Canonical extensions of conditional probabilities and compound conditionals

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Abstract. In this paper we show that the probability of conjunctions and disjunctions of conditionals in the recently introduced framework of Boolean algebras of conditionals are in full agreement with the corresponding operations of conditionals as defined in the approach developed by two of the authors to conditionals as three-valued objects, with betting-based semantics, and specified as suitable random quantities. We do this by first proving that the canonical extension of a full conditional probability on a finite algebra of events to the corresponding algebra of conditionals is compatible with taking subalgebras of events.

Keywords: Boolean algebras of conditionals · Conditional probability · Conjunction and disjunction of conditionals.

1 Introduction

Conditionals play a key role in different areas of logic and probabilistic reasoning, and they have been studied from many points of view, see, e.g., [1,2,3,5,6,8,15,16,18,19,20] In a recent paper [7], an algebraic setting for measure-free conditionals has been put forward. More precisely, given a finite Boolean algebra $\mathbf{A}$ of events, the authors build another (much bigger but still finite) Boolean algebra $\mathcal{C}(\mathbf{A})$ where *basic conditionals*, i.e. objects of the form $(A|B)$ for $A \in \mathbb{A}$ and $B \in \mathbb{A}' = \mathbb{A}\setminus\{\bot\}$, can be freely combined with the usual Boolean operations, yielding compound conditional objects, while they are required to satisfy a set of natural properties. Moreover, the set of atoms of $\mathcal{C}(\mathbf{A})$ are fully identified and it is shown they are in a one-to-one correspondence with sequences of pairwise different atoms of $\mathbf{A}$ of maximal length. Finally, it is also shown that any positive probability P on the set of events from $\mathbf{A}$ can be *canonically* extended to a probability μ_P on the algebra of conditionals $\mathcal{C}(\mathbf{A})$ in such a way that the probability $\mu_P(a|b)$ of a basic conditional coincides with the conditional probability $P(a|b) = P(a \wedge b)/P(b)$. This is done by suitably defining the probability of each atom of $\mathcal{C}(\mathbf{A})$ as a certain product of conditional probabilities.

However, we remark that in [7] explicit definitions of conjunction and disjunction of conditionals are not explicitly given. Rather, any compound conditional comes determined by the disjunction of those atoms in $\mathcal{C}(\mathbf{A})$ that lie below it. Similarly, the probability of any compound conditional is computed as the sum of the probabilities of the atoms below the conditional. But no operational and systematic procedure to do these computations avoiding a combinatorial explosion is provided in [7].

In this paper, after this introduction and some preliminaries in Section 2, we will first show that the canonical extension of a positive probability on $\mathbf{A}$ to the algebra of conditionals $\mathcal{C}(\mathbf{A})$ can be generalised to the case when we start from a conditional probability (in the axiomatic sense) on $\mathbb{A} \times \mathbb{A}'$. This is done in Section 3. Then in Section 4 we show that, if $\mathbf{B}$ is a subalgebra of events of $\mathbf{A}$ and P a conditional probability on $\mathbb{A} \times \mathbb{A}'$, then the restriction of the canonical extension μ_P on $\mathcal{C}(\mathbf{A})$ to $\mathcal{C}(\mathbf{B})$ is, in fact, the canonical extension of the restriction of P on $\mathbb{B} \times \mathbb{B}'$. This will allow us to prove in Section 5 that the probability of the conjunction coincides with McGee and Kaufmann's expressions obtained within the approach developed by two of the authors to conditionals as three-valued objects, with betting-based semantics, and specified as suitable random quantities. We also obtain the probability of the disjunction and the probability sum rule, in agreement with the approach given in [10]. We conclude in Section 6 with some remarks and prospects for future work.

2 Preliminaries

In this section we recall basic notions and results from [7] where, for any Boolean algebra of events $\mathbf{A} = (\mathbb{A}, \wedge, \vee, \bar{\ }, \bot, \top)$, a Boolean algebra of conditionals, denoted $\mathcal{C}(\mathbf{A})$, is built. We will also denote a conjunction $A \wedge B$ simply by AB. Intuitively, a Boolean algebra of conditionals over $\mathbf{A}$ allows *basic conditionals*, i.e. objects of the form $(A|B)$ for $A \in \mathbb{A}$ and $B \in \mathbb{A}' = \mathbb{A}\backslash\{\bot\}$, to be freely combined with the usual Boolean operations up to certain extent.

In mathematical terms, the formal construction of the algebra of conditionals $\mathcal{C}(\mathbf{A})$ is done as follows. One first considers the free Boolean algebra $\mathbf{Free}(\mathbb{A}|\mathbb{A}') = (Free(\mathbb{A}|\mathbb{A}'), \sqcap, \sqcup, \bar{\ }, \bot, \top)$ generated by the set $\mathbb{A}|\mathbb{A}' = \{(A|B) : A \in \mathbb{A}, B \in \mathbb{A}'\}$. Then, one considers the smallest congruence relation $\equiv_{\mathfrak{C}}$ on $\mathbf{Free}(\mathbb{A}|\mathbb{A}')$ satisfying the following natural properties:

(C1) $(B|B) \equiv_{\mathfrak{C}} \top$, for all $B \in \mathbb{A}'$;
(C2) $(A_1|B) \sqcap (A_2|B) \equiv_{\mathfrak{C}} (A_1A_2|B)$, for all $A_1, A_2 \in \mathbb{A}$, $B \in \mathbb{A}'$;
(C3) $\overline{(A|B)} \equiv_{\mathfrak{C}} (\bar{A}|B)$, for all $A \in \mathbb{A}$, $B \in \mathbb{A}'$;
(C4) $(AB|B) \equiv_{\mathfrak{C}} (A|B)$, for all $A \in \mathbb{A}$, $B \in \mathbb{A}'$;
(C5) $(A|B) \sqcap (B|C) \equiv_{\mathfrak{C}} (A|C)$, for all $A \in \mathbb{A}$, $B, C \in \mathbb{A}'$ such that $A \leqslant B \leqslant C$.

Finally, the algebra $\mathcal{C}(\mathbf{A})$ is defined as follows.

Definition 1. *For every Boolean algebra* $\mathbf{A}$*, the* Boolean algebra of conditionals *of* $\mathbf{A}$ *is the quotient structure* $\mathcal{C}(\mathbf{A}) = \mathbf{Free}(\mathbb{A}|\mathbb{A})/_{\equiv_{\mathfrak{C}}}$.

Since $\mathcal{C}(\mathbf{A})$ is a *quotient* of $\mathbf{Free}(\mathbb{A}|\mathbb{A})$, elements of $\mathcal{C}(\mathbf{A})$ are equivalence classes, but without danger of confusion, one can henceforth identify classes $[t]_{\equiv_{\mathfrak{C}}}$ with one of its representative elements, in particular, by t itself. Conditionals of the form $(A|\top)$ will also be simply denoted as A.

A basic observation is that if $\mathbf{A}$ is finite, $\mathcal{C}(\mathbf{A})$ is finite as well, and hence atomic. Indeed, if $\mathbf{A}$ is a Boolean algebra with n atoms $\mathrm{at}(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$, i.e. $|\mathrm{at}(\mathbf{A})| = n$, it is shown in [7] that the atoms of $\mathcal{C}(\mathbf{A})$ are in one-to-one correspondence with sequences $\overline{\alpha} = \langle \alpha_{i_1}, \ldots, \alpha_{i_{n-1}} \rangle$ of $n-1$ pairwise different atoms of $\mathbf{A}$, each of these sequences giving rise to an atom $\omega_{\overline{\alpha}}$ of $\mathcal{C}(\mathbf{A})$ defined as the following conjunction of $n-1$ basic conditionals:

$$\omega_{\overline{\alpha}} = (\alpha_{i_1}|\top) \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \cdots \sqcap (\alpha_{i_{n-1}}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{n-2}}), \tag{1}$$

It is then clear that $|\mathrm{at}(\mathcal{C}(\mathbf{A}))| = n!$.

