Logics of Formal Inconsistency Based on Distributive Involutive Residuated Lattices

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Abstract

The aim of this paper is to develop an algebraic and logical study of certain paraconsistent systems, from the family of the Logics of Formal Inconsistency (LFIs), that are definable from the degree-preserving companions of logics of distributive involutive residuated lattices (dIRLs) with a consistency operator, the latter including as particular cases, Nelson logic (NL), involutive monoidal t-norm based logic (IMTL) or Nilpotent Minimum logic (NM). To this end, we first algebraically study enriched distributive involutive residuated lattices with suitable consistency operators. In fact, we consider three classes of consistency operators, leading respectively to three subquasivarieties of such expanded residuated lattices. We characterise the simple and subdirectly irreducible members of these quasivarieties, and we extend Sendlewski's representation results for the case of Nelson lattices with consistency operators. Finally we define and axiomatise the logics of three quasivarieties of dIRLs and their corresponding degree-preserving companions, that belong to the family of LFIs.

Keywords. Logics of formal inconsistency; paraconsistent logics; degree-preserving logics; distributive involutive residuated lattices; Nelson lattices.

1 Introduction

The aim of this paper is to develop an algebraic and logical study of paraconsistent systems definable from the degree-preserving companions of logics of distributive involutive residuated lattices with a consistency operator. The initial motivation comes from different considerations relating paraconsistency and Nelson's Constructive logic with strong negation.

In the 1950's, Constructive logic with strong negation, nowadays commonly known under the name of *Nelson logic* (even also called **N3**), was formulated by Nelson and Markov as a result of certain philosophical objections to the intuitionistic negation pointed out by Rasiowa [46], see also her celebrated book [47, Ch. XII]. The criticism concerned its disadvantageous non-constructive property, namely that the derivability of the formula $\neg(\alpha \wedge \beta)$ in an intuitionistic propositional calculus does not imply that at least one of the formulas $\neg\alpha$, $\neg\beta$ is derivable.

Although *Nelson algebras*, the algebraic semantics of Nelson logic developed by Rasiowa [46, 47], were not originally presented as a subclass of residuated lattices, in 2008 Spinks and Veroff [50, 51] have shown that Nelson logic is indeed a substructural logic. More precisely, they show that Nelson algebras are termwise equivalent to certain involutive, bounded, commutative and integral residuated lattices, called *Nelson (residuated) lattices*. Busaniche and Cignoli [7] have further contributed to the algebraic study of Nelson lattices.

A paraconsistent version of Nelson logic was first introduced in [1], where the authors observe that the weaker system obtained from Nelson logic by deleting the axiom schema $\varphi \to (\sim\varphi \to \psi)$ could be used to reason under inconsistency without incurring in a trivial logic. Semantics for this paraconsistent version of Nelson logic have been studied by Odintsov in [41, 42], who calls the logic **N4**, in terms of what is known in the literature as *Fidel structures* [26] and *twist-structures* [25, 54]. Later on, Busaniche and Cignoli [8] provided another algebraic semantics under the umbrella of non-integral commutative residuated lattices with involution. More recently, Carnielli and Rodrigues [15] show that **N4** is in fact equivalent to a paraconsistent logic of evidence, called BLE (for Basic Logic of Evidence), for which they provide a semantics based on non truth-functional evaluations.

Our initial interest also was on paraconsistent variants of Nelson logic, but taking a different road. Indeed, our idea was to consider paraconsistent logics from the family of Logics of Formal Inconsistecy (LFIs), introduced by Carnielli and Marcos in 2000 (see e.g. [13]), and also studied e.g. by Avron [2, 3]. LFIs constitute a generalization of da Costa's C-systems [21, 22]. The main characteristic of these logics is that they internalize in the object language a notion of *consistency* by means of a specific connective $\circ$ (primitive or definable) in the following sense: although LFI's are not explosive in general, meaning that for at least a formula φ the theory $\{\varphi, \sim\varphi\}$ is consistent, the connective $\circ$ allows to recover the explosion property from a formula ψ and its negation $\sim\psi$ whenever they are retained to be *consistent*, that is to say, whenever ψ falls under the scope of $\circ$. In other words, even if $\{\varphi, \sim\varphi\}$ is not explosive, $\{\varphi, \sim\varphi, \circ\varphi\}$ trivialises because $\circ\varphi$ states that φ is consistent. It is worth noticing that, in fact, Carnielli and Rodrigues have already introduced in [15] a LFI based on paraconsistent Nelson logic **N4**, called LET_J (for Logic of Evidence and Truth), by adding a consistency-like operation to BLE, their paraconsistent and paracomplete Basic Logic of Evidence.

There is however another approach to define LFIs based on Nelson logic, and more generally, on logics of involutive residuated lattices. In general, given a quasivariety of bounded residuated lattices $\mathbb{Q}$, its corresponding logic is given by the usual (non-paraconsistent) truth-preserving notion of logical consequence $\models_{\mathbb{Q}}$, that is, a formula φ follows from a set of formulas Γ, written $\Gamma \models_{\mathbb{Q}} \varphi$, if $e(\varphi) = 1$ whenever $e(\psi) = 1$ for all $\psi \in \Gamma$ and for all evaluations e on every algebra $\mathbf{A}$ in the quasivariety $\mathbb{Q}$. A weaker notion of consequence companion of $\models$ is the one called *degree-preserving* logical consequence, where a formula φ follows from a set of formulas Γ, written $\Gamma \models_{\mathbb{Q}}^{\leq} \varphi$ if $e(\varphi) \geq a$ whenever $e(\psi) \geq a$ for all $\psi \in \Gamma$ and for all evaluations e on every algebra $\mathbf{A} \in \mathbb{Q}$ and every element $a \in A$. This weaker notion of logical consequence, firstly introduced by [55], has been further investigated in e.g. [28, 6, 27]. The point is that, as observed in [23] and unlike the truth-preserving logics, the degree-preserving companions of a large class of fuzzy logics (i.e. logics of varieties of prelinear residuated lattices), in particular those with an involutive negation, are paraconsistent. Interestingly, this is also the case of Nelson logic and, more generally, the logics of varieties of involutive residuated lattices. However, although the degree-preserving companions of these logics are paraconsistent, they are not expressive enough to define a consistency connective $\circ$ in its own language (see [23, Corollary 2 and Example 2] and Remark 3.6 below), and hence they are not LFIs. A further step was made in [20] where the authors introduce a wide class of LFIs by first expanding (non pseudo-complemented) fuzzy logics with a consistency operator while preserving the semi-linearity of the logics, and then considering their corresponding degree-preserving companions.

Given all these antecedents, our initial aim was to follow a similar approach to [20] to expand the language of Nelson logic by a primitive consistency connective $\circ$, and to add axioms and rules encoding suitable postulates in such a way that the corresponding degree-preserving companion of the logic arising in this way be a LFI. However, in doing so, we realised that, essentially, all the definitions and results we got for Nelson logic and lattices, as underlying logical and algebraic framework, remain valid for the more general framework of involutive, distributive (bounded, commutative and integral) residuated lattices and their logics. Therefore, in this paper we have finally chosen to present our algebraic and logical results in this more general setting and only particularise to Nelson logic when necessary.

The organization of this paper is as follows. In Section 2, the basic notions about varieties of involutive residuated lattices and their logics are recalled. Section 3 contains the main algebraic results and it is divided into three subsections, in each of which we will introduce a specific type of consistency operator and we will present results concerning the so arising quasivarieties. In Section 4 we consider the particular case of Nelson lattices with a consistency operator, for which we prove structural results and their relation to Heyting algebras with dual pseudocomplement. In particular, in Subsection 4.2 we also present additional results on subdirectly irreducible and simple Nelson lattices with a consistency operator. Section 5 considers the logical counterparts of the classes of algebras studied in Section 3 and their associated Logics of Formal

Inconsistency over the degree-preserving companions of the former. In the last subsection of Section 5, we will point out that, besides being LFIs, our logics are *Logics of Formal Undeterminedness*[1] (LFUs) as well, and we analyze the relationships between these two notions. We conclude in Section 6 collecting some final remarks and our future work on the topics of this paper.

2 Preliminary notions

Recall that a *commutative, integral, bounded residuated lattice*, that we will simply call *residuated lattice*, is an algebra $\mathbf{A} = \langle A, \wedge, \vee, *, \to, 0, 1\rangle$ of type $(2, 2, 2, 2, 0, 0)$ such that $\langle A, *, 1\rangle$ is a commutative monoid, $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with least element 0 and greatest element 1, and such that the following residuation condition holds: $x * y \leq z$ iff $x \leq y \to z$, where x, y, z denote arbitrary elements of A and $\leq$ is the order given by the lattice structure. Since we assume the neutral element of the monoid reduct coincides with the greatest element of its lattice reduct, we have that: $x \leq y$ iff $x \to y = 1$.

It is well-known that the class $\mathbb{RL}$ of residuated lattices forms a variety, which is related to different and well-known varieties studied in substructural and fuzzy logics literature. In particular, $\mathbb{RL}$ coincides with the variety of FL_{ew}-algebras of [29]. According to the denotational conventions of [29], FL refers to the "Full Lambek calculus", which is the base system and associated algebras, and subindices indicate several axiomatic extensions with properties such as exchange (e) or weakening (w).

A residuated lattice is called *distributive* if its lattice reduct is a distributive lattice, that is to say, if it satisfies the distributivity equations:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{Dist}$$

The variety of distributive residuated lattices will be denoted by $d\mathbb{RL}$

A residuated lattice is called *involutive* if it satisfies the double negation equation:

$$\sim\sim x = x, \tag{Inv}$$

where $\sim x$ is $x \to 0$. In every involutive residuated lattice, it is possible to prove that $x * y = \sim(x \to \sim y)$ and $x \to y = \sim(x * \sim y)$. Clearly involutive residuated lattices form a variety that will be denoted by $\mathbb{IRL}$.[2]

The variety of distributive and involutive residuated lattices (dIRL-algebras for short) will be henceforth denoted by $d\mathbb{IRL}$.

In this paper we will also consider some proper subvarieties of $d\mathbb{IRL}$, in particular:

[1] A logic of Formal Undeterminedness is a paracomplete logic with a unary *determinedness* operator that controls the law of the Excluded Middle, it is a sort of dual notion of a LFI [38].

[2] In the literature, involutive residuated lattices have been called *Involutive* FL_{ew}*-algebras* (IFL_{ew}-algebras), see e.g. [33], and they are also known as the algebras of the *affine Linear Logic without exponentials*, see e.g. [5]. In this paper we shall adopt the notation IRL-algebras to denote them without danger of confusion.

- The variety $\mathbb{NL}$ of *Nelson residuated lattices*, defined within $\mathbb{IRL}$ by the so called Nelson equation:

$$(((x * x) \to y) \wedge ((\sim y * \sim y) \to \sim x)) \to (x \to y) = 1. \quad \text{(Nel)}$$

- The variety $\mathbb{IMTL}$ of *involutive monoidal t-norm based algebras* (or IMTL-algebras for short), which can be defined as the proper subvariety of $\mathbb{IRL}$ (and of $d\mathbb{IRL}$) of those algebras satisfying the *prelinearity equation*:

$$(x \to y) \vee (y \to x) = 1. \quad \text{(Prel)}$$

- The variety $\mathbb{NM}$ of *nilpotent minimum* algebras (or NM-algebras for short), which is identified as proper subvariety of $\mathbb{IMTL}$ by the following equation:

$$(x * y \to 0) \vee (x \wedge y \to x * y) = 1, \quad \text{(NM)}$$

  or equivalently, as shown in [7], the subvariety of $\mathbb{NL}$ of those algebras satisfying the prelinearity equation.

Notice that Nelson, IMTL and NM-algebras can be axiomatized directly within $\mathbb{IRL}$ without explicitly requiring the distributivity equations to hold.

In Figure 1 we represent the graph of the above considered subvarieties of $\mathbb{RL}$ together with their characteristic axioms.

2.1 Truth-preserving and degree-preserving logics of residuated lattices

The substructural logic which is complete with respect to the variety of (bounded, commutative, integral) residuated lattices is the so-called Full Lambek calculus extended with exchange (commutativity) and weakening (integrality), FL_{ew}, see e.g. [29].[3] The language of FL_{ew} consists of denumerably many propositional variables $p_1, p_2, \cdots$, binary connectives $\wedge, \vee, \&, \to$, and the truth constant $\bot$. Formulas, which will be denoted by lower case greek letters $\phi, \psi, \cdots$, are defined by induction as usual. Further connectives and constants are definable; in particular, $\neg\phi$ stands for $\phi \to \bot$, $\top$ stands for $\neg\bot$, and $\psi \leftrightarrow \phi$ stands for $(\psi \to \phi) \wedge (\phi \to \psi)$. A Hilbert-style calculus for FL_{ew} has the following set of axioms:

Ax1. $(\phi \to \psi) \to ((\psi \to \gamma) \to (\phi \to \gamma))$,

Ax2. $(\gamma \to \phi) \to ((\gamma \to \psi) \to (\gamma \to (\phi \wedge \psi)))$,

Ax3. $(\psi \wedge \phi) \to \psi$, and Ax4. $(\psi \wedge \phi) \to \phi$,

Ax5. $\psi \to (\psi \vee \phi)$, and Ax6. $\phi \to (\psi \vee \phi)$,

[3] An equivalent system, called *Monoidal logic*, was previously defined and studied by Höhle in [31].

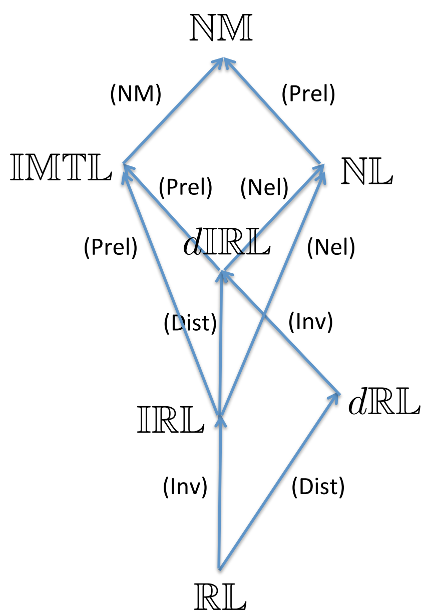


Figure 1: Diagram of main varieties of algebras in this paper and their relationships.

Ax7. $(\psi \to \gamma) \to ((\phi \to \gamma) \to ((\psi \vee \phi) \to \gamma))$,

Ax8. $(\psi \& \phi) \to (\phi \& \psi)$,

Ax9. $(\psi \& \phi) \to \psi$,

Ax10. $(\psi \to (\phi \to \gamma)) \to ((\psi \& \phi) \to \gamma)$,

Ax11. $((\psi \& \phi) \to (\psi \to (\phi \to \gamma))$,

Ax12. $\bot \to \psi$, and Ax13. $\psi \to \top$.

The only inference rule of FL_{ew} is modus ponens:

(MP) $\dfrac{\psi, \psi \to \phi}{\phi}$

The logic IRL of involutive residuated lattices is the axiomatic extension of FL_{ew} with the double negation axiom

Ax14. $\neg\neg\phi \to \phi$,

and the logic dIRL of distributive and involutive residuated lattices is the axiomatic extension of IRL with the following axiom

Ax15. $\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$.

This paper will be mainly concerned with the logic dIRL and some of its axiomatic extensions. Hence, distributivity will always be assumed to hold. In particular we will consider *Nelson logic* NL obtained by extending dIRL by the Nelson axiom:

Ax16. $(((\psi \& \psi) \to \phi) \wedge ((\neg\phi \& \neg\phi) \to \neg\psi)) \to (\psi \to \phi)$.

The logic IMTL introduced in [24] is the axiomatic extension of the logic MTL (the logic of the variety of prelinear residuated lattices) with the axiom (Ax14) or, equivalently, obtained by extending IRL by the prelinearity axiom:

Ax17. $(\phi \to \psi) \vee (\phi \to \psi)$.

Finally, the *nilpotent minimum* logic NM, introduced also in [24], is the axiomatic extension of IMTL by the axiom

Ax18. $(\phi \& \psi \to \bot) \vee (\phi \wedge \psi \to \phi \& \psi)$.

Equivalently, NM can be obtained as the axiomatic extension of Nelson logic NL by the prelinearity axiom.

We will denote by $\vdash_{\mathsf{L}}$ the notion of proof for each logic $\mathsf{L} \in \{\mathsf{IRL}, \mathsf{dIRL}, \mathsf{IMTL}, \mathsf{NL}, \mathsf{NM}\}$ defined as usual from the corresponding sets of axioms described above and the inference rule of Modus Ponens (MP).

Each of these logics L is algebraizable (as all of them are axiomatiac extensions of FL_{ew}), and thus it has an equivalent algebraic semantics given by the corresponding variety of L-algebras introduced before, and which brings the same name. This means that the truth-preserving (finitary) consequence relation $\models_{\mathsf{L}}$ induced by the variety of L-algebras, defined as:

$$\Gamma \models_{\mathsf{L}} \varphi \quad \text{iff} \quad \begin{array}{l}\text{for every L-algebra } \boldsymbol{A} \text{ and every } \boldsymbol{A}\text{-evaluation } e,\\ \text{if } e(\psi) = 1 \text{ for every } \psi \in \Gamma, \text{ then } e(\varphi) = 1,\end{array}$$

is such that $\vdash_{\mathsf{L}}$ is sound and complete w.r.t. $\models_{\mathsf{L}}$.

