Bounded and multi-adjoint lattice algebraizable logics

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Abstract

Nowadays is critical to complement Artificial Intelligence (AI) systems, such as those based on Deep Learning and Generative AI, by robust and trustworthy methodologies like different useful mathematical tools, such as (fuzzy) logic, formal concept analysis, rough set theory, etc. Multi-adjoint algebras are flexible structures considered in many of these mathematical tools, which are fundamental for obtaining traceable and reliable information from data sets. Along this line, this paper aims at introducing an algebraizable logic having the quasi-variety of these algebras as its associated equivalent algebraic semantics. To do so, we first introduce a logic associated with bounded lattices with an implication and show it is algebraizable in the sense of Blok-Pigozzi. We then expand this base logic to another algebraizable logic able to properly capture the multi-adjoint framework. Furthermore, we also consider different extensions of the logics and their properties analysed.

Keywords: Bounded poset, multi-adjoint algebra, algebraizable logic, fuzzy logic

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1. Introduction

Generative AI and Deep Learning, among others, are essential approaches to design automated artificial intelligent systems. However, these systems require complementary tools to provide them with fundamental features, such as trustworthiness, reliability, and traceability. The use of mathematical tools based on logic and associated methods can give these valuable characteristics. An important step in this challenge is to axiomatize the algebraic structure underpinning these tools and taking advantage of the relationship between (properties in) logic and algebras, so that the understanding of these algebras can be used to better understand the logic at hand, and vice versa.

Multi-adjoint algebras [14, 15] are general structures which have given a high degree of flexibility to the mathematical tools in which they have been used, such as formal concept analysis [13, 12, 28], rough set theory [17, 27], fuzzy relational equations [18, 24] and logic programming [29, 30, 32]. In this paper, we will begin with the basic algebraic structure where multi-adjoint algebras are usually defined, that is, bounded lattices. Then, taking inspiration in previous works [11, 9] we consider a specific logic which encodes in somehow minimal way the properties of bounded lattices (with an implication) while being algebraizable in the sense of Blok-Pigozzi, obtaining what we define as a *bounded lattice algebraizable logic* (BLAL). This fact enables the acquisition of equations equivalent to the axioms and inference rules of BLAL, together with the determination of the quasi-variety of algebras associated with these structures. Thus, the class of algebras that deserve to be taken as the natural one associated with BLAL. We have proven that these algebras are expansions of bounded lattices, but more general than Heyting algebras, which are the algebraic semantics of Intuitionistic Propositional Calculus [3, 21, 23, 26]. Furthermore, we will also show that BLAL also belongs to the family of the Rasiowa implicative logics [33]. These comparisons allow us to contextualize the underlying quasi-variety from which the final expanded quasi-variety of algebras associated with multi-adjoint algebras is constructed.

Hence, the next step is to expand BLAL with a set of pairs of operators, together with corresponding inference rules, whose intended semantics are those of adjoint pairs in multi-adjoint algebras. In this way, a multi-adjoint algebraizable logic (MAL) is introduced whose algebraic counterpart is the class (quasivariety) of multi-adjoint algebras. This goes further than the results in [11] since we are able to prove that MAL is actually algebraizable in the sense of Blok-Pigozzi. As a straightforward consequence, we obtain a general algebraic completeness result for MAL with respect to multi-adjoint algebras.

As mentioned above, multi-adjoint algebras are structures successfully used as very general and flexible domains of truth-values in different applications. On the other hand, it is very common when dealing with imprecise or fuzzy properties to work with linearly ordered scales of graded truth. Therefore, it can be interesting to consider how (BLAL and) MAL can be enforced to capture reasoning with linearly ordered multi-adjoint algebras. By using general results in the literature on semilinear logics (i.e. algebraizable logics complete with respect to a class of linearly-ordered algebras), we show how BLAL and MAL can be minimally extended to guarantee to be complete with respect to their associated subclasses of linearly-ordered algebras.

Finally, let us mention in that in [8], we already introduced a logic for multi-adjoint algebras called MGL_Δ defined on top of an expansion of Gödel fuzzy logic with the so-called Baaz-Monteiro projection operator Δ, also showing completeness results with respect to the class of linearly ordered MGL_Δ-algebras. The last goal of this paper is to show that the approach developed in this paper is more general than the one in [8] since we can prove that MGL_Δ can be obtained as a very simple axiomatic extension of the semilinear variant of MAL.

The structure of the paper is as follows. After this introduction, in Section 2 we provide necessary preliminaries with basic notions on Blok-Pigozzi theory of algebraizable logics and on multi-adjoint lattices, to make the paper self-contained as much as possible. In Section 3 we introduce the logic BLAL as well as its associated quasivariety of algebras, while in Section 4 we make explicit the relationship of BLAL to both Rasiowa's implicative logic and Intuitionistic propositional logic. In Section 5 we present MAL, the expansion of BLAL with generic pairs of left-adjoint operators. In Section 6 we consider the minimal semilinear extensions of both BLAL and MAL. Finally, in Section 7 we show that the logic MGL_Δ introduced in [8] can be recovered from the semilinear extension of MAL by enforcing a Boolean behaviour of the implication operator. We conclude with some final remarks and prospects for future work.

2. Preliminaries

This section presents a basic introduction of agebraizable logics and the basic notions of multi-adjoint algebras.

2.1. Algebraizable logics

The field of (abstract) algebraic logic studies the algebraization of deductive systems arising as an abstraction of the well-known Lindenbaum-Tarski algebra. The idea is to translate the axioms and rules of a deductive system into the equational logic of a suitable class of algebras. The strongest correspondence is when the deductive system and the equational logic can be mutually translated into each other. The logic in this case is called *algebraizable* . One of the consequences of algebraizability is that a complete algebraic semantics (called equivalent algebraic semantics) can be defined from the deductive system via the so-called defining equations and equivalence formulas which underlie the two translations.

Let us start recalling the notion of an algebraizable logic and their conditions, following the theory by Blok and Pigozzi. All the cited results in this subsection are from [2], although we follow the presentation by Jansana [22].

Let $\mathbf{K}$ be a class of algebras, that is, a collection of similar algebraic structures. Its equational consequence relation $\models_{\mathbf{K}}$ is defined as follows: for every set of equations $\{\varphi_i = \psi_i\}_{i\in I} \cup \{\varphi = \psi\}$,

$$\{\varphi_i = \psi_i\}_{i\in I} \models_{\mathbf{K}} \varphi = \psi \quad \text{iff} \quad \begin{array}{l}\text{for every } \mathbf{A} \in \mathbf{K} \text{ and every valuation } v \text{ on } \mathbf{A},\\ \text{if } v(\varphi_i) = v(\psi_i) \text{ for all } i \in I, \text{ then } v(\varphi) = v(\psi)\end{array}$$

The translations are given by the set of defining equations and the set of equivalence formulas. A set of equations $Eq(p) = \{\delta_i(p) = \varepsilon_i(p)\}_{i\in\{1,\dots,n\}}$ in one variable p defines a translation from formulas to sets of equations: each formula φ is translated into the set of equations $Eq(\varphi)$. In the following, if Γ is a set of formulas, we will write $Eq(\Gamma) = \bigcup_{\varphi\in\Gamma} Eq(\varphi)$. Similarly, a set of formulas[1] $p\underline{\Delta}q = \{p\underline{\Delta}_j q\}_{j\in\{1,\dots,m\}}$ in two variables p and q defines a translation from equations to sets of formulas: each equation $\varphi = \psi$ is translated into the set of formulas $\varphi\underline{\Delta}\psi$.

[1]We have considered in this paper the symbol $\underline{\Delta}$ instead of the original symbol used in [2] in order to avoid some confusion with the well-known Baaz-Monteiro operator symbol [20] we will use in Section 7.

The formal definition of algebraizable logic is as follows.

Definition 1. A logic L is *algebraizable* if there is a class of algebras $\mathbf{K}$, a set of equations $Eq(p)$ in one variable p and a set of formulas $p\underline{\Delta}q$ in two variables p and q such that, for each set of formulae $\Gamma \cup \{\varphi\}$ in L we have:

(i) $\mathbf{K}$ is an Eq-algebraic semantics for L, that is, it holds that

$$\Gamma \vdash_L \varphi \quad \text{iff} \quad \text{for every } \mathbf{A} \in \mathbf{K} \text{ and every valuation } v \text{ on } \mathbf{A}, \\ \text{if } v[\Gamma] \subseteq Eq(\mathbf{A}), \text{then } v(\varphi) \in Eq(\mathbf{A})$$

where $Eq(\mathbf{A})$ is the set of elements of $\mathbf{A}$ that satisfy all the equations in $Eq(p)$.

(ii) $p = q \models_{\mathbf{K}} Eq(p\underline{\Delta}q)$ and $Eq(p\underline{\Delta}q) \models_{\mathbf{K}} p = q$.

A class of algebras $\mathbf{K}$ for which there are sets $Eq(p)$ and $p\underline{\Delta}q$ with these two properties is said to be an *equivalent algebraic semantics* for L. The set of formulas $\underline{\Delta}$ is called a set of *equivalence formulas* and the set of equations Eq is a set of *defining equations*.

Using our notational conventions, condition (i) above in the definition of algebraizable logic can be simply reformulated as

(i') $\Gamma \vdash_L \varphi$ if and only if $Eq(\Gamma) \models_{\mathbf{K}} Eq(\varphi)$

for all sets of formulae $\Gamma \cup \{\varphi, \psi\}$ in L; while condition (ii) amounts to say that equivalent formulas in L are translated into equalities in $\mathbf{K}$, i.e. that the set of equations $Eq(p\underline{\Delta}q)$ holds true in any algebra $\mathbf{A} \in \mathbf{K}$ when replacing the variables p, q by elements $a, b \in A$ respectively, if and only if $a = b$.

