The result above immediately shows that the variety of $\Lambda^\star_{n+1}$-algebras is the equivalent algebraic semantics of the logic $\Lambda^\star_{n+1}$. Indeed, by definition, for every finite set of formulas $\Gamma \cup \{\varphi\}$ over Σ, we have that $\Gamma \models_{\Lambda^\star_{n+1}} \varphi$ iff for every $\mathbf{Ł}^*_{n+1}$-evaluation e, $e(\psi) = 1$ for every $\psi \in \Gamma$ implies $e(\varphi) = 1$ iff, by Theorem 4.11, for every $\Lambda^\star_{n+1}$-algebra $\mathbf{B}$ and every $\Lambda^\star_{n+1}$-evaluation e, $e(\psi) = 1$ for every $\psi \in \Gamma$ implies $e(\varphi) = 1$. This observation, together with Lemma 4.2, leads to the following result.

Corollary 4.12. *The variety $\mathbb{\Lambda}^\star_{n+1}$ of $\Lambda^\star_{n+1}$-algebras is the equivalent algebraic semantics of the logics $\Lambda^*_{n+1,i}$ for every $1 \leq i \leq n$.*

One of the anonymous referees communicated to us an alternative proof of this corollary using techniques of Abstract Algebraic Logic (AAL), presented in the Appendix. The referee also suggested to look at further interesting algebraic properties of the variety of $\Lambda^\star_{n+1}$-algebras that can be derived from the fact that it is generated by $\mathbf{Ł}^*_{n+1}$. For instance, let us consider the following ternary term:

$$t(x, y, z) = (\Delta(x \Leftrightarrow y) \wedge z) \vee (\neg\Delta(x \Leftrightarrow y) \wedge x).$$

It is very easy to check that in $\mathbf{Ł}^*_{n+1}$, for every $a, b, c \in Ł_{n+1}$, it holds that

$$t(a, b, c) = \begin{cases} c, & \text{if } a = b \\ a, & \text{if } a \neq b. \end{cases}$$

This means that $t(x, y, z)$ is a discriminator term for $\mathbf{Ł}^*_{n+1}$, and thus $\mathbf{Ł}^*_{n+1}$ is simple (see [5, Lemma 9.2]) and the variety of $\Lambda^\star_{n+1}$-algebras, that is generated by the algebra $\mathbf{Ł}^*_{n+1}$, is a discriminator variety. By [5, Theorem 9.4], this has another nice algebraic consequence.

Corollary 4.13. *The variety of $\Lambda^\star_{n+1}$-algebras is a discriminator variety. Therefore, it is also an arithmetical variety, i.e. it is both congruence-distributive and congruence-permutable.*

4.3 A uniform axiomatization of the logics $\Lambda^*_{n+1,i}$

Now, we present a uniform axiomatization for the logics $\Lambda^*_{n+1,i}$. Let us remark that the calculus we are going to present in this section provides an alternative axiomatization to the one that can be obtained by translating the algebraic equations defining the variety of Λ^*_{n+1}-algebras.

The Hilbert calculi will be defined over the signature $\Sigma = (\vee, \sim, \star, \bot, \top)$ of the matrix logics $\Lambda^*_{n+1,i}$, an expansion of the signature Σ_0 mentioned in the Introduction. In this signature, the following derived connectives will be useful:

- $\alpha \wedge \beta := {\sim}({\sim}\alpha \vee {\sim}\beta)$
- Δ_a, for each $a \in \text{Ł}_{n+1}$, as defined in (the proof of) Proposition 3.11, replacing all the occurrences of $\neg$ and $*$ by $\sim$ and $\star$ respectively[8]
- $\chi_a\alpha := \Delta_a\alpha \wedge {\sim}\Delta_{a^+}\alpha$, if $0 < a < 1$, where $a^+ = a + (1/n)$
- $\chi_0\alpha := {\sim}\Delta_{1/n}\alpha$; $\chi_1\alpha := \Delta_1\alpha$
- $\alpha \Rightarrow_c \beta := \bigvee_{0\leq i\leq n}(\chi_{i/n}(\alpha) \wedge \Delta_{i/n}(\beta))$
- $-_{i/n}\alpha := {\sim}\Delta_{i/n}\alpha$
- $\alpha \to_{i/n} \beta := -_{i/n}\alpha \vee \beta = {\sim}\Delta_{i/n}\alpha \vee \beta$
- $\alpha \leftrightarrow_{i/n} \beta := (\alpha \to_{i/n} \beta) \wedge (\beta \to_{i/n} \alpha)$

In order to keep notation lighter, and without risk of confusion, the subscript i/n will be omitted from the symbols $\to_{i/n}$ and $\leftrightarrow_{i/n}$.

Definition 4.14. The Hilbert calculus $\mathsf{AX}^*_{n+1,i}$ for the logic $\Lambda^*_{n+1,i}$, defined over the signature Σ, is given as follows:

Axiom schemas: those of CPL (propositional classical logic) restricted to the signature $(\vee, \to)$[9] plus the following ones, where $a, b \in \{0, 1/n, \ldots, (n-1)/n, 1\}$:

(Ax1) $(\alpha \leftrightarrow \beta) \to ({\sim}\alpha \leftrightarrow {\sim}\beta)$

(Ax2) ${\sim}{\sim}\alpha \leftrightarrow \alpha$

(Ax3) ${\sim}(\alpha \vee \beta) \to {\sim}\alpha$

(Ax4) ${\sim}\alpha \to ({\sim}\beta \to {\sim}(\alpha \vee \beta))$

(Ax5) $\Delta_a\Delta_b\alpha \leftrightarrow \Delta_b\alpha$

(Ax6) $\Delta_a\alpha \vee {\sim}\Delta_a\alpha$

(Ax7) $\Delta_{a^+}\alpha \to \Delta_a\alpha$

(Ax8) $\Delta_a(\alpha \vee \beta) \leftrightarrow (\Delta_a\alpha \vee \Delta_a\beta)$

(Ax9) $\Delta_{\neg a}{\sim}\alpha \leftrightarrow {\sim}\Delta_{a^+}\alpha, \quad \text{if } \ a < 1$

(Ax10) $\Delta_{i/n}\alpha \to \alpha$

(Ax11) $\Delta_a\alpha \to \Delta_{*a}\star\alpha$

(Ax12) $\Delta_{(*a)^+}\star\alpha \to \Delta_{a^+}\alpha$

(Ax13) $\bot \to \alpha$

(Ax14) $\alpha \to \top$

Inference rule:

$$\text{(MP)} \quad \frac{\alpha \quad \alpha \to \beta}{\beta}$$

It is easy to prove that the usual axioms involving $\wedge$ of positive classical propositional logic CPL$^+$, over $(\wedge, \vee, \to)$, can be derived in the system $\mathsf{AX}^*_{n+1,i}$ by means of the axioms (Ax1)-(Ax4), thus the logic $\Lambda^*_{n+1,i}$ in fact contains CPL$^+$. Moreover, it is worth noting that the system $\mathsf{AX}^*_{n+1,i}$ satisfies the deduction-detachment theorem w.r.t. the implication $\to$, namely:

$$\Gamma \cup \{\alpha\} \vdash_{\mathsf{AX}^*_{n+1,i}} \beta \text{ iff } \Gamma \vdash_{\mathsf{AX}^*_{n+1,i}} \alpha \to \beta,$$

[8] Recall that, by definition, $\Delta_1\alpha = (\star)^n\alpha$ and $\Delta_0\alpha = (\star)^n\alpha \vee {\sim}(\star)^n\alpha$.

[9] Namely, the schemas $\alpha \to (\alpha \vee \beta)$, $\beta \to (\alpha \vee \beta)$, $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$, $\alpha \to (\beta \to \alpha)$, $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$, and $\alpha \vee (\alpha \to \beta)$.

for every set of formulas $\Gamma \cup \{\alpha, \beta\}$. Indeed, it is well-known that any logic presented by means of a Hilbert calculus containing a binary connective $\to$ such that the schemas

(A1): $\alpha \to (\beta \to \alpha)$
(A2): $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

are derivable, and where (MP) (w.r.t. $\to$) is the only inference rule, satisfies the deduction-detachment theorem w.r.t. $\to$. In addition, $\mathsf{AX}^*_{n+1,i}$ satisfies the following metaproperty (sometimes called *proof by cases*):

$$\Gamma, \alpha \vdash_{\mathsf{AX}^*_{n+1,i}} \gamma \text{ and } \Gamma, \beta \vdash_{\mathsf{AX}^*_{n+1,i}} \gamma \text{ implies that } \Gamma, \alpha \vee \beta \vdash_{\mathsf{AX}^*_{n+1,i}} \gamma.$$

This is a consequence of the deduction-detachment theorem and CPL. Besides, the conjunction $\wedge$ (defined as above) satisfies in this logic the classical properties, namely: $\alpha \to (\beta \to (\alpha \wedge \beta))$, $(\alpha \wedge \beta) \to \alpha$, and $(\alpha \wedge \beta) \to \beta$. This can be easily proven by using axioms (Ax1)-(Ax4) and MP.

Also, observe that Axiom (Ax6), together with items (i) and (x) in Lemma 4.15 below, capture the fact the Δ_a's connectives are Boolean in the sense that formulas built from expressions $\Delta_a\varphi$ with connectives $\vee, \sim, \to$ behave as in classical logic, and thus one can classically reason with them. Formulas of this kind will be called *Boolean.* We will provide a formal justification of this statement a bit later.

Next lemma gathers some interesting theorems of $\mathsf{AX}^*_{n+1,i}$ that follow from the above axiomatics.

Lemma 4.15. *The following are theorems of $\mathsf{AX}^*_{n+1,i}$, where $a, b \in \{0, 1/n, \ldots, (n-1)/n, 1\}$:*

(i) $\Delta_a\alpha \to ({\sim}\Delta_a\alpha \to \beta)$

(ii) $\Delta_{a^+}\alpha \to (\Delta_{\neg a}{\sim}\alpha \to \beta)$

(iii) $\Delta_{a^+}\alpha \vee \Delta_{\neg a}{\sim}\alpha$

(iv) $\Delta_a\alpha \leftrightarrow \Delta_a{\sim}{\sim}\alpha$

(v) $\alpha \to \Delta_{i/n}\alpha$

(vi) $(\alpha \wedge -_{i/n}\alpha) \to \beta$

(vii) $\chi_a(\alpha \vee \beta) \to (\chi_a\alpha \vee \chi_a\beta)$

(viii) $\bigvee_{a \in L_{n+1}} \chi_a\alpha$

(ix) $(\chi_a\alpha \wedge \chi_b\alpha) \to \beta$*, for* $a \neq b$

(x) $(\Delta_a\alpha \to \Delta_b\beta) \leftrightarrow ({\sim}\Delta_b\beta \to {\sim}\Delta_a\alpha)$

(xi) $(\chi_a\alpha \wedge \chi_b\beta) \to \chi_{\max(a,b)}(\alpha \vee \beta)$

(xii) $\chi_a\alpha \leftrightarrow \chi_{\neg a}{\sim}\alpha$

(xiii) $\chi_a\alpha \to \chi_{*a}\star\alpha$

(xiv) $\chi_a\alpha \to \chi_{\Delta_b(a)}\Delta_b\alpha$

(xv) $\Delta_a\alpha \leftrightarrow \vee_{b \geq a}\, \chi_b\alpha$

(xvi) $\Delta_b\alpha \to \Delta_a\alpha$*, if* $b \geq a$

Proof. The proofs of all the cases are as follows.

(i) By definition of $\to$, we have $\Delta_a\alpha \to ({\sim}\Delta_a\alpha \to \beta) = {\sim}\Delta_{i/n}\Delta_a\alpha \vee ({\sim}{\sim}\Delta_{i/n}\Delta_a\alpha \vee \beta)$, and by applying (Ax5), (Ax1) and (Ax2) (as well as CPL), the latter is equivalent to $({\sim}\Delta_a\alpha \vee \Delta_a\alpha) \vee \beta$, which is clearly a theorem of $\mathsf{AX}^*_{n+1,i}$ by axiom (Ax6) and CPL.