Next we will recall some properties holding in $\mathcal{C}(\mathbf{A})$ that will be useful for next sections. For each subvector $(i_1, \ldots, i_k)$ of $(1, \ldots, n)$ we set

$$\omega_{i_1 \cdots i_k} = \alpha_{i_1} \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \cdots \sqcap (\alpha_{i_k}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{k-1}}), \tag{2}$$

that is, $\omega_{i_1 \cdots i_k}$ denotes an initial conjunction of k components of the atom $\omega_{i_1 \cdots i_{n-1}}$. Indeed, as $(\alpha_{i_n}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{n-1}}) = (\alpha_{i_n}|\alpha_{i_n}) = \top$, for each permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, we obtain the following atom of $\mathcal{C}(\mathbf{A})$:

$$\omega_{i_1 \cdots i_n} = \omega_{i_1 \cdots i_{n-1}} = \alpha_{i_1} \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \cdots \sqcap (\alpha_{i_{n-1}}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{n-2}}).$$

We hence recall that, from [7, Proposition 4.3], for each k, the conjunctions $\omega_{i_1 \cdots i_k}$'s constitute a partition of the algebra $\mathcal{C}(\mathbf{A})$. In particular this implies that $\bigsqcup_{(i_1,\ldots,i_k) \in \Pi_{\{j_1,\ldots,j_k\}}} \omega_{i_1 \cdots i_k} = \top$, where $\Pi_{\{j_1,\ldots,j_k\}}$ is the set of all permutations $(i_1, \ldots, i_k)$ of the set $\{j_1, \ldots, j_k\}$.

Now, consider a *positive* probability on the algebra of plain events $P : \mathbf{A} \to [0, 1]$. It is shown in [7] that P can be extended to a probability $\mu_P : \mathcal{C}(\mathbf{A}) \to [0, 1]$ on the Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$, called *canonical extension*, such that $\mu_P(\text{“}(A|B)\text{”})$, the probability of a basic conditional $(A|B)$, coincides with the conditional probability of A given B, i.e. $\mu_P(\text{“}(A|B)\text{”}) = P(A|B) = P(A \wedge B)/P(B)$. In particular, $\mu_P(\text{“}(A|\top)\text{”}) = P(A|\top) = P(A)$ for any $A \in \mathbf{A}$. Actually, the probability μ_P is first defined on the atoms of $\mathcal{C}(\mathbf{A})$ as follows: for any atom $\omega_{i_1 \cdots i_{n-1}} = \alpha_{i_1} \sqcap (\alpha_{i_2}|\bar{\alpha}_{i_1}) \sqcap \cdots \sqcap (\alpha_{i_{n-1}}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{n-2}})$, its probability is defined as the following product of conditional probabilities:

$$\mu_P(\omega_{i_1 \cdots i_{n-1}}) = P(\alpha_{i_1}) \cdot P(\alpha_{i_2}|\bar{\alpha}_{i_1}) \cdots P(\alpha_{i_{n-1}}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{n-2}}).$$

Then μ_P is extended to the whole algebra $\mathcal{C}(\mathbf{A})$ of conditionals by additivity. Moreover, it is shown in [7] that for any k, the following factorization holds:

$$\begin{aligned} &\mu_P(\omega_{i_1 \cdots i_k}) = \textstyle\sum_{(i_{k+1},\ldots,i_n) \in \Pi_{\{1,\ldots,n\} \setminus \{i_1,\ldots,i_k\}}} \mu_P(\omega_{i_1 \cdots i_{n-1}}) = \\ &= P(\alpha_{i_1}) \cdot P(\alpha_{i_2}|\bar{\alpha}_{i_1}) \cdots P(\alpha_{i_k}|\bar{\alpha}_{i_1} \cdots \bar{\alpha}_{i_{k-1}}). \end{aligned} \tag{3}$$

We finally notice that, as observed above, since for each k the conjunctions $\omega_{i_1\cdots i_k}$'s constitute a partition of $\mathcal{C}(\mathbf{A})$, the sum of the probabilities over all of them is 1, that is:

$$1 = \sum_i P(\alpha_i) = \sum_i \mu_P(\omega_i) = \sum_{i\neq j} \mu_P(\omega_{ij}) = \cdots = \sum_{(i_1,\ldots,i_n)\in\Pi_{\{1,\ldots,n\}}} \mu_P(\omega_{i_1\cdots i_{n-1}}).$$

3 Canonical extension of a conditional probability

In the definition of the canonical extension μ_P on $\mathcal{C}(\mathbf{A})$, a crucial assumption is that P is positive, i.e. that $P(\alpha) > 0$ for every $\alpha \in \mathrm{at}(\mathbf{A})$, otherwise $\mu_P(\omega)$ can be undefined for some $\omega \in \mathrm{at}(\mathcal{C}(\mathbf{A}))$ (it would be of the form 0/0). A way to overcome this problem is, instead of starting with a (unconditional) probability on $\mathbf{A}$, to start with a *conditional probability* on $\mathbb{A} \times \mathbb{A}'$ in the axiomatic sense, that is to say, a binary map $P : \mathbb{A} \times \mathbb{A}' \to [0,1]$, where $\mathbb{A}' = \mathbb{A}\backslash\{\bot\}$, such that

(CP1) For all $B \in \mathbb{A}'$, $P(\cdot|B) : \mathbb{A} \to [0,1]$ is a finitely additive probability on $\mathbf{A}$;
(CP2) For all $A \in \mathbb{A}$ and $B \in \mathbb{A}'$, $P(A|B) = P(A \wedge B|B)$;
(CP3) For all $A \in \mathbb{A}$, $B, C \in \mathbb{A}'$, if $A \leqslant B \leqslant C$, then $P(A|B) = P(A|B) \cdot P(B|C)$.

Remark 1. Recall that requesting $P : \mathbb{A} \times \mathbb{A}' \to [0,1]$ to satisfy the above three postulates assures that P is a coherent conditional probability assessment in the sense of de Finetti to all the conditional objects $(A|B)$, with $A, B \in \mathbb{A}$ and $B \neq \bot$. In fact, a conditional probability assessment on an arbitrary family of (basic) conditional events $P(A_1|B_1) = x_1, \ldots, P(A_n|B_n) = x_n$, is coherent iff it can be extended to a conditional probability (in the above sense) on $\mathbb{A} \times \mathbb{A}'$ ([4].

Then, given a conditional probability $P : \mathbb{A} \times \mathbb{A}' \to [0,1]$, we can proceed as in the previous section and first define a mapping μ_P on $\mathrm{at}(\mathcal{C}(\mathbf{A}))$ as follows: for any atom $\omega = (\alpha_1|\top) \sqcap (\alpha_2|\bar{\alpha}_1) \sqcap \ldots \sqcap (\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2})$,

$$\mu_P(\omega) = P(\alpha_1|\top) \cdot P(\alpha_2|\bar{\alpha}_1) \cdot \ldots \cdot P(\alpha_{n-1}|\bar{\alpha}_1 \cdots \bar{\alpha}_{n-2}), \tag{4}$$

One can check that μ_P so defined is a probability distribution on $\mathrm{at}(\mathcal{C}(\mathbf{A}))$.

Proposition 1. $\sum_{\omega\in\mathrm{at}(\mathcal{C}(\mathbf{A}))} \mu_P(\omega) = 1$.

Proof. Although one could adapt here the proof of [7, Lemma 6.8], we provide below a direct proof. Let $\mathrm{at}(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$. First of all, for any subset of atoms $\{\beta_1, \ldots, \beta_k\} \subseteq \mathrm{at}(\mathbf{A})$, with $k < n$, by the law of total probabilities,

$$\textstyle\sum_{\beta\in\mathrm{at}(\mathbf{A})\backslash\{\beta_1,\ldots,\beta_k\}} P(\beta|\bar{\beta}_1 \cdots \bar{\beta}_k) = 1.$$

For $k = 1$ it is clear that $\sum_{\alpha\in\mathrm{at}(\mathbf{A})} P(\alpha|\top) = 1$, and for $k = 2$, we have $P(\alpha|\top) = \sum_{\beta\neq\alpha} P(\alpha|\top) \cdot P(\beta|\bar{\alpha})$. More generally, for any $k < 1$ we have:

$$P(\beta_1)\cdot\ldots\cdot P(\beta_k|\bar{\beta}_1 \cdots \bar{\beta}_{k-1}) = \sum_{\beta\notin\{\beta_1,\ldots,\beta_k\}} P(\beta_1)\cdot\ldots\cdot P(\beta_k|\bar{\beta}_1 \cdots \bar{\beta}_{k-1})\cdot P(\beta|\bar{\beta}_1 \cdots \bar{\beta}_k).$$

Then, we can write: $1 = \sum_{\beta_1} P(\beta_1|\top) = \sum_{\beta_1}\sum_{\beta_2\neq\beta_1} P(\beta_1|\top)\cdot P(\beta_2|\bar{\beta}_1) = \cdots = \sum_{\langle\beta_1,\ldots,\beta_n\rangle\in Seq(\mathbf{A})} P(\beta_1|\top)\cdot P(\beta_2|\bar{\beta}_1)\cdots P(\beta_{n-1}|\bar{\beta}_1\cdots\bar{\beta}_{n-2}) = \sum_{\overline{\alpha}\in Seq(\mathbf{A})} \mu_P(\omega_{\overline{\alpha}}) = \sum_{\omega\in \mathrm{at}(\mathcal{C}(\mathbf{A}))} \mu_P(\omega)$. □

Then, we can extend μ_P to a probability on the whole algebra $\mathcal{C}(\mathbf{A})$ in the usual way by additivity, as in the previous case: for any $T \in \mathcal{C}(\mathbf{A})$, $\mu_P(T) = \sum_{\omega\leqslant T}\mu_P(\omega)$. We will keep referring to μ_P as the *canonical extension* of P.