Moreover, for each such a logic L, in [28, 6] the authors introduce a companion logic denoted $\mathsf{L}^{\leq}$, whose associated consequence relation, that will be denoted as $\models_{\mathsf{L}}^{\leq}$, has the following semantics: for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_{\mathsf{L}}^{\leq} \varphi \quad \text{iff} \quad \begin{array}{l}\text{for every } \mathsf{L}\text{-algebra } \boldsymbol{A}, \text{ every } a \in A, \text{ and every } \boldsymbol{A}\text{-evaluation } e,\\ \text{if } a \leq e(\psi) \text{ for every } \psi \in \Gamma, \text{ then } a \leq e(\varphi).\end{array}$$

By obvious reasons, $\mathsf{L}^{\leq}$ is known as the companion logic of L *preserving degrees of truth*, or the *degree-preserving companion* of L. It is not difficult to show that L and $\mathsf{L}^{\leq}$ have the same valid formulas (i.e. $\models_{\mathsf{L}} \varphi$ iff $\models_{\mathsf{L}}^{\leq} \varphi$), and that, for every finite set of formulas $\Gamma \cup \{\varphi\}$, the following property holds:

$$\Gamma \models_{\mathsf{L}}^{\leq} \varphi \text{ iff } \models_{\mathsf{L}} \Gamma^{\wedge} \to \varphi,$$

where $\Gamma^{\wedge}$ means $\gamma_1 \wedge \ldots \wedge \gamma_k$ if $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ (when Γ is empty then $\Gamma^{\wedge}$ is taken as $\top$).

As it regards to axiomatization, if L is an axiomatic extension of FL_{ew}, then the logic $\mathsf{L}^{\leq}$ admits a Hilbert-style axiomatization having the same axioms as L and the following deduction rules [6]:

(Adj-$\wedge$) from φ and ψ derive $\varphi \wedge \psi$

(MP-r) if $\vdash_{\mathsf{L}} \varphi \to \psi$, then from φ and $\varphi \to \psi$, derive ψ

Note that (MP-r) is a *restricted form* of the Modus Ponens rule, as it is only applicable when $\varphi \to \psi$ is a theorem of L. Thus, although L and $\mathsf{L}^{\leq}$ share the same theorems, $\mathsf{L}^{\leq}$ is in fact a weaker logic than L.

If the set of theorems of L is decidable, then the above systems of axioms and rules provides a recursive Hilbert-style axiomatization of $\mathsf{L}^{\leq}$.

In the more general case of L being not an axiomatic extension but a finitary Rasiowa-implicative expansion of FL_{ew} (i.e. L may have new inference rules and hence its algebraic semantics may be a sub-quasivariety of $\mathbb{RL}$), the definition of the degree-preserving companion $\mathsf{L}^{\leq}$ keeps being the same as above. However, the axiomatisation needs to be tuned. Assume the new inference rules of L are:

($\mathbf{R}_i$) from Γ_i derive φ_i,

for $i \in I$. Then, following the same idea of [6, Th. 2.12], one can show the following generalised result about the axiomatisation of $\mathsf{L}^{\leq}$.

Proposition 2.1. *Let L be an expansion of FL_{ew}, with the above set of new inference rules $\{(R_i)\}_{i \in I}$. Then $\mathsf{L}^{\leq}$ is axiomatized by the axioms of L, the inference rules (Adj-$\wedge$) and (MP-r), and the following restricted inference rules:*

($\mathbf{R}_i$-r) *If $\vdash_{\mathsf{L}} \Gamma_i$, then from Γ_i derive φ_i*

for each $i \in I$.

The proof is completely analogous to the one in [6] in the context of expansions of MTL and it is omitted.

3 Distributive involutive residuated lattices expanded by a consistency operator

Paraconsistency is the study of logics having a negation operator $\sim$ that are *not explosive* with respect to that negation; that is to say, logics for which there exists at least a formula ϕ such that the theory $\{\phi, {\sim}\phi\}$ does not entail any other formula. Therefore, a paraconsistent logic is a logic having at least a contradictory but non-trivial theory.

Among the plethora of paraconsistent logics proposed in the literature, the Logics of Formal Inconsistency (LFIs) (see, for instance, [13, 12]), play an important role, since they internalize in the object language the very notion of *consistency*[4] by means of a specific connective, primitive or not. This generalizes the strategy of da Costa, who introduced in [22] the well-known hierarchy of systems C_n , for $n > 0$. Briefly said, LFIs have a non-explosive negation $\sim$, as well as a (primitive or derived) *consistency connective* $\circ$ which allows to recover the explosion law in a controlled way.

Let Σ be a propositional signature which contains a negation $\sim$ and a primitive or defined unary connective $\circ$, let $\mathcal{V} = \{p_1, p_2, \ldots\}$ be a denumerable set of propositional variables, and let $\mathsf{L} = \langle \Sigma, \vdash \rangle$ be a Tarskian, finitary and structural logic defined over Σ and $\mathcal{V}$. Then, according to e.g. [12], we have the following definition.

Definition 3.1. L is said to be a *Logic of Formal Inconsistency* with respect to $\sim$ and $\circ$ if the following holds:

(i) $\varphi, \sim\varphi \nvdash \psi$ for some φ and ψ;

(ii) there are two formulas α and β such that

  (ii.a) $\circ\alpha, \alpha \nvdash \beta$;

  (ii.b) $\circ\alpha, \sim\alpha \nvdash \beta$;

(iii) $\circ\varphi, \varphi, \sim\varphi \vdash \psi$ for every φ and ψ.

Moreover, in [12] the authors also consider the following stronger notion of LFIs.

Definition 3.2. L is said to be a *strong Logic of Formal Inconsistency* with respect to $\sim$ and $\circ$ if the following holds:

(i) if p and q are two different propositional variables then

  (i.a) $p, \sim p \nvdash q$

  (i.b) $\circ p, p \nvdash q$

  (i.c) $\circ p, \sim p \nvdash q$

(ii) $\circ\varphi, \varphi, \sim\varphi \vdash \psi$ for every φ and ψ.

Our aim is to consider different possibilities of defining (strong) LFIs over the degree-preserving logic companion of the logic dIRL. To this end, we will first study suitable expansions of dIRL-algebras by a new consistency operator $\circ$ in such a way that their degree-preserving companions are LFIs. Actually, we will consider three different axiomatic definitions of varying strength. Similar ideas of having LFIs over a degree-preserving logic companion of other class of algebras can be seen in [20] and [11].

Let $\mathbf{A}$ be an involutive residuated lattice and let $\circ : A \to A$ be a unary operation on $\mathbf{A}$. With the above goal in mind, we consider different properties we may ask to the $\circ$ operation in order to be a suitable consistency operator.

[4] A formula ϕ is named consistent in a paraconsistent logic when $\{\phi, \sim\phi\}$ is an explosive theory.

It is clear that the minimal properties to require to $\circ$ to be a consistency operator are:

($\circ$0) $\circ(1) = \circ(0) = 1$

($\circ$1) $x \wedge {\sim}x \wedge \circ(x) = 0$

However, these properties turn out to be a weak specification in the sense that, for a given distributive involutive residuated lattice $\mathbf{A}$, one can define many operations satisfying the above sets of properties, in particular, one can always define a minimal operation by letting $\circ(x) = 0$ for all $x \in A \setminus \{0, 1\}$. A natural way out is to require that $\circ$ provides the maximum value in A such that ($\circ$1) is satisfied and hence we can define $\circ(x)$ to be the $\max\{z \in A \mid x \wedge {\sim}x \wedge z = 0\}$. In a sense, such an operator, if it exists, can be considered as the least committed one satisfying ($\circ$1). This is formally achieved by considering the following additional requirement:

($\circ$2) if $x \wedge {\sim}x \wedge y = 0$ then $y \leq \circ(x)$

Since Boolean elements are the prototypical examples of consistent and explosive elements, another reasonable property one can further require to $\circ$ is to be a *Boolean operator*, that is to say, that for each $x \in A$ to require $\circ(x) \vee {\sim}\circ(x) = 1$. This can be achieved in at least two ways.

A first possibility is to define $\circ(x)$ as the maximum, among the set $B(\mathbf{A}) = \{x \in A \mid x \wedge {\sim}x = 0\}$ of Boolean elements of $\mathbf{A}$, satisfying the condition ($\circ$1) above. In other words, to take $\circ(x) = \max\{z \in B(\mathbf{A}) \mid x \wedge {\sim}x \wedge z = 0\}$. This amounts to replace ($\circ$2) by the following two new conditions:

($\circ$3) if $x \wedge {\sim}x \wedge y = 0$ and $y \wedge {\sim}y = 0$ then $y \leq \circ(x)$

($\circ$4) $\circ(x) \vee {\sim}\circ(x) = 1$

A second possibility is to consider an operator $\circ$ such that $\circ(x)$ is, at the same time, both the $\max\{z \in A \mid x \wedge {\sim}x \wedge z = 0\}$ and a Boolean element. In fact this latter requirement differs from $\max\{z \in B(\mathbf{A}) \mid x \wedge {\sim}x \wedge z = 0\}$. This is achieved by asking $\circ$ to satisfy the above conditions ($\circ$0), ($\circ$1), ($\circ$2) and ($\circ$4).

Actually, in this paper, we will study expansions of involutive residuated lattices with these three types of consistency operators $\circ$. Namely, besides satisfying ($\circ$0) and ($\circ$1), we will consider operators:

(i) additionally satisfying ($\circ$2), that we will call *maximal consistency operators* (max-consistency operators for short), in Subsection 3.1;

(ii) additionally satisfying ($\circ$3) and ($\circ$4), that we will call *maximal Boolean consistency operators* (maxB-consistency operators for short), in Subsection 3.2; and

(iii) additionally satisfying ($\circ$2) and ($\circ$4), that we will call *Boolean and maximal consistency operators* (Bmax-consistency operators, for short), in Subsection 3.3.

As one can already anticipate, Bmax-consistency operators are those operators that are both max- and maxB-consistency operators. We will formally show this in Subsection 3.3.

The following observations on congruences, filters and subdirect product decompositions will be useful in the rest of the paper.

As is well-known, an *implicative filter* of a (bounded) residuated lattice $\mathbf{A}$ is a subset $F \subseteq A$ such that $1 \in F$ and it is closed under modus ponens: $x \in F$ and $x \to y \in F$ imply $y \in F$.[5] For each implicative filter F, the binary relation $\theta(F)$ defined by $(x, y) \in \theta(F)$ if and only if $x \to y, y \to x \in F$ is a congruence of the residuated lattice $\mathbf{A}$, and $F = \{z \in A : (z, 1) \in \theta(F)\}$. This is actually a one-one correspondence between the lattice of congruences and the lattice of implicative filters for the variety of bounded residuated lattices.

Moreover, since the classes of expansions of distributive involutive residuated lattices with the above types of consistency operators involve not only equations but also quasi-equations, we will also deal with quasivarieties. In a quasivariety, congruences that allow for the decomposition of an algebra as a sudirect product of subdirectly irreducible components are required to satisfy an additional condition: the quotient of an algebra by a congruence has to belong to the quasivariety, see e.g. [45]. This condition is automatically satisfied in varieties but not in quasivarieties. Congruences satisfying this condition are usually called *Q-congruences*. Similarly, filters that are in a one-one correspondence between Q-congruences are implicative filters 'closed' by the quasiequations of the quasivariety, and are called *Q-filters*.

3.1 Distributive involutive residuated lattices with max-consistency operators

We start by formally defining the first class of distributive involutive residuated lattices with a consistency operator, namely with a max-consistency operator.

Definition 3.3. A *distributive involutive residuated lattice with a* max-*consistency operator* (an dIRL$_c$*-algebra* for short) is a pair $(\mathbf{A}, \circ)$ where $\mathbf{A}$ is a distributive involutive residuated lattice and $\circ : A \to A$ satisfies the following two conditions: for all $x, y, z \in A$,

($\circ$1) $x \wedge \sim x \wedge \circ(x) = 0$

($\circ$2) if $x \wedge \sim x \wedge y = 0$ then $y \leq \circ(x)$

From the definition, it is clear that the class of dIRL$_c$-algebras is a quasivariety.

It is easy to check that conditions ($\circ$1) and ($\circ$2) faithfully capture the expected behavior max-consistency operators as described above.

[5] Recall that an implicative filter of a residuated lattice $\mathbf{A}$ can be equivalently defined as a subset $F \subseteq A$ such that $1 \in F$ and is closed by the monoidal operation $*$ of the residuated lattice.

Lemma 3.4. *Let* **A** *be a* dIRL*-algebra, and let* $\circ$ *be a unary operation on* A*. Then* $(\mathbf{A}, \circ)$ *is a* dIRL_c*-algebra iff, for any* $x \in A$*,* $\circ(x) = \max\{z \in A \mid x \wedge {\sim}x \wedge z = 0\}$*.*

Proof. Clearly, for a given $x \in A$, the set $\{z \in A \mid x \wedge {\sim}x \wedge z = 0\}$ is closed by $\vee$, it has $\circ(x)$ as an upper bound by ($\circ$2), and moreover $\circ(x)$ belongs to that set by ($\circ$1). □

Conditions ($\circ$1) and ($\circ$2) are also enough to ensure that condition ($\circ$0) also holds in a dIRL_c-algebra. This and other easy properties of dIRL_c-algebras are displayed in the next lemma.

Lemma 3.5. *The following properties hold in a* dIRL_c*-algebra* $(\mathbf{A}, \circ)$*:*

(i) $\circ(x) = \circ({\sim}x) = \circ(x \wedge {\sim}x) = \circ(x \vee {\sim}x)$

(ii) $\circ(x) = 1$ *iff* x *is Boolean, in particular* $\circ(1) = \circ(0) = 1$

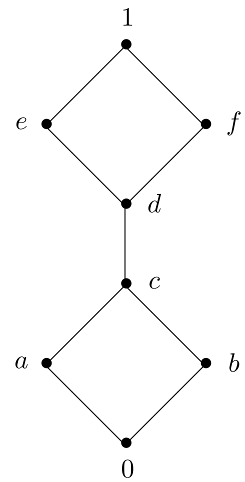
Proof. (i) follows from Lemma 3.4 just noticing that ${\sim}{\sim}x = x$, and $\circ(1) = 1$ follows from ($\circ$2) by taking $x = y = 1$. (ii) also follows from Lemma 3.4 by noticing that $x \wedge {\sim}x = 0$ for any Boolean element x. □

As an example of a finite dIRL_c-algebra, let $L = \{0, a, b, c, d, e, f, 1\}$ and consider the lattice $(L, \wedge, \vee)$ represented in the upper part of Fig. 2. Then the algebra $\mathbf{L} = (L, \wedge, \vee, *, \to, \sim, 0, 1)$, where $*$ and $\sim$ are those specified in the tables of Fig. 2 and where $x \to y = {\sim}(x * {\sim}y)$, is a distributive involutive residuated lattice (it is indeed a finite *Nilpotent Minimum* algebra) and $(\mathbf{L}, \circ)$ is a dIRL_c-algebra.

Remark 3.6. It follows from Lemma 3.4 that if a max-consistency operator is definable in an involutive residuated lattice it is uniquely determined. Moreover, it also follows that the max-consistency operator is always definable in every finite involutive residuated lattice. However it is not always the case in infinite involutive residuated lattices. Actually, also according to Lemma 3.4 above, such an operator is definable if and only if all elements of the involtive residuated lattice of the form $x \wedge {\sim}x$ admit a pseudo-complement. The following are examples of infinite distributive involutive residuated lattices in which the consistency operator cannot be defined.

(1) Let **A** be the an algebra over $[0, 1] \setminus \{\frac{1}{2}\}$ with the Nilpotent Minimum operations: $x * y = \min(x, y)$ if $x + y > 1$ and $x * y = 0$ otherwise; and ${\sim}x = 1 - x$. Let $A^+ = \{x \in A \mid x > 1/2\}$ and $A^- = \{x \in A \mid x < 1/2\}$ be the sets of positive and negative elements of **A** respectively. Further, let **B** be the Nilpotent Minimum subalgebra of $\mathbf{A} \times \mathbf{A}$ defined on the sublattice $B = (A^+ \times A^+) \cup (A^- \times A^-)$. Take an element $(x, 1) \in B$ such that $x \in A^+$. An easy computation shows that $\circ((x, 1))$ does not exist.

(2) Let **A** be the 1-generated free MV-algebra [18], i.e., up to isomorphism, the algebra of continuous piecewise linear functions with integer coefficients form



$*$	0	a	b	c	d	e	f	1	$\sim x$	$\circ x$
0	0	0	0	0	0	0	0	0	1	1
a	0	0	0	0	0	a	0	a	f	b
b	0	0	0	0	0	0	b	b	e	a
c	0	0	0	0	0	a	e	c	d	0
d	0	0	0	0	d	e	d	d	c	0
e	0	a	0	a	e	e	d	e	b	a
f	0	0	b	e	d	d	a	f	a	b
1	0	a	b	c	d	e	f	1	1	1

Figure 2: A finite lattice of eight elements together with three operations $*, \sim$ and $\circ$.