(iv) The case $a = 0$ is obviously true, by definition of Δ_0. Suppose now that $a > 0$. From (Ax9), $\Delta_{\neg b}{\sim}\alpha \leftrightarrow {\sim}\Delta_{b^+}\alpha$ is a theorem, for every $0 \leq b < 1$. By taking $b = a^- = a - 1/n$ we get $\Delta_{\neg(a^-)}{\sim}\alpha \leftrightarrow {\sim}\Delta_a\alpha$, and so $\Delta_a\alpha \leftrightarrow {\sim}\Delta_{\neg(a^-)}{\sim}\alpha$, by (Ax1), (Ax2) and CPL. Noticing that $\neg(a^-) = (\neg a)^+$, ${\sim}\Delta_{\neg(a^-)}{\sim}\alpha$ is ${\sim}\Delta_{(\neg a)^+}{\sim}\alpha$. By applying (Ax9) again to this last formula, and taking into account that $\neg\neg a = a$, we finally have the following chain of equivalences: $\Delta_a\alpha \leftrightarrow {\sim}\Delta_{(\neg a)^+}{\sim}\alpha \leftrightarrow \Delta_a{\sim}{\sim}\alpha$.

(v) It directly follows by definition of $\to$: $\alpha \to \Delta_{i/n}\alpha = {\sim}\Delta_{i/n}\alpha \vee \Delta_{i/n}\alpha$, the latter being a theorem by (Ax6).

(vi) Notice that $\alpha \wedge -_{i/n}\alpha = \alpha \wedge {\sim}\Delta_{i/n}\alpha$ and, due to (v), this implies $\Delta_{i/n}\alpha \wedge {\sim}\Delta_{i/n}\alpha$, which implies any β by (i).

(vii) If $a = 1$ the result follows by (Ax8). If $a = 0$ then $\chi_a(\alpha \vee \beta) = {\sim}\Delta_{1/n}(\alpha \vee \beta)$, which implies ${\sim}(\Delta_{1/n}\alpha \vee \Delta_{1/n}\beta)$, by (Ax8), (Ax1) and CPL. The latter implies ${\sim}\Delta_{1/n}\alpha$, by (Ax3), and this implies ${\sim}\Delta_{1/n}\alpha \vee {\sim}\Delta_{1/n}\beta$, by CPL. Suppose now that $0 < a < 1$. Then, $\chi_a(\alpha \vee \beta) = \Delta_a(\alpha \vee \beta) \wedge {\sim}\Delta_{a^+}(\alpha \vee \beta)$ is equivalent to $(\Delta_a\alpha \vee \Delta_a\beta) \wedge {\sim}(\Delta_{a^+}\alpha \vee \Delta_{a^+}\beta)$, by (Ax8) and (Ax1). The latter is equivalent to $(\Delta_a\alpha \vee \Delta_a\beta) \wedge {\sim}\Delta_{a^+}\alpha \wedge {\sim}\Delta_{a^+}\beta$, by definition of $\wedge$ and (Ax1)-(Ax4). But this is equivalent to $(\Delta_a\alpha \wedge {\sim}\Delta_{a^+}\alpha \wedge {\sim}\Delta_{a^+}\beta) \vee (\Delta_a\beta \wedge {\sim}\Delta_{a^+}\alpha \wedge {\sim}\Delta_{a^+}\beta)$, by CPL. By using CPL again, this formula implies $(\Delta_a\alpha \wedge {\sim}\Delta_{a^+}\alpha) \vee (\Delta_a\beta \wedge {\sim}\Delta_{a^+}\beta)$, that is, $\chi_a\alpha \vee \chi_a\beta$.

(viii) By item (i) and CPL it is easy to see that $\bigvee_{0 \leq a \leq 1} \chi_a\gamma$ is equivalent to $\Delta_1\gamma \vee \Delta_{(n-1)/n}\gamma \vee \ldots \vee \Delta_{1/n}\gamma \vee {\sim}\Delta_{1/n}\gamma$, and the latter is a theorem of $\mathsf{AX}^*_{n+1,i}$, by (Ax6) and the properties of $\vee$ coming from CPL.

(ix) W.l.o.g., suppose $a > b$. By definition, $\chi_a\alpha \wedge \chi_b\alpha = (\Delta_a\alpha \wedge {\sim}\Delta_{a^+}\alpha) \wedge (\Delta_b\alpha \wedge {\sim}\Delta_{b^+}\alpha)$. Since $a > b$ then $a \geq b^+$ and so $\Delta_a\alpha \to \Delta_{b^+}\alpha$ is a theorem, by (Ax7) and CPL. Hence, by CPL once again, $\chi_a\alpha \wedge \chi_b\alpha$ implies $\Delta_{b^+}\alpha \wedge {\sim}\Delta_{b^+}\alpha$, which implies β by (i). From this $(\chi_a\alpha \wedge \chi_b\alpha) \to \beta$ is a theorem, for any β.

(x) Let $\Gamma = \{\Delta_a\alpha \to \Delta_b\beta, {\sim}\Delta_b\beta\}$. By (i) it is easy to see that $\Gamma, \Delta_a\alpha \vdash {\sim}\Delta_a\alpha$. Clearly $\Gamma, {\sim}\Delta_a\alpha \vdash {\sim}\Delta_a\alpha$ and so, by proof by cases, $\Gamma, \Delta_a \vee {\sim}\Delta_a\alpha \vdash {\sim}\Delta_a\alpha$. From this, $\Gamma \vdash {\sim}\Delta_a\alpha$, by (Ax6). By the deduction-detachment theorem, $(\Delta_a\alpha \to \Delta_b\beta) \to ({\sim}\Delta_b\beta \to {\sim}\Delta_a\alpha)$ is a theorem. The proof that $({\sim}\Delta_b\beta \to {\sim}\Delta_a\alpha) \to (\Delta_a\alpha \to \Delta_b\beta)$ is a theorem is analogous, but now by considering the set $\Gamma' = \{{\sim}\Delta_b\beta \to {\sim}\Delta_a\alpha,\ \Delta_a\}$.

(xi) W.l.o.g., we can assume $a \leq b$. Suppose also that $0 < a \leq b < 1$. Then, $\chi_a\alpha \wedge \chi_b\beta = \Delta_a\alpha \wedge {\sim}\Delta_{a^+}\alpha \wedge \Delta_b\beta \wedge {\sim}\Delta_{b^+}\beta$. Since $a \leq b$ then $a^+ \leq b^+$. By (Ax7) and CPL, $\Delta_{b^+}\alpha \to \Delta_{a^+}\alpha$ is a theorem. By (x), ${\sim}\Delta_{a^+}\alpha \to {\sim}\Delta_{b^+}\alpha$ is a theorem. Using this, (Ax8) and CPL, $\chi_a\alpha \wedge \chi_b\beta$ implies $\Delta_b(\alpha \vee \beta) \wedge {\sim}\Delta_{b^+}\alpha \wedge {\sim}\Delta_{b^+}\beta$. This implies $\Delta_b(\alpha \vee \beta) \wedge {\sim}(\Delta_{b^+}\alpha \vee \Delta_{b^+}\beta)$, which implies $\Delta_b(\alpha \vee \beta) \wedge {\sim}\Delta_{b^+}(\alpha \vee \beta) = \chi_b(\alpha \vee \beta)$, by (Ax8), (Ax1) and CPL. The cases involving $a = 0$ or $b = 1$ can be proved analogously, and are left to the reader.

(xii) If $a = 0$ the result follows by (Ax9), namely: ${\sim}\Delta_{1/n}\alpha$ is equivalent to $\Delta_1{\sim}\alpha$. If $a = 1$ then $\chi_1\alpha = \Delta_1\alpha$, which is equivalent to $\Delta_1{\sim}{\sim}\alpha$, by (iv). By the first part of the proof of this item, this is equivalent to ${\sim}\Delta_{1/n}{\sim}\alpha$, that is, $\chi_0{\sim}\alpha$. Now, suppose that $0 < a < 1$. Then, $\chi_a\alpha = \Delta_a\alpha \wedge {\sim}\Delta_{a^+}\alpha$. Observe that, since $a = \neg\neg a$, $\Delta_a\alpha$ is equivalent to ${\sim}\Delta_{(\neg a)^+}{\sim}\alpha$, by (Ax9) and item (iv). By (Ax9) again, ${\sim}\Delta_{a^+}\alpha$ is equivalent to $\Delta_{\neg a}{\sim}\alpha$. Therefore, by CPL, $\chi_a\alpha$ is equivalent to ${\sim}\Delta_{(\neg a)^+}{\sim}\alpha \wedge \Delta_{\neg a}{\sim}\alpha$, that is, to $\chi_{\neg a}{\sim}\alpha$.

(xiii) Note first that $(\Delta_{(*a)^+}\star\alpha \to \Delta_{a^+}\alpha) \leftrightarrow ({\sim}\Delta_{a^+}\alpha \to {\sim}\Delta_{(*a)^+}\star\alpha)$, by item (x). Then, from (Ax11), (Ax12) and CPL, we get $(\Delta_a\alpha \to \Delta_{*a}\star\alpha) \wedge ({\sim}\Delta_{a^+}\alpha \to {\sim}\Delta_{(*a)^+}\star\alpha)$ is a theorem. By using CPL once again, we get that the latter formula implies $(\Delta_a\alpha \wedge {\sim}\Delta_{a^+}\alpha) \to (\Delta_{*a}\star\alpha \wedge {\sim}\Delta_{(*a)^+}\star\alpha)$. From this, $\chi_a\alpha \to \chi_{*a}\star\alpha$ is a theorem, by definition.

(xiv) Immediate from (xii) and (xiii) and the definition of the Δ_a's operations and connectives as sequences of $\star$'s and $\sim$'s.

(xv) By the proof of item (vii), $\bigvee_{b \geq 0} \chi_b\alpha$ is equivalent to $\Delta_1\alpha \vee \Delta_{(n-1)/n}\alpha \vee \ldots \vee \Delta_{1/n}\alpha \vee {\sim}\Delta_{1/n}\alpha$. Thus, if $a = 0$ then the result holds, since $\Delta_0\alpha$ and $\bigvee_{b \geq 0} \chi_b\alpha$ are both theorems. Suppose now that $a = k/n > 0$. By reasoning as in item (viii) it is easy to prove that

$\bigvee_{b\geq a} \chi_b\alpha$ is equivalent to $\Delta_1\alpha \vee \Delta_{(n-1)/n}\alpha \vee \ldots \vee \Delta_{k/n}\alpha$. By (Ax7) and CPL it follows that $\Delta_b\alpha \to \Delta_{k/n}\alpha$ is a theorem, for $b \geq a$. From this $\bigvee_{b\geq a} \chi_b\alpha$ is equivalent to $\Delta_{k/n}\alpha$, by CPL.

(xvi) It directly follows by an iterative application of (Ax7).

□

The following shows that the logic $\mathsf{AX}^*_{n+1,i}$ proves two basic properties of the unary connective $\star$: that $\star\alpha$ is smaller than α and that $\star$ preserves the order given by $\Rightarrow_c$.

Proposition 4.16. *The following formulas are theorems of* $\mathsf{AX}^*_{n+1,i}$*:*
(1) $\star\alpha \Rightarrow_c \alpha$*;*
(2) $(\alpha \Rightarrow_c \beta) \to (\star\alpha \Rightarrow_c \star\beta)$*.*

Proof. (1) From Lemma 4.12(xiii), $\chi_a(\alpha) \vdash_{\mathsf{AX}^*_{n+1,i}} \chi_{*a}(\star\alpha)$. By CPL, it follows that $\chi_a(\alpha) \vdash_{\mathsf{AX}^*_{n+1,i}} \bigvee_{b\geq *a} \chi_b(\alpha)$. But $\bigvee_{b\geq *a} \chi_b(\alpha) \vdash_{\mathsf{AX}^*_{n+1,i}} \Delta_{*a}(\alpha)$, by Lemma 4.12(xv), hence $\chi_a(\alpha) \vdash_{\mathsf{AX}^*_{n+1,i}} \Delta_{*a}(\alpha)$. By CPL, $\chi_a(\alpha) \vdash_{\mathsf{AX}^*_{n+1,i}} \chi_{*a}(\star\alpha) \wedge \Delta_{*a}(\alpha)$. By using CPL once again, $\chi_a(\alpha) \vdash_{\mathsf{AX}^*_{n+1,i}} \bigvee_b \chi_b(*\alpha) \wedge \Delta_b(\alpha)$, that is, $\chi_a(\alpha) \vdash_{\mathsf{AX}^*_{n+1,i}} \star\alpha \Rightarrow_c \alpha$. Using proof-by-cases, $\bigvee_a \chi_a(\alpha) \vdash_{\mathsf{AX}^*_{n+1,i}} \star\alpha \Rightarrow_c \alpha$. But then $\vdash_{\mathsf{AX}^*_{n+1,i}} \star\alpha \Rightarrow_c \alpha$, by Lemma 4.12(viii).
(2) By Lemma 4.12(xiii), (Ax11), and by CPL, $\chi_a(\alpha) \wedge \Delta_a(\beta) \vdash_{\mathsf{AX}^*_{n+1,i}} \chi_{*a}(\star\alpha) \wedge \Delta_{*a}(\star\beta)$. By CPL, $\chi_a(\alpha) \wedge \Delta_a(\beta) \vdash_{\mathsf{AX}^*_{n+1,i}} \bigvee_b \chi_b(\star\alpha) \wedge \Delta_b(\star\beta)$, that is, $\chi_a(\alpha) \wedge \Delta_a(\beta) \vdash_{\mathsf{AX}^*_{n+1,i}} (\star\alpha \Rightarrow_c \star\beta)$. Using proof-by-cases and the definition of $\Rightarrow_c$ it follows that $(\alpha \Rightarrow_c \beta) \vdash_{\mathsf{AX}^*_{n+1,i}} (\star\alpha \Rightarrow_c \star\beta)$. The result follows by the deduction-detachment theorem w.r.t. $\to$. □

Next we prove that Boolean formulas behave as in classical propositional logic. First, we need a preliminary lemma with some further derivations in $\mathsf{AX}^*_{n+1,i}$.