To conclude this section, we check that Equation (3) keeps holding in this more general setting. Indeed, concerning the canonical extension on the conjunctions $\omega_{i_1\cdots i_k}$'s, we first observe that, as $\omega_{1\cdots n-2\, n-1} \sqcup \omega_{1\cdots n-2\, n} = \omega_{1\cdots n-2}$, from (4) it holds that:

$$\begin{aligned}
&\mu_P(\omega_{1\cdots n-2}) = \mu_P(\omega_{1\cdots n-2\, n-1}) + \mu_P(\omega_{1\cdots n-2\, n}) = \\
&= P(\alpha_1)P(\alpha_2|\bar{\alpha}_1)\cdots P(\alpha_{n-2}|\bar{\alpha}_1\cdots\bar{\alpha}_{n-3})[P(\alpha_{n-1}|(\alpha_{n-1}\vee\alpha_n))+ \\
&+P(\alpha_n|(\alpha_{n-1}\vee\alpha_n))] = P(\alpha_1)P(\alpha_2|\bar{\alpha}_1)\cdots P(\alpha_{n-2}|\bar{\alpha}_1\cdots\bar{\alpha}_{n-3}).
\end{aligned}$$

Likewise $\mu_P(\omega_{i_1\cdots i_{n-2}}) = P(\alpha_{i_1})P(\alpha_{i_2}|\bar{\alpha}_{i_1})\cdots P(\alpha_{i_{n-2}}|\bar{\alpha}_{i_1}\cdots\bar{\alpha}_{i_{n-3}})$. Then, by backward iteration, for each $k \leqslant n-1$, it holds that

$$\mu_P(\omega_{i_1\cdots i_k}) = P(\alpha_{i_1})P(\alpha_{i_2}|\bar{\alpha}_{i_1})\cdots P(\alpha_{i_k}|\bar{\alpha}_{i_1}\cdots\bar{\alpha}_{i_{k-1}}). \quad (5)$$

The question of whether μ_P actually extends P, in the sense that, for any basic conditional $(A|B) \in \mathcal{C}(\mathbf{A})$, it holds $\mu_P(``(A|B)") = P(A|B)$ is deferred to Theorem 2 in next the section.

4 The canonical extension for subalgebras

In this section we examine the restriction of the canonical extension μ_P for conditional subalgebras of $\mathcal{C}(\mathbf{A})$. Then, we let $\mathbf{A}$ be a finite algebra whose set of atoms is $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let $i < n$, and for $i = 1, \ldots, n-1$, let

$$\beta_j = \begin{cases} \alpha_j & \text{if } j < i \\ \alpha_i \vee \alpha_{i+1} & \text{if } j = i \\ \alpha_{j+1} & \text{if } j > i+1 \end{cases}$$

and let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by $\beta_1, \ldots, \beta_{n-1}$, so that $\mathrm{at}(\mathbf{B}) = \{\beta_1, \ldots, \beta_{n-1}\}$. Now let us consider $P : \mathbb{A}\times\mathbb{A}' \to [0,1]$ a conditional probability and $\mu_P : \mathcal{C}(\mathbf{A}) \to [0,1]$ its canonical extension to $C(\mathbf{A})$. Further, let $P' : \mathbb{B}\times\mathbb{B}' \to [0,1]$ be the restriction of P to $\mathbb{B}\times\mathbb{B}'$, and let $\mu_{P'} : \mathcal{C}(\mathbf{B}) \to [0,1]$ its canonical extension to $\mathcal{C}(\mathbf{B})$. The question is whether $\mu_{P'}$ is the restriction of μ_P to $\mathcal{C}(\mathbf{B})$. Next theorem shows this is actually the case.

We set $\omega'_{j_1\cdots j_{n-2}} = (\beta_{j_1}|\top)\sqcap(\beta_{j_2}|\bar{\beta}_{j_1})\sqcap\cdots\sqcap(\beta_{j_{n-2}}|\bar{\beta}_{j_1}\cdots\bar{\beta}_{j_{n-3}})$ and we recall that $\mu_{P'}(\omega'_{j_1\cdots j_{n-2}}) = P(\beta_{j_1}|\top)P(\beta_{j_2}|\bar{\beta}_{j_1})\cdot\cdots\cdot P(\beta_{j_{n-2}}|\bar{\beta}_{j_1}\cdots\bar{\beta}_{j_{n-3}})$. In the next result we show that $\mu_P(\omega'_{j_1\cdots j_{n-2}}) = \mu_{P'}(\omega'_{j_1\cdots j_{n-2}})$.

Theorem 1. *For each atom $\omega'_{j_1\cdots j_{n-2}} \in at(\mathbf{B})$, the following holds:*

$$\begin{aligned}
&\mu_P(\omega'_{j_1\cdots j_{n-2}}) = \mu_P((\beta_{j_1}|\top)\sqcap(\beta_{j_2}|\bar{\beta}_{j_1})\sqcap\cdots\sqcap(\beta_{j_{n-2}}|\bar{\beta}_{j_1}\cdots\bar{\beta}_{j_{n-3}})) = \\
&= P(\beta_{j_1}|\top)P(\beta_{j_2}|\bar{\beta}_{j_1})\cdots P(\beta_{j_{n-2}}|\bar{\beta}_{j_1}\cdots\bar{\beta}_{j_{n-3}}) = \mu_{P'}(\omega'_{j_1\cdots j_{n-2}}).
\end{aligned}$$

Proof. The proof is omitted due to lack of space (it can be provisionally found in the Appendix).

Remark 2. We observe that, for each conditional subalgebra $\mathcal{C}(\mathbf{B})$ of $\mathcal{C}(\mathbf{A})$, by a suitable iterated application of Theorem 1, it can be proved that $\mu_P(\omega') = \mu_{P'}(\omega')$, for every $\omega' \in \mathrm{at}(\mathbf{B})$.

As an illustration of Theorem 1, let us consider the following simple example. Let $\mathbf{A}$ be an algebra with four atoms $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Now let us consider the partition defined by the elements $\beta_1 = \alpha_1, \beta_2 = \alpha_2$ and $\beta_3 = \alpha_3 \vee \alpha_4$, and let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated these three elements so that so that $\{\beta_1, \beta_2, \beta_3\}$ become the atoms of $\mathbf{B}$. As above, let P be a conditional probability on $\mathbb{A} \times \mathbb{A}'$, and let P' its restriction to $\mathbb{B} \times \mathbb{B}'$. According to Theorem 1, let us practically show that $\mu_{P'}$ is the restriction of μ_P on $\mathcal{C}(\mathbf{B})$. We have to show that, for any pairwise different $i, j \in \{1, 2, 3\}$, the following condition holds:

$$\mu_P((\beta_i|\top) \sqcap (\beta_j|\bar{\beta}_i)) = P(\beta_i) \cdot P(\beta_j|\bar{\beta}_i) = \mu_{P'}((\beta_i|\top) \sqcap (\beta_j|\bar{\beta}_i)).$$