$[0,1]$ to $[0,1]$ and with operations defined as follows: for all $f, g \in A$ and for all $x \in [0,1]$,

$$(f \oplus g)(x) = \min\{1, f(x) + g(x)\} \text{ and } \sim f(x) = 1 - f(x).$$

Consider the function $f : [0,1] \to [0,1]$ defined by the stipulation $f(x) = 0$ for all $x \in [0,1/3) \cup (2/3,1]$, while $f(x) = \min\{3x-1, -3x+2\}$ on $[1/3,2/3]$. A direct computation shows that $\sim f(x) = 1$ for all $x \in [0,1/3) \cup (2/3,1]$ and $f(x) = \max\{3x-1, -3x+2\}$ on $[1/3,2/3]$ so that $f \leq \sim f$ and hence $f \wedge \sim f = f$. Therefore the set $\{g \in A \mid f \wedge \sim f \wedge g = 0\} = \{g \in A \mid f \wedge g = 0\}$. However, notice that $\sup\{g \in A \mid f \wedge g = 0\}$ is a function $h : [0,1] \to [0,1]$ such that $h(x) = 0$ for all $x \in [0,1/3) \cup (2/3,1]$ and $h(x) = 1$ for all $x \in [1/3,2/3]$ and hence it is not continuous and hence it does not belong to A. As a consequence $\circ(f)$ is not definable.

Taking into account the observation above on filters in quasivarieties, a Q-filter F of a dIRL$_c$-algebra $(\mathbf{A}, \circ)$, besides being implicative, has to additionally satisfy the following two conditions:

(F1) if $x \to y, y \to x \in F$ then $\circ x \to \circ y, \circ y \to \circ x \in F$,

(F2) if $x \vee \sim x \vee \sim y \in F$ then $y \to \circ x \in F$.

We shall call such a filter a $\circ$-*filter*. Note that from (F1) it follows in particular that $\circ$-filters are closed by $\circ$: if $x \in F$ then $\circ x \in F$ as well. Since in every dIRL$_c$-algebra $(\mathbf{A}, \circ)$, $\circ$-filters bijectively correspond to Q-congruences by the maps

$$F \mapsto \Theta_F = \{(a,b) \in A \times A \mid (a \to b) \wedge (b \to a) \in F\}$$

and

$$\Theta \mapsto F_\Theta = \{a \in A \mid a\Theta 1\},$$

we will henceforth say that an dIRL$_c$-algebra $(\mathbf{A}, \circ)$ is *simple* if it only has two Q-congruences, and hence if it only has two (trivial) $\circ$-filters: $\{1\}$ and A. Furthermore, a quasivariety of dIRL$_c$-algebras will be said to be *semisimple* provided that all its subdirectly irreducible elements are simple in the above sense.

Lemma 3.7. *Let* $\mathbf{A}$ *be a subdirectly irreducible distributive involutive residuated lattice and define* $\circ : A \to A$ *as* $\circ(1) = \circ(0) = 1$ *and* $\circ(x) = 0$ *otherwise. Then* $(\mathbf{A}, \circ)$ *is a simple* dIRL$_c$*-algebra.*

Proof. Let us start showing that $\circ$ defined as in the statement is a max-consistency operator. First of all it is immediate to see that ($\circ$1) holds, then let us hence show ($\circ$2). To this end assume that $x \wedge {\sim}x \wedge y = 0$, or equivalently that ${\sim}(x \wedge {\sim}x \wedge y) = {\sim}x \vee x \vee {\sim}y = 1$. Since $\mathbf{A}$ is subdirectly irreducible, by [44, Theorem 4.1] one has that $({\sim}x \vee x) \vee {\sim}y = 1$ if either $x \vee {\sim}x = 1$, or ${\sim}y = 1$. If the former is the case, then x is Boolean. Since $\mathbf{A}$ is subdirectly irreducible, it is hence directly indecomposable. Therefore, [36, Proposition 1.5] implies $x \in \{0, 1\}$. Then, by definition $\circ(x) = 1$, whence necessarily $\circ(x) \geq y$. Conversely, if ${\sim}y = 1$, then $y = 0 \leq \circ(x)$.

Finally assume, without loss of generality, that A contains at least an element x distinct from 0 and 1, for otherwise $\mathbf{A}$ would be the two element Boolean algebra and $(\mathbf{A}, \circ)$ would be obviously simple. Furthermore, let $F \neq \{1\}$ be a $\circ$-filter of $(\mathbf{A}, \circ)$. Then F must contain x and since F is closed under $\circ$, by definition of $\circ$, $\circ(x) = 0 \in F$. Thus, $F = A$ and $(\mathbf{A}, \circ)$ is simple. □

The next result provides a generalization to the case of dIRL$_c$-algebras of the well-known result showing that a residuated lattice $\mathbf{A}$ is directly indecomposable iff $B(\mathbf{A}) = \{0, 1\}$, see [36, Proposition 1.5].

Theorem 3.8. *Let* $(\mathbf{A}, \circ)$ *be a* dIRL$_c$*-algebra. Then* $(\mathbf{A}, \circ)$ *is directly indecomposable iff* $B(\mathbf{A}) = \{0, 1\}$.

Proof. The right-to-left direction is clear. Indeed, if $(\mathbf{A}, \circ)$ were product of two (or more) dIRL$_c$-algebras, say $(\mathbf{A}_1, \circ)$ and $(\mathbf{A}_2, \circ)$, denoting by $1_1, 0_1$ the top and the bottom elements of the first and by $1_2, 0_2$ the top and the bottom elements of the second, then $(1_1, 1_2)$, $(0_1, 0_2)$, $(1_1, 0_2)$ and $(0_1, 1_2)$ would be four distinct Boolean elements of $\mathbf{A}$ contradicting the fact that $B(\mathbf{A}) = \{0, 1\}$.

Hence, let us prove the left-to-right direction. First of all observe that if $z \in B(\mathbf{A})$, for all $x \in A$ $x \wedge z = x * z$. The proof is again by reduction ad absurdum. Assume there is $z \in B(\mathbf{A}) \setminus \{0, 1\}$, and let us see that $[z)$ is a $\circ$-filter. It is clear that it is an implicative filter. Let us check that $[z)$ satisfies the conditions of a $\circ$-filter:

- First we prove that $z \leq (x \to y), (y \to x)$ implies $z \leq (\circ x \to \circ y), (\circ y \to \circ x)$.

  Since $z \leq (x \to y)$ we have $x \wedge z = x * z \leq y$ since z is Boolean. Hence ${\sim}y \leq {\sim}(x \wedge z) = {\sim}x \vee {\sim}z$. Also, since $z \leq y \to x$, we have $y \leq z \to x$. Then we have the following inequalities:

  $y \wedge {\sim}y \wedge (z \wedge \circ x) \leq y \wedge ({\sim}x \vee {\sim}z) \wedge z \wedge \circ x =$
  $(y \wedge {\sim}x \wedge z \wedge \circ x) \vee (y \wedge {\sim}z \wedge z \wedge \circ x) \leq$
  $((z \to x) \wedge {\sim}x \wedge z \wedge \circ x) \vee 0 = ((z * (z \to x)) \wedge {\sim}x \wedge \circ x) \leq x \wedge {\sim}x \wedge \circ x = 0$

  Therefore, we have $z \wedge \circ x \leq \circ y$, that is, $z \leq \circ x \to \circ y$. Analogously, one can prove $z \leq \circ y \to \circ x$.

- Second, we prove that $z \leq x \vee {\sim}x \vee {\sim}y$ implies $z \leq y \to \circ x$.

  Note that $x \vee {\sim}x \vee {\sim}y = {\sim}(x \wedge {\sim}x \wedge y)$. Then we have:

  $x \wedge {\sim}x \wedge (z \wedge y) = (x \wedge {\sim}x \wedge y) \wedge z = (x \wedge {\sim}x \wedge y) * z \leq$
  $(x \wedge {\sim}x \wedge y) * {\sim}(x \wedge {\sim}x \wedge y) = 0.$

  Therefore, $z * y = z \wedge y \leq \circ x$, hence, $z \leq y \to \circ x$.

Analogously we can prove that $[{\sim}z)$ is a $\circ$-filter. Then if $B(\mathbf{A}) \setminus \{0, 1\}$ is non-empty, there exists $z \in B(\mathbf{A}) \setminus \{0, 1\}$ and the $\circ$-filters $[z)$ and $[{\sim}z)$ are such that $[z) \cap [{\sim}z) = \{1\}$, while the filter generated by $[z) \cup [{\sim}z)$ contains z, contains ${\sim}z$ and hence it contains $z \wedge {\sim}z = 0$. Thus it coincides with the entire A. Clearly their associated Q-congruences, say Θ_z and $\Theta_{\sim z}$, permute: $\Theta_{\sim z}(\Theta_z) = \Theta_z(\Theta_{\sim z})$. Indeed $(a, b) \in \Theta_{\sim z}(\Theta_z)$ iff there exists $c \in A$ such that $(a \to c) \wedge (c \to a) \geq z$ and $(c \to b) \wedge (b \to c) \geq {\sim}z$ iff, by the commutativity of $\wedge$ and the reflexive property of congruences, $(b, a) = (a, b) \in \Theta_z(\Theta_{\sim z})$. Thus, they form a nontrivial pair of complementary factor congruences and $(\mathbf{A}, \circ)$ is not directly indecomposable. $\square$

Thus, since every subdirectly irreducible algebra is also directly indecomposable, we also have the following result.

Corollary 3.9. *Let* $(\mathbf{A}, \circ)$ *be a subdirectly irreducible* dIRL$_c$*-algebra. Then* $B(\mathbf{A}) = \{0, 1\}$.

When the algebra $\mathbf{A}$ is finite or is a *connected or disconnected rotation* of a residuated lattice, we can prove more. Connected and disconnected rotations of residuated lattices were studied by Jenei in [32]. The paper [10] (see also [53]) studies varieties of algebras obtained as connected or disconnected rotations of residuated lattices from a purely algebraic perspective. There, the authors introduce a variety of algebras, denoted by $\mathbb{IMVR}3$, whose directly indecomposable elements are obtained as *generalized 3-rotations* (in the terminology of [10]) of residuated lattices. Within directly indecomposable IMVR3-algebras we can hence identify structures with a negation fixpoint (corresponding to Jenei's connected rotations) and structures without a negation fixpoint (corresponding to Jenei's disconnected rotations). As for distributivity, in [10, Remark 3.9] it is observed that any IMVR3-algebra obtained as a generalized 3-rotation of a residuated lattice $\mathbf{R}$ is distributive iff so is $\mathbf{R}$. We will henceforth say that an algebra is a dIMVR3-*algebra* if it is a distributive IMVR3-algebra and the corresponding variety will be denoted by $d\mathbb{IMVR}3$. Thus, every algebra of $d\mathbb{IMVR}3$ is a commutative, integral, bounded, involutive and distributive residuated lattice, i.e., $d\mathbb{IMVR}3$ as a subvariety of $d\mathbb{IRL}$.

By [10, Theorem 4.6], the domain of every directly indecomposable IMVR3-algebra $\mathbf{A}$ can be partitioned in two sets,

$$A^{+} = \{x : x \geq \sim x\} \text{ and } A^{-} = \{x : x \leq \sim x\}.$$

Furthermore, every directly indecomposable IMVR3-algebra satisfies the following conditions: if $x \in A^{+}$ and $y \in A^{-}$, then (1) $x \geq y$ and (2) $y^2 = y * y = 0$.

In what follows, dIRL_c-algebras $(\mathbf{A}, \circ)$ where $\mathbf{A} \in d\mathbb{IMVR}3$ will be called a $\mathrm{dIMVR3}_c$-algebra.

The next theorem characterizes, in particular, those subdirectly irreducible dIRL_c-algebras $(\mathbf{A}, \circ)$ in which $\mathbf{A}$ is either a finite or a $\mathrm{dIMVR3}_c$-algebra. It is worth pointing out that the following characterization does not hold in general for infinite structures. In Subsection 4.2, we will give an example of an infinite dIRL_c-algebra which is subdirectly irreducible but not simple.

Theorem 3.10. *Let* $(\mathbf{A}, \circ)$ *be a finite* dIRL_c*-algebra or an arbitrary* $\mathrm{dIMVR3}_c$*-algebra. Then the following conditions are equivalent:*

(i) $B(\mathbf{A}) = \{0, 1\}$,

(ii) $(\mathbf{A}, \circ)$ *is a directly indecomposable* dIRL_c*-algebra,*

(iii) $(\mathbf{A}, \circ)$ *is a subdirectly irreducible* dIRL_c*-algebra,*

(iv) $(\mathbf{A}, \circ)$ *is a simple* dIRL_c*-algebra.*

As a consequence, the quasivariety of $\mathrm{dIMVR3}_c$*-algebras is semisimple.*

Proof. Due to Theorem 3.8 and Corollary 3.9, we are left to prove only that (iii) implies (iv), since (iv) implies (iii) in general.

In the case of $\mathbf{A}$ being finite, towards a contradiction, assume $(\mathbf{A}, \circ)$ is not simple. Hence, there is a $\circ$-filter F such that $F \neq A$ and $F \neq \{1\}$. Since A is finite, there is $0 < a \in A \setminus \{1\}$ such that $F = [a)$. Since F is a $\circ$-filter, $\circ(a) \in F$ and thus it must be $a \leq \circ(a)$. But then we have $0 = a \wedge \sim a \wedge \circ(a) = a \wedge \sim a$, that is, a is Boolean, and hence $B(\mathbf{A}) \neq \{0, 1\}$ and hence $(\mathbf{A}, \circ)$ is not subdirectly irreducible by Corollary 3.9.

Assume now $(\mathbf{A}, \circ)$ is a subdirectly irreducible, and hence a directly indecomposable, $\mathrm{dIMVR3}_c$-algebra. We have to show that $(\mathbf{A}, \circ)$ is simple. By Theorem 3.8, $(\mathbf{A}, \circ)$ is directly indecomposable iff $B(\mathbf{A}) = \{0, 1\}$. Therefore, by [36, Proposition 1.5], $\mathbf{A}$ is directly indecomposable as a dIMVR3-algebra. Moreover, [10, Theorems 4.6] shows that if a dIMVR3-algebra $\mathbf{A}$ is directly indecomposable, its domain, as mentioned above, can be expressed as $A = A^{+} \cup A^{-}$, where $A^{+} = \{x : x \geq \sim x\}$ and $A^{-} = \{x : x \leq \sim x\}$, and satisfying $x \geq y$ whenever

$x \in A^+$ and $y \in A^-$, and $y^2 = 0$ for every $y \in A^-$. Now, if $x \in A^+$ then ${\sim}x \in A^-$ and thus, $0 = x \wedge {\sim}x \wedge \circ(x) = {\sim}x \wedge \circ(x)$, from which it follows that $\circ(x) \in A^-$ as well, and hence $(\circ(x))^2 = 0$. Since any $\circ$-filter F of $(\mathbf{A}, \circ)$ containing an element $x \neq 1$ has to contain $\circ(x)$ as well, it has to contain $(\circ(x))^2$ as well, i.e. F must be such that $0 \in F$. Therefore F must be the whole algebra domain A. Thus we have shown that $(\mathbf{A}, \circ)$ is a simple dIRL$_c$-algebra. □

In the light of the Theorem 3.10 above, it is clear that the lattice of Figure 2 together with the operations of the table of Figure 2 is an example of a finite dIRL$_c$-algebra whose Boolean elements are 0 and 1, and hence it is directly indecomposable, subdirectly irreducible and simple.

3.2 Distributive involutive residuated lattices with a maxB-consistency operator

We now start considering consistency operators that map the elements of an dIRL-algebra into Boolean elements of the same. The next definition introduces distributive involutive residuated lattices with a maxB-consistency operator.

Definition 3.11. A *distributive involutive residuated lattice with a* maxB-*consistency operator* (or dIRL$_c^{\mathsf{mB}}$-*algebra* for short) is a pair $(\mathbf{A}, \circ)$ where $\mathbf{A}$ is a dIRL-algebra and $\circ : A \to A$ satisfies the following conditions: for all $x, y, z \in A$,

($\circ$1) $x \wedge {\sim}x \wedge \circ(x) = 0$

($\circ$3) if $x \wedge {\sim}x \wedge y = 0$ and $y \wedge {\sim}y = 0$, then $y \leq \circ(x)$

($\circ$4) $\circ(x) \wedge {\sim}\circ(x) = 0$

Again, from this definition it follows that the class of dIRL$_c^{\mathsf{mB}}$-algebras is a quasivariety. Moreover, and similarly to the case of max-consistency operators, also now the conditions ($\circ$1), ($\circ$3) and ($\circ$4) capture the expected behavior of maxB-consistency operators as described at the beginning of this Section 3.

Lemma 3.12. *Let* $\mathbf{A}$ *be a distributive involutive residuated lattice, and let* $\circ$ *be unary operation on* A. *Then* $(\mathbf{A}, \circ)$ *is an* dIRL$_c^{\mathsf{mB}}$*-algebra iff, for any* $x \in A$,

$$\circ(x) = \max\{z \in B(\mathbf{A}) \mid x \wedge {\sim}x \wedge z = 0\}.$$

Proof. Clearly, for a given $x \in A$, the set $\{z \in B(\mathbf{A}) \mid x \wedge {\sim}x \wedge z = 0\}$ is closed by $\vee$, it has $\circ(x)$ as an upper bound by ($\circ$3), and moreover $\circ(x)$ belongs to that set by ($\circ$1) and ($\circ$4). □

From the definition it is easy to prove the following properties of maxB-consistency operators.