Lemma 4.17. *(1)* $\mathsf{AX}^*_{n+1,i}$ *proves* $\star\alpha \to \alpha$*.*
(2) If α *is Boolean, then* $\mathsf{AX}^*_{n+1,i}$ *proves* $\chi_0\alpha \vee \chi_1\alpha$*.*
(3) Further, if α *is Boolean, then* $\mathsf{AX}^*_{n+1,i}$ *proves* $\alpha \to \star\alpha$*.*

Proof. (1) By definition $\star\alpha \to \alpha = \sim\Delta_{i/n}\star\alpha \vee \alpha$. We reason by cases:
Let $a \geq i/n$. Then $\chi_a\alpha \vdash \Delta_{i/n}\alpha$, and $\vdash \Delta_{i/n}\alpha \leftrightarrow \alpha$, therefore, $\chi_a\alpha \vdash \sim\Delta_{i/n}\star\alpha \vee \alpha$.
Let $a < i/n$. Then $\chi_a\alpha \vdash \chi_{*a}\star\alpha$, and $\chi_{*a}\star\alpha = \Delta_{*a}\star\alpha \wedge \sim\Delta_{(*a)^+}\star\alpha$. But an easy computation shows that if $a < i/n$, then $(*a)^+ \leq i/n$, and hence, by (Ax7) and (x) of Lemma 4.15, we have that $\sim\Delta_{(*a)^+}\star\alpha \vdash \sim\Delta_{i/n}\star\alpha$. By CPL, we have therefore $\chi_a\alpha \vdash \sim\Delta_{i/n}\star\alpha \vee \alpha$.
Finally, by (viii) of Lemma 4.15 we get the desired result.

(2) By induction. If $\alpha = \Delta_a\beta$ (base case), observe that $\chi_a\beta \vdash \chi_{\Delta(a)}\Delta\beta$, but $\Delta(a) \in \{0,1\}$, hence $\chi_a\beta \vdash \chi_0\Delta\beta \vee \chi_1\Delta\beta$. The other cases are proved analogously, noticing that all connectives are closed on the set of classical values $\{0,1\} \subseteq \text{Ł}_{n+1}$.

(3) We have to prove that, if α is Boolean, then $\mathsf{AX}^*_{n+1,i}$ proves $\varphi = \alpha \to \star\alpha$.
We prove it by induction.

- $\alpha = \Delta_a\beta$ (base case). Then we have to prove $\varphi = \Delta_a\beta \to \star\Delta_a\beta$.

  By (Ax5), $\Delta_a\beta$ is equivalent to $\Delta_1\Delta_a\beta$, i.e. $\star \overset{n}{\ldots} \star\Delta_a\beta$. But now, using repeatedly (1) above $n-1$ times, it follows that $\star \overset{n}{\ldots} \star\Delta_a\beta$ implies $\star\Delta_a\beta$.

- $\alpha = \sim\beta$, with β Boolean. Then $\varphi = (\sim\Delta_{i/n}\sim\beta) \vee (\star\sim\beta)$.

  By (xii) and (xiii) of Lemma 4.15, $\chi_0\beta \vdash \chi_1\sim\beta$ and then $\chi_0\beta \vdash \chi_1(\star\sim\beta)$ as well. By (xvi) of Lemma 4.15 and by definition of χ_1 it follows that $\chi_1(\star\sim\beta) \vdash \Delta_{i/n}(\star\sim\beta)$. Hence, $\chi_0\beta \vdash \Delta_{i/n}(\star\sim\beta)$ and so $\chi_0\beta \vdash \star\sim\beta$, by (Ax10). On the other hand, $\chi_1\beta \vdash \chi_0\sim\beta$ by (xii) of Lemma 4.15. That is, $\chi_1\beta \vdash \sim\Delta_{1/n}\sim\beta$, by definition of χ_0. But then $\chi_1\beta \vdash \sim\Delta_{i/n}\sim\beta$, by (xvi) and (x) of Lemma 4.15. From this it follows that $\chi_0\beta \vee \chi_1\beta \vdash (\sim\Delta_{i/n}\sim\beta) \vee (\star\sim\beta)$. But $\mathsf{AX}^*_{n+1,i}$ proves $\chi_0\beta \vee \chi_1\beta$, by (2), hence φ is a theorem.

- The remaining cases $\alpha = \beta \vee \gamma$, with β, γ Boolean and $\alpha = \star\beta$ with β Boolean can be proved by cases in a similar way.

$\square$

Proposition 4.18. *The sublanguage of Boolean formulas obeys the axioms of classical propositional logic.*

Proof. Since all the formulas obey the axioms of CPL$^+$, over $(\wedge, \vee, \rightarrow)$, it is enough to check that, if α and β are Boolean formulas, then the formula $(\alpha \rightarrow \sim\beta) \rightarrow (\beta \rightarrow \sim\alpha)$ is a theorem of $\mathsf{AX}^*_{n+1,i}$. We first prove by induction that (Ax5) can be generalized to

(Ax5') $\Delta_a\alpha \leftrightarrow \alpha$, if α is Boolean.

The Base case is axiom (Ax5). Then we consider the following inductive steps:

- $\alpha = \sim\beta$. In this case $\Delta_a\alpha = \Delta_a\sim\beta$, and, replacing $\neg a$ by a in (Ax9), we get that the latter is equivalent to $\sim\Delta_{(\neg a)^+}\beta$, and by I.H., this is equivalent to $\sim\beta$.
- $\alpha = \beta_1 \vee \beta_2$. In this case, $\Delta_a\alpha = \Delta_a(\beta_1 \vee \beta_2)$, that by (Ax8) is equivalent to $(\Delta_a\beta_1) \vee (\Delta_a\beta_2)$, and by I.H., this is equivalent to $\beta_1 \vee \beta_2$.
- $\alpha = \star\beta$. In this case $\Delta_a\alpha = \Delta_a\star\beta$. Let b the smallest element of Ł_{n+1} such that $a \leq (*b)^+$, then, by (Ax12), $\Delta_a\star\beta$ is equivalent to $\Delta_{b^+}\beta$, and by I.H., this is equivalent to β, and by (1) and (3) of Lemma 4.17, β is equivalent to $\star\beta$.

Then let α and β be Boolean. By definition, $\alpha \rightarrow \sim\beta = \sim\Delta_{i/n}\alpha \vee \sim\beta$, and due to the above (Ax5'), the latter is equivalent to $\sim\alpha \vee \sim\Delta_{i/n}\beta$, that, by definition, is in fact $\beta \rightarrow \sim\alpha$. $\square$

Finally we prove soundness and completeness of the logic $\mathsf{AX}^*_{n+1,i}$.

Proposition 4.19 (Soundness of $\mathsf{AX}^*_{n+1,i}$)**.** *The calculus $\mathsf{AX}^*_{n+1,i}$ is sound w.r.t. the matrix $\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$, that is: $\Gamma \vdash_{\mathsf{AX}^*_{n+1,i}} \varphi$ implies that $\Gamma \models_{\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle} \varphi$, for every finite set of formulas $\Gamma \cup \{\varphi\}$.*

Proof. Straightforward, taking into account the definitions of the terms Δ_a's and χ_a's in Proposition 3.11. $\square$

Since $\mathsf{AX}^*_{n+1,i}$ is a finitary Tarskian logic, completeness can be proved by using maximal consistent sets of formulas. Thus, as a consequence of the well-known Lindenbaum-Łos theorem, if $\Gamma \nvdash_{\mathsf{AX}^*_{n+1,i}} \varphi$ then Γ can be extended to a maximal set Υ such that $\Upsilon \nvdash_{\mathsf{AX}^*_{n+1,i}} \varphi$. We will call the set Υ *maximal non-trivial with respect to* φ in $\mathsf{AX}^*_{n+1,i}$.

In the following proposition, we list the main properties of maximal consistent sets in $\mathsf{AX}^*_{n+1,i}$.

Proposition 4.20. *Let Υ be a set of formulas which is maximal non-trivial w.r.t. some formula φ in $\mathsf{AX}^*_{n+1,i}$. Then:*

1. *Υ is closed, i.e., $\Upsilon \vdash_{\mathsf{AX}^*_{n+1,i}} \psi$ iff $\psi \in \Upsilon$, for every formula ψ*
2. *$\alpha \vee \beta \in \Upsilon$ iff either $\alpha \in \Upsilon$ or $\beta \in \Upsilon$*
3. *$\alpha \wedge \beta \in \Upsilon$ iff $\alpha, \beta \in \Upsilon$*
4. *$-_{i/n}\alpha \in \Upsilon$ iff $\alpha \notin \Upsilon$*
5. *$\alpha \rightarrow \beta \in \Upsilon$ iff either $\alpha \notin \Upsilon$ or $\beta \in \Upsilon$*
6. *$\alpha \leftrightarrow \beta \in \Upsilon$ iff either $\alpha, \beta \in \Upsilon$ or $\alpha, \beta \notin \Upsilon$*
7. *For every formula α, one and only one of the conditions '$\chi_a\gamma \in \Upsilon$', holds, for $a \in L_{n+1}$.*
8. *$\chi_a\alpha \in \Upsilon$ iff $\chi_{\neg a}\sim\alpha \in \Upsilon$.*
9. *If $\chi_a\alpha \in \Upsilon$ then $\chi_{*a} \star \alpha \in \Upsilon$.*

Proof. 1. This holds by construction of the maximal non-trivial sets

2. The 'only if' part follows by the fact that Υ is maximal non-trivial w.r.t. φ, and by taking into account that $\mathsf{AX}^*_{n+1,i}$ satisfies proof by cases (recall the observations after Definition 4.14). Indeed, if $\alpha \notin \Upsilon$ and $\beta \notin \Upsilon$ then $\Upsilon, \alpha \vdash \varphi$ and $\Upsilon, \beta \vdash \varphi$, hence $\Upsilon, \alpha \vee \beta \vdash \varphi$. From this, $\alpha \vee \beta \notin \Upsilon$.

3. In order to prove that $\alpha \wedge \beta = {\sim}({\sim}\alpha \vee {\sim}\beta) \in \Upsilon$ implies that $\beta \in \Upsilon$ it is necessary to use (Ax1), showing that ${\sim}({\sim}\beta \vee {\sim}\alpha) \in \Upsilon$, and so apply (Ax3) and (Ax2).

4. Suppose $-_{i/n}\alpha \in \Upsilon$, i.e. ${\sim}\Delta_{i/n}\alpha \in \Upsilon$. Then, by (i) of Lemma 4.15 it follows that $\Delta_{i/n}\alpha \notin \Upsilon$ and, by (v) of the same Lemma, it must be $\alpha \notin \Upsilon$ as well. Conversely, assume $-_{i/n}\alpha \notin \Upsilon$, that is, ${\sim}\Delta_{i/n}\alpha \notin \Upsilon$. Then, by (Ax6), $\Delta_{i/n}\alpha \in \Upsilon$, and hence $\alpha \in \Upsilon$, by (Ax11). That is: $\alpha \notin \Upsilon$ implies that $-_{i/n}\alpha \in \Upsilon$.