Both conditions (i) and (ii) imply the condition:

(iii) $\Gamma \models_K \varphi = \psi$ if and only if $\{\chi\underline{\Delta}\gamma \mid \chi = \gamma \in \Sigma\} \vdash_L \varphi\underline{\Delta}\psi$

for all $\Sigma \subseteq Eq$ and $\varphi = \psi \in Eq$. Thus, by condition (i') an algebraizable logic L is faithfully interpreted into the equational logic of its equivalent algebraic semantics by means of translating formulas into sets of equations given by a set of defining equations Eq, and by condition (iii) the equational logic of the equivalent algebraic semantics is faithfully interpreted into the logic L by translating equations into sets of formulas given by an equivalence set of formulas Λ. Moreover, both translations are inverses of each other modulo logical equivalence.

The following is a syntactical characterisation of when a logic is algebraizable.

Theorem 2. *A deductive system or logic L, with consequence relation $\vdash_L$, is algebraizable if, and only if, there exist a system $\underline{\Delta}$ of formulas in two variables and a system Eq of equations $\delta = \varepsilon$ in a single variable such that the following conditions (i)-(v) hold for each three formulae φ, ψ, χ:*

(i) $\vdash_L \varphi \underline{\Delta} \varphi$

(ii) $\varphi \underline{\Delta} \psi \vdash_L \psi \underline{\Delta} \varphi$

(iii) $\varphi \underline{\Delta} \psi, \psi \underline{\Delta} \chi \vdash_L \varphi \underline{\Delta} \chi$

(iv) $\{\varphi_i \underline{\Delta} \psi_i \mid i \in \{1, \ldots, n\}\} \vdash_L c(\varphi_1, ..., \varphi_n) \underline{\Delta} c(\psi_1, ..., \psi_n)$, *for every n-ary connective c.*

(v) $\varphi \dashv\vdash_L \delta(\varphi) \underline{\Delta} \varepsilon(\varphi)$, *that is,* $\varphi \vdash_L \delta(\varphi) \underline{\Delta} \varepsilon(\varphi)$ *and* $\delta(\varphi) \underline{\Delta} \varepsilon(\varphi) \vdash_L \varphi$

Here, $\underline{\Delta}$ and $\delta = \varepsilon$ are systems of equivalence formulas and defining equations for L.

Finally, given an algebraizable logic L, the following theorem describes a simple axiomatisation of its unique equivalent quasivariety semantics.

Theorem 3. *Let L be an algebraizable logic defined by a set of axioms Ax and a set of inference rules Ir, with equivalence formulas $\underline{\Delta}$ and a system Eq of equations $\delta = \varepsilon$. Then, the unique equivalent quasivariety semantics for L is axiomatized by the following identities:*

(i) for each axiom $\varphi \in Ax$, the identity $\delta(\varphi) = \varepsilon(\varphi)$, and

(ii) $\delta(p \underline{\Delta} p) = \varepsilon(p \underline{\Delta} p)$,

together with the following quasi-identities:

(iii) for each inference rule '$\{\psi_1, ..., \psi_m\} \vdash \varphi$' $\in Ir$,

$$\textit{if} \quad \bigwedge_{j=1}^{m} \delta(\psi_j) = \varepsilon(\psi_j), \ \textit{then} \quad \delta(\varphi) = \varepsilon(\varphi)$$

(iv) and, if $\delta(p \underline{\Delta} q) = \varepsilon(p \underline{\Delta} q)$, then $p = q$.

2.2. Multi-adjoint algebras

Order-left multi-adjoint algebras are the underlying algebraic structure from which the propositional logic frameworks studied in [11, 9] were defined. This section is only devoted to recall the definitions of order-left adjoint pair and (bounded) order-left multi-adjoint algebra, due to a comprehensive study on multi-adjoint algebras was already given in [15, 16]. One of the key features of order-left adjoint pairs relies on their conjunctions are neither commutative nor associative, and as consequence, they can be employed in general frameworks where both properties be not a requirement.

Definition 4. Let $(P, \preceq)$ be a poset and $\&, \swarrow : P \times P \to P$ binary operators in P satisfying the following conditions:

- The adjoint property holds for all $x, y, z \in P$, that is:

$$x \,\&\, y \preceq z \quad \text{if and only if} \quad x \preceq z \swarrow y$$

- The conjunction $\&$ is order-preserving in the second argument, that is:

$$\text{if} \quad x, y_1, y_2 \in P \ \text{ and } \ y_1 \preceq y_2, \ \text{ then } \ x \,\&\, y_1 \preceq x \,\&\, y_2$$

Then, we say $(\&, \swarrow)$ is an *order-left adjoint pair* with respect to P.

Applying the adjoint property, we obtain that the conjunction of an order-left adjoint pair is actually order-preserving in both arguments. This monotonicity property yields to adequately capture the semantics interpretation of the fuzzy modus ponens proposed by Hájek in [20].

Order-left multi-adjoint algebras are algebraic structures composed of a poset together with a finite family of order-left adjoint pairs. These algebras bring an enhanced degree of flexibility to those frameworks where they can be used, such as formal concept analysis [13, 12, 28], rough set theory [17, 27], fuzzy relational equations [18, 24] and logic programming [29, 30, 32].

Definition 5. Let $(P, \preceq)$ be a poset. An *order-left multi-adjoint algebra* is a tuple $(P, \preceq, \&_1, \swarrow^1, \ldots, \&_n, \swarrow^n)$ where $(\&_i, \swarrow^i)$, with $i \in \{1, \ldots, n\}$, is a finite family of order-left adjoint pairs with respect to P.

From now on, order-left multi-adjoint algebras will be defined on a bounded lattice $(L, \inf, \sup, 0, 1)$ rather than on a poset $(P, \preceq)$, thus obtaining a different algebraic structure $(L, \inf, \sup, 0, 1, \&_1, \swarrow^1, \ldots, \&_n, \swarrow^n)$, which will be called *bounded order-left multi-adjoint lattice*.

3. Bounded lattice algebraizable logic

This section is focused on introducing a logic associated with bounded lattices, which is algebraizable in the sense of Blok-Pigozzi [2] (see Section 2.1 for some details). Hence, this section will also prove that the introduced logic will satisfy Conditions (i)-(v) of Theorem 2. As an immediate consequence, we will have that the introduced logic is sound and complete with respect to its equivalent algebraic semantics [2].

This logic will be called *bounded lattice algebraizable logic* (BLAL) and the language (set of well-formed formulas) is built in the usual way from a countable set of propositional symbols Π together with the set of binary connective symbols $\{\to, \wedge, \vee\}$ and the constant $\bot$.

Definition 6. The *alphabet* of BLAL, denoted as $\mathfrak{A}_{\mathrm{BLAL}}$, is composed of the set of connective symbols $\{\to, \wedge, \vee\}$, the logic symbol $\bot$, the auxiliary symbols "(", ")" and "," and a set of propositional symbols Π.

The *language* of BLAL, denoted as $\mathcal{L}_{\mathfrak{A}_{\mathrm{BLAL}}}$, is given by the set of well-formed formulas (wff), which is inductively defined from an alphabet $\mathfrak{A}_{\mathrm{BLAL}}$ as follows: $\bot$ and p are wff, where $p \in \Pi$; If φ and ψ are wff, then $(\varphi \to \psi)$, $(\varphi \wedge \psi)$, and $(\varphi \vee \psi)$, are wff; nothing else is a formula.

Throughout this paper we will also consider two definable connectives: the negation connective $\neg$ defined as $\neg\varphi := \varphi \to \bot$, and the equivalence connective $\leftrightarrow$ defined as $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Next, the main logic considered in this paper is introduced next, which is a small modification of the logic given in [9] (adding a new inference rule and avoiding (for now) the axioms associated with the adjoint operators).

Definition 7. The *bounded lattice algebraizable logic* (BLAL) is the axiomatic system defined by the following axioms:

L1. $(\varphi \wedge \psi) \to \varphi$

L2. $(\varphi \wedge \psi) \to \psi$

L3. $(\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to (\varphi \wedge \psi)))$

L4. $\varphi \to (\varphi \vee \psi)$

L5. $\psi \to (\varphi \vee \psi)$

L6. $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$

L7. $\varphi \to \varphi$

L8. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

L9. $(\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

L10. $\bot \to \varphi$

L11. $\varphi \to (\bot \to \bot)$

and the following two inference rules:

MP. $\varphi, \varphi \to \psi \vdash \psi$

IR. $\varphi \vdash (\varphi \to \varphi) \to \varphi$

Notice that Axioms **L1-L3** and **L4-L5** encode basic properties of the lattice conjunction and disjunction connectives respectively, while axioms **L7-L11** encode minimal properties of the implication, like identity, transitivity, and boundary conditions, intended to provide a model for the lattice order. On the other hand, the first inference rule is the well-known modus ponens, while the rule **IR** is needed for the algebraizability of the logic as it will be seen. The notion of proof and provable formula (also called *theorem*) in BLAL are defined as usual [20, 31]. Next, for the reader's convenience, we recall the definition of provable formula with respect to a *theory* (set of wffs).

Definition 8. Given a theory Γ, a formula φ is provable from Γ, denoted as $\Gamma \vdash_{\mathrm{BLAL}} \varphi$, if there is a sequence $\varphi_1, \ldots, \varphi_{n-1}, \varphi_n = \varphi$ of formulas such that each φ_i is either an axiom of BLAL, a formula of Γ, or follows from some preceding φ_j, φ_k (with $j, k < i$) by **MP**, or by a preceding φ_j (with $j < i$) by the inference rule **IR**.