The cases $(\beta_i|\top) \sqcap (\beta_3|\bar{\beta}_i)$ with $i \in \{1, 2\}$ can be easily verified by exploiting (5). Let us consider the case $(\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3)$, the other case $(\beta_3|\top) \sqcap (\beta_2|\bar{\beta}_3)$ is analogous. We have to compute the probability $\mu_P((\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3))$. First of all, note that $(\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3) = (\alpha_3 \vee \alpha_4|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2)$, so we have to compute the probability $\mu_P((\alpha_3 \vee \alpha_4|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2))$, and for that, we have to find the compound conditionals ω of $\mathcal{C}(\mathbf{A})$ such that $\omega \leqslant (\alpha_3 \vee \alpha_4|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2)$. It is not difficult to check that $(\alpha_3 \vee \alpha_4|\top) \sqcap (\alpha_1|\alpha_1 \vee \alpha_2) = \omega_{31} \sqcup \omega_{341} \sqcup \omega_{41} \sqcup \omega_{431}$. Then, by recalling (5), we have:

$$\begin{aligned}
&\mu_P((\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3)) = P(\omega_{31}) + P(\omega_{341}) + P(\omega_{41}) + P(\omega_{431}) = \\
&= P(\alpha_3) \cdot P(\alpha_1|\bar{\alpha}_3) + P(\alpha_3) \cdot P(\alpha_4|\bar{\alpha}_3) \cdot P(\alpha_1|\bar{\alpha}_3\bar{\alpha}_4) + \\
&+ P(\alpha_4) \cdot P(\alpha_1|\bar{\alpha}_4) + P(\alpha_4) \cdot P(\alpha_3|\bar{\alpha}_4) \cdot P(\alpha_1|\bar{\alpha}_3\bar{\alpha}_4) = \\
&= P(\alpha_3) \cdot P(\alpha_1|\bar{\alpha}_3\bar{\alpha}_4) \cdot P(\bar{\alpha}_3\bar{\alpha}_4|\bar{\alpha}_3) + P(\alpha_3) \cdot P(\alpha_4|\bar{\alpha}_3) \cdot P(\alpha_1|\bar{\alpha}_3\bar{\alpha}_4) + \\
&+ P(\alpha_4) \cdot P(\alpha_1|\bar{\alpha}_3\bar{\alpha}_4) \cdot P(\bar{\alpha}_3\bar{\alpha}_4|\bar{\alpha}_4) + P(\alpha_4) \cdot P(\alpha_3|\bar{\alpha}_4) \cdot P(\alpha_1|\bar{\alpha}_3\bar{\alpha}_4) = \\
&= P(\alpha_1|\bar{\alpha}_3\bar{\alpha}_4) \cdot [P(\alpha_3) \cdot (P(\bar{\alpha}_3\bar{\alpha}_4|\bar{\alpha}_3) + P(\alpha_4|\bar{\alpha}_3)) + \\
&+ P(\alpha_4) \cdot (P(\bar{\alpha}_3\bar{\alpha}_4|\bar{\alpha}_4) + P(\alpha_3|\bar{\alpha}_4))] = \\
&= P(\alpha_1|\alpha_1 \vee \alpha_2) \cdot [P(\alpha_3) \cdot P(\alpha_1 \vee \alpha_2 \vee \alpha_4|\bar{\alpha}_3) + P(\alpha_4) \cdot P(\alpha_1 \vee \alpha_2 \vee \alpha_3|\bar{\alpha}_4)] = \\
&= P(\alpha_1|\alpha_1 \vee \alpha_2) \cdot (P(\alpha_3) + P(\alpha_4)) = P(\alpha_1|\alpha_1 \vee \alpha_2) \cdot P(\alpha_3 \vee \alpha_4) = \\
&= P(\beta_3) \cdot P(\beta_1|\bar{\beta}_3) = \mu_{P'}((\beta_3|\top) \sqcap (\beta_1|\bar{\beta}_3)).
\end{aligned}$$

In the next result we give a proof of [7, Theorem 6.13] where P is (not a positive probability on $\mathbf{A}$, but) a conditional probability on $\mathbb{A} \times \mathbb{A}'$.

Theorem 2. *Let P be a conditional probability on $\mathbb{A} \times \mathbb{A}'$ and μ_P its canonical extension to $\mathcal{C}(\mathbf{A})$. Then, for every basic conditional $(A|H) \in \mathcal{C}(\mathbf{A})$, it holds that $\mu_P(A|H) = P(A|H)$.*

Proof. Let $(A|H) \in \mathcal{C}(\mathbf{A})$ and $\mathbf{B}$ the subalgebra of $\mathbf{A}$ generated by the partition $\{\beta_1, \beta_2, \beta_3\} = \{AH, \bar{A}H, \bar{H}\}$. Let $P' : \mathbb{B} \times \mathbb{B}' \to [0, 1]$ be the restriction of P to $\mathbb{B} \times \mathbb{B}'$, and let $\mu_{P'} : \mathcal{C}(\mathbf{B}) \to [0, 1]$ its canonical extension to $\mathcal{C}(\mathbf{B})$. Of course $P'(A|H) = P(A|H)$. We notice that $A|H = \omega'_{12} \sqcup \omega'_{13} \sqcup \omega'_{31}$, where $\omega'_{12} = \beta_1 \sqcap (\beta_2|\bar{\beta}_1) = AH \sqcap (\bar{A}H|(\bar{A}H \vee \bar{H}))$, $\omega'_{13} = \beta_1 \sqcap (\beta_3|\bar{\beta}_1) = AH \sqcap (\bar{H}|(\bar{A}H \vee \bar{H}))$, and $\omega'_{31} = \beta_3 \sqcap (\beta_1|\bar{\beta}_3) = \bar{H} \sqcap (A|H)$. Then, by Theorem 1, it holds that

$$\begin{aligned} \mu_P(A|H) &= \mu_P(\omega'_{12}) + \mu_P(\omega'_{13}) + \mu_P(\omega'_{31}) = \mu_{P'}(\omega'_{12}) + \mu_{P'}(\omega'_{13}) + \mu_{P'}(\omega'_{31}) = \\ &= P(AH)P(\bar{A}H|(\bar{A}H \vee \bar{H})) + P(AH)P(\bar{H}|(\bar{A}H \vee \bar{H})) + P(\bar{H})P(A|H) = \\ &= P(AH) + P(\bar{H})P(A|H) = P(H)P(A|H) + P(\bar{H})P(A|H) = P(A|H). \qquad \square \end{aligned}$$

We now generalize the above result to a general element of a conditional subalgebra of $\mathcal{C}(\mathbf{A})$.

Theorem 3. *Given a conditional probability P on $\mathbb{A}\times\mathbb{A}'$, let P' be its restriction to $\mathbb{B}\times\mathbb{B}'$, where $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. For each $\mathcal{C} \in \mathcal{C}(\mathbf{B})$ it holds that*

$$\mu_P(\mathcal{C}) = \mu_{P'}(\mathcal{C}).$$

Proof. Indeed, by observing that $\mathcal{C} = \bigsqcup_{\omega' \leqslant \mathcal{C}} \omega'$, by Theorem 1 it follows that $\mu_P(\mathcal{C}) = \sum_{\omega' \leqslant \mathcal{C}} \mu_P(\omega') = \sum_{\omega' \leqslant \mathcal{C}} \mu_{P'}(\omega') = \mu_{P'}(\mathcal{C})$. $\square$

Theorem 3 shows that μ'_P is the restriction of μ_P to $\mathcal{C}(\mathbf{B})$. This result allows a local approach in order to study properties, as done in the next section.

Remark 3. Given three events A, B, C, with $A \leqslant B \leqslant C$, by (CP3) it holds that $P(A|C) = P(A|B)P(B|C)$. Moreover, by recalling $(C5)$, we observe that

$$(A|B) = [(A|B) \sqcap (B|C)] \sqcup [(A|B) \sqcap (\bar{B}|C)] = (A|C) \sqcup [(A|B) \sqcap (\bar{B}|C)]$$

and hence by Theorem 2

$$P(A|B) = P(A|C) + \mu_P[(A|B) \sqcap (\bar{B}|C)]. \tag{6}$$

Then, as $P(A|B) - P(A|C) = P(A|B) - P(A|B)P(B|C)$, it follows that

$$\mu_P[(A|B) \sqcap (\bar{B}|C)] = P(A|B)P(\bar{B}|C). \tag{7}$$

As we can see, (7) shows that the "independence" between $A|B$ and $B|C$, when $A \leqslant B \leqslant C$, still holds between $A|B$ and $\bar{B}|C$. In particular, given any events E and H, by applying (7) with $A = EH, B = H$,and $C = \top$, as $\bar{H}|\top = \bar{H}$, we obtain

$$\mu_P((E|H) \sqcap \bar{H}) = P(E|H)P(\bar{H})\,. \tag{8}$$

Formula (8) will be generalized in Theorem 4, where $\bar{H}$ is replaced by any K such that $HK = \bot$.