Proposition 3.13. *The following properties hold in a* dIRL$_c^{\mathsf{mB}}$*-algebra* $(\mathbf{A}, \circ)$:

(i) $\circ(x) = \max\{z \in B(\mathbf{A}) \mid z \leq x \vee {\sim}x\}$

x	$\diamond x$	$\circ x$
0	1	1
a	0	b
b	0	a
c	0	0
d	0	0
e	0	a
f	0	b
1	1	1

Table 1: Two operators for the Nelson lattice of Figure 2.

(ii) $\circ(x) = \circ({\sim}x) = \circ(x \wedge {\sim}x) = \circ(x \vee {\sim}x)$

(iii) $\circ(x) = 1$ *iff* $x \in B(\mathbf{A})$, *in particular* $\circ(1) = \circ(0) = 1$

(iv) $\circ\circ(x) = 1$.

Notice that if $(\mathbf{A}, \circ)$ is a dIRL$_c$-algebra and $(\mathbf{A}, \diamond)$ is dIRL$_c^{\mathsf{mB}}$-algebra, then from Lemmas 3.4 and 3.12 it is clear that $\diamond(x) \leq \circ(x)$ for all $x \in A$.

In the Nilpotent Minimum algebra of Figure 2, the corresponding mB-consistency operator $\diamond$ comes defined as $\diamond(0) = \diamond(1) = 1$ and $\diamond(x) = 0$ otherwise. Since this is a different operation from the operation $\circ$ in the table of Figure 2, it readily follows that the classes of dIRL$_c$-algebras and dIRL$_c^{\mathsf{mB}}$-algebras are different. Indeed, consider again the two unary operations in Table 1. It turns out that $(\mathbf{L}, \circ)$ is a dIRL$_c$-algebra but not a dIRL$_c^{\mathsf{mB}}$-algebra, while $(\mathbf{L}, \diamond)$ is a dIRL$_c^{\mathsf{mB}}$-algebra but not a dIRL$_c$-algebra.

Remark 3.14. Like in the case of a max-consistency operator over a dIRL-algebra, a maxB-consistency operator, if it exists, is unique. Actually, according to Lemma 3.12, the maxB-consistency operator $\circ$ is definable on a distributive involutive residuated lattice if and only if all the elements of the form $x \wedge {\sim}x$ have a minimum Boolean element above them. Since the value of $\circ(x)$ is a maximum of Boolean elements, $\circ$ is always definable in any distributive involutive residuated lattice $\mathbf{A}$ such that $B(\mathbf{A})$ is finite. However, the definability of the $\circ$ operator is not guaranteed when $B(\mathbf{A})$ is infinite.

In the quasivariety of dIRL$_c^{\mathsf{mB}}$-algebras, the corresponding notion of Q-filter is that of an implicative filter F further satisfying the following two conditions:

(F1) if $x \to y, y \to x \in F$ then $\circ x \to \circ y, \circ y \to \circ x \in F$

(F3) if $x \vee {\sim}x \vee {\sim}y \in F$ and $y \vee {\sim}y \in F$ then $y \to \circ x \in F$

We will call them $\circ_b$-filters.

From this we can provide a characterization of subdirectly irreducible dIRL$_c$-algebras. To begin with, we can observe that the same proof of Lemma 3.7 also applies to the case of $\mathrm{dIRL}_c^{\mathsf{mB}}$-algebras with a unique atom and hence to all finite and subdirectly irreducible dIRL-algebras with a mB-consistency operator as well.

Lemma 3.15. *Let* $\mathbf{A}$ *be a subdirectly irreducible distributive involutive residuated lattice with a unique atom and define* $\circ : A \to A$ *as* $\circ(1) = \circ(0) = 1$ *and* $\circ(x) = 0$ *otherwise. Then* $(\mathbf{A}, \circ)$ *is a simple* $\mathrm{dIRL}_c^{\mathsf{mB}}$*-algebra.*

Moving from max- to maxB-consistency operators allows us to improve the results shown in Theorem 3.8 and Theorem 3.10 and, as anticipated, to characterize all subdirectly irreducible $\mathrm{dIRL}_c^{\mathsf{mB}}$-algebras.

Theorem 3.16. *Let* $(\mathbf{A}, \circ)$ *be a* $\mathrm{dIRL}_c^{\mathsf{mB}}$*-algebra, then the following conditions are equivalent:*

(i) $B(\mathbf{A}) = \{0, 1\}$

(ii) $(\mathbf{A}, \circ)$ *is a directly indecomposable* $\mathrm{dIRL}_c^{\mathsf{mB}}$*-algebra*

(iii) $(\mathbf{A}, \circ)$ *is a subdirectly irreducible* $\mathrm{dIRL}_c^{\mathsf{mB}}$*-algebra*

(iv) $(\mathbf{A}, \circ)$ *is a simple* $\mathrm{dIRL}_c^{\mathsf{mB}}$*-algebra*

Proof. That (iv) implies (iii) and (iii) implies (ii) is clear. It is hence left to show that (ii) implies (i) and (i) implies (iv).

(ii) implies (i). The same proof of Theorem 3.8 applies.

(i) implies (iv). If $B(\mathbf{A}) = \{0, 1\}$ then a simple computation shows that $\circ(0) = \circ(1) = 1$ and $\circ(x) = 0$ otherwise since for all $x \in A \setminus \{0, 1\}$, $x \wedge {\sim}x \neq 0$. Then every $\circ_b$-filter F containing an element $x \notin \{0, 1\}$ is equal to A and thus A is simple. □

As a direct consequence of this result we have the following corollary.

Corollary 3.17. *The quasi-variety of* $\mathrm{dIRL}_c^{\mathsf{mB}}$*-algebras is semisimple.*

3.3 Distributive involutive residuated lattices with a Bmax-consistency operator

Finally, in this subsection, we consider distributive involutive residuated lattices expanded with a Bmax-consistency operator.

Definition 3.18. A *distributive involutive residuated lattice with a* Bmax*-consistency operator* (or $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebra* for short) is a dIRL_c-algebra $(\mathbf{A}, \circ)$ satisfying the additional condition: for all $x \in A$,

($\circ$4) $\circ(x) \vee {\sim}\circ(x) = 1$.

From the very definition, it is clear that the class of $\mathrm{dIRL}_c^{\mathsf{Bm}}$-algebras constitutes a subquasivariety of the quasivariety of dIRL_c-algebras.

We also have a characterisation of Bmax-consistency operators similar to the case of max-consistency operators with the obvious modification.

Lemma 3.19. *Let* $\mathbf{A}$ *a distributive involutive residuated lattice, and let* $\circ$ *be a unary operation on* A*. Then* $(\mathbf{A}, \circ)$ *is a* $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebra iff, for any* $x \in A$,

$$\circ(x) = \max\{z \in A \mid x \wedge \sim x \wedge z = 0\} \text{ and } \circ(x) \wedge \sim\circ(x) = 0.$$

Actually, as anticipated, Bmax-consistency operators are just max-consistency operators that are maxB-consistency operators as well.

Lemma 3.20. *The quasivariety of* $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebras is the intersection of the quasivariety of* dIRL_c*-algebras and the quasivariety of* $\mathrm{dIRL}_c^{\mathsf{mB}}$*-algebras.*

Proof. Let $(\mathbf{A}, \circ)$ be a $\mathrm{dIRL}_c^{\mathsf{Bm}}$-algebra. We have to show that $\circ$ is a maxB-consistency operator as well. Since $\circ(x)$ is Boolean, $\circ(x) \in \{z \in B(\mathbf{A}) \mid x \wedge \sim x \wedge z = 0\}$, that is, we also have $\circ(x) = \max\{z \in B(\mathbf{A}) \mid x \wedge \sim x \wedge z = 0\}$. Conversely, if $\circ$ is both a max- and maxB-consistency operator, then it is clear that $\circ(x) = \max\{z \in A \mid x \wedge \sim x \wedge z = 0\}$ and that $\circ(x)$ is Boolean, that is, $\circ$ is a Bmax-consistency operator. □

The following proposition collects basic properties of $\mathrm{dIRL}_c^{\mathsf{Bm}}$-algebras.

Proposition 3.21. *For a given* $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebra* $(\mathbf{A}, \circ)$ *and* $x \in A$*, we have:*

(i) $\circ\circ(x) = 1$.

(ii) $\circ(x) = 1$ *iff* $x \wedge \sim x = 0$ *iff* x *is a Boolean element.*

Again, an analogous result to Lemma 3.7 and Lemma 3.15 for dIRL_c-algebras and $\mathrm{dIRL}_c^{\mathsf{mB}}$-algebras resp. also holds for $\mathrm{dIRL}_c^{\mathsf{Bm}}$.

Lemma 3.22. *Let* $\mathbf{A}$ *be a subdirectly irreducible* dIRL*-algebra with a unique atom and let* $\circ$ *the unary operation on* A *defined as:* $\circ(1) = \circ(0) = 1$ *and* $\circ(x) = 0$ *otherwise. Then* $(\mathbf{A}, \circ)$ *is a simple* $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebra.*

Since $\mathrm{dIRL}_c^{\mathsf{Bm}}$-algebras are dIRL_c-algebras fulfilling the additional equation ($\circ$4), the corresponding notions of Q-filters and Q-congruences are the same, and we can obtain similar results as those obtained for dIRL_c-algebras and $\mathrm{dIRL}_c^{\mathsf{mB}}$-algebras. In particular, the following result characterizes subdirectly irreducible $\mathrm{dIRL}_c^{\mathsf{Bm}}$-algebras.

Theorem 3.23. *Let* $(\mathbf{A}, \circ)$ *be a* $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebra, then the following conditions are equivalent:*

(i) $B(\mathbf{A}) = \{0, 1\}$

(ii) $(\mathbf{A}, \circ)$ *is a directly indecomposable* $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebra*

(iii) $(\mathbf{A}, \circ)$ *is a subdirectly irreducible* $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebra*

(iv) $(\mathbf{A}, \circ)$ *is a simple* $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebra*

Proof. The same proof of Theorem 3.16 applies. □

3.4 The prelinear case: IMTL-algebras

We now turn the attention to the particular case of prelinear dIRL-algebras with a consistency operator $\circ$. As shown in [24], involutive residuated lattices satisfying the prelinearity equation

$$(x \to y) \vee (y \to x) = 1$$

are precisely the so-called *involutive MTL-algebras* (or *IMTL-algebras* for short). Let us start showing a first result concerning IMTL-algebras expanded by a Bmax-consistency operator.

Lemma 3.24. *Let* $\mathbf{A}$ *be a* IMTL*-algebra that is not a chain. Then there are two elements* $x, y \neq 1$ *such that* $x \vee y = 1$*.*

Proof. If $\mathbf{A}$ is not a chain, let a, b two uncomparable elements. Then $x = a \to b$ and $y = b \to a$ must be different from 1, and by prelinearity, $x \vee y = (a \to b) \vee (b \to a) = 1$. □

Let us call $\mathrm{IMTL}_c^{\mathsf{Bm}}$-algebras those $\mathrm{dIRL}_c^{\mathsf{Bm}}$-algebras $(\mathbf{A}, \circ)$ such that $\mathbf{A}$ is a IMTL-algebra.

Proposition 3.25. *Let* $(\mathbf{A}, \circ)$ *be a* $\mathrm{IMTL}_c^{\mathsf{Bm}}$*-algebra. Then the following statements are equivalent:*

(i) $B(\mathbf{A}) = \{0, 1\}$

(ii) $\mathbf{A}$ *is an* IMTL*-chain*

(iii) $(\mathbf{A}, \circ)$ *is a directly indecomposable* $\mathrm{IMTL}_c^{\mathsf{Bm}}$*-algebra*

(iv) $(\mathbf{A}, \circ)$ *is a subdirectly irreducible* $\mathrm{IMTL}_c^{\mathsf{Bm}}$*-algebra*

(v) $(\mathbf{A}, \circ)$ *is a simple* $\mathrm{IMTL}_c^{\mathsf{Bm}}$*-algebra*

Proof. Thanks to Theorem 3.23 it is sufficient to prove the equivalence between (i) and (ii).

(i) implies (ii). By Theorem 3.23, $B(\mathbf{A}) = \{0, 1\}$. By Lemma 3.24 if $\mathbf{A}$ is not a chain there exist elements $x, y \neq 1$ such that $x \vee y = 1$. Moreover $x \wedge {\sim}x \neq 0$, since if $x \wedge {\sim}x = 0$ then x would be a Boolean element, a contradiction with the assumption that $x \neq 1, 0$. Moreover $x \wedge {\sim}x \wedge {\sim}y \leq {\sim}x \wedge {\sim}y = {\sim}(x \vee y) = 0$, and thus $\circ(x) \geq {\sim}y$. Since $B(\mathbf{A}) = \{0, 1\}$ and ${\sim}y \neq 0$, it follows that $\circ(x) = 1$, which contradicts the definition of $\circ$.

In order to prove that (ii) implies (i), let us show that (ii) implies (v) whence the result will follow from Theorem 3.23. If $\mathbf{A}$ is an IMTL-chain, then (v) obviously follows because, over a chain, $\circ$ is defined as $\circ(0) = \circ(1) = 1$ and $\circ(x) = 0$ otherwise, and thus, a $\circ$-filter F that contains an element $x \neq 1, 0$ must contain $\circ(x) = 0$, hence $F = A$, whence $(\mathbf{A}, \circ)$ is a simple. □

Let us remark that Proposition 3.25 above does not apply to IMTL-algebras expanded by max and maxB-consistency operators. Indeed, consider the IMTL-algebra $\mathbf{A}$ of Figure 2 expanded by either its (unique) max or maxB-consistency operator $\circ$. By Theorem 3.10 and Theorem 3.16, $(\mathbf{A}, \circ)$ is directly indecomposable, subdirectly irreducible, simple and $B(\mathbf{A}) = \{0, 1\}$, however its lattice reduct is not totally ordered. Moreover, it is clear that the same finite algebra $\mathbf{A}$ does not admit a Bmax-consistency operator, while on the other hand, it is always definable on a chain.

It is worth pointing out that Theorem 3.10 applies to this peculiar case and, in particular, to those IMTL_c-algebra whose underlying IMTL-algebra is either finite or is obtained as rotation of a prelinear semihoop. The latter class of structures have been largely studied in [40].

Now, we turn our attention to the variety of $\mathrm{IMTL}_{\circledast}$-algebras that was defined in [20] and let us analyze analogies and differences between them and IMTL-algebras with a consistency operator $\circ$. Indeed, in [20] the authors consider expansions of IMTL-algebras with a consistency operator that, to avoid a notational clash, we will denote here by $\circledast$. The equations for $\circledast$ are stronger than the ones we introduced in this paper and in fact they allow us to prove that the variety of $\mathrm{IMTL}_{\circledast}$ is *semilinear*, that is to say, it is generated by its totally ordered members.

More precisely, a pair $(\mathbf{A}, \circledast)$ is a $\mathrm{IMTL}_{\circledast}$-algebra if $\mathbf{A}$ is an IMTL-algebra and the following equations for $\circledast$ hold:

(B1) $x \wedge \neg x \wedge \circledast(x) = 0$

(B2) $\circledast(x \leftrightarrow y) \leq (\circledast x \leftrightarrow \circledast y)$

(B3) $\circledast(x \vee y) \leq \circledast(x) \vee y$

(B4) $\circledast(0) = \circledast(1) = 1$

The variety of $\mathrm{IMTL}_{\circledast}$-algebras is shown in [20] to be semisimple, and the simple $\mathrm{IMTL}_{\circledast}$-algebras are those defined over IMTL-chains, i.e. all $\mathrm{IMTL}_{\circledast}$-algebras are subdirect products of chains, and chains are simple. As also shown in [20], if $\mathbf{A}$ is a IMTL-chain, the unique $\circledast$ operator that makes $(\mathbf{A}, \circledast)$ a $\mathrm{IMTL}_{\circledast}$-algebra is the one defined as $\circledast(0) = \circledast(1) = 1$ and $\circledast(x) = 0$ otherwise. Hence $\mathrm{IMTL}_{\circledast}$-, dIRL_c-, $\mathrm{dIRL}_c^{\mathsf{mB}}$- and $\mathrm{dIRL}_c^{\mathsf{Bm}}$-algebras over IMTL-chains share the same consistency operators.

The following is a consequence of Proposition 3.25.

Corollary 3.26. *The class of* $\mathrm{IMTL}_c^{\mathsf{Bm}}$*-algebras is a variety that coincides with the variety of* $\mathrm{IMTL}_{\circledast}$*-algebras. Thus, both are semilinear and semisimple.*

Proof. Let us start showing that the quasivariety of $\mathrm{IMTL}_c^{\mathsf{Bm}}$-algebras is indeed a variety. To this end notice that by Proposition 3.25, the class of $\mathrm{IMTL}_c^{\mathsf{Bm}}$-algebras is generated by its totally ordered members. Moreover, following [20], for every $\mathrm{IMTL}_c^{\mathsf{Bm}}$-algebra $(\mathbf{A}, \circ)$, the unary operation $\Delta : A \to A$ defined by $\Delta(x) = x \wedge \circ(x)$ coincides with the Baaz-Monteiro projection operator

[4, 39]. These facts show that the algebraic logic of $\text{IMTL}_c^{\mathsf{Bm}}$-algebras is a Δ-core fuzzy logic in the sense of [19] and hence, for every finite set of formulas $\{\psi_1, \ldots, \psi_k, \phi\}$, it holds $\{\psi_1, \ldots, \psi_k\} \vdash \phi$ iff $\vdash (\Delta\psi_1 \wedge \ldots \wedge \Delta\psi_k) \to \phi$ (see [30, Theorem 2.4.14]). Algebraically, this means that every quasi-equation describing $\text{IMTL}_c^{\mathsf{Bm}}$-algebras can be written equationally, whence it is indeed a variety.