We will use the notation $\varphi \dashv\vdash_{\mathrm{BLAL}} \psi$ as a shorthand for $\varphi \vdash_{\mathrm{BLAL}} \psi$ and $\psi \vdash_{\mathrm{BLAL}} \varphi$.

In the following we prove that BLAL is algebraizable in the sense of Blok-Pigozzi, for which we will make use of different results given in the aforementioned paper [9]. From now on, we will consider $p\underline{\Delta}q = \{p \to q, q \to p\}$, $\delta(p) = p$ and $\varepsilon(p) = p\underline{\Delta}p = p \to p$, for all propositional symbol p in Π.

Lemma 9. *BLAL satisfies conditions (i), (ii), (iii) and (iv) for* $c \in \{\wedge, \to\}$ *of Theorem 2.*

Proof. Let us check the properties consecutively. Before that, we have, by definition, that $\varphi \underline{\Delta} \varphi = \varepsilon(\varphi) = \varphi \to \varphi$.

(i) $\vdash_{\mathrm{BLAL}} \varphi \underline{\Delta} \varphi$. Indeed, $\vdash_{\mathrm{BLAL}} \varphi \to \varphi$, by Axiom **L7**.

(ii) $\varphi \underline{\Delta} \psi \vdash_{\mathrm{BLAL}} \psi \underline{\Delta} \varphi$. This amounts to prove that $\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{BLAL}} \varphi \to \psi$, and $\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{BLAL}} \psi \to \varphi$, which obviously hold.

(iii) $\varphi \underline{\Delta} \psi, \psi \underline{\Delta} \chi \vdash_{\mathrm{BLAL}} \varphi \underline{\Delta} \chi$. We have to prove that $\varphi \to \psi, \psi \to \varphi, \psi \to \chi, \chi \to \psi \vdash_{\mathrm{BLAL}} \{\varphi \to \chi, \chi \to \varphi\}$, which also holds due Axiom **L8**.

(iv) We check now that $\varphi_1 \underline{\Delta} \psi_1, \varphi_2 \underline{\Delta} \psi_2 \vdash_{\mathrm{BLAL}} c(\varphi_1, \varphi_2) \underline{\Delta} c(\psi_1, \psi_2)$, for $c \in \{\wedge, \to\}$.

(1) Considering $c = \wedge$, it is enough to prove the following:

- $\varphi_1 \to \psi_1 \vdash_{\mathrm{BLAL}} \varphi_1 \wedge \varphi_2 \to \psi_1 \wedge \varphi_2$
- $\varphi_2 \to \psi_2 \vdash_{\mathrm{BLAL}} \psi_1 \wedge \varphi_2 \to \psi_1 \wedge \psi_2$

which hold by Proposition 14 in [9].

(2) Now, if $c = \to$, it is enough to prove the following:

- $\varphi_1 \to \psi_1 \vdash_{\mathrm{BLAL}} (\psi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_2)$
- $\varphi_2 \to \psi_2 \vdash_{\mathrm{BLAL}} (\varphi_1 \to \varphi_2) \to (\varphi_1 \to \psi_2)$

where the first arises from Axiom **L8** and **MP**, and the second from Axiom **L9** and **MP**. □

To check that condition (iv) of Theorem 2 also holds for $c = \vee$, we first need to check that the derivability $\varphi \to \psi \vdash_{\mathrm{BLAL}} (\varphi \vee \chi) \to (\psi \vee \chi)$ holds in BLAL.

Lemma 10. *The following derivations hold in BLAL:*

$$\varphi \to \psi \quad \vdash_{BLAL} \quad (\varphi \vee \chi) \to (\psi \vee \chi) \quad (L\vee)$$
$$\varphi \to \psi \quad \vdash_{BLAL} \quad (\chi \vee \varphi) \to (\chi \vee \psi) \quad (R\vee)$$

Proof. By Axiom **L8**, we have

$$\vdash_{\text{BLAL}} (\varphi \to \psi) \to ((\psi \to (\psi \vee \chi)) \to (\varphi \to (\psi \vee \chi)))$$

Then, from $\varphi \to \psi$, by **MP**, we have

$$\varphi \to \psi \vdash_{\text{BLAL}} (\psi \to (\psi \vee \chi)) \to (\varphi \to (\psi \vee \chi))$$

And now, by Axiom **L4** and **MP**, we have

$$\varphi \to \psi \vdash_{\text{BLAL}} (\varphi \to (\psi \vee \chi)) \quad (*)$$

By Axiom **L6**, $(\varphi \to (\psi \vee \chi)) \to (((\chi \to (\psi \vee \chi)) \to ((\varphi \vee \chi) \to (\psi \vee \chi))$ is a theorem of BLAL. Then, by (*) and **MP**, we get

$$\varphi \to \psi \vdash_{\text{BLAL}} (((\chi \to (\psi \vee \chi)) \to ((\varphi \vee \chi) \to (\psi \vee \chi))$$

Finally, by Axiom **L5** and **MP**, we get

$$\varphi \to \psi \vdash_{\text{BLAL}} ((\varphi \vee \chi) \to (\psi \vee \chi))$$

The proof of $\varphi \to \psi \vdash_{\text{BLAL}} (\chi \vee \varphi) \to (\chi \vee \psi)$ holds analogously. □

Now, we can prove that condition (iv) of Theorem 2 for the disjunction connective $\vee$ indeed holds in BLAL.

Lemma 11. *BLAL satisfies condition (iv) of Theorem 2 for $c = \vee$.*

Proof. We have to check that $\varphi_1 \underline{\Delta} \psi_1, \varphi_2 \underline{\Delta} \psi_2 \vdash_{\text{BLAL}} c(\varphi_1, \varphi_2) \underline{\Delta} c(\psi_2, \psi_2)$, for $c = \vee$. For this, it is enough to check that:

- $\varphi_1 \to \psi_1 \vdash_{\text{BLAL}} \varphi_1 \vee \varphi_2 \to \psi_1 \vee \varphi_2$
- $\varphi_2 \to \psi_2 \vdash_{\text{BLAL}} \psi_1 \vee \varphi_2 \to \psi_1 \vee \psi_2$

which hold by using the derivability properties (L$\vee$), (R$\vee$) from Lemma 10 and **MP**. □

Before finally proceeding to check the last condition (v) of Theorem 2, let us show a couple of relevant properties of BLAL.

Lemma 12. *BLAL proves the formula $(\psi \to \psi) \to (\varphi \to \varphi)$.*

Proof. We consider the following steps:

(i) $(\bot \to \varphi) \to ((\varphi \to \bot) \to (\varphi \to \varphi))$ is an instance of Axiom **L9**, and since $(\bot \to \varphi)$ is Axiom **L10**, by **MP**, we have

$$\vdash_{\mathrm{BLAL}} (\varphi \to \bot) \to (\varphi \to \varphi)$$

(ii) On the other hand, $(\bot \to \varphi) \to ((\bot \to \bot) \to (\bot \to \varphi))$ is an instance of Axiom **L8**, and since $(\bot \to \varphi)$ is Axiom **L10**, by **MP**, we have

$$\vdash_{\mathrm{BLAL}} (\bot \to \bot) \to (\varphi \to \bot)$$

(iii) Therefore, by transitivity (Axiom **L8**) and **MP**, from (i) and (ii), it follows that

$$\vdash_{\mathrm{BLAL}} (\bot \to \bot) \to (\varphi \to \varphi)$$

(iv) By Axiom **L11**, we have $\vdash_{\mathrm{BLAL}} (\psi \to \psi) \to (\bot \to \bot)$.

Finally, from (iii), (iv) and transitivity (Axiom **L8**), we obtain that

$$\vdash_{\mathrm{BLAL}} (\psi \to \psi) \to (\varphi \to \varphi)$$

□

As a consequence, we have that all formulas of the kind $\varphi \to \varphi$ are not only theorems of BLAL but also equivalent in this stronger sense:

$$(\varphi \to \varphi) \to (\psi \to \psi) \dashv\vdash_{\mathrm{BLAL}} (\psi \to \psi) \to (\varphi \to \varphi)$$

Therefore, besides having the *falsum* constant $\bot$ in the language, it also makes sense in BLAL to consider

$$\top := \varphi \to \varphi$$

as another constant, the *verum* constant.

From the above, we obtain an equivalent alternative to the inference rule **IR** in the following sense.

Corollary 13. *Let BLAL′ be the logic obtained from BLAL by replacing the rule* **IR** *by the rule*

$$\mathbf{IR'}. \quad \varphi \vdash (\psi \to \psi) \to \varphi$$

Then BLAL′ and BLAL define the same logic, i.e. $\vdash_{BLAL'} = \vdash_{BLAL}$.

PROOF. We have to prove that (i) **IR**' is derivable in BLAL, and (ii) **IR** is derivable in BLAL$'$. Condition (i) follows from Lemma 12 and Axiom **L8**, while condition (ii) straightforwardly holds since **IR** is a particular case of **IR**' by considering $\psi = \varphi$. □

The following result presents new interesting provable formulae.

Corollary 14. *BLAL proves the formula* $\varphi \to (\psi \to \psi)$*. In particular, it proves the formula* $\varphi \to (\varphi \to \varphi)$ *as well.*

PROOF. By Axiom **L11**, we know that the formula $\varphi \to (\bot \to \bot)$ is provable in BLAL, but Lemma 12 has shown that $(\bot \to \bot) \to (\psi \to \psi)$ is a theorem of BLAL. Thus, by transitivity (Axiom **L8**), we have that $\vdash_{\mathrm{BLAL}} \varphi \to (\psi \to \psi)$. Hence, in particular, taking $\psi = \varphi$, we also obtain that $\vdash_{\mathrm{BLAL}} \varphi \to (\varphi \to \varphi)$. □

Now, we can prove that BLAL satisfies condition (v) of Theorem 2.