5 Probability of the conjunction and the disjunction under canonical extension

In this section we start by showing a basic property for the probability of the conjunction and then, under canonical extension, we obtain the probability for the conjunction and the disjunction of two conditional events, which are related with analogous results given in the setting of coherence in [10,11,12,13,14]. In the next result we generalize formula (8).

Theorem 4. *Given an algebra* $\mathbf{A}$ *and any events* $A, H, K \in \mathbb{A}$, *with* $H \neq \bot$ *and* $HK = \bot$, *given a conditional probability* P *on* $\mathbb{A} \times \mathbb{A}'$ *and its canonical extension* μ_P *to* $C(\mathbf{A})$, *it holds*

$$\mu_P[K \sqcap (A|H)] = P(K)P(A|H)\,. \tag{9}$$

Proof. As $HK = \bot$, it holds that $H\bar{K} = H$, $H \vee \bar{K} = \bar{K}$, and $\bar{H}K = K$; then

$$\top = (AH \vee \bar{A}H \vee \bar{H}) \wedge (K \vee \bar{K}) = AH\bar{K} \vee \bar{H}K \vee \bar{A}H\bar{K} \vee \bar{H}\bar{K}.$$

We consider the partition $\{\beta_1, \ldots, \beta_4\}$, where

$$\beta_1 = AH\bar{K} = AH\,,\ \beta_2 = \bar{H}K = K\,,\ \beta_3 = \bar{A}H\bar{K} = \bar{A}H\,,\ \beta_4 = \bar{H}\bar{K},$$

and the associated subalgebra $\mathbf{B}$; moreover we consider the atoms $\omega'_{i_1 i_2 i_3}$'s of $\mathcal{C}(\mathbf{B})$. As $\omega'_{213} \sqcup \omega'_{214} = \omega'_{21}$, it holds that

$$K \sqcap (A|H) = \omega'_{213} \sqcup \omega'_{214} \sqcup \omega'_{241} = \omega'_{21} \sqcup \omega'_{241}\,.$$

Let P' be the restriction of P to $\mathbb{B} \times \mathbb{B}'$ and $\mu_{P'}$ its canonical extension to $\mathcal{C}(\mathbf{B})$. As $H\bar{K} = H$ it holds that $P(AH|\bar{K}) = P(A|H\bar{K})P(H|\bar{K}) = P(A|H)P(H|\bar{K})$. Then, from Theorem 2 and from (5) we obtain

$$\begin{aligned}
&\mu_P[K \sqcap (A|H)] = \mu_{P'}[K \sqcap (A|H)] = \mu_{P'}(\omega'_{21}) + \mu_{P'}(\omega'_{241}) = \\
&= P(\beta_2)P(\beta_1|\bar{\beta}_2) + P(\beta_2)P(\beta_4|\bar{\beta}_2)P(\beta_1|\bar{\beta}_2\bar{\beta}_4) = P(K)P(AH|\bar{K}) + \\
&+ P(K)P(\bar{H}|\bar{K})P(A|H) = P(K)[P(AH|\bar{K}) + P(\bar{H}|\bar{K})P(A|H)] = \\
&= P(K)[P(A|H)P(H|\bar{K}) + P(A|H)P(\bar{H}|\bar{K})] = P(K)P(A|H)\,. \quad \square
\end{aligned}$$

In the next result we obtain the probability for the conjunction $(A|H) \sqcap (B|K)$.

Theorem 5. *Given an algebra* $\mathbf{A}$ *and a conditional probability* P *on* $\mathbb{A} \times \mathbb{A}'$, *let* μ_P *be the canonical extension to* $\mathcal{C}(\mathbf{A})$. *For any conditional events* $A|H, B|K \in \mathcal{C}(\mathbf{A})$ *it holds that*

$$\begin{aligned}
&\mu_P[(A|H) \sqcap (B|K)] = \\
&= P(AHBK|(H \vee K)) + P(A|H)P(\bar{H}BK|(H \vee K)) + P(B|K)P(\bar{K}AH|(H \vee K)).
\end{aligned} \tag{10}$$

Proof. We consider the partition $\{\beta_1, \ldots, \beta_9\}$, where

$$\begin{aligned}
&\beta_1 = AHBK,\ \beta_2 = AH\bar{B}K,\ \beta_3 = AH\bar{K},\ \beta_4 = \bar{A}HBK, \\
&\beta_5 = \bar{A}H\bar{B}K,\ \beta_6 = \bar{A}H\bar{K},\ \beta_7 = \bar{H}BK,\ \beta_8 = \bar{H}\bar{B}K,\ \beta_9 = \bar{H}\bar{K},
\end{aligned} \tag{11}$$

and the associated subalgebra $\mathbf{B}$. Moreover, we consider the compound conditionals $\omega'_{i_1 \cdots i_k}$'s of $\mathcal{C}(\mathbf{B})$, $1 \leqslant k \leqslant 8$. Let P' be the restriction of P to $\mathbb{B} \times \mathbb{B}'$ and $\mu_{P'}$ its canonical extension to $\mathcal{C}(\mathbf{B})$. We recall that from Theorem 3, $\mu_P(\mathcal{C}) = \mu_{P'}(\mathcal{C})$, for every $\mathcal{C} \in \mathcal{C}(\mathbf{B})$. By exploiting the distributivity property, we decompose the conjunction $(A|H) \sqcap (B|K)$ as

$$\begin{aligned}
&(A|H) \sqcap (B|K) = [(A|H) \sqcap (B|K) \sqcap HK] \sqcup [(A|H) \sqcap (B|K) \sqcap \bar{H}K] \sqcup \\
&\sqcup [(A|H) \sqcap (B|K) \sqcap H\bar{K}] \sqcup [(A|H) \sqcap (B|K) \sqcap \bar{H}\,\bar{K}].
\end{aligned}$$

For the compound $(A|H) \sqcap (B|K) \sqcap HK$ it holds that

$$(A|H) \sqcap (B|K) \sqcap HK = ((A|H) \sqcap H) \sqcap ((B|K) \sqcap K) = AHBK = \beta_1 = \omega_1',$$

with $\mu_P((A|H) \sqcap (B|K) \sqcap HK) = \mu_P(AHBK) = P(AHBK)$.
For the compound $(A|H) \sqcap (B|K) \sqcap \bar{H}K$ it holds that

$$(A|H) \sqcap (B|K) \sqcap \bar{H}K = (A|H) \sqcap ((B|K) \sqcap K) \sqcap \bar{H} = (A|H) \sqcap \bar{H}BK,$$

and, by Theorem 4, $\mu_P((A|H) \sqcap (B|K) \sqcap \bar{H}K) = \mu_P((A|H) \sqcap \bar{H}BK) = P(A|H)P(\bar{H}BK)$.
Likewise, for the compound $(A|H) \sqcap (B|K) \sqcap H\bar{K}$ it holds that

$$(A|H) \sqcap (B|K) \sqcap H\bar{K} = (B|K) \sqcap AH\bar{K},$$

and, by Theorem 4, $\mu_P((A|H) \sqcap (B|K) \sqcap H\bar{K}) = P(B|K)P(AH\bar{K})$.
Thus, by observing that $H \vee K = HK \vee \bar{H}K \vee H\bar{K}$, we obtain

$$\begin{aligned} &\mu_P[(A|H) \sqcap (B|K) \sqcap (H \sqcup K)] = P(ABHK) + P(\bar{H}BK)P(A|H)+ \\ &+P(AH\bar{K})P(B|K) = P(H \vee K)[P(ABHK|(H \vee K))+ \\ &+P(\bar{H}BK|(H \vee K))P(A|H) + P(AH\bar{K}|(H \vee K))P(B|K)] = zP(H \vee K), \end{aligned} \tag{12}$$

where

$$z = P(ABHK|(H\vee K))+P(\bar{H}BK|(H\vee K))P(A|H)+P(AH\bar{K}|(H\vee K))P(B|K). \tag{13}$$