Finally, the claim follows from Proposition 3.25 above. Indeed, both the variety of $\text{IMTL}_c^{\mathsf{Bm}}$-algebras and the variety of $\text{IMTL}_{\circledast}$-algebras are generated by its totally ordered members. We already observed that, on IMTL-chains, the unique Bmax-consistency operator $\circ$ coincides with the $\circledast$-operator: it maps all non-Boolean elements to 0, and 0 and 1 to 1. Thus the two varieties are generated by the same structures and hence they coincide. □

In fact the previous result can be proved in a slightly more general setting. To this end consider the *semilinear extension* of the quasi-varieties of dIRL_c, $\text{dIRL}_c^{\mathsf{mB}}$ and $\text{dIRL}_c^{\mathsf{Bm}}$-algebras. That is to say the quasi-varieties obtained from the above ones by replacing ($\circ$2) in Definition 3.3 by the quasi-equation

$$\text{if } (x \wedge {\sim}x \wedge y) \vee z = 0 \text{ implies } (y \to \circ(x)) \vee z = 1$$

and ($\circ$3) in Definition 3.11 by

$$\text{if } (x \wedge {\sim}x \wedge y) \vee z = 0 \text{ and } (y \wedge {\sim}y) \vee z = 0 \text{ implies } (y \to \circ(x)) \vee z = 1.$$

By general results it follows that every algebra in the semilinear extension of the quasi-varieties of dIRL_c, $\text{dIRL}_c^{\mathsf{mB}}$ and $\text{dIRL}_c^{\mathsf{Bm}}$-algebras can be represented as a subdirect product of totally ordered structures in the same classes [17]. Thus, the following corollary easily holds.

Corollary 3.27. *The semilinear extensions of* dIRL_c*,* $\text{dIRL}_c^{\mathsf{mB}}$ *and* $\text{dIRL}_c^{\mathsf{Bm}}$ *-algebras form varieties and they all coincide with the variety of* $\text{IMTL}_{\circledast}$*-algebras.*

Proof. Over a dIRL-chain, max, Bmax and maxB consistency operators coincide and therefore the sub-quasivariety generated by them are the same and coincide with the variety of $\text{IMTL}_{\circledast}$-algebras. □

Nevertheless there are IMTL_c-algebras that are not subdirect product of chains as the one in Figure 2 that is simple, hence subdirectly irreducible, but it is not totally ordered. Therefore the quasi-varieties of IMTL_c- and $\text{IMTL}_c^{\mathsf{mb}}$-algebras are not semilinear.

Of course the quasivarietiy of $\text{IMTL}_c^{\mathsf{mB}}$-algebras is semisimple but it is an open problem whether the quasivarietiy of IMTL_c-algebras is semisimple or whether there exist subdirectly irreducible algebras that are not simple, as in the general case of dIRL or NL algebras.

As a sort of summary, Figure 3 provides a graphical representation of the main quasivarieties of algebras expanded with a consistency operator that we have considered in this section, where dashed arrows stand for expansions and solid arrows denote extensions.

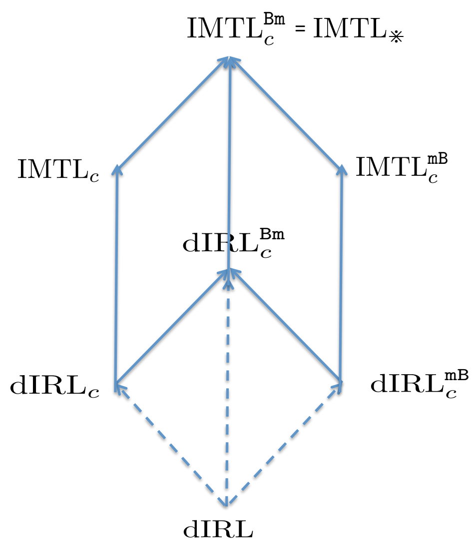


Figure 3: Diagram of main classes of algebras with a consistency operator we consider in this paper and their relationships as expansions of dIRL.

4 The particular case of Nelson lattices with a max-consistency operator

In this section we will focus on expansions of Nelson lattices by means of a max-consistency operator. In particular we will take advantage of well developed structural properties for these algebras in terms of *twist-products* to both investigate more in details subdirectly irreducible structures and also to show that Theorem 3.10 does not hold in the general setting of infinite structures.[6]

Nelson lattices can be represented by means of Heyting algebras and their *Boolean filters*, that is to say, filters F of a Heyting algebra $\mathbf{H}$ such that the quotient $\mathbf{H}/F$ is a Boolean algebra. This relation has been investigated by several authors and main contributions in this sense have been provided by Fidel [25], Vakarelov [54] and Sendlewski [49].

In the following two subsections, after some needed brief preliminaries, we are going to prove similar representations for Nelson lattices expanded by consistency operators (Subsection 4.1) and, as anticipated in Subsection 3.1, we will also show more results on subdirectly irreducible algebras (Subsection 4.2).

[6]Twist-product have been developed also for, in general unbounded, involutive residuated lattices (see [9]). However these results, although being more general, does not lead to a complete representation of all (distributive) involutive residuated lattices. Thus, here we preferred to restrict to Nelson lattices since in this setting we can obtain stronger results.

It is worth to notice that prelinear Nelson lattices precisely correspond to NM-algebras and moreover the twist-product preserves prelinearity. In other words, each NM-algebra **A** can be uniquely represented by a prelinear Heyting algebra **G** (aka a Gödel algebra) and a Boolean filter of **G** [7, Theorem 6.19]. As a consequence, all the results of this section (with the clear exception of the counterexample we show in Subsection 4.2) apply to NM-algebras with a consistency operator.

Conforming to the previous notation, we will call NL_c-, $\mathrm{NL}_c^{\mathsf{mB}}$- and $\mathrm{NL}_c^{\mathsf{Bm}}$-algebras to the Nelson lattices expanded, respectively, with max-, maxB- and Bmax-consistency operators.

To start with, let us recall the following result, whose formulation is taken from [8].

Theorem 4.1 ([49])**.** *Given a Heyting algebra* $\mathbf{H} = (H, \wedge, \vee, \to_H, 0, 1)$ *and a Boolean filter* F *of* $\mathbf{H}$ *let*

$$N(\mathbf{H}, F) := \{(x, y) \in H \times H : x \wedge y = 0 \text{ and } x \vee y \in F\}.$$

Then we have:

(i) $\mathbf{N}(\mathbf{H}, F) = (N(\mathbf{H}, F), \vee, \wedge, *, \to, \sim, 0, 1)$ *is a Nelson lattice, where operations are defined as follows:*

$$\begin{aligned}
(x, y) \vee (s, t) &= (x \vee s, y \wedge t),\\
(x, y) \wedge (s, t) &= (x \wedge s, y \vee t),\\
(x, y) * (s, t) &= (x \wedge s, (x \to_H t) \wedge (s \to_H y)),\\
(x, y) \to (s, t) &= ((x \to_H s) \wedge (t \to_H y), x \wedge t),\\
\sim(x, y) &= (y, x),\\
1 &= (1, 0),\\
0 &= (0, 1).
\end{aligned}$$

(ii) *If* F_1*,* F_2 *are Boolean filters of* $\mathbf{H}$*, then* $\mathbf{N}(\mathbf{H}, F_1)$ *is a subalgebra of* $\mathbf{N}(\mathbf{H}, F_2)$ *if and only if* $F_1 \subseteq F_2$*.*

Following the tradition (see [8] for instance), we will include, for every Heyting algebra **H**, the improper filter H among the Boolean filters of **H**. In what follows, the Nelson lattice $\mathbf{N}(\mathbf{H}, H)$ will be simply denoted by $\mathbf{N}(\mathbf{H})$.

The following result, independently proved by Fidel in [25] and by Vakarelov in [54], can hence be stated as a corollary of Theorem 4.1 above.

Corollary 4.2. *For every Nelson lattice* **A** *there is a Heyting algebra* **H** *such that* **A** *is isomorphic to a subalgebra of* $\mathbf{N}(\mathbf{H})$*.*

In what follows we will need a further requirement for the Boolean filter of a Heyting algebra in order to extend the previous representation to Nelson lattices with consistency operators. Let us hence recall from [48] that a unary operation $\urcorner$ on a Heyting algebra **H** is called a *dual pseudocomplement*, if the following equations are satisfied:

(D1) $x \vee \neg(x \vee y) = x \vee \neg y$,

(D2) $x \vee \neg 1 = x$,

(D3) $\neg\neg 1 = 1$.

In any Heyting algebra $\mathbf{H}$, if the dual pseudocomplement of $x \in H$ exists, then it is defined as

$$\neg x = \min\{z \in H \mid z \vee x = 1\}. \tag{1}$$

Recall from [48] that in every Heyting algebra with dual pseudocomplement, congruences bijectively correspond to *normal filters*, that is to say, implicative filters that satisfy the following further requirement: if $x \in F$, then $\neg\neg x \in F$ as well.

We will make use of the following easy result that is known in the literature but we provide the proof for the sake of self-containedness.

Lemma 4.3. *A filter F of a Heyting algebra $\mathbf{H}$ is Boolean iff every $a \in H$, $a \vee \neg a \in F$.*

Proof. Assume that F is Boolean, and hence $\mathbf{H}/F$ is a Boolean algebra. Thus for all $[a]_F \in H/F$, $[a]_F \vee \neg[a]_F = [1]_F$, that is, $[a \vee \neg a]_F = [1]_F$ and hence $a \vee \neg a \in F$. Conversely, if $a \vee \neg a \in F$ for all $a \in H$, the quotient $\mathbf{H}/F$ satisfies $[1]_F = [a \vee \neg a]_F = [a]_F \vee \neg[a]_F$ for all $a \in H$. Therefore $\mathbf{H}/F$ is a Boolean algebra whence F is Boolean. □

4.1 Representation of Nelson lattices with consistency operators

Thanks to the representation of Nelson lattices in terms of Heyting algebras and Boolean filters, if $\mathbf{A} = \mathbf{N}(\mathbf{H}, F)$ is a Nelson lattice and $\circ : A \to A$ is a max-consistency operator, by virtue of Lemma 3.4, $\circ$ can be equivalently reformulated in the following way: for all $(a, b) \in A$,

$$\circ(a, b) = \max\{(z, z') \in A \mid a \vee b \vee z' = 1\}. \tag{2}$$

Indeed, taking into account how the operations are defined in $\mathbf{N}(\mathbf{H}, F)$ and that $a \wedge b = 0$, we have the following chain of equalities:

$$\begin{aligned}
\circ(a, b) &= \max\{(z, z') \in A \mid (a, b) \wedge \neg(a, b) \wedge (z, z') = (0, 1)\} \\
&= \max\{(z, z') \in A \mid (a, b) \wedge (b, a) \wedge (z, z') = (0, 1)\} \\
&= \max\{(z, z') \in A \mid (a \wedge b \wedge z, b \vee a \vee z') = (0, 1)\} \\
&= \max\{(z, z') \in A \mid (0, a \vee b \vee z') = (0, 1)\} \\
&= \max\{(z, z') \in A \mid a \vee b \vee z' = 1\}.
\end{aligned}$$

We are now going to show a representation for NL_c-algebras in terms of Heyting algebras, Boolean filters and the dual-pseudocomplement. First, we need to prove the following.

Lemma 4.4. *Let $\mathbf{H}$ be a Heyting algebra, F Boolean filter of $\mathbf{H}$. If the dual pseudo-complement of $a \vee b$, $\llcorner(a \vee b)$, exists for all those a, b in H such that $a \wedge b = 0$ and $a \vee b \in F$, then the* max*-consistency operator $\circ(a, b)$ exists in the Nelson lattice $\mathbf{N}(\mathbf{H}, F)$ and $\circ(a, b) = (\neg d, d)$, where $d = \llcorner(a \vee b)$.*

Proof. Recalling the above equation (2), consider the set $D = \{(z, z') \in N(H, F) \mid a \vee b \vee z' = 1\}$. It is clear that, by definition, if $\max D$ exists, then $\circ(a, b) = \max D$. Besides, we can write $D = \{(z, z') \in H \times H \mid z \wedge z' = 0,\ z \vee z' \in F, a \vee b \vee z' = 1\}$.

Now, for each $z' \in H$ such that $a \vee b \vee z' = 1$, let us define $D_{z'} = \{(z, z') \mid z \in H,\ z \wedge z' = 0,\ z \vee z' \in F\}$. Hence, it is clear that

$$D = \bigcup \{D_{z'} \mid z' \in H, a \vee b \vee z' = 1\}$$

and $D_{z'} \neq \emptyset$ for each $z' \in H$ such that $a \vee b \vee z' = 1$. Moreover, it is possible to prove that $\max D_{z'}$ exists and $\max D_{z'} = (\neg z', z')$. Indeed, if $(z, z') \in D_{z'}$ then $(z, z') \leq (\neg z', z')$ because, if $(z, z') \in D_{z'}$ then $z \wedge z' = 0$ and then, $z \leq \neg z'$. Therefore, $\max D_{z'} = (\neg z', z')$. On the other hand, we can see that $\max\{\max D_{z'} \mid z' \in H, a \vee b \vee z' = 1\}$ exists and equals $(\neg d, d)$, where, recalling Equation (1), $d = \llcorner(a \vee b)$. Indeed, $\max\{\max D_{z'} \mid z' \in H, a \vee b \vee z' = 1\} = \max\{(\neg z', z') \mid z' \in H, a \vee b \vee z' = 1\} = (\neg \min\{z' \mid a \vee b \vee z' = 1\}, \min\{z' \mid a \vee b \vee z' = 1\}) = (\neg d, d)$, where $d = \llcorner(a \vee b)$.

Finally, we are going to show that $\max D$ exists and in fact $\max D = (\neg d, d)$. We have to prove the following two conditions:

(I) $(\neg d, d) \in D$ and

(II) $(f, g) \leq (\neg d, d)$ for any $(f, g) \in D$.

As for condition (I), we have to check that (i) $\neg d \wedge d = 0$, which is obvious, (ii) $\neg d \vee d \in F$, that follows from Lemma 4.3, and (iii) $a \vee b \vee d = 1$, that also follows because $d = \llcorner(a \vee b)$. As for condition (II), assume $(f, g) \in D$. Then $f \wedge g = 0$, $f \vee g \in F$ and $a \vee b \vee g = 1$. From $f \wedge g = 0$ it follows that $f \leq \neg g$, and from $a \vee b \vee g = 1$ and (1) it follows that $g \geq \llcorner(a \vee b)$ and $f \leq \neg g \leq \neg(\llcorner(a \vee b))$. Therefore, $(f, g) \leq (\neg(\llcorner(a \vee b)), \llcorner(a \vee b))$. □

Therefore, every NL_c-algebra can be represented as follows.

Theorem 4.5. *Every NL_c-algebra is of the form $(\mathbf{N}(\mathbf{H}, F), \circ)$ where $\mathbf{H}$ is a Heyting algebra and F is a Boolean filter of H such that the dual pseudo-complement exists in $\mathbf{H}$ for all the elements of F, and for all $(a, b) \in N(H, F)$, $\circ(a, b) = (\neg\llcorner(a \vee b), \llcorner(a \vee b))$.*

Proof. Let us define $G = \{a \vee b \mid a, b \in H, (a, b) \in A\} = \{a \vee b \mid a, b \in H, a \wedge b = 0, a \vee b \in F\}$ and let us prove that $G = F$. It is clear that, by definition of G, $G \subseteq F$. Now, let $y \in F$ and write it as $y = y \vee 0$. Then, $y \wedge 0 = 0$ and that $y \vee 0 = y \in F$, hence $y \in G$. Hence, $F \subseteq G$.

Therefore, by Lemma 4.4, if the dual pseudocomplement $\llcorner$ exists for all $x \in F$, the max-consistency operator $\circ$ exists in $\mathbf{N}(\mathbf{H}, F)$ and it is in the form $\circ(a, b) = (\neg\llcorner(a \vee b), \llcorner(a \vee b))$. Conversely, if $\circ$ exists in $\mathbf{N}(\mathbf{H}, F)$, then $\llcorner$ exists for all $a \vee b \in G$ and hence it exists for all the elements of the Boolean filter F by the previous argument. □

The last theorem shows under which conditions a Heyting algebra $\mathbf{H}$ and a Boolean filter F of $\mathbf{H}$ induces an NL$_c$-algebra over the Nelson lattice $\mathbf{N}(\mathbf{H}, F)$. It is clear that in every finite Heyting algebra all the elements have a dual pseudocomplement, whence all the elements of each of its Boolean filters also have. However, this might not be the case for infinite algebras. In order to see an example, consider the infinite Heyting algebra $\mathbf{H}$ depicted in Figure 4. It is clear that the element x_1 does not have a dual pseudocomplement, while all other elements do have.Then, take the implicative filter $F = \{y_i : i \in N\} \setminus \{0\}$. It is easy to see that F is Boolean since the quotient algebra $\mathbf{H}/F$ is the Boolean algebra $\{0, 1\}$. Thus, although the dual pseudocomplenet is not defined on the whole domain H, Theorem 4.5 ensures that $(\mathbf{N}(\mathbf{H}, F), \circ)$ is a NL$_c$-algebra.