Lemma 15. $\varphi \dashv\vdash_{BLAL} \delta(\varphi)\underline{\Delta}\varepsilon(\varphi)$

PROOF. Considering that $p\underline{\Delta}q = \{p \to q, q \to p\}$, $\delta(p) = p$ and $\varepsilon(p) = p\underline{\Delta}p = p \to p$, checking $\varphi \dashv\vdash_{\mathrm{BLAL}} \delta(\varphi)\underline{\Delta}\varepsilon(\varphi)$ amounts to check:

(i) $\varphi \vdash_{\mathrm{BLAL}} \varphi\underline{\Delta}(\varphi \to \varphi)$, that is, $\varphi \vdash_{\mathrm{BLAL}} \{\varphi \to (\varphi \to \varphi), (\varphi \to \varphi) \to \varphi\}$, and

(ii) $\varphi \to (\varphi \to \varphi), (\varphi \to \varphi) \to \varphi \vdash_{\mathrm{BLAL}} \varphi$.

By Corollary 14, the formula $\varphi \to (\varphi \to \varphi)$ is a theorem of BLAL. Furthermore, by the inference rule **IR** we have $\varphi \vdash_{\mathrm{BLAL}} (\varphi \to \varphi) \to \varphi$. Therefore, we obtain statement (i). Finally, (ii) arises by applying **MP** since $(\varphi \to \varphi)$ is Axiom **L7**. □

As a consequence of Lemmas 9, 11 and 15, we have that BLAL is algebraizable and, by Theorem 3, we have an axiomatization of the quasi-variety that is its equivalent algebraic semantics.

Theorem 16. *BLAL is algebraizable, and its equivalent algebraic semantics is the quasi-variety* $\mathbb{BLAL}$ *of algebras*[2] $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ *satisfying the following equations for each* $x, y, z \in A$*:*

[2]Notice that, we have abused the notation by using the same symbols for the language as for the operators in the algebra.

E1. $(x \wedge y) \to x = 1$

E2. $(x \wedge y) \to y = 1$

E3. $(z \to x) \to ((z \to y) \to (z \to (x \wedge y))) = 1$

E4. $x \to (x \vee y) = 1$

E5. $y \to (x \vee y) = 1$

E6. $(x \to z) \to ((y \to z) \to ((x \vee y) \to z))) = 1$

E7. $x \to x = 1$

E8. $(x \to y) \to ((y \to z) \to (x \to z)) = 1$

E9. $(y \to z) \to ((x \to y) \to (x \to z)) = 1$

E10. $0 \to x = 1$

E11. $x \to (0 \to 0) = 1$

and the following quasi-equations:

QE1. *If* $x = 1$ *and* $(x \to y) = 1$*, then* $y = 1$

QE2. *If* $x = 1$*, then* $(x \to x) \to x = 1$

QE3. *If* $x \to y = 1$ *and* $y \to x = 1$*, then* $x = y$

Proof. It is clear that BLAL satisfies now all the conditions (i)-(v) of Theorem 2 and thus it is algebraizable. Moreover, using $p \underline{\Delta} q = \{p \to q, q \to p\}$, $\delta(p) = p$ and $\varepsilon(p) = p \underline{\Delta} p = p \to p$, its equivalent algebraic semantics is given by class of algebras satisfying the above set of equations and quasi-equations.

The quasi-equation **QE1** corresponds to the Modus Ponens rule **MP**, **QE2** to the inference rule **IR**, and **QE3** to the quasi-equation (iv) in Theorem 3. Notice that, with the proposed mappings δ and ε, item (*ii*) of Theorem 3 does not provide any equation. □

Next we provide an example of an algebra in $\mathbb{BLAL}$.

Example 17. A particular algebra in the quasi-variety $\mathbb{BLAL}$ is the algebra $\mathbf{M}_3 = (M_3, \wedge, \vee, \to, 0, 1)$, where $(M_3, \wedge, \vee, 0, 1)$ is the (non-distributive) lattice over the domain $M_3 = \{0, a, b, c, 1\}$, where 0 is the bottom, 1 is the top, and a, b, c are incomparable, see Figure 1), and $\to$ is the order implication defined by setting $x \to y = 1$ if $x \leq y$ and $x \to y = 0$ otherwise.

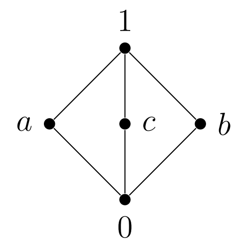

Figure 1: Lattice $\mathbf{M}_3$

In fact, the definition of BLAL-algebras can be simplified by dropping the quasi-equation **QE2**.

Lemma 18. *The quasi-equation* **QE2** *is derivable from the axioms and the rest of rules.*

Proof. By Equation **E7** we have $x \to x = 1$, and by assumption $x = 1$. Hence $(x \to x) \to x = 1 \to 1$, but again by Equation **E7**, $1 \to 1 = 1$, so $(x \to x) \to x = 1$. □

It is worth pointing out that, unlike the fact that the quasi-equation **QE2** can be omitted from the definition of BLAL-algebras as the above lemma shows, we cannot remove the rule **(IR)** from the definition of the logic BLAL. In fact, if we remove **(IR)** from the axiomatic system of BLAL, it can be shown that the resulting logic, called BLAL$^-$, is no longer algebraizable. See the Appendix for a proof.

Finally, after introducing the natural notion of semantical truth-evaluation for BLAL, we finish this section by making explicit the soundness and completeness result for BLAL that follows as a direct consequence of its algebraizability.

Definition 19. Let $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ be a BLAL-algebra. An **A**-*evaluation* of BLAL formulas is a mapping $e\colon \mathcal{L}_{\mathfrak{A}_{\mathrm{BLAL}}} \to A$ defined inductively from the propositional symbols of the language as:

$$\begin{aligned} e(p) &\in A, \text{ for all propositional symbol } p \in \Pi \\ e(\bot) &= 0 \\ e(\varphi \to \psi) &= e(\varphi) \to e(\psi) \\ e(\varphi \wedge \psi) &= e(\varphi) \wedge e(\psi) \\ e(\varphi \vee \psi) &= e(\varphi) \vee e(\psi) \end{aligned}$$

As usual, we will say that a formula φ of the language $\mathfrak{A}_{\mathrm{BLAL}}$ is a tautology if, for any BLAL-algebra $\mathbf{A}$, such formula is true under any $\mathbf{A}$-evaluation, that is, if $e(\varphi) = 1$ for any $\mathbf{A}$-evaluation e. Moreover, a formula φ is said to be a logical consequence of a set of formulas Γ, written $\Gamma \models_{\mathrm{BLAL}} \varphi$, if, for any BLAL-algebra $\mathbf{A}$ and every $\mathbf{A}$-evaluation e, $e(\varphi) = 1$ whenever $e(\psi) = 1$ for all $\psi \in \Gamma$.

Theorem 20 (Soundness and completeness of BLAL). *For any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathfrak{A}_{BLAL}$, we have $\Gamma \vdash_{BLAL} \varphi$ iff $\Gamma \models_{BLAL} \varphi$.*

4. Some properties of BLAL and BLAL-algebras

In this section we show three interesting properties about the logic BLAL and its associated quasivariety of BLAL-algebras that can help in positioning them in the landscape of well known non-classical algebraic logics.

4.1. BLAL-algebras are expansions of bounded lattices

It is worth noticing that if $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BLAL-algebra, then its $\rightarrow$-free reduct, that is, the structure $(A, \wedge, \vee, 0, 1)$ where the implication is removed, is a bounded lattice.

Indeed, the presence of the quasi-equation **QE3** forces the ordering induced by the implication ($x \leq_I y$ if $x \rightarrow y = 1$) to match with the ordering induced by the lattice operators ($x \leq_L y$ if $x \wedge y = x$, or $x \vee y = y$), and as a consequence, the equivalence with the equality, i.e. $x = y$ iff $x \rightarrow y = 1$ and $y \rightarrow x = 1$.

Proposition 21. *Given a BLAL-algebra $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$, we obtain for each $x, y \in A$ that*

$$x \rightarrow y = 1 \quad \textit{if and only if} \quad x \wedge y = x$$

Proof. If $x \rightarrow y = 1$, then by Equation **E7**, Equation **E3** (taking $z = x$) and applying the **QE1** twice, we obtain that $x \rightarrow x \wedge y = 1$, that together with Equation **E1** and **QE3**, we get $x \wedge y = x$.

Now, we assume that $x \wedge y = x$. Then, by Equation **E1**, we obtain $(x \rightarrow y) = (x \wedge y \rightarrow y) = 1$. □

Conversely, any bounded lattice $(A, \wedge, \vee, 0, 1)$ can be enriched with an implication operator $\rightarrow$ in such a way that the expanded algebra $(A, \wedge, \vee, \rightarrow, 0, 1)$ is a BLAL-algebra. In particular, it is easy to check that this is case when we consider $\rightarrow$ to be the order implication defined as $x \rightarrow y = 1$ if $x \leq y$ and $x \rightarrow y = 0$ otherwise. In particular, it is worth noticing that the underlying lattice of a BLAL-algebra needs not be distributive. For instance, the algebra $\mathbf{M}_3$ from Example 17 over the diamond lattice recalled in Figure 1 is a BLAL-algebra on top of a non-distributive lattice.

Although we have seen that implications in a BLAL-algebra characterize the ordering in the underlying lattice (Proposition 21), this property does not uniquely characterize the implications in BLAL-algebras. Next, we will see that bounded lattices $\mathbf{A} = (A, \wedge, \vee, 0, 1)$ exist on top of which we can define a binary operator $\rightarrow: A \times A \rightarrow A$ satisfying the condition $x \leq y$ iff $x \rightarrow y = 1$, and moreover such that $\rightarrow$ is non-increasing in the first variable and non-decreasing in the second, but $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is not a BLAL-algebra.