For the compound $(A|H) \sqcap (B|K) \sqcap \bar{H}\,\bar{K}$ it can be verified that

$$\begin{aligned} (A|H) \sqcap (B|K) \sqcap \bar{H}\,\bar{K} = {} & \omega_{91}' \sqcup \omega_{971}' \sqcup \omega_{972}' \sqcup \omega_{973}' \sqcup \omega_{9781}' \sqcup \omega_{9782}' \sqcup \omega_{9783}' \sqcup \\ & \sqcup \omega_{931}' \sqcup \omega_{934}' \sqcup \omega_{937}' \sqcup \omega_{9361}' \sqcup \omega_{9364}' \sqcup \omega_{9367}'. \end{aligned} \tag{14}$$

By Theorem 4, it holds that
$\mu_P(\omega_{91}') = \mu_P(\bar{H}\bar{K} \sqcap AHBK|(H \vee K)) = P(\bar{H}\bar{K})P(AHBK|(H \vee K))$. Moreover, as

$$\beta_1|(\bar{\beta}_7\bar{\beta}_9) \sqcap \beta_2|(\bar{\beta}_7\bar{\beta}_9) \sqcap \beta_3|(\bar{\beta}_7\bar{\beta}_9) = (\beta_1 \vee \beta_2 \vee \beta_3)|(\bar{\beta}_7\bar{\beta}_9) = AH|(H \vee \bar{H}\bar{B}K),$$

from (5) it holds that

$$\begin{aligned} &\mu_P(\omega_{971}' \sqcup \omega_{972}' \sqcup \omega_{973}') = \mu_P(\omega_{971}') + \mu_P(\omega_{972}') + \mu_P(\omega_{973}') = \cdots = \\ &= P(\bar{H}\bar{K})P(\bar{H}BK|(H \vee K))P(AH|(H \vee \bar{H}\bar{B}K)). \end{aligned}$$

Likewise, as

$$\beta_1|(\bar{\beta}_7\bar{\beta}_8\bar{\beta}_9) \sqcap \beta_2|(\bar{\beta}_7\bar{\beta}_8\bar{\beta}_9) \sqcap \beta_3|(\bar{\beta}_7\bar{\beta}_8\bar{\beta}_9) = (\beta_1 \vee \beta_2 \vee \beta_3)|(\bar{\beta}_7\bar{\beta}_8\bar{\beta}_9) = A|H,$$

it holds that

$$\begin{aligned} &\mu_P(\omega_{9781}' \sqcup \omega_{9782}' \sqcup \omega_{9783}') = \mu_P(\omega_{9781}') + \mu_P(\omega_{9782}') + \mu_P(\omega_{9783}') = \cdots = \\ &= P(\bar{H}\bar{K})P(\bar{H}BK|(H \vee K))P(\bar{H}\bar{B}K|(H \vee \bar{H}\bar{B}K))P(A|H). \end{aligned}$$

Then, by observing that $P(AH|(H \vee \bar{H}\bar{B}K)) = P(A|H)P(H|(H \vee \bar{H}\bar{B}K))$, it follows that

$$\begin{aligned}&\mu_P(\omega'_{971} \sqcup \omega'_{972} \sqcup \omega'_{973}) + \mu_P(\omega'_{9781} \sqcup \omega'_{9782} \sqcup \omega'_{9783}) = \\ &= P(\bar{H}\bar{K})P(\bar{H}BK|(H \vee K))[P(AH|(H \vee \bar{H}\bar{B}K)) + \\ &+ P(\bar{H}\bar{B}K|(H \vee \bar{H}\bar{B}K))P(A|H)] = P(\bar{H}\bar{K})P(\bar{H}BK|(H \vee K))P(A|H).\end{aligned}$$

Likewise $\mu_P(\omega'_{931} \sqcup \omega'_{934} \sqcup \omega'_{937}) + \mu_P(\omega'_{9361} \sqcup \omega'_{9364} \sqcup \omega'_{9367}) = \cdots = P(\bar{H}\bar{K})P(AH\bar{K}|(H \vee K))P(B|K)$. Thus, by recalling (13) and (15), it follows that

$$\begin{aligned}&\mu_P[(A|H) \sqcap (B|K) \sqcap (\bar{H}\bar{K})] = P(\bar{H}\bar{K})[P(AHBK|(H \vee K)) + \\ &+ P(\bar{H}BK|(H \vee K))P(A|H) + P(\bar{K}AH|(H \vee K))P(B|K)] = zP(\bar{H}\bar{K}).\end{aligned} \tag{15}$$

Finally, by also recalling (12) , it follows that

$$\begin{aligned}&\mu_P[(A|H) \sqcap (B|K)] = \mu_P[(A|H) \sqcap (B|K) \sqcap (H \vee K)] + \\ &+ \mu_P[(A|H) \sqcap (B|K) \sqcap (\bar{H}\bar{K})] = zP(H \vee K) + zP(\bar{H}\bar{K}) = z = \\ &P(AHBK|(H \vee K)) + P(A|H)P(\bar{H}BK|(H \vee K)) + P(B|K)P(\bar{K}AH|(H \vee K)).\end{aligned}$$

□

As shown in (12) and in (15), $(A|H) \sqcap (B|K)$ is "independent" from $H \vee K$ and from $\bar{H}\bar{K}$. Notice that formula (10) coincides with the prevision of the conjunction $\mathcal{C} = (A|H) \wedge (B|K)$, introduced in the setting of coherence as the following conditional random quantity (see, e.g.,[10,12])

$$\mathcal{C} = [AHBK + P(A|H)(\bar{H}BK|(H \vee K)) + P(B|K)(AH\bar{K}|(H \vee K))]|(H \vee K), \tag{16}$$

where (conditional) events and their indicators are denoted by the same symbol. Moreover, when $P(H \vee K) > 0$, formula (10) becomes

$$\mu_P[(A|H) \sqcap (B|K)] = \frac{P(AHBK) + P(A|H)P(\bar{H}BK) + P(B|K)P(\bar{K}AH)}{P(H \vee K)},$$

that is the formula obtained by McGee ([17]) and Kaufmann ([15]). We also note that, when $HK = \bot$ and hence $\bar{H}BK = BK, AH\bar{K} = AH$, from (10) it follows that ([9,21])

$$\begin{aligned}&\mu_P[(A|H) \sqcap (B|K)] = P(A|H)P(BK|(H \vee K)) + P(B|K)P(AH|(H \vee K)) = \\ &= P(A|H)P(B|K)P(K|(H \vee K)) + P(A|H)P(B|K)P(H|(H \vee K)) = \\ &= P(A|H)P(B|K).\end{aligned}$$

In the next result we obtain the probability of the disjunction $(A|H) \sqcup (B|K)$.

Theorem 6. *Given an algebra* $\mathbf{A}$ *and a conditional probability* P *on* $\mathbb{A} \times \mathbb{A}'$*, let* μ_P *be the canonical extension to* $\mathcal{C}(\mathbf{A})$*. For any conditional events* $A|H, B|K \in \mathcal{C}(\mathbf{A})$ *it holds that*

$$\begin{aligned}&\mu_P[(A|H) \sqcup (B|K)] = \\ &= P((AH \vee BK)|(H \vee K)) + P(A|H)P(\bar{H}\,\bar{B}K|(H \vee K)) + P(B|K)P(\bar{A}H\bar{K}|(H \vee K)).\end{aligned} \tag{17}$$

Proof. We observe that

$$\mu_P((A|H) \sqcup (B|K)) = \mu_P[(A|H) \sqcap (B|K) \sqcup (\bar{A}|H) \sqcap (B|K) \sqcup (A|H) \sqcap (\bar{B}|K)] = \\ = \mu_P[(A|H) \sqcap (B|K)] + \mu_P[(\bar{A}|H) \sqcap (B|K)] + \mu_P[(A|H) \sqcap (\bar{B}|K)]. \tag{18}$$