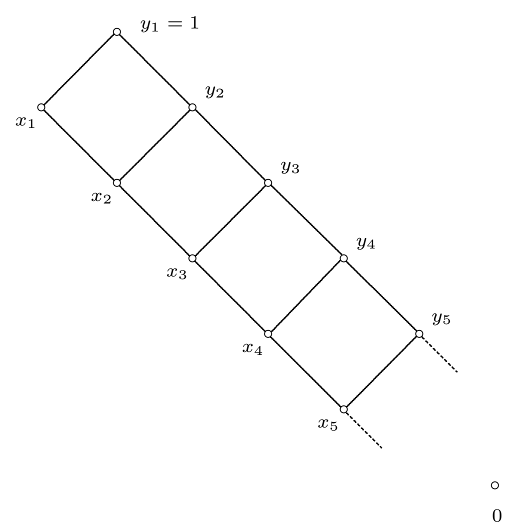


Figure 4: A Heyting algebra that is not dually pseudo-complemented.

The final result of this subsection is a direct consequence of Theorem 4.5 above and it extends Sendewski's Theorem (see Theorem 4.1 (ii)) to Nelson lattices expanded by a consistency operators.

Theorem 4.6. *Let* $\mathbf{H}$ *be a Heyting algebra and let* F_1, F_2 *be two Boolean filters of* $\mathbf{H}$. *Then* $(\mathbf{N}(\mathbf{H}, F_1), \circ)$ *is a NL$_c$- subalgebra of* $(\mathbf{N}(\mathbf{H}, F_2), \circ)$ *iff* $F_1 \subseteq F_2$.

Proof. It is an immediate consequence of Theorem 4.5 and of the fact that the value of $\circ(a, b)$ only depends on a and b. □

Let us close this subsection with a brief comment aimed at clarifying what is the situation for the two remaining cases: $\mathrm{NL}_c^{\mathsf{mB}}$- and $\mathrm{NL}_c^{\mathsf{Bm}}$-algebras.

(1) As for $\mathrm{NL}_c^{\mathsf{mB}}$-algebras, it is easy to see that the Boolean range of the maxB-consistency operator is characterized, in terms of twist-product, by requiring in Theorem 4.5 that the dual pseudocomplement $\urcorner$ exists for all the elements of the Boolean filter F and, in addition, it maps all elements of F into a Boolean element of H.

(2) As for $\mathrm{NL}_c^{\mathsf{Bm}}$-algebras, on the other hand, we need to define a new unary operator on a Heyting algebra $\mathbf{H}$, that we denote by $\urcorner^b$, called a *Boolean dual pseudocomplement*, defined by the following condition: for all $x \in H$,

$$\urcorner^b x = \min\{z \in B(\mathbf{H}) \mid x \vee z = 1\}.$$

Notice that, similarly to a dual pseudocomplement which assigns a Boolean value to an element of a Heyting algebra, $\urcorner^b$ also ranges on the Boolean subalgebra $B(\mathbf{H})$ of $\mathbf{H}$. But notice that the operator $\urcorner^b$ need not always exist since, for a fixed x in a Heyting algebra $\mathbf{H}$, the set $\{z \in B(\mathbf{H}) \mid x \vee z = 1\}$ might not have a minimum in $B(\mathbf{H})$. Again, a variant of Theorem 4.5 holds for $\mathrm{NL}_c^{\mathsf{Bm}}$-algebras stating that if $\mathbf{H}$ is a Heyting algebra and is F a Boolean filter of $\mathbf{H}$, $\urcorner^b x$ exists for all $x \in F$ iff the maxB-consistency operator $\circ$ exists in $\mathbf{N}(\mathbf{H}, F)$. In case it exists, the Bmax-consistency operator $\circ$ is definable in $N(\mathbf{H}, F)$ as follows: for every (a, b), $\circ(a, b) = (\neg\urcorner^b(a \vee b), \urcorner^b(a \vee b))$.

4.2 More on subdirectly irreducible and simple NL_c-algebras

In the light of the results of the previous subsection, we are now in position to add some results on subdirectly irreducible NL_c-algebras. In particular this subsection is devoted to show that Theorem 3.10 cannot be extended to infinite Nelson lattices with a max-consistency operator. To this end, consider a Heyting algebra with dual pseudocomplement $\mathbf{H} = (H, \wedge, \vee, \to_H, \urcorner, 0, 1)$ and a normal Boolean filter F_B of $\mathbf{H}$ (recall how normal filters are defined at the beginning this section). Since $\mathbf{H}$ has a dual pseudocomplement, also F_B has it as well, and hence the max-consistency operator $\circ$ can be defined on $\mathbf{N}(\mathbf{H}, F_B)$ as in Lemma 4.4: for all $(a, b) \in N(\mathbf{H}, F_B)$, $\circ(a, b) = (\neg\urcorner(a \vee b), \urcorner(a \vee b))$. Now define the following subset of $N(\mathbf{H}, F)$:

$$G = \{(a, b) \in N(\mathbf{H}, F_B) \mid a \in F_B\}. \tag{3}$$

Then the following holds.

Lemma 4.7. *The set G is a $\circ$-filter of $(\mathbf{N}(\mathbf{H}, F_B), \circ)$. Furthermore if F_B is proper then so is G.*

Proof. Let us start showing that G is an implicative filter, and to this end let us prove that G is closed under $*$. If $(a, b), (a', b') \in G$, then $a, a' \in F_B$, whence $a \wedge a' \in F_B$ as well. Therefore, $(a, b) * (a', b') = (a \wedge a', (a \to_H b') \wedge (a' \to_H b)) \in G$.

Second, let us prove that G satisfies conditions (F1) and (F2). As for the former, assume that $(a,b) \to (a',b'), (a',b') \to (a,b) \in G$ and let us show that $\circ((a,b) \to (a',b')), \circ((a',b') \to (a,b)) \in G$. For all $(x,y),(z,k) \in N(\mathbf{H}, F_B)$, $(x,y) \to (z,k) \in G$ iff $(x \to_H x'), (y' \to_H y) \in F_B$, therefore

$$a \to_H a', b' \to_H b, a' \to_H b, b \to_H b' \in F_B.$$

In addition $a \vee b, a' \vee b' \in F_B$ because $(a,b),(a',b') \in N(\mathbf{H}, F_B)$, whence $\neg\neg(a \vee b), \neg\neg(a' \vee b') \in F_B$ since F_B is normal. Therefore one immediately has (i) $\neg\neg(a \vee b) \to_H \neg\neg(a' \vee b') \in F_B$ and (ii) $\neg\neg(a' \vee b') \to_H \neg\neg(a \vee b) \in F_B$. Furthermore, by hypothesis $a \leftrightarrow_H a', b \leftrightarrow_H b' \in F_B$, whence (iii) $\neg(a \vee b) \leftrightarrow_H \neg(a' \vee b') \in F_B$. Therefore, $\circ(a,b) \to \circ(a',b') = (\neg\neg(a \vee b), \neg(a \vee b)) \to_H (\neg\neg(a' \vee b'), \neg(a' \vee b')) = (\neg\neg(a \vee b) \to \neg\neg(a' \vee b'), \neg\neg(a \vee b) \wedge \neg(a' \vee b')) \in G$ by (i) and by (ii) $\circ(a',b') \to \circ(a,b) \in G$ as well. Thus (F1) holds.

As for (F2), assume that $(a,b) \vee \neg(a,b) \vee \neg(c,d) \in G$, or equivalently by definition of G, that $a \vee b \vee d \in F_B$. Thus, in order to prove that $(c,d) \to \circ(a,b) \in G$, we need to show: (i) $c \to_H \neg\neg(a \vee b) \in F_B$ and (ii) $\neg(a \vee b) \to_H d \in F_B$. Now, since $(a,b) \in N(\mathbf{H}, F_B)$, $a \vee b \in F_B$ and $\neg\neg(a \vee b) \in F_B$ because F_B is normal. Therefore, $c \to_H \neg\neg(a \vee b) \in F_B$. As to prove (ii), notice that every Heyting algebra satisfies $\neg x \vee y \leq x \to_H y$, and hence $\neg(a \vee b) \to_H d \geq \neg\neg(a \vee b) \vee d \geq \neg\neg(a \vee b) \in F_B$.

Finally let us observe that if F_B is proper, that is, if $\{1\} \neq F_B \neq H$, then by definition $\{1\} \neq G \neq N(\mathbf{H}, F_B)$ and hence G is proper as well. □

Lemma 4.8. *Let* $\mathbf{H}$ *be a Heyting algebra with dual pseudocomplement and* F_B *a Boolean filter of* $\mathbf{H}$*. Then, for every proper* $\circ$*-filter* G *of* $\mathbf{N}(\mathbf{H}, F_B)$*, the set* $F(G) = \{a \in F_B \mid \exists b \in H, (a,b) \in G\}$ *is a proper normal filter of* $\mathbf{H}$*.*

Proof. The fact that $F(G)$ is an implicative filter, is immediate. Thus, let us assume that $a \in F(G)$. By definition, it means that $(a,b) \in G$ for some $b \in H$. Since $(a,b) \leq (a,0)$ and $(a,0)$ clearly belongs to $N(\mathbf{H}, F_B)$, $(a,0) \in F(G)$ as well. Therefore, $\circ(a,0) = (\neg\neg a, \neg a) \in G$ because G is a $\circ$-filter, and hence $\neg\neg a \in F(G)$ proving that $F(G)$ is normal.

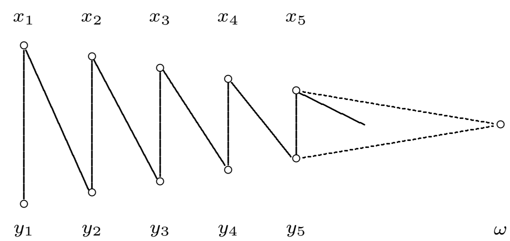
It is easy to see that if G is proper, then $F(G)$ is proper as well. □

Theorem 4.9. *Let* $\mathbf{H}$ *be a Heyting algebra with dual pseudocomplement and* F_B *a Boolean filter of* $\mathbf{H}$*. Then* $G = \{(a,b) \in N(\mathbf{H}, F_B) \mid a \in F_B\}$ *is a proper* $\circ$*-filter of* $(\mathbf{N}(\mathbf{H}, F_B))$ *iff* F_B *is proper and normal. Furthermore, if* F_B *is the minimal filter of* $\mathbf{H}$*, then* G *is the minimal filter of* $\mathbf{N}(\mathbf{H}, F_B)$*.*

Proof. In the light of Lemma 4.7 and Lemma 4.8, in order to prove the first part of the claim, it is enough to observe that if G is defined as in (3) and $F(G)$ is as in Lemma 4.8, then indeed $F(G) = F_B$. Therefore, let us prove the last claim and assume that G is not minimal. Thus, let R be a $\circ$-filter of $\mathbf{N}(\mathbf{H}, F_B)$ such that $R \subset G$. Then $F(R) \subset F(G) = F_B$ and F_B would not be minimal. A contradiction. □

As an application of this last result, let us now show that Theorem 3.10 fails for general infinite structures, i.e. there exist NL_c-algebras that are subdirectly irreducible but not simple. To this end, consider the following example that we have elaborated from an insight provided by Taylor in a personal communication and that can be found in [52, Figure 8.1 of pag. 99].

Example 4.10. Let $Z = \{\omega\} \cup X \cup Y$ being $X = \{x_i : i \in N\}$ and $Y = \{y_i : i \in N\}$ and the following diagram of the partial order on Z:



Let us topologize X in the following manner: $U \subseteq X$ is open iff either $\omega \notin U$, or $\omega \in U$ and $X \setminus U$ is finite. Topologise Y similarly. Therefore, U is open in Z if only if $U \cap X$ and $U \cap Y$ are open in X and Y, respectively. Let H be the set of coplen upsets of Z. A direct computation shows those are the following subsets of Z:

Type (1): subsets U not containing ω and containing finite subsets of elements of X and Y and satisfying the condition that if $y_i \in U$ then $x_i, x_{i-1} \in U$ (in particular if $y_1 \in U$ then $x_1 \in U$ as the particular case of $i = 1$).

Type (2): subsets U containing ω and containing all elements of X and Y except, at most, a finite subset of them and satisfying the condition that if $y_i \in U$ then $x_i, x_{i-1} \in U$.

Using the standard construction for Priestley duality is possible to see that $\mathbf{H} = (H, \cap, \cup, \rightarrow_H, \neg, \emptyset, Z)$ is a Heyting algebra in which the dual pseudocomplementation $\urcorner$ is indeed definable. Moreover, let us observe the following facts:

- the pseudocomplement is defined as $\neg D = D \rightarrow_H \emptyset$.
- the atoms of $\mathbf{H}$ are the subsets $\{x_i\}$ for $i \in \mathbb{N}$.
- the antiatoms of $\mathbf{H}$ are the subsets $\{\omega\} \cup X \cup (Y \setminus \{y_i\})$ for every $i \in \mathbb{N}$.

As proved in [48], the congruences of $\mathbf{H}$ are in 1-1 correspondence with normal filters, that is, lattice filters F closed by a combination of the two pseudocomplementations, i.e., if $U \in F$, then $\neg\urcorner(U) \in F$.

By the result about Pristley duality we can prove that the normal filters of $\mathbf{H}$ are: (i) the one containing only the maximum $\{Z\}$, (ii) the one containing all subsets of type (2) and, (iii) the full algebra $\mathbf{H}$. Then $\mathbf{H}$ has only three congruences that form a three element chain and thus it is subdirectly irreducible but not simple.

In order to be self-contained let us sketch the proof of this fact:

(1) Compute the normal filter F generated by the antiatom $D = \{\omega\} \cup X \cup (Y \setminus \{y_1\})$. Since $\neg D = \{y_1, x_1\}$ and $\neg\neg(D) = \{\omega\} \cup X \cup (Y \setminus \{y_1, y_2\})$, we deduce that $\{\omega\} \cup X \cup (Y \setminus \{y_2\}) \in F$ and, recursively, we obtain that all antiatoms belong to F, and so F consists of all subsets of type (2)

(2) Compute the normal filter generated by any subset of type (1). It is easy to see that it is the full Heyting algebra.

(3) Then the set of congruences of $\mathbf{H}$ has exactly three congruences: the identity, the one corresponding to the normal filter F, and the full algebra.

Also notice that the normal filter F defined in (1) above is indeed Boolean, because an easy computation shows that the quotient $\mathbf{H}/\equiv_F$ is the two-element Boolean algebra.

Then let us consider the NL_c-algebra $(\mathbf{N}(\mathbf{H}, F), \circ)$. Notice that $N(\mathbf{H}, F) = \{(D, E) \in H \times H : D \cap E = \emptyset, D \cup E \in F\}$ is indeed the set of pairs (D, E) such that D is of type (1) and $E \subseteq \neg D$. By Theorem 4.9, the $\circ$-filters of $(\mathbf{N}(\mathbf{H}, F), \circ)$ are the singleton $\{(Z, \emptyset)\}$, the set $\mathcal{F} = \{(D, E) \in R(H, F) : D \in F\}$ and the full Nelson lattice $N(\mathbf{H}, F)$. Thus the $\circ$-congruences of $(\mathbf{N}(\mathbf{H}, F), \circ)$ form a three-element chain and hence our NL_c-algebra is subdirectly irreducible but not simple.

Remark 4.11. The residuated lattice of the last example is not a IMTL algebra, i.e. is not prelineal, it does not satisfy the equation $(x \to y) \vee (y \to x) = 1$ as the following example shows: Take $A = \{x_i\}$ and $B = \{x_{i-1}\}$, two atoms of the Nelson lattice. Then an easy computation shows that:

- $A \to B = \{w\} \cup X \cup Y \setminus \{x_i, y_i, y_{i+1}\}$,
- $A \to B = \{w\} \cup X \cup Y \setminus \{x_{i-1}, y_i, y_{i-1}\}$,

Therefore $(A \to B) \vee (B \to A) = \{w\} \cup X \cup Y \setminus \{y_i\}$, which is not the maximum of the lattice, and thus the residuated lattice is not prelinear.

5 Adding consistency operators to the logic dIRL and its paraconsistent companions

Let us recall from Section 2.1 the logic dIRL, the logic corresponding to the variety of distributive involutive residuated lattices, that can be presented as the axiomatic extension of FL_{ew} with the following axioms:

(Inv) $\phi \to \neg\neg\phi$,

(Dist) $\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$

and its degree-preserving companion $\mathsf{dIRL}^{\leq}$. Since the residual negation $\neg$ in dIRL is involutive, and hence it does not prove the pseudo-complementation axiom, i.e. $\not\vdash_{\mathsf{NL}} \varphi \wedge \neg\varphi \to \bot$, the logic $\mathsf{dIRL}^{\leq}$ satisfies

$$\varphi, \neg\varphi \not\vdash_{\mathsf{dIRL}^{\leq}} \bot$$

and hence it is paraconsitent with respect to the residual negation $\neg$. However, $\mathsf{dIRL}^{\leq}$ is not a *logic of formal inconstency* (LFI) since we cannot define, in its language, a consistency connective $\circ$ satisfying

$$\circ\varphi, \varphi, \neg\varphi \vdash_{\mathsf{dIRL}^{\leq}} \psi$$

for every φ and ψ. This has been the main motivation in the previous sections for the algebraic study of expansions of dIRL-algebras with suitable consistency operators. With this background, in this section we will consider the logical counterparts of those expansions, namely the expansion of the logic dIRL with three classes of consistency operators $\circ$ whose degree-preserving companions will turn out to be LFIs.