5. By definition, $\alpha \to \beta = -_{i/n}\alpha \vee \beta$. Then, by item (2), $\alpha \to \beta \in \Upsilon$ iff either $-_{i/n}\alpha \in \Upsilon$ or $\beta \in \Upsilon$, iff either $\alpha \notin \Upsilon$ or $\beta \in \Upsilon$, by (4).

6. Easily follows from (3) and (5).

7. By (viii) of Lemma 4.15 and by (1), $\bigvee_{0 \le a \le 1} \chi_a\alpha \in \Upsilon$. By (2), $\chi_a\alpha \in \Upsilon$ for some $a \in \text{Ł}_{n+1}$. By (ix) of Lemma 4.15, there are no $a \neq b$ such that $\chi_a\alpha, \chi_b\alpha \in \Upsilon$, since $\varphi \notin \Upsilon$. From this, $\chi_a\alpha \in \Upsilon$ for one and only one $a \in \text{Ł}_{n+1}$.

8. It follows from (xii) of Lemma 4.15 and by (5).

9. If directly follows from (xiii) of Lemma 4.15 together with (5).

□

Lemma 4.21 (Truth Lemma for $\mathsf{AX}^*_{n+1,i}$). *Let Υ be a maximal set of formulas non-trivial with respect to φ in $\mathsf{AX}^*_{n+1,i}$. Consider the mapping e_Υ of formulas to Ł_{n+1} defined as follows: for each formula α,*

$$e_\Upsilon(\alpha) = a \quad \textit{if} \quad \chi_a\alpha \in \Upsilon.$$

Then, e_Υ is a $\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$-evaluation.

Proof. First, observe that e_Υ is well-defined, i.e. every formula gets a unique value. This is an immediate consequence of (7) of Proposition 4.20. We have to prove that the following conditions are satisfied for every formulas α and β:

(i) $e_\Upsilon(\alpha \vee \beta) = \max(e_\Upsilon(\alpha), e_\Upsilon(\beta))$. Indeed, let $c = e_\Upsilon(\alpha \vee \beta)$. By definition, $\chi_c(\alpha \vee \beta) \in \Upsilon$, and so $\chi_c(\alpha) \vee \chi_c(\beta) \in \Upsilon$, by (vii) of Lemma 4.15. By (2) of Proposition 4.20, either $\chi_c(\alpha) \in \Upsilon$ or $\chi_c(\beta) \in \Upsilon$. That is, either $e_\Upsilon(\alpha) = c$ or $e_\Upsilon(\beta) = c$. By way of contradiction, suppose e.g. $e_\Upsilon(\alpha) = d > c$ and $e_\Upsilon(\beta) = c$. Then $\chi_c(\alpha) \in \Upsilon$ and $\chi_d(\alpha) \in \Upsilon$ and so, by (xi) of Lemma 4.15, $\chi_d(\alpha \vee \beta) \in \Upsilon$. Hence $e_\Upsilon(\alpha \vee \beta) = d > c$, contradiction. From this, $d \le c$ and $c = \max(e_\Upsilon(\alpha), e_\Upsilon(\beta))$.

(ii) $e_\Upsilon({\sim}\alpha) = 1 - e_\Upsilon(\alpha)$. Indeed, let $c = e_\Upsilon(\alpha)$, that is, $\chi_c\alpha \in \Upsilon$. By (8) of Proposition 4.20, $\chi_{1-c}{\sim}\alpha \in \Upsilon$, i.e. $e_\Upsilon({\sim}\alpha) = 1 - c$.

(iii) $e_\Upsilon(\star\alpha) = *(e_\Upsilon(\alpha))$. Indeed, let $c = e_\Upsilon(\alpha)$. By definition, $\chi_c\alpha \in \Upsilon$. By (9) of Proposition 4.20, $\chi_{*c}\star\alpha \in \Upsilon$, i.e. $e_\Upsilon(\star\alpha) = *c$.

This ends the proof. □

Finally, we can state and prove the completeness result for $\mathsf{AX}^*_{n+1,i}$.

Theorem 4.22 (Completeness of $\mathsf{AX}^*_{n+1,i}$). *The calculus $\mathsf{AX}^*_{n+1,i}$ is complete w.r.t. $\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle$, that is: $\Gamma \vDash_{\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle} \varphi$ implies that $\Gamma \vdash_{\mathsf{AX}^*_{n+1,i}} \varphi$, for every finite set of formulas $\Gamma \cup \{\varphi\}$.*

Proof. Let $\Gamma \cup \{\varphi\}$ be a set of formulas of $\mathsf{AX}^*_{n+1,i}$ such that $\Gamma \nvdash_{\mathsf{AX}^*_{n+1,i}} \varphi$. By Lindenbaum-Łos, there exists a set Υ maximal non-trivial with respect to φ in $\mathsf{AX}^*_{n+1,i}$ such that $\Gamma \subseteq \Upsilon$. Let e_Υ be the evaluation defined as in the Truth Lemma 4.21. Then, it follows that, for every formula α: $e_\Upsilon(\alpha) \in F_{i/n}$ iff $\chi_1\alpha \vee \chi_{(n-1)/n}\alpha \vee \ldots \vee \chi_{i/n}\alpha \in \Upsilon$, by the Truth Lemma 4.21. Moreover, by (xiv) of Lemma 4.15, this is equivalent to the condition $\Delta_{i/n}\alpha \in \Upsilon$. By (Ax10) and by (v) of Lemma 4.15, the latter is equivalent to the condition $\alpha \in \Upsilon$. That is, for every formula α we have: $e_\Upsilon(\alpha) \in F_{i/n}$ iff $\alpha \in \Upsilon$. Therefore, e_Υ is an evaluation such that $e_\Upsilon[\Gamma] \subseteq F_{i/n}$ but $e_\Upsilon(\varphi) \notin F_{i/n}$, since $\varphi \notin \Upsilon$. This means that $\Gamma \nvDash_{\langle \mathbf{Ł}^*_{n+1}, F_{i/n}\rangle} \varphi$. □

5 Subalgebras of $Ł^*_{n+1}$ and Gödel algebras with an involutive negation and a $\star$ operation

In this section we present an alternative approach to capture the behaviour of the square operator in structures obtained by adding a unary operator $\star$ to Gödel chains with an involutive negation. In order to do so, in a first subsection we characterize the subalgebras of a $\mathbf{Ł}^*_{n+1}$ algebra. The second subsection is devoted to the study of structures obtained by adding a unary operation $\star$ to Gödel algebras with an involutive negation in general. There, for each natural n, we will axiomatically characterize the class of algebras such that the implication-free reducts of its chains (of length at most $n+1$) are isomorphic to a subalgebra of $\mathbf{Ł}^*_{m+1}$ for some m, possibly different from n. We will call such algebras *representable*.[10]

From now on, we will denote by $[0,1]^*_{MV}$ the algebra defined over the real unit interval by the Łukasiewicz operations $\wedge, \vee, \neg, *, 0$, and 1, i.e. where $*x = \max(2x-1, 0)$ and $\neg x = 1-x$.

5.1 Finite subalgebras of $[0,1]^*_{MV}$

We start by noticing that, for every $n > 1$, $\mathbf{Ł}^*_{n+1}$ and its subalgebras are subalgebras of $[0,1]^*_{MV}$. Conversely, as it will be shown in Proposition 5.7, every finite subalgebra of $[0,1]^*_{MV}$ is a subalgebra of some $\mathbf{Ł}^*_{n+1}$ (although, as seen in Example 3.10(1), it is not necessarily of the form $\mathbf{Ł}^*_{m+1}$ for some $m \leq n$). Then, studying the subalgebras of $\mathbf{Ł}^*_{n+1}$ (for any $n > 1$) turns out to be equivalent to study the finite subalgebras of $[0,1]^*_{MV}$.

For what follows it is useful to introduce the notion of *skeleton* of an element of a finite subalgebra of $[0,1]^*_{MV}$. In order for the next definition be precise, let us notice that Definition 3.2, introducing the procedure $\mathtt{P}$, can be easily adapted to any finite subalgebra $\mathbf{A}$ of $[0,1]^*_{MV}$.

Definition 5.1. Let $\mathbf{A}$ be a finite subalgebra of $[0,1]^*_{MV}$, let a be a positive element of $A \setminus \{1\}$ and let $\mathtt{P}(\mathbf{A}, a) = [a_1, \ldots, a_k]$, with $a_{k+1} = a_j$ for some $1 \leq j \leq k$. Then we define the *skeleton of a in* $\mathbf{A}$, denoted by $Sk(\mathbf{A}, a)$, as the finite string of symbols $[o_1, \ldots, o_k]$, where $o_i \in \{*, \neg\}$ is such that $o_i(a_i) = a_{i+1}$ for all $i = 1, \ldots, k$ and thus $o_k(a_k) = a_j$.

By definition of $\mathtt{P}$, one can notice that the skeleton of every element $a \in A$ is a string of symbols of the form

$$[*^{n_1}, \neg, *^{n_2}, \ldots, \neg, *^{n_k}],$$

with $k > 1$ and $n_1, \ldots, n_{k-1} > 0$, where $*^{n_i}$ is a shorthand for '$*, \overset{n_i}{\ldots}, *$', i.e. the string with n_i repetitions of $*$. Moreover, if $n_k = 0$ then we assume the string of symbols reduces to $[*^{n_1}, \neg, *^{n_2}, \ldots, \neg]$. In what follows, we will call this kind of strings *sk-sequences*.

Let us notice that, as in the case of $\mathbf{Ł}^*_{n+1}$-algebras, if $\mathbf{A}$ is a finite strictly simple subalgebra of $[0,1]^*_{MV}$ and $\mathbf{c}$ is the coatom of $\mathbf{A}$, then $\mathtt{P}(\mathbf{A}, \mathbf{c}) = [a_1, \ldots, a_k]$ is such that $a_k = \neg\mathbf{c}$, i.e., $\mathtt{P}(\mathbf{A}, \mathbf{c})$ ends with the atom of $\mathbf{A}$. Thus, $Sk(\mathbf{A}, \mathbf{c}) = [o_1, \ldots, o_k]$ is such that $o_k = \neg$.

The following result presents a slight generalization of the above argument.

Proposition 5.2. *A finite subalgebra $\mathbf{A}$ of $[0,1]^*_{MV}$ is strictly simple iff $\mathbf{A} = \langle a \rangle^*$ for a positive element $a \in A$ and $\mathtt{P}(\mathbf{A}, a) = [a_1, \ldots, a_k]$ with $a_{k+1} = a$.*

Proof. Left-to-right. It is obvious that if $\mathbf{A}$ is strictly simple, then for any $a \in A \setminus \{0,1\}$, $A = \langle a \rangle^*$ (for otherwise, $\langle a \rangle^*$ would be a proper subalgebra of $\mathbf{A}$). Moreover, if $A = \langle a \rangle^*$ but $\mathtt{P}(\mathbf{A}, a) = [a_1, \ldots, a_k]$ with $a_{k+1} = a_i$ for $i > 1$, then $\langle a_i \rangle^* \subsetneq A$ (since $a \notin \langle a_i \rangle^*$) and $\mathbf{A}$ would not be strictly simple.

Right-to-left. If $\mathbf{A} = \langle a \rangle^*$ for some positive element $a \in A \setminus \{1\}$ and $\mathtt{P}(\mathbf{A}, a) = [a_1, \ldots, a_k]$ with $a_{k+1} = a$, then every positive element of $\mathbf{A}$ belongs to $\mathtt{P}(\mathbf{A}, a)$ and, since $a_{k+1} = a$ for any a_i we have $\mathtt{P}(\mathbf{A}, a_i) = [b_1, \ldots, b_k]$ with $a_i = b_1 = b_{k+1}$, i.e. $\mathtt{P}(\mathbf{A}, a_i)$ is a cyclic permutation of the sequence $\mathtt{P}(\mathbf{A}, a)$. Therefore, for any positive element $a_i \in A$, $\mathbf{A} = \langle a_i \rangle^*$ and $\mathbf{A}$ has no subalgebras, i.e., it is strictly simple. $\square$

Example 5.3. Consider Example 3.10 (1). There we have

$$\mathbf{A} = \langle 8/9 \rangle^* = \{0, 1/9, 2/9, 4/9, 5/9, 7/9, 8/9, 1\}$$

[10] Note that these classes of algebras are different from the varieties of $\Lambda^\star_{n+1}$-algebras introduced in Section 4.2, which, for each n, are generated by the single chain $\mathbf{Ł}^*_{n+1}$.

with $\mathbf{c} = 8/9$ and $\mathtt{P}(\mathbf{A}, 8/9) = [8/9, 7/9, 5/9, 1/9]$. Then, $Sk(\mathbf{A}, 8/9) = [*, *, *, \neg]$. Observe that $8/9$ is the solution of the equation $\neg(*^3(x)) = x$. Indeed using the semantics of $*, \neg$ in $[0,1]^*_{MV}$, the equation $\neg(*^3(x)) = x$ can be written as $1 - (2(2(2x-1)-1)-1) = x$ which has a unique solution $x = 8/9$. Notice also that $Sk(\mathbf{A}, 7/9) = [*, *, \neg, *]$ and $Sk(\mathbf{A}, 5/9) = [*, \neg, *, *]$ are cyclic permutations of $Sk(\mathbf{A}, \mathbf{c})$.