From Theorem 5, besides (10), one has $\mu_P[(\bar{A}|H) \sqcap (B|K)] =$ $= P(\bar{A}HBK|(H \vee K)) + P(\bar{A}|H)P(\bar{H}BK|(H \vee K)) + P(B|K)P(\bar{K}\bar{A}H|(H \vee K))$ and $\mu_P[(A|H) \sqcap (\bar{B}|K)] = P(AH\bar{B}K|(H \vee K)) + P(A|H)P(\bar{H}\bar{B}K|(H \vee K)) + P(\bar{B}|K)P(\bar{K}AH|(H \vee K))$. As it can be verified, it holds that $AH \vee BK = AHBK \vee AH\bar{B}K \vee \bar{A}HBK \vee AH\bar{K} \vee \bar{H}BK$; then, by recalling (18), it follows that equation (17) is satisfied. □

We observe that (17) coincides with the prevision of the disjunction of two conditional events obtained in the framework of conditional random quantities in [10]. We also observe that De Morgan Laws are satisfied in $\mathcal{C}(\mathbf{A})$, therefore, $\mu_P(\overline{(A|H) \sqcup (B|K)}) = \mu_P((\bar{A}|H) \sqcap (\bar{B}|K))$ and $\mu_P(\overline{(A|H) \sqcap (B|K)}) = \mu_P((\bar{A}|H) \sqcup (\bar{B}|K))$ in agreement with formulas (10) and (17). We remark that

$$\mu_P((A|H) \sqcup (B|K)) = P(A|H) + P(B|K) - \mu_P((A|H) \sqcap (B|K)),$$

which coincides with the prevision sum rule obtained in [9,10]. Finally, an aspect to be deepened concerns the notion of iterated conditional, say $(B|K)|(A|H)$, and its probability. If we define $\mu_P((B|K)|(A|H)) =_{def} \frac{\mu_P(A|H) \sqcap (B|K))}{\mu_P(A|H)}$, then, under the hypothesis $P(A|H) > 0$, it holds that

$$\mu_P((B|K)|(A|H)) = \frac{P(AHBK|(H \vee K)) + P(A|H)P(\bar{H}BK|(H \vee K)) + P(B|K)P(\bar{K}AH|(H \vee K))}{P(A|H)}, \tag{19}$$

which is the prevision of the iterated conditional $(B|K)|(A|H)$, obtained in the setting of coherence in ([9, Section 6]). Under the further assumption $P(H \vee K) > 0$, formula (19) coincides with the result given in ([15, Thm 3]).

6 Conclusions

In this paper we have advanced in the study of conditionals in the setting of the Boolean algebras of conditionals as proposed in [7]. More precisely, given a finite Boolean algebra of events $\mathbf{A}$, we have first considered the canonical extension μ_P of a conditional probability P on $\mathbb{A} \times \mathbb{A}'$ to the Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$. Our first main result establishes that the process of canonical extensions commutes, in a sense made precise in Section 4, with taking subalgebras of $\mathbf{A}$. This fact then allows us to show that μ_P extends P over basic conditionals, and in turn to get an operational computation of the probability of a conjunction and a disjunction of conditionals, in agreement with previous approaches in the literature, in particular with the one developed by Gilio and Sanfilippo by formalising conditionals as random quantities [10].

As for future work, encouraged by the above obtained results, we plan to deepen into the relationship between the approach based on Boolean algebras of conditionals, together with canonical extensions of conditional probabilities on events, and the approach based on interpreting compound and iterated conditionals as random quantities.

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7 Appendix

Proof of Theorem 1:

Proof. Without loss of generality, we examine the case $(j_1,\dots,j_{n-2}) = (1,\dots,n-2)$. Note that

$$\begin{aligned}
&\omega'_{12\cdots n-2} = (\beta_1|\top) \sqcap \cdots \sqcap \ (\beta_{n-2}|\bar{\beta}_1 \wedge \cdots \wedge \bar{\beta}_{n-3}) = \\
&= (\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_i \vee \alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n) \sqcap \\
&\sqcap (\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) = \\
&= W^{(i+1)}_{1\cdots i-1\, i\, i+2\cdots n-1} \sqcup W^{(i+1)}_{1\cdots i-1\, i+1\, i+2\cdots n-1},
\end{aligned} \tag{20}$$

where $W^{(i+1)}_{1\cdots i-1\, i\, i+2\cdots n-1}$ and $W^{(i+1)}_{1\cdots i-1\, i+1\, i+2\cdots n-1}$ are defined as

$$\begin{aligned}
&W^{(i+1)}_{1\cdots i-1\, i\, i+2\cdots n-1} = (\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) \sqcap \\
&\sqcap (\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n),
\end{aligned}$$

and

$$\begin{aligned}
&W^{(i+1)}_{1\cdots i-1\, i+1\, i+2\cdots n-1} = (\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n) \sqcap \\
&\sqcap (\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n),
\end{aligned}$$

respectively. By first examining in (20) the term $W^{(i+1)}_{1\cdots i-1\, i\, i+2\cdots n-1}$, we observe that

$$\begin{aligned}
&W^{(i+1)}_{1\cdots i-1\, i\, i+2\cdots n-1} = (\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) \sqcap \\
&\sqcap (\alpha_{i+1} \vee \cdots \vee \alpha_n)|(\alpha_{i+1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) = \\
&= \omega_{1\cdots n-1} \sqcup (\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) \sqcap \\
&\sqcap (\alpha_{i+2} \vee \cdots \vee \alpha_n|\alpha_{i+1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n).
\end{aligned}$$

By recalling (C5), it holds that

$$(\alpha_{i+2} \vee \cdots \vee \alpha_n)|(\alpha_{i+1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n) = (\alpha_{i+2}|\alpha_{i+1} \vee \cdots \vee \alpha_n);$$

then, we obtain

$$W^{(i+1)}_{1\cdots i-1\, i\, i+2\cdots n-1} = \omega_{1\cdots n-1} \sqcup W^{(i+2)}_{1\cdots i-1\, i\, i+2\cdots n-1},$$

where

$$\begin{aligned}
&W^{(i+2)}_{1\cdots i-1\, i\, i+2\cdots n-1} = (\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) \sqcap \\
&\sqcap (\alpha_{i+2}|\alpha_{i+1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_{i+3}|\alpha_{i+3} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n).
\end{aligned}$$

Concerning $W^{(i+2)}_{1\cdots i-1\, i\, i+2\cdots n-1}$, we observe that

$$\begin{aligned}
&W^{(i+2)}_{1\cdots i-1\, i\, i+2\cdots n-1} = (\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) \sqcap \\
&\sqcap (\alpha_{i+2}|\alpha_{i+1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_{i+1} \vee \alpha_{i+3} \vee \cdots \vee \alpha_n|\alpha_{i+1} \vee \alpha_{i+3} \vee \cdots \vee \alpha_n) \sqcap \cdots \\
&\cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) = \cdots = \omega_{1\cdots i\, i+2\, i+1\, i+3\cdots n-1} \sqcup W^{(i+3)}_{1\cdots i-1\, i\, i+2\cdots n-2}.
\end{aligned}$$

Then, by iteration, we obtain

$$W^{(i+1)}_{1\cdots i-1\,i\,i+2\cdots n-1} = \omega_{1\cdots n-1} \sqcup \omega_{1\cdots i\,i+2\,i+1\,i+3\cdots n-1} \sqcup W^{(i+3)}_{1\cdots i-1\,i\,i+2\cdots n-2} = \cdots =$$
$$= \omega_{1\cdots n-1} \sqcup \omega_{1\cdots i\,i+2\,i+1\,i+3\cdots n-1} \sqcup \cdots \sqcup \omega_{1\cdots i\,i+2\,\cdots\,n-2 i+1 n-1} \sqcup W^{(n-1)}_{1\cdots i-1\,i\,i+2\cdots n-1},$$

where

$$W^{(n-1)}_{1\cdots i-1\,i\,i+2\cdots n-1} = \omega_{1\cdots i\,i+2\,\cdots n-1 i+1} \sqcup \omega_{1\cdots i\,i+2\,\cdots n-1 n} =$$
$$= (\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) \sqcap$$
$$\sqcap (\alpha_{i+2}|\alpha_{i+1} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{i+1} \vee \alpha_{n-1} \vee \alpha_n) = \omega_{1\cdots i\,i+2\,\cdots n-1}.$$

Notice that in the previous iteration we exploited the relation

$$W^{(k)}_{1\cdots i-1\,i\,i+2\cdots n-1} = \omega_{1\cdots i\,i+2\cdots k\,i+1\,k+1\cdots n-1} \sqcup W^{(k+1)}_{1\cdots i-1\,i\,i+2\cdots n-2}, \quad k = i+1, \ldots n-2.$$