5.1 dIRL-logic with a max-consistency operator

In this section we start by defining the expansion of the logic dIRL with a new connective $\circ$ whose algebraic interpretation corresponds to a max-consistency operator.

Definition 5.1. The logic dIRL_c is syntactically defined as the expansion of dIRL in a language which incorporates a new unary connective $\circ$ with the following additional axiom:

(A1) $\neg(\varphi \wedge \neg\varphi \wedge \circ\varphi)$

and inference rules:

$$\text{(CNG)} \quad \frac{\varphi \leftrightarrow \psi}{\circ\varphi \leftrightarrow \circ\psi} \qquad\qquad \text{(Max)} \quad \frac{\varphi \vee \neg\varphi \vee \neg\psi}{\psi \to \circ\varphi} \quad .$$

Clearly, the axiom (A1) and the rule (Max) are the logical counterparts of conditions ($\circ$1) and ($\circ$2) in dIRL$_c$-algebras respectively, while the rule (CNG) enforces $\circ$ to be congruent w.r.t. logical equivalence.

Some observations follow:

(i) Both $\circ\top$ and $\circ\bot$ are derivable in dIRL_c, it suffices in (Max) to take $\psi = \top$ and then $\varphi = \top$ (resp. $\varphi = \bot$).

(ii) The rule of necessitation for $\circ$:

$$\frac{\varphi}{\circ\varphi}$$

is also derivable. Indeed, assuming φ, we can derive $\varphi \to \top$ and $\top \to \varphi$ in dIRL_c. As a matter of fact, $\varphi \to \top$ and $\varphi \to (\psi \to \varphi)$ both are theorems of RL (and hence a theorem of dIRL_c a fortiori). Thus, $\varphi \vdash_{\mathsf{dIRL}_c} \varphi \leftrightarrow \top$.

Now, by (CNG) with $\psi = \top$, $\varphi \leftrightarrow \top \vdash_{\mathsf{dIRL}_c} \circ\varphi \leftrightarrow \circ\top$ and since by the above (i) $\circ\top$ is a theorem of dIRL_c, by $\circ\varphi \leftrightarrow \circ\top$ and $\circ\top$ we get $\circ\varphi$ by modus ponens.

Lemma 5.2. *The following derivabilities hold in* dIRL_c*:*

(i) $\varphi \vee \neg\varphi \vdash_{\mathsf{dIRL}_c} \circ\varphi$

(ii) $\vdash_{\mathsf{dIRL}_c} \circ\varphi \to \neg\varphi \vee \varphi$

(iii) $\varphi \vee \neg\varphi \dashv\vdash_{\mathsf{dIRL}_c} \circ\varphi$

Proof. (i) It follows by using the rule (Max) with $\psi = \top$.

(ii) Notice that in distributive involutive residuated lattices the inequality $\sim x \vee y \leq x \to y$ holds true, and hence, since dIRL is sound and complete with respect to dIRL-algebras, $(\neg\varphi \vee \psi) \to (\varphi \to \psi)$ is a theorem in dIRL. Therefore, since $\neg(\varphi \wedge \neg\varphi \wedge \circ\varphi)$ is logically equivalent to $\neg\varphi \vee \varphi \vee \neg \circ \varphi$ and, by the previous observation, in dIRL_c the latter logically implies $\circ\varphi \to \neg\varphi \vee \varphi$. Hence $\mathsf{dIRL}_c \vdash \circ\varphi \to \neg\varphi \vee \varphi$.

(iii) Direct from (i) and (ii).

□

It is easy to check that, due to the presence of the rule (CNG) for $\circ$, dIRL_c is a Rasiowa implicative logic, hence it is algebraizable, and its equivalent algebraic semantics is given by the quasi-variety of dIRL_c-algebras studied in Section 3.1.

Proposition 5.3. dIRL_c *is strongly complete w.r.t. the class of* dIRL_c*-algebras.*

Now, thanks to results on dIRL_c-algebras in Section 3.1, we can prove that dIRL_c is a conservative expansion of dIRL.

Proposition 5.4. dIRL_c *is a conservative expansion of* dIRL*, that is, if* $\Gamma \cup \{\varphi\}$ *is a set of formulas in the language of* dIRL*, i.e. without the connective* $\circ$*, then* $\Gamma \vdash_{\mathsf{dIRL}_c} \varphi$ *iff* $\Gamma \vdash_{\mathsf{dIRL}} \varphi$*.*

Proof. The right-to-left direction is obvious. Thus, assume $\Gamma \nvdash_{\mathsf{dIRL}} \varphi$, then there is a subdirectly irreducible dIRL-algebra $\mathbf{A}$ and an $\mathbf{A}$-evaluation of formulas e such that $e(\psi) = 1$ for all $\psi \in \Gamma$ and $e(\varphi) < 1$. By Lemma 3.7, the unary operation $\circ$ on A defined as $\circ(1) = \circ(0) = 1$ and $\circ(x) = 0$ otherwise, makes $(\mathbf{A}, \circ)$ a simple dIRL_c-algebra. Then e can be extended in the obvious way to an $(\mathbf{A}, \circ)$-evaluation e' agreeing with e on the formulas not containing the connective $\circ$. Therefore, we have $e'(\psi) = 1$ for all $\psi \in \Gamma$ and $e'(\varphi) < 1$, that is, $\Gamma \nvdash_{\mathsf{dIRL}_c} \varphi$. □

Now we move to the logic $\mathsf{dIRL}_c^{\leq}$, the degree-preserving companion of the logic dIRL_c.

Definition 5.5. The degree-preserving companion of logic dIRL_c is the logic $\mathsf{dIRL}_c^{\leq}$ defined by the following axioms and rules:

- Axioms of $\mathsf{dIRL}_c^{\leq}$ are those of dIRL_c
- Rules of $\mathsf{dIRL}_c^{\leq}$ are:

(Adj-$\wedge$) from φ and ψ derive $\varphi \wedge \psi$

(MP-r) if $\vdash_{\mathrm{dIRL}_c} \varphi \to \psi$, then from φ and $\varphi \to \psi$, derive ψ

(CNG-r) if $\vdash_{\mathrm{dIRL}_c} \varphi \leftrightarrow \psi$, then from $\varphi \leftrightarrow \psi$ derive $\circ\varphi \leftrightarrow \circ\psi$

(Max-r) if $\vdash_{\mathrm{dIRL}_c} \varphi \vee \neg\varphi \vee \neg\psi$, then from $\varphi \vee \neg\varphi \vee \neg\psi$ derive $\psi \to \circ\varphi$

We will denote by $\vdash^{\leq}_{\mathsf{dIRL}_c}$ its corresponding notion of proof. By Proposition 2.1, this axiomatization is sound and complete w.r.t. the semantical consequence relation $\models^{\leq}_{\mathsf{dIRL}_c}$, defined in the obvious way.

Observe that the rules (MP-r), (CNG-r) and (Max-r) are *restricted* in the sense that the premises are required to satisfy an extra condition, to be theorems of the logic.

Now we can check that $\mathsf{dIRL}^{\leq}_c$ is in fact a strong LFI in the sense of Definition 3.2.

Proposition 5.6. *The logic $\mathsf{dIRL}^{\leq}_c$ is a strong Logic of Formal Inconsistency w.r.t. the negation $\neg$ and the consistency operator $\circ$.*

Proof. By the above mentioned completeness of $\mathsf{dIRL}^{\leq}_c$ w.r.t. $\models^{\leq}_{\mathsf{dIRL}_c}$, it is enough to check the conditions (i.a), (i.b), (i.c) and (ii) in Definition 3.2 for the semantical consequence relation $\models^{\leq}_{\mathsf{dIRL}_c}$. If p, q are two different propositional variables, then we have:

(i.a) any evaluation e such that $e(p) > 0$ and $e(q) = 0$ is such that $e(p \wedge \neg p) > e(q)$ and hence $p, \neg p \not\models^{\leq}_{\mathsf{dIRL}_c} q$;

(i.b) any evaluation e such that $e(p) = e(\circ p) = 1$ and $e(q) = 0$ is such that $e(p \wedge \circ p) > e(q)$, and hence $p, \circ p \not\models^{\leq}_{\mathsf{dIRL}_c} q$;

(i.c) any evaluation e such that $e(\neg p) = e(\circ p) = 1$ and $e(q) = 0$ is such that $e(\neg p \wedge \circ p) > e(q)$, and hence $\neg p, \circ p \not\models^{\leq}_{\mathsf{dIRL}_c} q$; and

(ii) any evaluation e is such that $e(p \wedge \neg p \wedge \circ p) = 0$ and hence $p, \neg p, \circ p \models^{\leq}_{\mathsf{dIRL}_c} \bot$. □

In the context of LFIs, it is a desirable property to recover the classical reasoning by means of the consistency connective $\circ$ (see [13]). Specifically, let CPL be classical propositional logic. If L is a given LFI such that its reduct to the language of CPL is a sublogic of CPL, then a DAT (Derivability Adjustment Theorem) for L with respect to CPL is as follows: for every finite set of formulas $\Gamma \cup \{\varphi\}$ in the language of CPL, there exists a finite set of formulas Θ in the language of L, whose variables occur in formulas of $\Gamma \cup \{\varphi\}$, such that

$$\text{(DAT)} \quad \Gamma \vdash_{\mathsf{CPL}} \varphi \ \text{ iff } \ \circ(\Theta) \cup \Gamma \vdash_{\mathsf{L}} \varphi.$$

When the operator $\circ$ enjoys the *propagation property* in the logic L with respect to a set X of classical connectives, i.e. when

$$\circ\varphi_1, \ldots, \circ\varphi_n \vdash_{\mathsf{L}} \circ\#(\varphi_1, \ldots, \varphi_n),$$

for every n-ary connective $\# \in X$ and formulas $\varphi_1, \ldots, \varphi_n$ built with connectives from X, then the DAT takes the following, simplified form: for every finite set of formulas $\Gamma \cup \{\varphi\}$ in the language of CPL,

$$\text{(PDAT)} \quad \Gamma \vdash_{\mathsf{CPL}} \varphi \ \text{ iff } \ \{\circ p_1, \ldots, \circ p_m\} \cup \Gamma \vdash_{\mathsf{L}} \varphi$$

where $\{p_1, \ldots, p_m\}$ is the set of propositional variables occurring in $\Gamma \cup \{\varphi\}$.

In particular, checking whether $\mathsf{dIRL}_c^{\leq}$ satisfies the propagation property for the connectives $X = \{\bot, \wedge, \&, \to\}$ amounts to check the following conditions:

$$\begin{cases} \vdash_{\mathsf{dIRL}_c} \circ\bot \\ \vdash_{\mathsf{dIRL}_c} (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \# \psi), \quad \text{for each binary } \# \in X \end{cases}$$

In Proposition 5.8 below we will show that the operator $\circ$ of dIRL_c satisfies the propagation property with respect to all connectives of logic dIRL.[7] Our proof is algebraic and for that we will need the claims proved in the next lemma.

Lemma 5.7. *The following equations hold in the quasivariety of* dIRL_c-*algebras:*

(i) $x \wedge {\sim}x \wedge {\sim}y \wedge \circ(x \vee (y \wedge z)) = 0$

(ii) $(x * y) \vee (({\sim}x \vee {\sim}\circ(x)) \wedge y) = y.$

Proof. (i) By ($\circ$1), we know that the equation $(x \vee y) \wedge {\sim}(x \vee y) \wedge \circ(x \vee y) = 0$ holds, that is, we have $(x \vee y) \wedge {\sim}x \wedge {\sim}y \wedge \circ(x \vee y) = 0$. But since $x \leq (x \vee y)$ we also have that $x \wedge {\sim}x \wedge {\sim}y \wedge \circ(x \vee y) = 0$.

Now, by replacing y by $y \wedge z$ in the former equation, we get $x \wedge {\sim}x \wedge {\sim}(y \wedge z) \wedge \circ(x \vee (y \wedge z)) = 0$, and since ${\sim}(y \wedge z) \geq {\sim}y$, we finally get $x \wedge {\sim}x \wedge {\sim}y \wedge \circ(x \vee (y \wedge z)) = 0$.

(ii) We start from the equation $x \vee {\sim}x \vee {\sim}\circ(x) = 1$, we multiply by y in both sides and get the following equations:

$y * (x \vee {\sim}x \vee {\sim}\circ(x)) = y,$
$(x * y) \vee y * ({\sim}x \vee {\sim}\circ(x)) = y,$
$y = (x * y) \vee (y * ({\sim}x \vee {\sim}\circ(x))) \leq (x * y) \vee (y \wedge ({\sim}x \vee {\sim}\circ(x))) \leq y \vee y = y.$

Hence $(x * y) \vee (y \wedge ({\sim}x \vee {\sim}\circ(x))) = y.$

□

Proposition 5.8. *The logic* dIRL_c *satisfies the following conditions:*

- $\vdash_{\mathsf{dIRL}_c} \circ\bot$, *and*
- $\vdash_{\mathsf{dIRL}_c} (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \# \psi)$, *for* $\# \in \{\wedge, \vee, \&, \to\}$,

and thus, in $\mathsf{dIRL}_c^{\leq}$ *the consistency connective* $\circ$ *satisfies the propagation properties w.r.t. all connectives of the logic* dIRL.

[7] The proof of the propagation property for the monoidal conjunction & is in fact a re-elaboration of a proof found by the automated theorem-prover Prover9 [37].

Proof. The first condition is satisfied since $\circ\bot$ is a theorem of the logic, as observed above after Def. 5.1. The proof is algebraic, so we have to prove that, for any dIRL$_c$-algebra $(\mathbf{A}, \circ)$, the following condition holds: $\circ(x) \wedge \circ(y) \leq \circ(x\#y)$, for all $\# \in \{\wedge, \vee, *\}$. As for the lattice operations $\wedge$ and $\vee$, we have

$$\begin{aligned}
\circ(x \wedge y) \quad &= \max\{z \mid (x \wedge y) \wedge {\sim}(x \wedge y) \wedge z = 0\} \\
&= \max(\max\{z \mid x \wedge y \wedge {\sim}x \wedge z = 0\}, \max\{z \mid x \wedge y \wedge {\sim}y \wedge z = 0\}) \\
&\geq \max(\max\{z \mid x \wedge {\sim}x \wedge z = 0\}, \max\{z \mid y \wedge {\sim}y \wedge z = 0\}) \\
&= \circ(x) \vee \circ(y) \geq \circ(x) \wedge \circ(y). \\
\circ(x \vee y) \quad &= \max\{z \mid (x \vee y) \wedge {\sim}(x \vee y) \wedge z = 0\} \\
&= \max(\max\{z \mid x \wedge {\sim}x \wedge {\sim}y \wedge z = 0\}, \max\{z \mid y \wedge {\sim}x \wedge {\sim}y \wedge z = 0\}) \\
&\geq \max(\max\{z \mid x \wedge {\sim}x \wedge z = 0\}, \max\{z \mid y \wedge {\sim}y \wedge z = 0\}) \\
&= \circ(x) \vee \circ(y) \geq \circ(x) \wedge \circ(y).
\end{aligned}$$

Next, let us show that $\circ(x) \wedge \circ(y) \leq \circ(x * y)$. By ($\circ$2), it is enough to prove that $(x * y) \wedge {\sim}(x * y) \wedge \circ(x) \wedge \circ(y) = 0$. To do so, by (i) of Lemma 5.7, we can start from the equation

$$u \wedge {\sim}u \wedge {\sim}v \wedge \circ(u \vee (v \wedge t)) = 0,$$

and make the following substitutions: replace u by $x * y$, v by ${\sim}(x \wedge \circ(x))$ and t by y, and get

$$x * y \wedge {\sim}(x * y) \wedge {\sim}({\sim}(x \wedge \circ(x))) \wedge \circ(x * y \vee ({\sim}(x \wedge \circ(x)) \wedge y)) = 0,$$

that is, $x * y \wedge {\sim}(x * y) \wedge x \wedge \circ(x) \wedge \circ(x * y \vee ({\sim}(x \wedge \circ(x)) \wedge y)) = 0$. But $x * y \leq x$ and by (ii) of Lemma 5.7, $x * y \vee ({\sim}(x \wedge \circ(x)) \wedge y) = y$, and hence we finally get $x * y \wedge {\sim}(x * y) \wedge \circ(x) \wedge \circ(y) = 0$, as claimed.

Finally, since by (ii) of Proposition 3.13 we have $\circ(x) = \circ({\sim}x)$, and $x \to y = {\sim}(x * {\sim}y)$, the propagation property for $\to$, i.e. $\circ(x) \wedge \circ(y) \leq \circ(x \to y)$, directly follows as well.

□

Finally, we are interested in investigating whether we can expect some form close to (PDAT) for the logic $\mathsf{dIRL}_c^{\leq}$. Next theorem provides a result in this direction showing that it is possible to recover classical logic derivations inside the LFI logic $\mathsf{dIRL}_c^{\leq}$.