Example 22. Consider the chain with three elements $A_3 = \{0, 1/2, 1\}$ with the usual ordering $(0 \leq 1/2 \leq 1)$ and the implication operator $\rightarrow$ on A_3 defined as

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ 1/2, & \text{if } x = 1/2, y = 0 \\ 0, & \text{otherwise} \end{cases}$$

for all $x, y \in A_3$. We clearly have that $\rightarrow$ is non-increasing in the first variable and non-decreasing in the second, and the condition $x \leq y$ iff $x \rightarrow y = 1$ holds true in A. However, it does not satisfy the whole set of equations associated with a BLAL-algebra. For instance, if $x = 0, y = 1, z = 1/2$, Equation **E3** is not satisfied. Also Equations **E8** and **E9** are not satisfied either considering, for example, $x = y = 0, z = 1/2$, and $x = 1/2, y = 0, z = 1/2$, respectively.

4.2. BLAL and Rasiowa implicative logics

The logic BLAL is not only algebraizable but it also belongs to the family of implicative logics in the sense of Rasiowa (also known as Rasiowa implicative logics) [33].

Definition 23. A logic $\vdash$ (with binary connectives $\wedge, \vee, \rightarrow$) is a *Rasiowa Implicative logic* (or a *standard system of implicative extensional propositional calculus*) if it satisfies the following conditions:

I1. $\vdash \varphi \to \varphi$

I2. $\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$

I3. for $* \in \{\wedge, \vee, \to\}$,
$\varphi_1 \to \varphi_2, \varphi_2 \to \varphi_1, \psi_1 \to \psi_2, \psi_2 \to \psi_1 \vdash \varphi_1 * \psi_1 \to \varphi_2 * \psi_2$

I4. $\varphi, \varphi \to \psi \vdash \psi$

I5. $\varphi \vdash \psi \to \varphi$

The announced result is proved below.

Proposition 24. *BLAL is a Rasiowa implicative logic.*

PROOF. **I1** and **I2** directly follow from Axioms **L7** and **L8**, respectively, together with **MP**.

I3. This is (iv) of Lemma 9 and Lemma 11.

I4. This is **MP**.

I5. By Axiom **L11**, we have $\psi \to (\bot \to \bot)$, and by the inference rule **IR**, $\varphi \vdash (\bot \to \bot) \to \varphi$, hence by Axiom **L8**, it follows $\varphi \vdash \psi \to \varphi$. □

Indeed, being a Rasiowa implicative logic is a stronger notion than being algebraizable, since every Rasiowa implicative logic is algebraizable [5] but not viceversa.[3]

4.3. BLAL-algebras versus Heyting algebras

It is a well-known fact that the variety of Heyting algebras is the algebraic semantics of Intuitionistic Propositional Calculus (IPC) [3, 21]. We will see next that BLAL-algebras are more general structures than Heyting algebras. Recall that an algebra $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra if the following conditions are satisfied:

- $(A, \wedge, \vee, 0, 1)$ is a bounded lattice
- $(\wedge, \to)$ is a residuated pair, that is, for any $x, y, z \in A$, $x \wedge y \leq z$ if and only if $x \leq y \to z$, hence the implication is the residuum of the lattice conjunction.

[3]D'Ottaviano and da Costa's 3-valued paraconsistent logic J_3 is an example of an algebraizable logic which is not a Rasiowa implicative logic [6, Example 2.9.8].

These algebras satisfy for example the following identities [20]:

$$x \to (y \to z) = (x \wedge y) \to z = y \to (x \to z)$$

for all $x, y, z \in A$. The following result shows that a Heyting algebra is a particular case of a BLAL-algebra.

Proposition 25. *If* $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ *is a Heyting algebra, then* $\mathbf{A}$ *is a BLAL-algebra.*

PROOF. The proof that Heyting algebras satisfy all Equations **E1-E11** and quasi-equations **QE1-QE3** straightforwardly arises from definition and properties above. □

The converse of Proposition 25 is not true in general, as in a BLAL-algebra the pair $(\wedge, \to)$ is not necessarily residuated. In fact, it is very easy to find BLAL-algebras which are not Heying algebras. For instance take the linearly ordered BLAL-algebra on the domain $A_3 = \{0, 1/2, 1\}$ with the natural order and with an implication $\to$ defined as $x \to y = 1$ if $x \leq y$, and $x \to y = 0$ otherwise. Clearly $(\wedge, \to)$ is not an adjoint pair, in particular $1/2 \wedge 1 = 1/2$ but $1/2 \not\leq 1 \to 1/2 = 0$. Hence $(A_3, \wedge, \vee, \to, 0, 1)$ is not a Heyting algebra. As a consequence, the class of Heyting algebras is a strict subclass of the class of BLAL-algebras, or equivalently, BLAL is a strictly weaker logic than IPC.

Since in a BLAL-algebra the order in the underlying lattice is definable by the implication as $x \leq y$ if, and only if, $x \to y = 1$, the residuation condition can be equivalently expressed by the following two quasi-equations:

(Res1) if $(x \wedge y) \to z = 1$ then $x \to (y \to z) = 1$

(Res2) if $x \to (y \to z) = 1$ then $(x \wedge y) \to z = 1$

We can show that condition (Res1) can be somehow simplified.

Proposition 26. *A BLAL-algebra* $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ *is a Heyting algebra if and only if* $\mathbf{A}$ *satisfies the quasi-equation (Res2) and the following equation:*

(id) $1 \to x = x$

In that case, $(A, \wedge, \vee, 0, 1)$ *is obviously then a distributive bounded lattice.*

Proof. First of all, it is very easy to check that in any Heyting algebra the equation (id) holds. Indeed, since $x \wedge 1 \leq x$, by residuation we obtain $x \leq 1 \to x$. On the other hand, since $1 \to x \leq 1 \to x$, again by residuation we have $(1 \to x) \wedge 1 \leq x$, i.e. $1 \to x \leq x$.

Now, we only have to prove that if a BLAL-algebra **A** satisfies (id) then it also satisfies (Res1), i.e. it satisfies that $(x \wedge y) \to z = 1$ implies $x \to (y \to z) = 1$. If $(x \wedge y) \to z = 1$ then, by Equation **E9** and the quasi-equation **QE1**, we have

$$(y \to (x \wedge y)) \to (y \to z) = 1 \tag{1}$$

Next, we show that $x \to (y \to (x \wedge y)) = 1$. Indeed, the formula $(y \to y) \to ((y \to x) \to (y \to (x \wedge y))) = 1$ is an instance of Equation **E3**, and $y \to y = 1$ holds by Equation **E7**, and so by **QE1**, we have

$$(y \to x) \to (y \to (x \wedge y)) = 1 \tag{2}$$

On the other hand, an instance of Equation **E8** is $(y \to 1) \to ((1 \to x) \to (y \to x)) = 1$. Since $y \to 1 = 1$ by Equation **E11**, applying the quasi-equation **QE1**, we obtain that $((1 \to x) \to (y \to x)) = 1$. Moreover, by (id) we have $1 \to x = x$, and by the quasi-equation **QE1** it follows that

$$x \to (y \to x) = 1 \tag{3}$$

From (2) and (3), Equation **E8** and the quasi-equation **QE1**, it finally follows that $x \to (y \to (x \wedge y)) = 1$, and by (1), Equation **E8** and **QE1** , we finally have $x \to (y \to z) = 1$.

Finally, only to recall that, as is well known, if $(A, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra, then its underlying lattice $(A, \wedge, \vee, 0, 1)$ is distributive. □

The following is then a direct consequence.

Corollary 27. *The extension of BLAL with the axiom*

$$((\varphi \to \varphi) \to \psi) \leftrightarrow \psi$$

and the rule

$$(Res2) \quad \varphi \to (\psi \to \chi) \vdash (\varphi \wedge \psi) \to \chi$$

where φ, ψ and χ are formulae of the BLAL-language, is an equivalent axiomatizaton of IPC.

Proof. It follows from the fact that $\top \leftrightarrow (\varphi \to \varphi)$ is a theorem of BLAL, Proposition 26 and from the equations and quasi-equations associated with BLAL, in particular, **QE3**. $\square$

On the other hand, Heyting algebras are also a subvariety of the more general variety of bounded, commutative and integral residuated lattices, also known as FL_{ew}-algebras (see e.g. [19]).

Lemma 28. *Let $\mathbf{A} = (A, \wedge, \vee, *, \to, 0, 1)$ be a FL_{ew}-algebra, i.e. a bounded, commutative and integral residuated lattice. Then the $*$-free reduct of $\mathbf{A}$, $(A, \wedge, \vee, \to, 0, 1)$ is a BLAL-algebra.*

More generally, let $\mathbf{A} = (A, \wedge, \vee, *, /, \backslash, 0, 1)$ be a FL_w-algebra, i.e. a bounded and integral residuated lattice, non-necessarily commutative, where $/$ and $\backslash$ are the right and left adjoint operators of the monoidal operator $*$ respectively. Then both the $\{*, /\}$-free reduct $(A, \wedge, \vee, \backslash, 0, 1)$ and the $\{*, \backslash\}$-free reduct $(A, \wedge, \vee, /, 0, 1)$ are BLAL-algebras.

Proof. The proof straightforwardly follows from the properties of residuated and adjoint implications, Proposition 21 and the corresponding definitions. $\square$

An interesting question we could ask ourselves is whether, given a BLAL-algebra $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$, we can always define a monoidal operator $\&$ on A such that $\mathbf{A} = (A, \wedge, \vee, \&, \to, \Rightarrow 0, 1)$ is a (possibly non-commutative) residuated lattice, where $\to$ and $\Rightarrow$ are the left and right adjoint implications of $\&$. The answer is negative as the following example shows.