We recall, from (5), that

$$\mu_P(\omega_{i_1\ldots i_k}) = P(\alpha_{i_1}) P(\alpha_{i_2}|\alpha_{i_2} \vee \cdots \vee \alpha_{i_n}) \cdots P(\alpha_{i_k}|\alpha_{i_k} \vee \cdots \vee \alpha_{i_n});$$

then

$$\mu_P(W^{(n-1)}_{1\cdots i-1\,i\,i+2\cdots n-1}) = \mu_P(\omega_{1\cdots i\,i+2\,\cdots n-1}) =$$
$$= P(\alpha_1) \cdots P(\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) P(\alpha_{i+2}|\alpha_{i+1} \vee \alpha_{i+2} \vee \cdots \vee \alpha_n) \cdots P(\alpha_{n-1}|\alpha_{i+1} \vee \alpha_{n-1} \vee \alpha_n) =$$
$$= P(\omega_{1\cdots i\,i+2\,\cdots n-2}) P(\alpha_{n-1}|\alpha_{i+1} \vee \alpha_{n-1} \vee \alpha_n).$$

Moreover, as

$$W^{(n-2)}_{1\cdots i-1\,i\,i+2\cdots n-1} = \omega_{1\cdots i\,i+2\cdots n-2\,i+1\,n-1} \sqcup W^{(n-1)}_{1\cdots i-1\,i\,i+2\cdots n-2},$$

it holds that

$$\mu_P(W^{(n-2)}_{1\cdots i-1\,i\,i+2\cdots n-1}) = \mu_P(\omega_{1\cdots i\,i+2\cdots n-2\,i+1\,n-1}) + \mu_P(W^{(n-1)}_{1\cdots i-1\,i\,i+2\cdots n-2}) =$$
$$= \mu_P(\omega_{1\cdots i\,i+2\,\cdots n-2})[P(\alpha_{i+1}|\alpha_{i+1} \vee \alpha_{n-1} \vee \alpha_n) \cdot P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) +$$
$$+ P(\alpha_{n-1}|\alpha_{i+1} \vee \alpha_{n-1} \vee \alpha_n)]. \tag{21}$$

From (CP3), it holds that

$$P(\alpha_{n-1}|\alpha_{i+1} \vee \alpha_{n-1} \vee \alpha_n) = P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) P(\alpha_{n-1} \vee \alpha_n|\alpha_{i+1} \vee \alpha_{n-1} \vee \alpha_n);$$

then (21) becomes

$$\mu_P(W^{(n-2)}_{1\cdots i-1\,i\,i+2\cdots n-1}) = \mu_P(\omega_{1\cdots i\,i+2\,\cdots n-2}) P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) =$$
$$= \mu_P(\omega_{1\cdots i\,i+2\,\cdots n-3}) P(\alpha_{n-2}|\alpha_{i+1} \vee \alpha_{n-2} \vee \alpha_{n-1} \vee \alpha_n) P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n).$$

Now,

$$W^{(n-3)}_{1\cdots i-1\,i\,i+2\cdots n-1} = \omega_{1\cdots i\,i+2\cdots n-3\,i+1\,n-2 n-1} \sqcup W^{(n-2)}_{1\cdots i-1\,i\,i+2\cdots n-2},$$

and

$$\mu_P(W^{(n-3)}_{1\cdots i-1\,i\,i+2\cdots n-1}) =$$
$$= \mu_P(\omega_{1\cdots i\,i+2\cdots n-3\,i+1\,n-2 n-1}) + \mu_P(W^{(n-2)}_{1\cdots i-1\,i\,i+2\cdots n-2}) =$$
$$= \mu_P(\omega_{1\cdots i\,i+2\,\cdots n-3})P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n)\cdot$$
$$\cdot[P(\alpha_{i+1}|\alpha_{i+1} \vee \alpha_{n-2} \vee \alpha_{n-1} \vee \alpha_n)P(\alpha_{n-2}|\alpha_{n-2} \vee \alpha_{n-1} \vee \alpha_n) + P(\alpha_{n-2}|\alpha_{i+1} \vee \alpha_{n-2} \vee \alpha_{n-1} \vee \alpha_n)]$$

which by exploiting the relation

$$P(\alpha_{n-2}|\alpha_{i+1} \vee \alpha_{n-2} \vee \alpha_{n-1} \vee \alpha_n) =$$
$$= P(\alpha_{n-2}|\alpha_{n-2} \vee \alpha_{n-1} \vee \alpha_n)P(\alpha_{n-2} \vee \alpha_{n-1} \vee \alpha_n|\alpha_{i+1} \vee \alpha_{n-2} \vee \alpha_{n-1} \vee \alpha_n),$$

becomes

$$\mu_P(W^{(n-3)}_{1\cdots i-1\,i\,i+2\cdots n-1}) = \mu_P(\omega_{1\cdots i\,i+2\,\cdots n-3})P(\alpha_{n-2}|\alpha_{n-2} \vee \alpha_{n-1} \vee \alpha_n)P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n).$$

By iterating the previous reasoning, for every $k = i+1, \dots, n-2$, it holds that

$$\mu_P(W^{(k)}_{1\cdots i-1\,i\,i+2\cdots n-1}) = \mu_P(\omega_{1\cdots i\,i+2\,\cdots k})P(\alpha_{k+1}|\alpha_{k+1} \vee \cdots \vee \alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n).$$

In particular, by recalling (5)

$$\mu_P(W^{(i+1)}_{1\cdots i-1\,i\,i+2\cdots n-1}) = \mu_P(\omega_{1\cdots i})P(\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) =$$
$$= P(\alpha_1|\top)\cdots P(\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n)P(\alpha_i|\alpha_i \vee \cdots \vee \alpha_n)\cdot$$
$$\cdot P(\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n),$$

that is

$$\mu_P(W^{(i+1)}_{1\cdots i-1\,i\,i+2\cdots n-1}) = \mu_P[(\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_i|\alpha_i \vee \cdots \vee \alpha_n)\sqcap$$
$$\sqcap(\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n)] =$$
$$= P(\alpha_1|\top)\cdots P(\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n)P(\alpha_i|\alpha_i \vee \cdots \vee \alpha_n)\cdot$$
$$\cdot P(\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n),$$

which shows that the factorization property of μ_P holds for $W^{(i+1)}_{1\cdots i-1\,i\,i+2\cdots n-1}$. Likewise, by coming back to (20), the factorization property of μ_P holds for $W^{(i+1)}_{1\cdots i-1\,i+1\,i+2\cdots n-1}$, that is

$$\mu_P(W^{(i+1)}_{1\cdots i-1\,i+1\,i+2\cdots n-1}) = \mu_P[(\alpha_1|\top) \sqcap \cdots \sqcap (\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n) \sqcap (\alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n)\sqcap$$
$$\sqcap(\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n) \sqcap \cdots \sqcap (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n)] = P(\alpha_1|\top)\cdots P(\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n)\cdot$$
$$\cdot P(\alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n)P(\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n).$$

Finally, still concerning (20), we obtain

$$\mu_P(\omega'_{12\cdots n-2}) = \mu_P(W^{(i+1)}_{1\cdots i-1\,i\,i+2\cdots n-1}) + \mu_P(W^{(i+1)}_{1\cdots i-1\,i+1\,i+2\cdots n-1}) =$$
$$P(\alpha_1|\top)\cdots P(\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n)\cdot[P(\alpha_i|\alpha_i \vee \cdots \vee \alpha_n) + P(\alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n)]\cdot$$
$$\cdot P(\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) =$$
$$P(\alpha_1|\top)\cdots P(\alpha_{i-1}|\alpha_{i-1} \vee \cdots \vee \alpha_n)\cdot P(\alpha_i \vee \alpha_{i+1}|\alpha_i \vee \cdots \vee \alpha_n)\cdot$$
$$\cdot P(\alpha_{i+2}|\alpha_{i+2} \vee \cdots \vee \alpha_n)\cdots P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n) =$$
$$= P(\beta_1|\top)\cdots P(\beta_i|\bar{\beta}_1 \wedge \cdots \wedge \bar{\beta}_{i-1})\cdots P(\beta_{n-2}|\bar{\beta}_1 \wedge \cdots \wedge \bar{\beta}_{n-3}) = \mu_{P'}(\omega'_{12\cdots n-2}).$$

□