The example above anticipates a general result that we are going to prove in the next proposition. Henceforth, if $R = [o_1, \ldots, o_k]$ is any sequence where every $o_i \in \{*, \neg\}$, we will adopt the notation f_R to indicate the unary function in $[0,1]$ defined as

$$f_R(x) = o_k(o_{k-1}(\ldots o_1(x)\ldots)).$$

In particular, any finite subalgebra $\mathbf{A}$ of $[0,1]^*_{MV}$ and any $a \in A$ will have an associated function $f_{Sk(\mathbf{A},a)}$. For instance, taking into account Example 5.3 above, one has

$$f_{Sk(\langle 8/9\rangle^*, 8/9)}(x) = \neg(*(*(*(x)))),$$

while

$$f_{Sk(\langle 8/9\rangle^*, 5/9)}(x) = *(*(\neg(*(x)))).$$

Proposition 5.4. *Let S be a sequence of symbols from $\{*, \neg\}$, and let f_S be its corresponding function defined as above. Then we have:*

(i) if S is a sk-sequence, the equation $f_S(x) = x$ has a unique and rational solution $x_S > 1/2$;

(ii) the equation $f_S(x) = d$ has a unique and rational solution for every rational number $0 < d < 1$.

Proof. First of all, observe that for any sequence S, as a function $f_S : [0,1] \to [0,1]$, f_S is continuous, and it is increasing if the number of negations $\neg$ involved is even, otherwise it is decreasing. Let us assume then that f_S involves an even number of negations, and hence f_S is increasing with $f_S(0) = 0$ and $f_S(1) = 1$. By composing the functions $*$ and $\neg$ in the required form, one can easily check that f_S is of the following form: there are rationals $a, b \in [0,1]$, with $0 \le a < b \le 1$ such that:

$$f_S(x) = \begin{cases} 0, & \text{if } 0 \le x \le a \\ (x-a)/(b-a), & \text{if } a \le x \le b \\ 1, & \text{if } b \le x \le 1 \end{cases}$$

As for (i), if S is a sk-sequence, by construction, the rational a is such that $1/2 \le a$. Therefore, it is clear that the equation $f_S(x) = x$ has as a unique rational solution $x_S = a/(1-b+a)$, satisfying $a < x_S < b$.

As for (ii), since f_S is always strictly increasing in the open interval (a, b), the graph $y = f_S(x)$ always intersects the horizontal line $y = d$ if $0 < d < 1$, and hence the equation $f_S(x) = d$ has always as unique solution $x_d = (b-a)d + a$.

If f_S involves an odd number of negations, then f_S is decreasing, with $f_S(0) = 1$ and $f_S(0) = 1$, and the arguments for (i) and (ii) are completely dual to the ones above. □

To graphically exemplify the above result, Figure 1 displays examples of functions f_S for a sk-sequence S containing odd and even occurrences of $\neg$ and how they intersect the diagonal in a single point.

The following result is an easy consequence of the previous Proposition 5.4.

Corollary 5.5. *Let $\mathbf{A}$ be a finite strictly simple subalgebra of $[0,1]^*_{MV}$, then*

(1) If a is a positive element of $A \setminus \{1\}$, then a is the unique rational solution of the equation $f_{Sk(\mathbf{A},a)}(x) = x$.

(2) $\mathbf{A}$ is completely determined by $Sk(\mathbf{A}, \mathbf{c})$, meaning that for any two different strictly simple subalgebras $\mathbf{A}$ and $\mathbf{A}'$ of $[0,1]^*_{MV}$, $Sk(\mathbf{A}, \mathbf{c}) \neq Sk(\mathbf{A}', \mathbf{c}')$, where $\mathbf{c}$ and $\mathbf{c}'$ denote the coatoms of $\mathbf{A}$ and $\mathbf{A}'$ respectively.

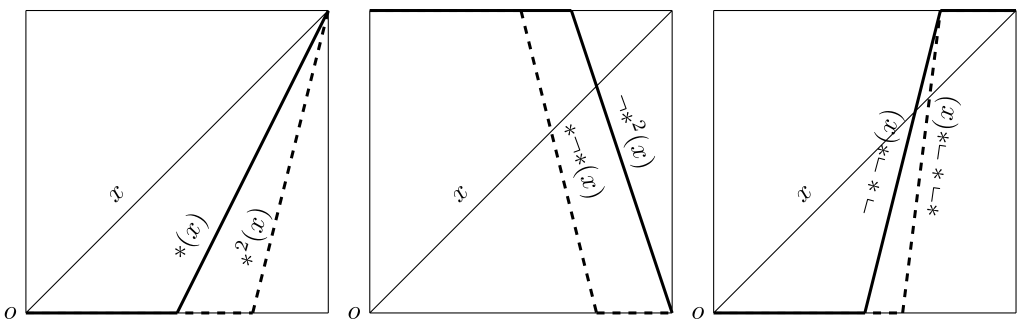

Figure 1: Examples of functions $f_S(x)$ with odd (central figure) and even (right-most figure) occurrences of $\neg$.

Proof. (1) By Proposition 5.2, $\mathbf{A} = \langle a \rangle^*$ and $f_{Sk(\mathbf{A},a)}(a) = a$. In other words a is a solution of $f_{Sk(\mathbf{A},a)}(x) = x$. Thus, by Proposition 5.4 (i), a is the unique and rational solution of the equation above.

(2) Suppose $Sk(\mathbf{A}, \mathbf{c}) = Sk(\mathbf{A}', \mathbf{c}') = S$, this means that $f_S(\mathbf{c}) = \mathbf{c}$ as well as $f_S(\mathbf{c}') = \mathbf{c}'$. But since by (1) the solution of the equation $f_S(x) = x$ is unique, we have that $\mathbf{c} = \mathbf{c}'$. Since $\mathbf{A}$ and $\mathbf{A}'$ are assumed to be strictly simple, we finally have $\mathbf{A} = \langle \mathbf{c} \rangle^* = \langle \mathbf{c}' \rangle^* = \mathbf{A}'$. □

In the corollary above, the hypothesis of $\mathbf{A}$ being strictly simple cannot be relaxed. Indeed, the following example proves that not any sk-sequence can be the skeleton of a strictly simple subalgebra of $[0, 1]^*_{MV}$.

Example 5.6. Consider the sk-sequence $S = [*, *, \neg, *, *, \neg]$ and suppose it is the skeleton of the coatom $\mathbf{c}$ of a strictly simple subalgebra $\mathbf{A}$ of $[0, 1]^*_{MV}$. Then $\mathbf{c}$ must be the rational solution of the equation $f_S(x) = x$, where $f_S(x) = 1 - 2(2(1 - (2(2x - 1) - 1) - 1) - 1)$. The unique solution of this equation is $4/5$. But $4/5$ is the coatom of $\mathbf{L}^*_{5+1}$, and $\mathtt{P}(\mathbf{L}^*_{5+1}, 4/5) = [4/5, 3/5, 1/5]$, whence $Sk(\mathbf{L}^*_{5+1}, 4/5) = [*, *, \neg]$, that is different from the initial sequence S.

Proposition 5.4 and Corollary 5.5 allows us to prove, as announced above, that the set of all finite subalgebras of $[0, 1]^*_{MV}$ coincides in fact with the set of subalgebras of all the $\mathbf{L}^*_{n+1}$ algebras.

Proposition 5.7. *The following conditions hold:*

(1) The subalgebra of $[0, 1]^*_{MV}$ *generated by an element* $a \in [0, 1]$ *is finite iff* a *is a rational number.*

(2) The finite subalgebras of $[0, 1]^*_{MV}$ *contain only rational numbers.*

(3) Any finite subalgebra of $[0, 1]^*_{MV}$ *is a subalgebra of some* $\mathbf{L}^*_{n+1}$.

Proof. (1) Left-to-right. Let $a \in [0, 1]$ and assume, without loss of generality, that a is positive, that is, $a > 1/2$ (clearly, if a was not positive one could consider its negation $\neg a > 1/2$).

If $\langle a \rangle^*$ is finite then $\mathtt{P}(\mathbf{A}, a) = [a_1, \ldots, a_k]$, where $a_{k+1} = a_i$ for some $i \leq k$. Then $\langle a_i \rangle^*$ is finite and strictly simple, and by Corollary 5.5, a_i is rational. But since $a_i \in \langle a \rangle^*$, there exists a term $f(x)$ as those considered in Proposition 5.4 such that $f(a) = a_i$, and by (ii) of Proposition 5.4, a has to be rational as well.

Right-to-left. Assume $a = n/d$ is a positive rational number of $[0, 1]$. Then, as we already observed in the proof of Lemma 3.19, the application of either $\neg$ or $*$ to a produces another rational number in $[0, 1]$ that, moreover, has the same denominator d. Indeed, $\neg a = 1 - (n/d) = (d - n)/d$ and $*(n/d) = 2n/d - 1 = (2n - d)/d$. Thus $\langle a \rangle^*$ is necessarily finite because there are only $d + 1$ rational numbers in $[0, 1]$ sharing the same denominator d.

(2) It is an easy consequence of (1).

(3) Let $\mathbf{A}$ be a finite subalgebra of $[0, 1]^*_{MV}$. Then all its elements are rational, and hence there must exist n such that $A \subseteq \{0, 1/n, \ldots, (n - 1)/n, 1\}$ (for instance take n as the l.c.m. of all denominators appearing in A), and therefore $\mathbf{A}$ must be a subalgebra of $\mathbf{L}^*_{n+1}$. □

In what follows, for every natural number n and every sequence R, we will denote by $(n)R$ the concatenation of R with itself n-times.

We say that a sequence S is *periodic* if it contains a strict subsequence R such that $S = (n)R$ for some $n \geq 2$. A sequence S will be called *non-periodic* if it is not periodic.

Proposition 5.8. *For every finite strictly simple subalgebra* $\mathbf{A}$ *of* $[0,1]^*_{MV}$, $Sk(\mathbf{A}, \mathbf{c})$ *is non-periodic.*

Proof. Assume by way of contradiction that $Sk(\mathbf{A}, \mathbf{c})$ is periodic and hence that there exists a subsequence R of $Sk(\mathbf{A}, \mathbf{c})$ such that $Sk(\mathbf{A}, \mathbf{c}) = (n)R$ for some $n \geq 2$. Since $\mathbf{A}$ is strictly simple, by Corollary 5.5, $\mathbf{c}$ is the unique rational solution of $f_{Sk(\mathbf{A},\mathbf{c})}(x) = x$. Denote it by r/n. Now, consider the equation $f_R(x) = x$ and let k/m its unique rational solution. Notice that k/m is also a solution of the equation $f_{Sk(\mathbf{A},\mathbf{c})}(x) = x$. In fact, $f_{Sk(\mathbf{A},\mathbf{c})}(x) = f_R(f_R(\ldots f_R(x)\ldots))$, and since $f_R(k/m) = k/m$, $f_{Sk(\mathbf{A},\mathbf{c})}(k/m) = k/m$. This implies that r/n and k/m are solutions of the same equation $f_{Sk(\mathbf{A},\mathbf{c})}(x) = x$. But the solution is unique and so $r/n = k/m$. Therefore, $\mathbf{A} = \langle k/m\rangle^*$ and $Sk(\mathbf{A}, \mathbf{c}) = R$ while we assumed that $Sk(\mathbf{A}, \mathbf{c}) = (n)R$ for $n \geq 2$. Contradiction. □

Finally, the next proposition presents additional properties of finite subalgebras of $[0,1]^*_{MV}$ that will be useful in the next section.