Example 29. Let $(A, \wedge, \vee, 0, 1)$ be a lattice and consider the operator $\to$ as the order implication defined as

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise,} \end{cases}$$

Then $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BLAL-algebra, such that $\to$ is the left adjoint implication of this conjunctor

$$x \,\&\, y = \begin{cases} y & \text{if } x \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

whose right adjoint implication is in turn

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x = 0 \\ y & \text{otherwise.} \end{cases}$$

However, $(A, \&, 1)$ is not a monoid because 1 is not a neutral element of $\&$ since, if $0 < x < 1$, we have $x \mathbin{\&} 1 = 1 \neq x$. Therefore $(A, \wedge, \vee, \&, \rightarrow, \Rightarrow, 0, 1)$ is not a residuated lattice, although $(A, \leq, \&, \rightarrow, 0, 1)$, where $\leq$ is the lattice order, is an order-left adjoint algebra as per Definition 5.

5. Multi-adjoint algebraizable logic

In this section, the logic BLAL studied above will be complemented with new connectives in order to capture multi-adjoint algebras [14, 15, 25]. Hence, we will extend the logic BLAL with a finite set of pairs of connectives $\{\&_i, \rightarrow_i\}_{i\in I}$ together with corresponding conditions to ensure that its algebraic semantics is given by extensions of BLAL-algebras with a set of left-adjoint pairs.

Let us define MAL (on the extended signature) as the expansion of BLAL with three new multi-adjoint rules for each $i \in I$ corresponding to axioms M1, M2, M3 in MLL in [9]:[4]

MR1^i. $\varphi \rightarrow (\psi \rightarrow_i \chi) \vdash (\varphi \&_i \psi) \rightarrow \chi$

MR2^i. $(\varphi \&_i \psi) \rightarrow \chi \vdash \varphi \rightarrow (\psi \rightarrow_i \chi)$

MR3^i. $\psi \rightarrow \chi \vdash (\varphi \&_i \psi) \rightarrow (\varphi \&_i \chi)$

It is worth noticing that the congruence rules for $\&_i$ and $\rightarrow_i$, i.e. the rules

- $\varphi \rightarrow \varphi', \psi \rightarrow \psi' \vdash \varphi \&_i \psi \rightarrow \varphi' \&_i \psi'$
- $\varphi \rightarrow \varphi', \psi \rightarrow \psi', \varphi' \rightarrow \varphi, \psi' \rightarrow \psi \vdash (\varphi \rightarrow_i \psi) \rightarrow (\varphi' \rightarrow_i \psi')$,

[4]Differently from the original formulation of MLL in [9], here we need to formulate the adjoint properties by rules instead of by axioms since the properties of the implication $\rightarrow$ operators in BLAL algebras do not guarantee the required conditions for the pairs of connectives $\{\&_i, \rightarrow_i\}_{i\in I}$ to be properly interpreted as left-adjoint pairs.

are provable in MAL, see [9]. Therefore, $\&_i$ and $\to_i$ satisfy Condition (iv) in Theorem 2. Whence, MAL is also algebraizable, and the corresponding quasi-variety of algebras $\mathbb{MAL}$ are algebras $\mathbf{A} = (A, \wedge, \vee, \to, \{\&_i, \to_i\}_{i \in I}, 0, 1)$ such that $(A, \wedge, \vee, \to, 0, 1)$ is a BLAL-algebra and the following conditions are satisfied for each $i \in I$ and each $x, y, z \in A$:

(AP1^i) $x \leq y \to_i z$ iff $x \&_i y \leq z$
(AP2^i) if $y \leq z$ then $x \&_i y \leq x \&_i z$

Actually condition (AP1^i) is exactly the condition needed to validate the above inference rules $\mathbf{MR1}^i$ and $\mathbf{MR2}^i$, while (AP2^i) is the condition related to $\mathbf{MR3}^i$. For instance $\mathbf{MR1}^i$ is valid in an algebra $\mathbf{A} = (A, \wedge, \vee, \to, \{\&_i, \to_i\}_{i \in I}, 0, 1)$, where $(A, \wedge, \vee, \to, 0, 1)$ is a BLAL-algebra, if and only if, for each evaluation e on $\mathbf{A}$ and every propositional variables p, q, r, if $e(p) \to (e(q) \to_i e(r)) = 1$ then $e(p) \&_i e(q) \to e(r) = 1$. Since in any BLAL-algebra we have $x \to y = 1$ iff $x \leq y$, the previous condition is equivalent to require that if $e(p) \leq e(q) \to_i e(r)$ then $e(p) \&_i e(q) \leq e(r)$, which amounts to the left-to-right direction of the equivalence in (AP1^i).

Therefore, we get the following general completeness result.

Theorem 30. *MAL is complete with respect to the class of MAL-algebras, that this, $\Gamma \vdash_{MAL} \varphi$ if and only if, for any MAL-algebra $\mathbf{A}$ and any evaluation e on $\mathbf{A}$, if $e(\psi) = 1$ for all $\psi \in \Gamma$, then $e(\varphi) = 1$ as well.*

Proof. Due to the derivability in the logic of the congruence rules for the new pairs of connectives $\&_i, \to_i$ mentioned above, by Theorem 2, the logic MAL is also algebraizable. Therefore, as a consequence of its algebraizability, it directly follows that it is complete with respect to its equivalent quasi-variety semantics, which is given by the class of MAL-algebras. □

Notice that, given a BLAL-algebra $(A, \wedge, \vee, \to, 0, 1)$, then the algebra $(A, \wedge, \vee, \to, \{\&, \to\}, 0, 1)$ built as in Example 29 is a MAL-algebra since in that example $\to$ is the left adjoint implication of $\&$.

6. Linearizing the semantics of BLAL and MAL

As declared in [5], one of distinguishing features of logics under the umbrella of the field known as Mathematical Fuzzy Logic (MFL) is that of being complete with respect to a semantics based on linearly ordered algebras, in accordance with the main thesis put forward in [1] that defends that fuzzy logics are those many-valued logics with a linearly-ordered set of truth-values.

An algebraizable logic L is called *semilinear* if it is complete with respect to the subclass composed of the linearly-ordered algebras of its associated quasivariety of L-algebras. This means, for instance, that in order to check whether a formula is a tautology of L it is enough to check that the formula is valid in all linearly-ordered L-algebras.

It is not difficult to check that BLAL (and hence MAL as well) is not semilinear. Indeed, consider the formula $(p \to q) \vee (q \to p)$, where p, q are propositional variables. This formula gets value 1 in all linear BLAL-algebras since in such algebras, for any pair of elements a, b, either $a \leq b$ or $b \leq a$. However, this is not necessarily the case in non-linearly ordered algebras. For instance, consider the following non-linearly ordered BLAL-algebra $\mathbf{A} = (\{0, a, b, 1\}, \wedge, \vee, \to, 0, 1)$, where the underlying lattice is given by the ordering: 1 is the top, 0 is the bottom, and a and b are incomparable, so $a \wedge b = 0$ and $a \vee b = 1$; and $\to$ is defined as

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in \{0, a, b, 1\}$. Then, we have that $(a \to b) \vee (b \to a) = 0$. Therefore, the interpretation that sends p to a and q to b does not satisfy the formula $(p \to q) \vee (q \to p)$.

Nonetheless, Cintula and Noguera prove a general result in [6, Corollary 6.2.8] about how to minimally extend a "well behaved" logic to become semilinear. By "well behaved", they mean a *weakly implicative* logic having a lattice *protodisjunction*. A weakly implicative logic [4, 6] is a generalisation of the notion of Rasiowa's implicative logic by dropping condition **I5** in Definition 23. On the other hand, in a weakly implicative logic $\vdash$ with implication $\to$, a binary connective $\vee$ is a lattice protodisjunction for $\to$ if the following conditions hold [6, Definition 4.1.1]:

$$\vdash \varphi \to \varphi \vee \psi, \quad \vdash \psi \to \varphi \vee \psi, \quad \text{and} \quad \varphi \to \psi, \chi \to \psi \vdash (\varphi \vee \chi) \to \psi.$$

Before formally recalling Cintula-Noguera's result, we introduce the following notation. If a rule R is of the form $\varphi_1, \ldots, \varphi_n \vdash \psi$, where $\varphi_1, \ldots, \varphi_n, \psi$ are formulae of the language, the $\vee$-form of R, denoted $\vee$-R, is the stronger rule $\varphi_1 \vee \chi, \ldots, \varphi_n \vee \chi \vdash \psi \vee \chi$, for an arbitrary formula χ of the language.

Theorem 31 ([6]). *Let L be a weakly implicative logic with a lattice protodisjunction* $\vee$ *and defined by a countable set of axioms Ax(L) and rules Rules(L). Then the least semilinear expansion of L, denoted as* L^ℓ*, is the expansion of L by the axiom*

(Lin) $(\varphi \to \psi) \vee (\psi \to \varphi)$

and the set of rules $\{\vee\text{-}R \mid R \in \mathit{Rules}(L)\}$.

Note that BLAL is a weakly implicative logic since it is a Rasiowa implicative logic. Moreover, in BLAL, the connective $\vee$ is a lattice protodisjunction since it satisfies Axioms **L4**, **L5** and **L6**. Therefore, we can directly apply the above Theorem 31 to axiomatise BLAL^ℓ, the least semilinear expansion of BLAL.

Theorem 32. *Let* $BLAL^\ell$ *be the extension of BLAL with the axiom (Lin) and the rules:*

$(\vee\text{-}MP)\quad \varphi \vee \chi, (\varphi \to \psi) \vee \chi \vdash \psi \vee \chi$
$(\vee\text{-}IR)\quad \varphi \vee \psi \vdash (\top \to \varphi) \vee \psi$

Then $BLAL^\ell$ *is the least semilinear extension of BLAL, and hence it is complete with respect to the class of linearly-ordered BLAL-algebras.*

Since BLAL^ℓ is an extension of BLAL (with an axiom and two inference rules), BLAL^ℓ keeps being algebraizable and Rasiowa implicative, and its equivalent algebraic semantics is given by the quasi-variety $\mathbb{BLAL}^\ell$ of BLAL-algebras $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ further satisfying the following equation and quasi-equations:

(E-Lin) $\quad (x \to y) \vee (y \to x) = 1$
($\vee$-QE1) $\quad x \vee z = 1, \quad (x \to y) \vee z = 1 \quad \Rightarrow \quad y \vee z = 1$
($\vee$-QE2) $\quad x \vee y = 1 \quad \Rightarrow \quad (1 \to x) \vee y = 1$

Next we give some examples BLAL^ℓ-algebras.

Example 33. We have seen that any bounded ordered lattice $(A, \leq, 0, 1)$ endowed with the implication $\to$ defined as $x \to y = 1$ if $x \leq y$, and $x \to y =$

0 otherwise, is a BLAL-algebra. Therefore, if $(A, \leq, 0, 1)$ is linearly ordered, $(A, \min, \max, \to, 0, 1)$ is a BLAL$^\ell$-algebra.

In accordance with Lemma 28, other examples of BLAL$^\ell$-algebras with other kinds of implications are, for instance, algebras defined on top of the real unit interval $[\mathbf{0}, \mathbf{1}]_* = ([0, 1], \min, \max, \to_*, 0, 1)$ where $\to_*$ is the residuum of a left-continuous t-norm $*$.

Finally, we notice that, if we now consider again the previously introduced full logic with left adjoint pairs MAL, the same process we have followed with BLAL in order to linearize its semantics, it amounts to extend MAL with axioms (Lin) and the $\vee$-form of the rules $\mathbf{MR1}^i$, $\mathbf{MR2}^i$ and $\mathbf{MR3}^i$.

Theorem 34. *Let MAL$^\ell$ be the extension of BLAL$^\ell$ with the rules for each $i \in I$:*

$$(\vee\text{−MR1}^i) \quad (\varphi \to (\psi \to_i \chi)) \vee \nu \vdash ((\varphi \&_i \psi) \to \chi) \vee \nu$$
$$(\vee\text{−MR2}^i) \quad ((\varphi \&_i \psi) \to \chi) \vee \nu \vdash (\varphi \to (\psi \to_i \chi)) \vee \nu$$
$$(\vee\text{−MR3}^i) \quad (\psi \to \chi) \vee \nu \vdash ((\varphi \&_i \psi) \to (\varphi \&_i \chi)) \vee \nu$$

Then MAL$^\ell$ is the least semilinear extension of MAL, and hence it is complete with respect to the class of linearly-ordered MAL-algebras.

Given a BLAL$^\ell$-algebra $(A, \wedge, \vee, \to, 0, 1)$, we have that the algebra $(A, \wedge, \vee, \to, \{\&, \to\}, 0, 1)$ obtained as in Example 29 is an MAL$^\ell$-algebra. The well-known Gödel, product and Łukasiwicz adjoint pairs also offer MAL$^\ell$-algebras on the trivial BLAL$^\ell$-algebra given by the unit interval and the usual ordering, together with the implication associated with the ordering, that is, the operator $\to$ defined as $x \to y = 1$ if $x \leq y$, and $x \to y = 0$ otherwise, for all $x, y \in [0, 1]$.

7. Booleanizing the implication: back to (linear) Heyting algebras

In Section 4.3, we have seen that BLAL is a weaker logic than IPC (and equivalently, the class of BLAL-algebras is strictly larger than the class of Heyting algebras). The same happens when we consider the semilinear case, i.e., BLAL$^\ell$ is a weaker logic than IPC extended with the prelinearity axiom (Lin), that is also known as Gödel logic, denoted G, in the setting of Mathematical Fuzzy logic.

Somehow surprisingly, in this section we show that, by forcing the implication in BLAL^ℓ to have a Boolean behaviour, we obtain a logic which is stronger than Gödel logic, namely, it is equivalent to the extension of Gödel logic with the so-called Baaz-Monteiro operator Δ [20]. This logic, called G_Δ, was precisely the underlying logic proposed in [8] to define on top of it a multi-adjoint logic in the same spirit as MAL^ℓ.

Let us then define the logic $\text{BLAL}^{\ell B}$ as the axiomatic extension of BLAL with the axiom

(Bool) $(\varphi \to \psi) \vee \neg(\varphi \to \psi)$

Because $\text{BLAL}^{\ell B}$ is indeed an axiomatic extension of BLAL^ℓ, it keeps being algebraizable and semilinear, i.e. complete with respect to the class of $\text{BLAL}^{\ell B}$-chains. Note that $\text{BLAL}^{\ell B}$-algebras are BLAL^ℓ-algebras where the equation

(E-Bool) $(x \to y) \vee \neg(x \to y) = 1$

holds. Now we can show that in linearly ordered $\text{BLAL}^{\ell B}$-algebras the implication $\to$ is indeed the *order* implication.

Proposition 35. *If* $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ *is a linearly-ordered* $BLAL^{\ell B}$*-algebra, then* $\to$ *is univocally defined as follows, for each* $x, y \in A$*,*

$$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

PROOF. By Proposition 21, we have that $x \leq y$ iff $x \to y = 1$. Hence, it remains to prove that, if $x \not\leq y$, then $x \to y = 0$. From the equivalence above, we can ensure that $x \to y < 1$. Hence, by E-Bool and since the algebra is linear, we necessarily have that $(x \to y) \to 0 = 1$. Finally, since $0 \to (x \to y) = 1$, by Equation **E10**, we can apply quasi-equation **QE3** to derive $x \to y = 0$. □

Therefore, for instance, if we are interested on $\text{BLAL}^{\ell B}$-algebras with the real unit interval $[0, 1]$ and the usual ordering as underlying lattice, there is only one such an algebra, which has as implication the order implication operator as defined in Proposition 35.

As a consequence, in $\text{BLAL}^{\ell B}$-algebras the so-called Baaz-Monteiro operator Δ is also definable as $\Delta(x) := 1 \to x$, since it is very easy to check that in linearly-ordered $\text{BLAL}^{\ell B}$-algebras we have:

$$\Delta(x) = 1 \to x = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, if we introduce in $\text{BLAL}^{\ell B}$ the definable unary connective Δ by putting $\Delta\varphi := \top \to \varphi$, it is easy to show that $\text{BLAL}^{\ell B}$ enjoys the following global form of deduction theorem.

Proposition 36. *For any set of formulas $\Gamma \cup \{\varphi, \psi\}$, the following condition holds:*

$$\Gamma, \varphi \vdash_{BLAL^{\ell B}} \psi \quad \textit{iff} \quad \Gamma \vdash_{BLAL^{\ell B}} \Delta\varphi \to \psi$$

Proof. By completeness of $\text{BLAL}^{\ell B}$, $\Gamma, \varphi \vdash_{\text{BLAL}^{\ell B}} \psi$ iff, for any $\text{BLAL}^{\ell B}$-chain $\mathbf{A}$ and any $\mathbf{A}$-evaluation e, if $e(\chi) = 1$ for any $\chi \in \Gamma$ and $e(\varphi) = 1$ then $e(\psi) = 1$. Taking into account the above semantics of the Δ connective in linearly-ordered algebras, the condition "if $e(\varphi) = 1$ then $e(\psi) = 1$" is equivalent to "$e(\Delta\varphi \to \psi) = 1$". Thus, it holds that $\Gamma, \varphi \vdash_{\text{BLAL}^{\ell B}} \psi$ iff, for any $\text{BLAL}^{\ell B}$-chain $\mathbf{A}$ and any $\mathbf{A}$-evaluation e, if $e(\chi) = 1$ for any $\chi \in \Gamma$ then $e(\Delta\varphi \to \psi) = 1$. Again by completeness of $\text{BLAL}^{\ell B}$, the later is equivalent to $\Gamma \vdash_{\text{BLAL}^{\ell B}} \Delta\varphi \to \psi$. □

It follows that in $\text{BLAL}^{\ell B}$, by using Δ, we can equivalently express rules as axioms, e.g. if we want to extend $\text{BLAL}^{\ell B}$ with a rule $\psi \vdash \varphi$, it is enough to axiomatically extend it with the axiom $\Delta\psi \to \varphi$. In particular, if we now consider again the previously introduced full semilinear logic with adjoint pairs MAL^{ℓ} and we extend it with the axiom (Bool), we can safely replace in this logic the rules ($\vee$-MR1^i), ($\vee$-MR2^i) and ($\vee$-MR3^i) by the following three axioms for each $i \in I$:

(AxMR1^i) $\Delta(\varphi \to (\psi \to_i \chi)) \to ((\varphi \&_i \psi) \to \psi)$
(AxMR2^i) $\Delta((\varphi \&_i \psi) \to \psi) \to (\varphi \to (\psi \to_i \chi))$
(AxMR3^i) $\Delta(\psi \to \chi) \to ((\varphi \&_i \psi) \to (\varphi \&_i \chi))$

Therefore we have the following completeness theorem.