Proposition 5.9. *Let* $\mathbf{A}$ *be a finite subalgebra of* $[0,1]^*_{MV}$. *Then:*

1. *For every positive* $a \in A \setminus \{1\}$, *either* $\mathbf{A}$ *is strictly simple (i.e.,* $\mathbf{A} = \langle a\rangle^*$ *and* $f_{Sk(\mathbf{A},a)} = a$*), or* $\langle a\rangle^*$ *contains a unique strictly simple subalgebra.*
2. *If* $\mathbf{B}, \mathbf{C}$ *are two different strictly simple subalgebras of* $\mathbf{A}$, *then* $Sk(\mathbf{B}, \mathbf{c}_B) \neq Sk(\mathbf{C}, \mathbf{c}_C)$, *where* $\mathbf{c}_B$ *and* $\mathbf{c}_C$ *respectively denote the coatom of* $\mathbf{B}$ *and the coatom of* $\mathbf{C}$.
3. *If* $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_r$ *are the strictly simple subalgebras of* $\mathbf{A}$ *then* $\{B_1^+, ..., B_r^+\}$ *is a partition of* A, *where* $B_i^+ = \{a \in A : \langle a\rangle^* \supseteq B_i\}$.

Proof. (1) If $\mathbf{A}$ is strictly simple, the claim follows from Corollary 5.5 (1). Thus assume $\mathbf{A}$ is not strictly simple and take a positive $a \in A \setminus \{1\}$. By Proposition 5.2 this implies that $\mathtt{P}(\mathbf{A}, a) = [a_1 = a, ..., a_k]$ with $a_{k+1} = a_i$ for some $i > 1$. Then it is obvious that $\mathtt{P}(\mathbf{A}, a_i) = [a_i, ..., a_k]$ with $a_{k+1} = a_i$. Then $B_a = \langle a_i\rangle^*$ is strictly simple and $B_a \subsetneq \langle a\rangle^*$. Moreover by construction, for each $a \in A$ the subalgebra B_a is the unique strictly simple subalgebra contained in $\langle a\rangle^\star$

(2) is an immediate consequence of Corollary 5.5 (2).

(3) Observe that (1) and (2) imply that the union of B_i^+'s is the whole domain A of the algebra $\mathbf{A}$. Obviously, two different strictly simple subalgebras B_i and B_j must be disjoint since, if $a \in (B_i \cap B_j)$ by Proposition 5.2 $B_i = B_j = \langle a\rangle^*$. On the other hand, if $a \in (B_i^+ \cap B_j^+)$ by the previous (2) $B_i = B_j$ and thus $B_i^+ = B_j^+$. □

We end this first subsection with the following observations.

Remark 5.10.

(1) Notice that in all finite subalgebras of $[0,1]^*_{MV}$, Gödel implication is definable as we did for every $\mathbf{Ł}^*_{n+1}$-algebra (see Section 3).

(2) The logic whose algebraic semantics is the variety generated by a finite subalgebra $\mathbf{A}$ of $[0,1]^*_{MV}$ can be axiomatized in the signature $\Sigma = (\vee, \sim, \star, \bot, \top)$ following the same method used for $\mathbf{Ł}^*_{n+1}$ in Subsection 4.3.

The first remark clearly relates subalgebras of $[0,1]^*_{MV}$ with Gödel chains with an involutive negation plus an $*$ operation. This relation is deepened in the next subsection.

5.2 Adding a $\star$-operator to involutive Gödel algebras

In this subsection we present an alternative algebraic approach to capture the behaviour of Łukasiewicz's square by adding a unary operator $\star$ to a Gödel algebra with an involution.

Let us hence define the following structures.

Definition 5.11. A *Gödel-algebra with an involution* $\sim$ *and an operator* $\star$ (*$IG^\star$-algebra* for short) is a triple $(\mathbf{A}, \sim, \star)$ where $(\mathbf{A}, \sim)$ is a Gödel algebra with involution and $\star$ is a unary operator on A satisfying the following equations:[11]

[11] As it was done with $\Lambda^\star_{n+1}$-algebras in Subsection 4.2, we will use the same symbols for the connectives $\wedge$, $\vee$, $\Rightarrow$ and $\star$ and for the respective operators in $IG^\star$-algebras.

($\star 1$) $\star x \leq x$

($\star 2$) $\Delta(x \Leftrightarrow \star x) = \Delta(x \vee \sim x)$

($\star 3$) $\Delta(x \Rightarrow \sim x) = \neg_G \star x$

($\star 4$) $\Delta(\sim x \Rightarrow x) \wedge \Delta(\sim y \Rightarrow y) \wedge \Delta(\star x \Rightarrow \star y) \leq \Delta(x \Rightarrow y)$

where $\Rightarrow$ stands for Gödel implication and $\neg_G$ for Gödel negation, and Δx and $x \Leftrightarrow y$ are abbreviations for $\neg_G \sim x$ and $(x \Rightarrow y) \wedge (y \Rightarrow x)$ respectively. If $\mathbf{A}$ is a Gödel algebra in the variety $\mathbb{G}_{n+1}$ (recall Section 2), we will say that $(\mathbf{A}, \sim, \star)$ is an $IG^\star_{n+1}$-algebra. The varieties of $IG^\star$-algebras and $IG^\star_{n+1}$-algebras, over the signature $\Sigma_+ = (\wedge, \vee, \Rightarrow, \sim, \star, \bot, \top)$, will be denoted by $\mathbb{IG}^\star$ and $\mathbb{IG}^\star_{n+1}$ respectively.

Observe that, analogously to the case of the varieties $\mathbb{A}^\star_{n+1}$ from Section 4.2, $\mathbb{IG}^\star$ and $\mathbb{IG}^\star_{n+1}$ are discriminator varieties with the same discriminator term $t(x, y, z)$, and thus they are arithmetical and semisimple as well.

Let us explain the equations above on an standard IG-algebra $([0,1]_G, \sim)$ where $\sim : [0,1] \to [0,1]$ is an involution with fixpoint $1/2$. First of all recall that in $([0,1]_G, \sim)$ the following conditions hold for all x: $x \vee \sim x = 1$ iff either $x = 1$ or $x = 0$; $x \Rightarrow \sim x = 1$ iff $x \leq \sim x$ iff x is negative, that is, $x \leq 1/2$; and $\Delta(\sim x \Rightarrow x) = 1$ iff $\sim x \leq x$ iff x is positive, meaning that $x \geq 1/2$.

($\star 1$) This axiom is self-explanatory, it requires $\star x$ be not greater than x.

($\star 2$) Since $\Delta z \in \{0, 1\}$ for all $z \in [0,1]$, the formula $\Delta(x \vee \sim x) = \Delta(x \Leftrightarrow \star x)$ states that $x = \star x$ iff either $x = 0$ or $x = 1$. Therefore, taking into account ($\star 1$), this means that $\star 0 = 0$, $\star 1 = 1$ and $\star x < x$ for all $x \in (0, 1)$.

($\star 3$) As we recalled above, $x \Rightarrow \sim x = 1$ iff $x \leq \sim x$ iff $x \leq 1/2$. Thus, $\Delta(x \Rightarrow \sim x) = 1$ if $x \leq 1/2$ and it is 0 otherwise. Moreover $\neg_G \star x = 1$ if $\star x = 0$ and $\neg_G \star x = 0$ if $\star x > 0$. Therefore $\Delta(x \Rightarrow \sim x) = \neg_G \star x$ states that $\star x = 0$ iff $x \leq 1/2$, or equivalently, $\star x > 0$ iff $x > 1/2$.

($\star 4$) The term $\Delta(\sim x \Rightarrow x) \wedge \Delta(\sim y \Rightarrow y) \wedge \Delta(\star x \Rightarrow \star y)$ only takes value 0 or 1. In particular, it takes value 1 iff $x, y \geq 1/2$ and $\star x \leq \star y$. Similarly, $\Delta(x \Rightarrow y) = 1$ iff $x \leq y$. Thus ($\star 4$) states that the $\star$ is strictly monotone (and thus, one-to-one) for positive elements: for all positive x, y, if $x > y$, then $\star x > \star y$. This, together with ($\star 3$), yields that $\star$ is non-decreasing in the whole interval $[0,1]$.

Now, we rise the question whether the equations introduced in Definition 5.11 above are enough for $\star$ to capture the standard behavior of the Łukasiewicz square operator $*$ on $[0,1]$. Equivalently, we are asking whether every countable $IG^\star$-chain embeds into the algebra

$$[0,1]^*_{GMV} = ([0,1], \wedge, \vee, \Rightarrow, \sim, *, 0, 1), \tag{2}$$

which is the expansion of $[0,1]^*_{MV}$ with Gödel implication $\Rightarrow$,[12] where $*x = \max\{0, 2x - 1\}$ and $\sim x = 1 - x$ is the involution.[13]

By definition, the variety $\mathbb{IG}^\star$ of $IG^\star$-algebras is prelinear. We begin investigating the finite linearly ordered algebras of $\mathbb{IG}^\star$. Basic properties are the following:

1. If $\mathbf{A} = (A, \wedge, \vee, \sim, *, 0, 1)$ is a finite subalgebra of $[0,1]^*_{MV}$, then $\Rightarrow$ is definable in $\mathbf{A}$ and $(A, \wedge, \vee, \Rightarrow, \sim, *, 0, 1)$ is a finite chain of $\mathbb{IG}^\star$. However, the variety generated by $[0,1]^*_{MV}$ is not the one generated by $[0,1]^*_{GMV}$, as $\Rightarrow$ is not definable in the infinite chain $[0,1]^*_{MV}$.

2. The procedure P described in Definition 3.2 can be easily adapted and used so as to define $\langle x \rangle^\star$, the subalgebra generated by an element x in any finite chain of $\mathbb{IG}^\star$.

3. For every finite IG*-chain $\mathbf{A}$ and every $a \in A$, the notion of skeleton $Sk(\mathbf{A}, a)$ is defined as for subalgebras of $[0,1]^*_{MV}$ in the previous subsection.

4. Proposition 5.9 (1) is also valid for finite $IG^\star$-chains.

[12] It is worth observing that Gödel implication $\Rightarrow$ *is not* definable in $[0,1]^*_{MV}$.

[13] In this subsection, without danger of confusion, we will use $\sim$ to denote the Łukasiewicz involution instead of $\neg$ to emphasize that we look at $[0,1]^*_{GMV}$ as a $IG^\star$-chain.

It is clear that any finite subalgebra of $[0,1]^*_{MV}$ can be embedded into a finite chain of $\mathbb{IG}^\star$. The converse is not true in general as the following examples show.

Example 5.12. Let $\mathbf{A}$ be the 6-element $IG^\star$-chain with support $A = \{1, a, b, \sim b, \sim a, 0\}$, where $0 < \sim a < \sim b < b < a < 1$, the operations $\wedge$, $\vee$, $\Rightarrow$ are defined according to the order, and $\star a = \sim b, \star b = \sim a$. This algebra is not embeddable in $[0,1]^*_{MV}$ because both elements a, b satisfy in $\mathbf{A}$ the equation $\sim\star\sim\star(x) = x$, while the corresponding equation in $[0,1]^*_{MV}$, $\neg*\neg*(x) = x$, has as a unique solution $x = 2/3$. The algebra generated by $2/3$ in $[0,1]^*_{GMV}$, $\langle 2/3\rangle^*$, has universe $\{1, 2/3, 1/3, 0\}$ and hence $\langle 2/3\rangle^*$ is not isomorphic to $\mathbf{A}$. Notice that, by definition of $\mathbf{A} \in \mathbb{IG}^\star$, $Sk(\mathbf{A}, a) = Sk(\mathbf{A}, b) = [\star, \sim, \star, \sim]$. This sequence is periodic and we have already proved in Proposition 5.8 that there is no strictly simple finite subalgebra of $[0,1]^*_{MV}$ with such a skeleton.

Also observe that in this algebra it is not possible to define the operators Δ_z for every $z \in A$ as defined in Proposition 3.11. In fact the algorithm given in the proof of that proposition does not terminate. As a consequence, the axiomatization given in Section 4 for the many-valued logic with semantics on a $\mathbf{L}^*_{n+1}$-chain is not generalizable to the case of a finite $IG^\star$-chain.

Example 5.13. Let $\mathbf{A}$ be the 14-element $IG^\star$-chain whose support is $A = \langle a\rangle^\star \cup \langle b\rangle^\star$, where $a > b$ and, for $x = a, b$, $\langle x\rangle^\star$ is made of the elements

$$1 > x = \sim\star\sim\star^2 x > \star x > \sim\star^2 x > \star^2 x > \sim\star x > \star\sim\star^2 x > 0$$

as in Figure 2, and the operations $\wedge$, $\vee$, $\Rightarrow$ are defined according to the order.