Theorem 37. *Let $MAL^{\ell B}$ be the axiomatic expansion of $BLAL^{\ell B}$ with axioms ($AxMR1^i$), ($AxMR2^i$) and ($AxMR3^i$) for each $i \in I$. Then $MAL^{\ell B}$ is sound and complete with respect to the class of linearly-ordered $BLAL^{\ell B}$-algebras expanded with left-adjoint pairs.*

Proof. The proof straightforwardly follows from the comments above.

Example 38. Every linearly-ordered lattice, together with the implication associated with the ordering and with any adjoint pair, is a $\text{MAL}^{\ell B}$-algebra. For example, if we consider the conjunctor $\&\colon [0,1] \times [0,1] \to [0,1]$ defined, for all $x, y \in [0,1]$, as:

$$\&(x,y) = x^2 y$$

and its left adjoint implication $\to_{\&}\colon [0,1] \times [0,1] \to [0,1]$ defined, for all $y, z \in [0,1]$, as:

$$y \to_{\&} z = \min\{1, \sqrt{z/y}\}$$

then the tuple $(A, \wedge, \vee, \to, \{\&, \to_{\&}\}, 0, 1)$ is a $\text{MAL}^{\ell B}$-algebra.

As a final remark, we pick up the relationship between $\text{BLAL}^{\ell B}$ and G_Δ, Gödel logic expanded with Baaz-Monteiro's Δ, announced at the beginning of the section. We will not go into details, but it turns out in $\text{BLAL}^{\ell B}$ one can faithfully interpret Gödel implication $\to_G$ by putting

$$\varphi \to_G \psi := (\varphi \to \psi) \vee \psi,$$

where $\to$ is the implication in $\text{BLAL}^{\ell B}$. Viceversa, in G_Δ one can faithfully interpret the $\text{BLAL}^{\ell B}$-implication as

$$\varphi \to_{\text{BLAL}^{\ell B}} \psi := \Delta(\varphi \to \psi),$$

where $\to$ is the implication in G_Δ. This gives a mutual faithful interpretation between $\text{BLAL}^{\ell B}$ and G_Δ, that keeps holding when we move to the logics $\text{MAL}^{\ell B}$ and MGL_Δ respectively, the latter introduced in [8] and defined as the expansion of G_Δ with new pairs of connectives $\&_i, \to_i$ together with axioms (AxMR1^i)-(AxMR3^i).

8. Conclusions and future work

In this paper we have provided a comprehensive logic-algebraic study of multi-adjoint algebras, pushing further the state of the art by previous works in this line [11, 9, 8]. To do so, we have first defined an algebraizable logical system BLAL, and proved next that it is indeed algebraizable in the sense of Blok-Pigozzi. We have also shown that its equivalent algebraic semantics is given by a quasi-variety of algebras which are bounded lattices enriched with a suitable implication operator, that belongs to the well known family of Rasiowa implicative logics, on the one hand, and are weaker than Heyting algebras, on the other hand. Then we have introduced the logic MAL as the expansion BLAL with left-adjoint pairs while keeping the good algebraic properties. We have also shown how to minimally extend both BLAL and MAL in order to become semilinear, and thus to be complete with respect to their corresponding subclasses of lineraly-ordered algebras. Finally, we have proved that our approach is general enough to encompass a previous alternative to define a well-behaved logic for multi-adjoint algebras in [8] based on the use of Gödel fuzzy logic.

In terms of future work, there are a number of issues that need to be further investigated. For instance, we plan to study possible completeness results for BLAL and MAL with respect to the so-called *standard* semantics, i.e. semantics provided by the subclasses of BLAL- and MAL-algebras defined on the real unit interval $[0, 1]$, their first-order extensions, as well as their decidability and complexity properties. Furthermore, we are also interested in studying whether these logics admit a Kripkean possible world semantics (in the style of IPC) that may help in the development of a tableaux method associated with multi-adjoint algebras. These advances will increase the applicability of logic in current trends of Artificial Intelligence (AI) of using for example the strategy outlined in different papers, such as on decision making and query answering [7, 10]. Thus, another challenge will be to apply the obtained results to real examples, complementing other (generative) AI tools.

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Appendix: non-algebraizability of BLAL$^-$

Let BLAL$^-$ be the logic which results by removing the rule (IR) from the axiomatic system of BLAL. We want to show that BLAL$^-$ is not algebraizable. To do so, we will make use of a characterisation result by Blok and Pigozzi of the algebraizability of a logic which is based on the so-called Leibnitz operator [2, Th. 5.1]. We need to recall some preliminary notions.

If $\mathbf{A}$ is any algebra on a set A, a congruence θ on $\mathbf{A}$ is an equivalence relation on A compatible with the operators in $\mathbf{A}$, i.e. assuming the operators are binary, it satisfies that if $x_1\theta y_1$ and $x_2\theta y_2$ then $(x_1 * x_2)\theta(y_1 * y_2)$, for each operator $*$ in $\mathbf{A}$ and for all $x_1, x_2, y_1, y_2 \in A$. We say that a congruence is compatible with a subset F of A, if $a\theta b$ and $a \in F$ implies that $b \in F$, for all $a, b \in A$.

Definition 39. For any algebra $\mathbf{A}$, the Leibnitz operator $\mathbf{\Omega_A}$ is the function with domain the set of all subsets of A such that, for any subset $F \subseteq A$, $\mathbf{\Omega_A}(F)$ is the largest congruence of $\mathbf{A}$ compatible with F.

Now, let L be a logic and $\mathbf{A}$ an algebra of the same signature that is, there is a one-to-one relation between the primitive connectives of the logic L and the operators in $\mathbf{A}$. Then a subset $F \subseteq A$ is a L-filter if the following condition holds: for any set of formulas $\Gamma \cup \{\varphi\}$, if $\Gamma \vdash_{\mathrm{L}} \varphi$ then, for every $\mathbf{A}$-evaluation e, if $e(\psi) \in F$ for all $\psi \in \Gamma$, then $e(\varphi) \in F$.

The following characterisation shows that in an algebraizable logic, filters and congruences in the algebras of the equivalent semantics quasivariety are in a one-to-one relation.

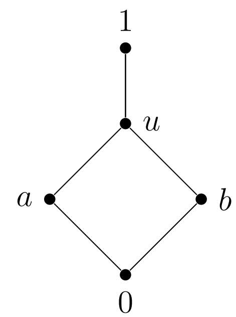

Figure 2: Bounded lattice $(A, \wedge, \vee, 0, 1)$ in the proof of Proposition 41

Theorem 40 ([2, Th. 5.1(i)]). *A logic L is algebraizable with equivalent semantics a quasivariety* $\mathbf{K}$ *iff for every algebra* $\mathbf{A} \in \mathbf{K}$ *the Leibniz operator* $\boldsymbol{\Omega}_{\mathbf{A}}$ *is an isomorphism between the lattice of L-filters of* $\mathbf{A}$ *and the lattice of* $\mathbf{K}$*-congruences of* $\mathbf{A}$*.*

Using this characterisation, we will show that the logic BLAL$^-$, resulting from the axiomatic system of BLAL by removing the rule (IR) from it, is not algebraizable.

Proposition 41. *BLAL$^-$ is not algebraizable.*

Proof. Consider the BLAL-algebra $\mathbf{A} = (A, \wedge, \vee, \to, 0, 1)$ on the domain $A = \{1, u, a, b, 0\}$ where $\wedge$ and $\vee$ are the meet and join operators corresponding to the lattice order given by $0 < a, b < u < 1$ and $\to$ is defined as: $x \to y = 1$ if $x \leq y$ and $x \to y = 0$ otherwise. See its Hasse diagram in Figure 2.

Note that the only congruences in $\mathbf{A}$ are $Id(A) = \{(x, x) \mid x \in A\}$ and $\nabla = A \times A$. Indeed, suppose θ is a congruence such that $x\theta y$ for some $x, y \in A$, with $x \neq y$. Without loss of generality, we can assume $x \not\leq y$. Then it must be $(x \to y)\theta(x \to x)$, which implies that $0\theta 1$. Hence, $(x \vee 0)\theta(x \vee 1)$ as well, that is, $x\theta 1$ for all $x \in A$. Therefore $\theta = \nabla = A \times A$.

Now, we will consider Theorem 40 to verify that BLAL$^-$ is not algebraizable. For this aim, consider the sets $F_1 = \{a, u, 1\}$ and $F_2 = \{b, u, 1\}$, and we will see that they are BLAL$^-$-filters in $\mathbf{A}$. Since any deduction from an arbitrary set of formulae Γ only involves, besides the formulas from Γ themselves, axioms of the logic and the use of Modus Ponens, the result follows from analyzing them. Since all the axioms are always evaluated to 1 by any

A-evaluation e (due to the soundness of BLAL), it is enough to check that F_1 and F_2 are closed by the MP rule, i.e. if $e(\varphi) \in F_i$ and $e(\varphi \to \psi) \in F_i$ then $e(\psi) \in F_i$, for $i = 1$ and $i = 2$. But $e(\varphi \to \psi) \in F_i$ only if $e(\varphi \to \psi) = 1$, and this holds iff $e(\varphi) \leq e(\psi)$. Hence, if $e(\varphi) \in F_i$ then $e(\psi) \in F_i$ as well, since F_i is upwards closed,for $i = 1$ and $i = 2$.

Note that the only congruence compatible with F_1 and F_2 is $Id(A)$ since ∇ is not. In fact, $(a, b) \in \nabla$ by definition, and $a \in F_1$ but $b \notin F_1$, and similarly, $b \in F_2$ but $a \notin F_2$. Therefore, $\mathbf{\Omega_A}(F_1) = \mathbf{\Omega_A}(F_2) = Id(A)$. Hence, the Leibnitz operator $\mathbf{\Omega_A}$ is not injective and, by Theorem 40, we obtain that BLAL$^-$ is not algebraizable. □