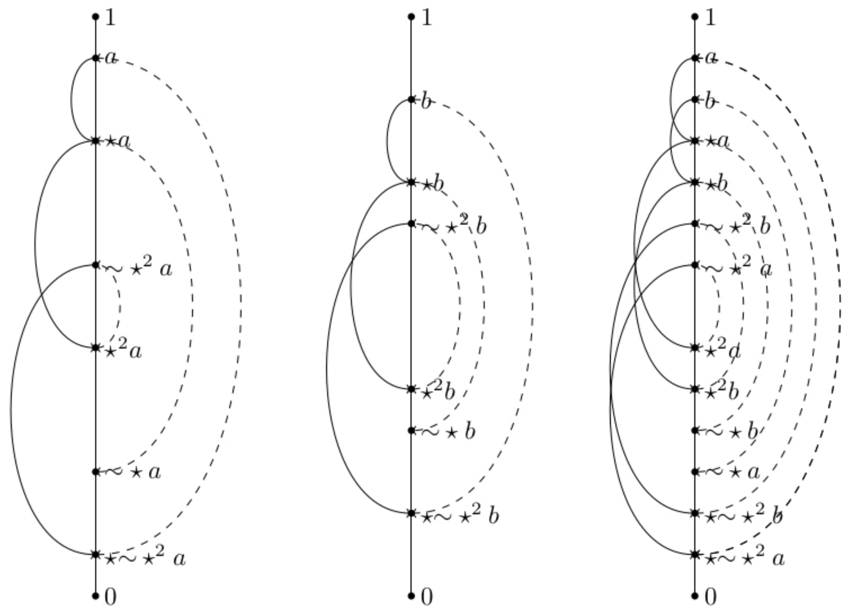

Figure 2: Graphical representation of the algebras $\langle a\rangle^*$, $\langle b\rangle^*$ and $\mathbf{A}$ from Example 5.13.

An easy computation shows that $\mathbf{A}$ is not embeddable into $[0,1]^*_{MV}$ since $\langle a\rangle^\star$ and $\langle b\rangle^\star$ are strictly simple and $Sk(\langle a\rangle^\star, a) = Sk(\langle b\rangle^\star, b) = [\star, \star, \sim, \star, \sim]$, while, by Proposition 5.9, in $[0,1]^*_{MV}$ there are no two different strictly simple subalgebras with the same skeleton.

In the light of the examples above, let us introduce the following definition.

Definition 5.14. A finite chain of $\mathbb{IG}^\star$ is called *representable* when its implication-free reduct is embeddable into $[0,1]^*_{MV}$, or in other words, when it is isomorphic to a finite subchain of $[0,1]^*_{MV}$.

Representable $IG^\star$-chains ($RIG^\star$-chains for short) form a proper subset of finite chains of $IG^\star$. Our next theorem characterizes $RIG^\star$-chains. Before showing this, we will need a first result that extends Proposition 5.9 to finite $IG^\star$-chains. To this end let us point out that, for every $IG^\star$-chain $\mathbf{A}$ which is not necessarily a subalgebra of $[0,1]^*_{MV}$ and for every $a \in A$, the procedure P (Definition

3.2) launched on a still produces a list of elements of A and it stops when it finds an element b already met at a previous step. Thus, analogously to the case of finite subalgebras of $[0,1]^*_{MV}$, in a finite $IG^\star$-chain $\mathbf{A}$ one can also easily define, for every $a \in A$, the *skeleton* of a in $\mathbf{A}$ and the strictly simple subalgebra $\langle b\rangle^\star$ associated to a (according to (1) of Prop. 5.9); and, moreover, for every strictly simple subalgebra $\mathbf{B}$ of $\mathbf{A}$, one can also define the set B^+ as in (3) of Prop. 5.9. With these preliminaries, the following proposition holds.

Proposition 5.15. *Let $\mathbf{A}$ be a finite $IG^\star$-chain and let $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_k$ the strictly simple subalgebras of $\mathbf{A}$. Then $\{B_1^+, \ldots B_k^+\}$ is a partition of $A \setminus \{0,1\}$. Furthermore, if $\mathbf{B}_i$ has a non-periodic skeleton, $\mathbf{B}_i$ is representable and each B_i^+, regarded as partial algebra, partially embeds into $[0,1]^*_{MV}$.*

Proof. The first part of the claim is proved, with no modification, by the same proof of Proposition 5.9. Indeed, in that proof, no assumption on the fact that $\mathbf{A}$ is subalgebra of $[0,1]^*_{MV}$ is made and hence it perfectly applies to this more general case.

As for the second part of the statement, assume that $\mathbf{B}_i$ has a non-periodic skeleton. Thus, in particular $Sk(\mathbf{B}_i, \mathbf{c}_i)$ for $\mathbf{c}_i$ being the coatom of $\mathbf{B}_i$. Thus, the equation $f_{Sk(\mathbf{B}_i,\mathbf{c}_i)}(x) = x$ has a unique rational solution r in $[0,1]^*_{MV}$. It is then easy to see that the finite subalgebra $\langle r\rangle^*$ of $[0,1]^*_{MV}$ is indeed isomorphic to $\mathbf{B}_i$ and the assignment $\lambda : b \mapsto r$ determines an embedding of $\mathbf{B}_i$ into $[0,1]^*_{MV}$.

Finally, in order to partially embed the partial algebra B_i^+ into $[0,1]^*_{MV}$ recall that $B_i^+ = \{a \in A : \langle a\rangle^\star \supseteq B_i\}$, or equivalently, $B_i^+ = B_i \cup \{a \in A \mid f_R(a) \in B_i$ for some finite sk-sequence $R\}$. Since we already showed that $\mathbf{B}_i$ embeds into $[0,1]^*_{MV}$, it is left to show how to map the elements a's such that $f_R(a) = b_a \in B_i$ for some sk-sequence R. Since $\mathbf{B}_i$ embeds, through a mapping λ, into the rational subalgebra of $[0,1]^*_{MV}$, the equation $f_R(x) = \lambda(b_a)$ has a rational solution, say r_a. Then, extend λ to a mapping sending each a of the above kind to r_a. The so obtained map clearly is a partial embedding of $\mathbf{B}_i^+$ into $[0,1]^*_{MV}$. □

Now, we are ready to characterize the representable $IG^\star$-chains.

Theorem 5.16. *A finite $IG^\star$-chain $\mathbf{A}$ is representable iff*

1. *For any strictly simple subalgebra $\mathbf{B}$ of $\mathbf{A}$ and for any positive $b \in B \setminus \{1\}$, $Sk(\mathbf{B}, b)$ is non-periodic, and*

2. *For each pair of strictly simple subalgebras $\mathbf{B}$ and $\mathbf{C}$ of $\mathbf{A}$ there are no positive elements $b \in B \setminus \{1\}$ and $c \in C \setminus \{1\}$ such that $Sk(\mathbf{B}, b) = Sk(\mathbf{C}, c)$.*

Proof. Left-to-right. If $\mathbf{A}$ is representable, then it is (isomorphic to) a subalgebra of $[0,1]^*_{MV}$. Therefore, (1) and (2) immediately follow from Proposition 5.8 and Proposition 5.9 (2) respectively.

Right-to-left. Assume (1) and (2) hold. (1) implies, by Proposition 5.15 that, for each strictly simple subalgebra $\mathbf{B}$ of $\mathbf{A}$ the partial algebra B^+ partially embeds into the rational subalgebra of $[0,1]^*_{MV}$ and hence it embeds into an $\mathbf{Ł}^*_{n_B+1}$ for some natural number n_B. Moreover, (2) implies that for two different strictly simple subalgebras $\mathbf{B}$ and $\mathbf{C}$ of $\mathbf{A}$, B^+ and C^+ do not partially embed into the same $\mathbf{Ł}^*_{n+1}$. In other words, for every strictly simple subalgebra $\mathbf{B}$ of $\mathbf{A}$ there exists a unique n_B and a unique partial embedding λ_B of B^+ into $Ł^*_{n_B+1}$. Let $k = \mathrm{lcm}\{n_B \mid \mathbf{B}$ is a strictly simple subalgebra of $\mathbf{A}\}$. Thus, each B^+ partially embeds into $\mathbf{Ł}^*_{k+1}$ by the same map λ which, adding $\lambda(0) = 0$ and $\lambda(1) = 1$ determines and embedding of $\mathbf{A}$ into $\mathbf{Ł}^*_{k+1}$. □

A direct inspection on the proof of Theorem 5.16 above suggests that points 1 and 2 of its statement can be equationally described. Indeed, in the following result, we will prove that for every n, representable $IG^\star_{n+1}$-algebras form a proper subvariety of $\mathbb{IG}^\star_{n+1}$.

In order to see it consider, for all $n \in \mathbb{N}$, for all sk-sequences $R = [o_1, \ldots, o_t]$ and for all natural numbers r such that $rt \leq n+1$, the following equations:

$$(R1n)\ \ {\sim}x \vee x \vee (x \Leftrightarrow y)\ \vee [\Delta(f_R(x) \Leftrightarrow y) \Rightarrow {\sim}\Delta(f_{(r-1)R}(y) \Leftrightarrow x)] = 1;$$

$$(R2n)\ \ {\sim}x \vee x \vee {\sim}y \vee y\ \vee [(\Delta(f_R(x) \Leftrightarrow x) \wedge (f_R(y) \Leftrightarrow y)) \Rightarrow \Delta(x \Leftrightarrow y)] = 1.$$

Theorem 5.17. *Let $\mathbf{A}$ be a finite $IG^\star$-algebra such that its G-reduct belongs to $\mathbb{G}(n+1)$. Then $\mathbf{A}$ is representable iff, for all sk-sequences $S = [o_1, \ldots, o_k]$ and for all natural numbers r such that $rk \leq n+1$, $\mathbf{A}$ satisfies $(R1n)$ and $(R2n)$.*

Proof. (Left-to-right). Assume $\mathbf{A}$ is not representable. Then, by Theorem 5.16, either: (1) $\mathbf{A}$ has a strictly simple subalgebra $\mathbf{B}$ such that $Sk(\mathbf{B}, b)$ is periodic, for a positive $b \in B \setminus \{1\}$, or (2) $\mathbf{A}$ has two strictly simple subalgebras $\mathbf{B}$ and $\mathbf{C}$ with positive elements $b \in B \setminus \{1\}$ and $c \in C \setminus \{1\}$ such that $Sk(\mathbf{B}, b) = Sk(\mathbf{C}, c)$.

Assume that (1) is the case and let $S = [o_1, \ldots, o_k]$ be the periodic skeleton of b in $\mathbf{B}$. Then there is an initial non-periodic sk-subsequence $R = [o_1, \ldots, o_t]$ of $Sk(b, \mathbf{B})$ and a natural number r such that $[o_1, \ldots, o_k]$ is the repetition r-times of $[o_1, \ldots, o_t]$, that is, $S = (r)R$. Call $c = f_R(b)$. Thus we have that $\sim b < 1$, $b < 1$ and $b \Leftrightarrow c < 1$ On the other hand $\Delta(f_R(b) \Leftrightarrow c) = 1$ holds by definition of c, and also $\Delta(f_{(r-1)R}(c) \Leftrightarrow b) = 1$ holds because $(r)[o_1, \ldots, o_t] = [o_1, \ldots, o_k]$ is the skeleton of b. Thus, $\sim\Delta(f_{(r-1)R}(c) \Leftrightarrow b) = 0$ and hence $(R1n)$ is not satisfied.

Hence, assume that (2) is the case. Since $\mathbf{B}$ and $\mathbf{C}$ are both strictly simple, $B\setminus\{0,1\} \cap C\setminus\{0,1\} = \emptyset$. Take positive elements $b \in B \setminus \{1\}$ and $c \in C \setminus \{1\}$. By hypothesis $Sk(\mathbf{B}, b) = Sk(\mathbf{C}, c) = S = [o_1, \ldots, o_k]$. Then one has $\Delta((f_S(b) \Leftrightarrow b) \wedge (f_S(c) \Leftrightarrow c)) = 1$ while $\Delta(b \Leftrightarrow c) = 0$. This shows that $(R2n)$ fails as well.

(Right-to-left). Let us assume that there exists a non-periodic sk-sequence $S = [o_1, \ldots, o_k]$ and a natural number r such that $rk \leq n$ and either $(R1n)$ fails or $(R2n)$ fails.

If $(R1n)$ fails, then there exist $x, y \in A$ different from 0 and 1 such that $x \neq y$, $f_R(x) \Leftrightarrow y = 1$ and $f_{(r-1)R}(y) \Leftrightarrow x = 1$. Thus $f_{(r)R}(x) \Leftrightarrow x = 1$, meaning that the subalgebra $\langle x\rangle^\star$ generated by x has a periodic skeleton. Thus $\mathbf{A}$ is not representable by Theorem 5.16.

If $(R2n)$ fails, then there are two distinct positive elements $b, c \in A\setminus\{1\}$ having the same skeleton. The strictly simple subalgebras $\langle b\rangle^\star$ and $\langle c\rangle^\star$ of $\mathbf{A}$ witness the fact that $\mathbf{A}$ is not representable again by Theorem 5.16. □

Remark 5.18. (1) As it was observed after the Proposition 3.11, in any finite RIG*-chain $\mathbf{A}$ it is possible to define the operators Δ_x for every $x \in A$. This implies that the axiomatization of the variety generated by the chain $\mathbf{Ł}^*_{n+1}$ given in Definition 4.5 and the proof of Theorem 4.11 can be easily generalized to axiomatize the variety generated by a single finite RIG*-chain.

(2) Theorem 5.17 gives an axiomatization of the variety generated by the representable $IG^\star$-chains whose length is less or equal to $n + 1$. This is the axiomatization of a variety generated by a finite family of chains, very different from the axiomatization in Definition 4.14 that gives the axiomatization of the variety generated by a single $RIG^\star$-chain.

Now we know that the answer to the question posed after Definition 5.11 is negative and we reformulate the question as whether every countable $RIG^\star$-chain embeds into the algebra $[0,1]^*_{GMV}$. In the next result we will denote by $\mathbb{RIG}^\star$ the variety generated by the finite representable $\mathbb{IG}^\star$-chains while $\mathbb{V}^\star$ will denote the variety generated by $[0,1]^*_{GMV}$.

Proposition 5.19. *The following statements are valid:*

1. *The variety $\mathbb{V}^\star$ has the finite model property.*
2. *The varieties $\mathbb{V}^\star$ and $\mathbb{RIG}^\star$ coincide.*
3. *The variety $\mathbb{V}^\star$ is axiomatized by the axioms of $\mathbb{IG}^\star$ plus the infinite set of axioms (R1n) and (R2n) for every $n \geq 2$.*

Proof. To prove (1), suppose that φ is not a tautology in $\mathbb{V}^\star$. Then there is an evaluation e into the chain $[0,1]^*_{GMV}$ such that $e(\varphi) < 1$. Then there is also a rational evaluation v (that is a good approximation of e) such that $v(\varphi) < 1$ and for any propositional variable p appearing in φ, $e(p)$ is rational. Since the subalgebra generated by the set of values $\{v(p)\colon p$ is a propositional variable appearing in $\varphi\} \subseteq [0,1]$ is finite, φ is not valid in a finite $RIG^\star$-chain.

On the other hand, (2) is immediate from (1) since both varieties are generated by finite subalgebras of $[0,1]^*_{GMV}$ which are the representable $\mathbb{IG}^\star$-chains.

Finally, (3) is a direct consequence of (2). □

6 Conclusions and final remarks

In this paper we have been concerned with the logical and algebraic analysis of the reduct of finite-valued Łukasiewicz logics over the signature $\Sigma = (\vee, \sim, \star, \bot, \top)$, where $\star$ represents the square operator $*x = x \odot x$, with $\odot$ being Łukasiewicz strong conjunction. Our main contributions are the following. First of all, we have characterized for which n of the corresponding structures $Ł^*_{n+1}$, over the $(n+1)$-element domain $\{0, 1/n, \ldots, 1\}$, the Łukasiewicz implication is definable, and thus for which n the algebra $Ł^*_{n+1}$ is term-equivalent to the MV-chain $Ł_{n+1}$. Second, we have studied the matrix logics arising from the $Ł^*_{n+1}$ structures with order filters. We have shown they are all algebraizable, we have described the resulting varieties of Λ^*_{n+1}-algebras that constitute their equivalent algebraic semantics, and provided a complete and uniform Hilbert-style axiomatization in a suitable signature that enjoys nice logical properties. And third, we have considered an alternative approach to capture the behaviour of the square operator in algebraic structures obtained by adding a unary operator $\star$ to n-valued Gödel chains with an involutive negation, and have identified the conditions under which they can be embedded into some $Ł^*_m$.

At this point we would like to make a couple of additional remarks we deem interesting to highlight. An interesting question is whether the well-known relationship between the finite-valued logics $Ł_n$ and the infinite-valued logic $Ł = \langle [0,1]_{MV}, \{1\} \rangle$ is preserved between the logics Λ^*_n and their corresponding $[0,1]$-valued version Λ^*. It is well-known that, with respect to their finitary consequence relations, Ł is the intersection of all finite-valued logics $Ł_n$, i.e. $\bigcap_n Ł_n = Ł$. It is not difficult to check that this relationship extends to our setting as follows:

- in the signature $\Sigma = (\vee, \sim, \star, \bot, \top)$, we have that $\bigcap_n \Lambda^*_n = \langle [0,1]^*_{MV}, \{1\} \rangle$, the latter standing for the matrix logic defined in the obvious way;
- while in the expanded signature $\Sigma_+ = (\wedge, \vee, \Rightarrow, \sim, \star, \bot, \top)$, we have that $\bigcap_n \Lambda^*_n = \langle [0,1]^*_{GMV}, \{1\} \rangle$.[14]

Another way to look at the relation $\bigcap_n Ł_n = Ł$ is that Łukasiewicz logic is complete with respect to the whole class of finite MV-chains. From the results of Subsection 3.3, we know that, if n is a prime number in Π (recall Definition 3.18), then the algebras $\mathbf{Ł}_{n+1}$ and $\mathbf{Ł}^*_{n+1}$ are term equivalent. Thus, we rise the question of whether primes from Π are enough to define a complete semantics for Ł. In order words, whether Ł is complete with respect to the set of finite chains $\mathbf{Ł}^*_{n+1}$ where $n \in \Pi$. Clearly, in order to provide an answer to the question above, we would need first to elucidate whether Π is an infinite set or not.

Our future work in this topic will also concern the variety $\mathbb{IG}^\star$, introduced in Subsection 5.2. In particular we will investigate whether $\mathbb{IG}^\star$ can be generated by *standard algebras*, that is to say, by $IG^\star$-chains on the real unit interval and if, moreover, $\mathbb{IG}^\star$ can be generated by its finite chains. The latter, then, would give the finite model property for its associated logic.

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[14] By the way, it is worth noting that a similar result holds by considering the signature $\Sigma_\Delta = (\wedge, \vee, \Delta, \sim, \star, \bot, \top)$. That is, recalling that the Δ operator is not definable in $[0,1]^*_{MV}$, if we let $[0,1]^*_{MV_\Delta} := ([0,1], \wedge, \vee, \Delta, \neg, *, 0, 1)$, then $\bigcap_n \Lambda^*_n = \langle [0,1]^*_{MV_\Delta}, \{1\} \rangle$ over the signature Σ_Δ.

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Appendix

In Subsections 4.1 and 4.2 we have presented properties of the logics $\Lambda^*_{n+1,i}$ that are commonly investigated by Abstract Algebraic Logic (AAL) means. Indeed, as it was suggested by one of the anonymous referees, adopting the AAL perspective, not only allows to provide an alternative proof for some of our results from a more abstract perspective, but it may also pave the way to address several issues that would be otherwise quite complex to tackle.

In this appendix, we hence present an alternative proof of the main result of Subsection 4.2 stated in Corollary 4.12, that is, the fact that, for every $n \in \mathbb{N}$, the variety $\mathbb{A}^*_{n+1}$ is, for all $i \leq n$, the equivalent algebraic semantics of the logic $\Lambda^*_{n+1,i}$. For the proof communicated to us by the aforementioned referee it is convenient to recall some basic notions and results form AAL. All the relevant definitions and background results can be found in [5, 21].

(1) Let L be an algebraizable logic. By $\mathrm{Alg}(\mathsf{L})$ we will denote the equivalent algebraic semantics of L.

(2) For every sentential logic L, the *intrinsic variety* of L is the variety generated by the Lindenbaum-Tarski algebra of L on countably many propositional variables.

(3) A congruence θ of an algebra $\mathbf{A}$ is *compatible* with a set $F \subseteq A$, if for all $a \in A$, $a \in F$ iff the congruence class of a modulo θ, a/θ, belongs to $F/\theta = \{f/\theta \mid f \in F\}$. Then, the *Leibniz congruence* of a matrix $\langle \mathbf{A}, F\rangle$ is the largest congruence $\Omega^{\mathbf{A}}F$ of $\mathbf{A}$ compatible with F. Finally, a matrix $\langle \mathbf{A}, F\rangle$ is *Leibniz reduced* if $\Omega^{\mathbf{A}}F$ is the identity.

Theorem 6.1. *For all $n \in \mathbb{N}$ and $i \leq n$, the variety generated by $Ł^*_{n+1}$, $\mathbb{A}^*_{n+1}$, is the equivalent algebraic semantics of $\Lambda^*_{n+1,i}$.*

Proof. For every natural number n, the logic Λ^*_{n+1} is, by its own definition, determined by the finite matrix $\langle Ł^*_{n+1}, \{1\}\rangle$. By Corollary 3.13 (ii), $Ł^*_{n+1}$ is simple, so that the congruence corresponding to $\{1\}$ is the Leibniz congruence of $\langle Ł^*_{n+1}, \{1\}\rangle$ that clearly is the identity. Therefore, by [21, Proposition 5.79], its intrinsic variety $\mathbb{V}(n+1)$ coincides with $\mathbb{A}^*_{n+1}$.

We are now going to show that, indeed, the equivalent algebraic semantics $\mathrm{Alg}(\Lambda^*_{n+1})$ of Λ^*_{n+1} coincides with $\mathbb{V}(n+1)$, and hence, also with $\mathbb{A}^*_{n+1}$. Notice that the latter implies the claim because, by Lemma 4.2, for all $i \leq n$ the logic $\Lambda^*_{n+1,i}$ is equivalent to Λ^*_{n+1}.

The intrinsic variety of any algebraizable logic contains its equivalent algebraic semantics. Thus, $\mathbb{V}(n+1) \supseteq \mathrm{Alg}(\Lambda^*_{n+1})$. Let us hence show that $\mathbb{V}(n+1) \subseteq \mathrm{Alg}(\Lambda^*_{n+1})$ as well.

By Corollary 4.13, $\mathbb{A}^*_{n+1} = \mathbb{V}(n+1)$ is congruence-distributive. Therefore, by Jónsson lemma ([29]), the subdirectly irreducible elements of $\mathbb{V}(n+1)$ are homomorphic images of subalgebras of its generator, in symbols, $\mathbb{V}(n+1)_{SI} \subseteq \mathrm{HS}(\text{Ł}^*_{n+1})$. Recall from Corollary 3.13 (ii) that every algebra in $S(\text{Ł}^*_{n+1})$ is simple, thus, in particular Ł^*_{n+1}, besides the trivial algebra, only has one homomorphic image, that is itself. Hence $\mathrm{HS}(\text{Ł}^*_{n+1}) = \mathrm{S}(\text{Ł}^*_{n+1})$

By Theorem 4.3, Λ^*_{n+1} is algebraizable and hence $\mathrm{Alg}(\Lambda^*_{n+1})$ is a quasivariety, whence it is closed under subalgebras, i.e.,

$$\mathbb{V}(n+1)_{SI} \subseteq \mathrm{S}(\text{Ł}^*_{n+1}) \subseteq \mathrm{Alg}(\Lambda^*_{n+1}).$$

Finally, since $\mathrm{Alg}(\Lambda^*_{n+1})$ is closed under subdirect products, by Birkhoff theorem ([5, Theorem 8.6]), we conclude that $\mathrm{SP}(\mathbb{V}(n+1)_{SI}) = \mathbb{V}(n+1) \subseteq \mathrm{Alg}(\Lambda^*_{n+1})$ and hence the equivalent algebraic semantics of Λ^*_{i+1} coincides with its intrinsic variety $\mathbb{V}(n+1)$. This settles the claim. $\square$