Theorem 5.9 (PDAT-like for $\mathsf{dIRL}_c^{\leq}$)**.** *Let $\Gamma \cup \{\varphi\}$ be a finite set of formulas in the language of* CPL *and let $\{p_1, \ldots, p_m\}$ the set of propositional variables appearing in $\Gamma \cup \{\varphi\}$. Then, there is a natural $k > 0$ such that:*

$$\Gamma \vdash_{\mathsf{CPL}} \varphi \quad \textit{iff} \quad \Gamma \vdash^{\leq}_{\mathsf{dIRL}_c} (\bigwedge_{i=1}^{m} \circ(p_i))^k \to \varphi.$$

Proof. Assume $\Gamma \vdash_{\mathsf{CPL}} \varphi$, or equivalently, $\vdash_{\mathsf{CPL}} \Gamma^{\wedge} \to \varphi$. First of all, observe that we also have

$$\{p_i \vee \neg p_i \ : \ i = 1, 2, \ldots, m\} \vdash_{\mathsf{dIRL}} \Gamma^{\wedge} \to \varphi,$$

since the class of Boolean subalgebras of distributive involutive residuated lattices is in fact the whole class of Boolean algebras. Then by the local deduction-detachment theorem of dIRL, there is a natural $k > 0$ such that $\vdash_{\mathsf{dIRL}} (\bigwedge_{i=1}^{m}(p_i \vee \neg p_i))^k \to (\Gamma^\wedge \to \varphi)$, and thus this theorem is also valid in dIRL_c. But this implies, by (iii) of Lemma 5.2, $\vdash_{\mathsf{dIRL}_c} (\bigwedge_{i=1}^{m}(\circ p_i))^k \to (\Gamma^\wedge \to \varphi)$ and hence $\Gamma \vdash_{\mathsf{dIRL}_c^{\leq}} (\bigwedge_{i=1}^{m} \circ(p_i))^k \to \varphi$ as well.

Conversely, assume $\vdash_{\mathsf{dIRL}_c} (\bigwedge_{i=1}^{m}(\circ p_i))^k \to (\Gamma^\wedge \to \varphi)$, and let e be any evaluation on the two-element Boolean algebra $\mathbf{2}$. Since $\mathbf{2}$ is a dIRL_c-algebra with $\circ(0) = \circ(1) = 1$, we have $e((\bigwedge_{i=1}^{m} \circ p_i)^k \to (\Gamma^\wedge \to \varphi)) = 1$. But then we necessarily have $e(\Gamma^\wedge \to \varphi) = 1$, because $e(\bigwedge_{i=1}^{m} \circ p_i) = 1$. Therefore $\Gamma^\wedge \to \varphi$ is a CPL-tautology and so $\Gamma \vdash_{\mathsf{CPL}} \varphi$. □

Note that, in the above formulation of the PDAT theorem for dIRL_c, the natural number k depends on the formulas $\Gamma \cup \{\varphi\}$ involved, and hence it is not possible to fix such exponent k in advance. However, if we replace the logic dIRL_c by any of its axiomatic extensions L validating the n-potency axiom

$$\varphi^n \to \varphi^{n+1}$$

for some n, then one can be more precise and state the PDAT theorem for $\mathsf{L}_c^{\leq}$: for any set of formulas $\Gamma \cup \{\varphi\}$, the following condition holds:

$$\Gamma \vdash_{\mathsf{CPL}} \varphi \;\text{ iff }\; \Gamma \vdash_{\mathsf{dIRL}_c}^{\leq} (\bigwedge_{i=1}^{m} \circ(p_i))^n \to \varphi.$$

In particular, when L is Nelson logic or its prelineal extension NM logic, which are 2-potent logics, we can take $n = 2$ in the previous expression.

5.2 dIRL-logic with a maxB-consistency operator

In this section we consider the logic $\mathsf{dIRL}_c^{\mathsf{mB}}$ corresonding to the quasivariety of $\mathrm{dIRL}_c^{\mathsf{mB}}$-algebras and its paraconsistent degree preserving companion $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$. The content is mostly parallel to the previous section, only some details change. In particular we get a simpler PDAT theorem due to the fact that the consistency connective is Boolean in this case.

Definition 5.10. The logic $\mathsf{dIRL}_c^{\mathsf{mB}}$ is the expansion of the logic dIRL, in a language which incorporates a new unary connective $\circ$, with the following additional axioms:

(A1) $\neg(\varphi \wedge \neg\varphi \wedge \circ\varphi)$
(A2) $\circ\varphi \vee \neg\circ\varphi$

and inference rules:

$$\text{(CNG)} \quad \frac{\varphi \leftrightarrow \psi}{\circ\varphi \leftrightarrow \circ\psi} \qquad\qquad \text{(MaxB)} \quad \frac{\neg(\varphi \wedge \neg\varphi \wedge \psi), \quad \psi \vee \neg\psi}{\psi \to \circ\varphi}$$

Analogously to the case of dIRL_c, thanks to Lemma 3.7, we also have that $\mathsf{dIRL}_c^{\mathsf{mB}}$ is a conservative expansion of dIRL.

We can check now that $\mathsf{dIRL}_c^{\mathsf{mB}}$ keeps validating the same properties in Lemma 5.2 for dIRL_c, plus a new one.

Lemma 5.11. *The logic* $\mathsf{dIRL}_c^{\mathsf{mB}}$ *enjoys the following properties:*

(i) $\varphi \vee \neg\varphi \vdash_{\mathsf{dIRL}_c^{\mathsf{mB}}} \circ\varphi$

(ii) $\mathsf{dIRL}_c^{\mathsf{mB}} \vdash \circ\varphi \to \neg\varphi \vee \varphi$

(iii) $\varphi \vee \neg\varphi \dashv\vdash_{\mathsf{dIRL}_c^{\mathsf{mB}}} \circ\varphi$

(iv) $\mathsf{dIRL}_c^{\mathsf{mB}} \vdash \circ\circ\varphi$.

Proof. (i), (ii) and (iii) are as in Lemma 5.2. As for (iv), using (Max) with $\psi = \top$, we have $\neg(\circ\varphi \wedge \neg\circ\varphi) \vdash \circ\circ\varphi$, but the premise is in fact Axiom (A2). □

Again, as in the case of the logic dIRL_c, it is easy to check that, due to the (Cong) rule for $\circ$, $\mathsf{dIRL}_c^{\mathsf{mB}}$ is a Rasiowa implicative logic and hence it is algebraizable. The equivalent algebraic semantics is now given by the quasi-variety of $\mathrm{dIRL}_c^{\mathsf{mB}}$-algebras.

We now introduce the degree-preserving companion of $\mathsf{dIRL}_c^{\mathsf{mB}}$.

Definition 5.12. The degree-preserving companion of logic $\mathsf{dIRL}_c^{\mathsf{mB}}$ is the logic $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ defined by the following axioms and rules:

- Axioms of $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ are those of $\mathsf{dIRL}_c^{\mathsf{mB}}$
- Rules of $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ are those of $\mathsf{dIRL}_c^{\leq}$ but replacing the rule **(Max-r)** by the following rule:

(MaxB-r) if $\vdash_{\mathsf{dIRL}_c^{\mathsf{mB}}} \neg(\varphi \wedge \neg\varphi \wedge \psi)$ and $\vdash_{\mathsf{dIRL}_c^{\mathsf{mB}}} \psi \vee \neg\psi$, then from $\neg(\varphi \wedge \neg\varphi \wedge \psi)$ and $\psi \vee \neg\psi$ derive $\psi \to \circ\varphi$

We will denote by $\vdash^{\leq}_{\mathsf{dIRL}_c^{\mathsf{mB}}}$ its corresponding notion of proof. As in the previous subsection, by Proposition 2.1, this axiomatization of $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ is sound and complete w.r.t. the semantical consequence relation $\models^{\leq}_{\mathsf{dIRL}_c^{\mathsf{mB}}}$, defined in the obvious way, and moreover $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ is a strong LFI as well. The proof is the same of Propostion 5.6 and it is omitted.

Proposition 5.13. *The logic* $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ *is a strong Logic of Formal Inconsistency w.r.t. to the negation* $\neg$ *and the consistency operator* $\circ$.

As in the case of the logic dIRL_c, the $\circ$ connective in the logic $\mathsf{dIRL}_c^{\mathsf{mB}}$ nicely propagates through the rest of connectives $X = \{\bot, \wedge, \vee, \&, \to\}$. This is shown next, after a previous lemma, formally similar to Lemma 5.8, but this one much easier to prove.

Lemma 5.14. *The following provability holds in* $\mathsf{dIRL}_c^{\mathsf{mB}}$*:*

$$\varphi \vee \neg\varphi, \psi \vee \neg\psi \vdash_{\mathsf{dIRL}_c^{\mathsf{mB}}} (\varphi \# \psi) \vee \neg(\varphi \# \psi)$$

for every binary connective $\# \in X$*.*

Proof. Using a semantic argument, it holds since the binary connectives $\wedge, \vee, \&, \to$ are closed under Boolean values in every evaluation over any $\mathrm{dIRL}_c^{\mathsf{mB}}$-algebra. □

Proposition 5.15. $\mathsf{dIRL}_c^{\mathsf{mB}}$ *satisfies the following conditions:*

- $\vdash_{\mathsf{dIRL}_c^{\mathsf{mB}}} \circ\bot$*, and*
- $\vdash_{\mathsf{dIRL}_c^{\mathsf{mB}}} (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \# \psi)$*, for every* $\# \in \{\wedge, \vee, \&, \to\}$*,*

and thus, in $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ *the consistency connective* $\circ$ *satisfies the propagation property w.r.t. all the connectives in* $X = \{\bot, \wedge, \vee, \&, \to\}$*.*

Proof. Straightforward from the above lemma, taking into account that, by (iii) of Lemma 5.11, $\varphi \vee \neg\varphi \dashv\vdash_{\mathsf{dIRL}_c^{\mathsf{mB}}} \circ\varphi$. □

This propagation property for $\circ$, together with the fact that $\circ\varphi$ is a Boolean formula for any φ, allows us to formulate the following PDAT theorem for $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ in simpler terms.

Theorem 5.16 (PDAT for $(\mathsf{NL}_c^{\mathsf{mB}})^{\leq}$)**.** *Let* $\Gamma \cup \{\varphi\}$ *be a finite set of formulas in the language of* CPL *and let* $\{p_1, \ldots, p_m\}$ *the set of propositional variables appearing in* $\Gamma \cup \{\varphi\}$*. Then*

$$\Gamma \vdash_{\mathsf{CPL}} \varphi \quad \textit{iff} \quad \{\circ p_1, \ldots, \circ p_m\} \cup \Gamma \vdash^{\leq}_{\mathsf{dIRL}_c^{\mathsf{mB}}} \varphi.$$

Proof. The proof is essentially the same as in Theorem 5.9 for $\mathsf{dIRL}_c^{\leq}$. The only difference is that now $(\bigwedge_{i=1}^{m} \circ(p_i))^n$ turns out to be equivalent to just $\bigwedge_{i=1}^{n} \circ(p_i)$ since this is a Boolean formula, and this conjunction can be moved back to the premise as the set of formulas $\{\circ p_1, \ldots, \circ p_m\}$. □

5.3 dIRL-logic with a Bmax-consistency operator

Finally, we consider in this section the logic resulting from expanding the logic of distributive involutive residuated lattices with a Bmax-consistency connective and its paraconsistent degree-preserving companion. Both of them are very similar to the case of maxB-consistency connective of previous section.

Definition 5.17. The logic $\mathsf{dIRL}_c^{\mathsf{Bm}}$ is the axiomatic extension of dIRL_c with the following additional axiom:

(A4) $\circ\varphi \vee \neg\circ\varphi$

It is clear that $\mathsf{dIRL}_c^{\mathsf{Bm}}$ is thus algebraizable, with equivalent algebraic semantics given by the quasivariety of $\mathrm{dIRL}_c^{\mathsf{Bm}}$-algebras.

Proposition 5.18. $\mathsf{dIRL}_c^{\mathsf{Bm}}$ *is strongly complete w.r.t the class of* $\mathrm{dIRL}_c^{\mathsf{Bm}}$*-algebras.*

Actually, by definition, the logic $\mathsf{dIRL}_c^{\mathsf{Bm}}$ is an extension of dIRL_c, but it is also an extension of $\mathsf{dIRL}_c^{\mathsf{mB}}$, as the dIRL_c's rule (Max) is stronger than the $\mathsf{dIRL}_c^{\mathsf{mB}}$'s rule (MaxB), and hence the latter is derivable in $\mathsf{dIRL}_c^{\mathsf{Bm}}$. As a consequence, all the properties that have been shown for dIRL_c and $\mathsf{dIRL}_c^{\mathsf{mB}}$ also hold for $\mathsf{dIRL}_c^{\mathsf{Bm}}$.

This has also impact on the properties of the corresponding degree-preserving companion of $\mathsf{dIRL}_c^{\mathsf{Bm}}$.

Definition 5.19. The degree-preserving companion of logic $\mathsf{dIRL}_c^{\mathsf{Bm}}$ is the logic $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$ defined as the axiomatic extension of $\mathsf{dIRL}_c^{\leq}$ with the axiom (A4).

Of course, as in the previous cases, the logic $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$ is a strong LFI (details are omitted). Moreover, it follows that the consistency connective $\circ$ in $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$ also satisfies the same propagation properties and the same PDAT theorem as for $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$.

Theorem 5.20 (PDAT for $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$)**.** *Let $\Gamma \cup \{\varphi\}$ be a finite set of formulas in the language of* CPL *and let $\{p_1, \ldots, p_m\}$ be the set of propositional variables appearing in $\Gamma \cup \{\varphi\}$. Then, the following condition holds:*

$$\Gamma \vdash_{\mathsf{CPL}} \varphi \quad \textit{iff} \quad \{\circ p_1, \ldots, \circ p_m\} \cup \Gamma \vdash^{\leq}_{\mathsf{dIRL}_c^{\mathsf{Bm}}} \varphi.$$

5.4 Logics of Formal Undeterminedness

In this final subsection, we would like to emphasize that the logics $\mathsf{dIRL}_c^{\leq}$, $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ and $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$ are not only LFIs but also *Logics of Formal Undeterminedness* (*LFUs*), the latter being introduced in [38].

LFUs are in a sense dual to LFIs, since in the same way that a consistency operator in a LFI controls the explosion in the presence of a contradiction, a "determinedness" operator $\circ$ in a LFU controls the law of the Excluded Middle, namely: given a paracomplete logic L, i.e. such that $\not\vdash_{\mathsf{L}} \varphi \vee {\sim}\varphi$, if $\circ$ is a determinedness operator in L, then $\circ\varphi \vdash_{\mathsf{L}} \varphi \vee {\sim}\varphi$. More formally, we have the following definition.

Definition 5.21. Let $\mathsf{L} = \langle \Sigma, \vdash \rangle$ be a Tarskian, finitary and structural logic defined over a signature Σ with a disjunction $\vee$, a negation $\sim$ and a primitive or defined unary connective $\circ$. Then L is said to be a *Logic of Formal Undeterminedness* with respect to $\sim$ and $\circ$ if the following holds:

(i) $\nvdash \varphi \vee {\sim}\varphi$, for some φ;

(ii) there is a formula φ such that

  (ii.a) $\circ\varphi \nvdash \varphi$;

  (ii.b) $\circ\varphi \nvdash {\sim}\varphi$;

(iii) $\circ\varphi \vdash \varphi \vee {\sim}\varphi$, for every φ.

It turns out that, as we will formally show below, the consistency operator $\circ$ of any of the logics logics dIRL_c, $\mathsf{dIRL}_c^{\mathsf{mB}}$ and $\mathsf{dIRL}_c^{\mathsf{Bm}}$ is a determinedness operator as well, and hence all these explosive logics are LFUs. Moreover, this is also the case for the paraconsistent logics $\mathsf{dIRL}_c^{\leq}$, $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ and $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$. This is clear for the explosive logics dIRL_c, $\mathsf{dIRL}_c^{\mathsf{mB}}$ and $\mathsf{dIRL}_c^{\mathsf{Bm}}$ due to (i) of Lemma 5.2, but also for the logics $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ and $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$ since in these logics $\circ\varphi$ is a Boolean formula. It is not so obvious in the case of the logic $\mathsf{dIRL}_c^{\leq}$, where checking condition (iii) above

$$\circ\varphi \vdash^{\leq}_{\mathsf{dIRL}_c} \varphi \vee {\sim}\varphi,$$

amounts to check the validity of the inference

$$\vdash_{\mathsf{dIRL}_c} \circ\varphi \to \varphi \vee {\sim}\varphi.$$

But this follows from observing that the formula $({\sim}\delta \vee \gamma) \to (\delta \to \gamma)$ is a theorem in dIRL, and that ${\sim}\circ\varphi \vee (\varphi \vee {\sim}\varphi)$ is a theorem of dIRL_c, as it is an equivalent rewriting of the axiom (A1).

In algebraic terms, we can analyze the above observations as follows: if a operator $\circ$ on a dIRL-algebra satisfies the basic equation we imposed for a consistency operator, that is

($\circ$1) $x \wedge {\sim}x \wedge \circ(x) = 0$

then the following inequality is valid as well:

($\circ d$) $\circ(x) \leq x \vee {\sim}x$.

As a matter of fact, we have the following lemma.

Lemma 5.22. *Let* $\mathbf{A}$ *be a dIRL-algebra and let* $\circ : A \to A$ *be a unary operation. Then* ($\circ$1) *implies* ($\circ d$).

Proof. In every residuated lattice $x * y \leq x \wedge y$ and hence, by ($\circ$1), one has $\circ(x) * (x \wedge {\sim}x) \leq \circ(x) \wedge (x \wedge {\sim}x) = 0$ and hence $\circ(x) \leq {\sim}(x \wedge {\sim}x)$. By the involutive property of $\sim$, then, $\circ(x) \leq {\sim}(x \wedge {\sim}x) = x \vee {\sim}x$. Therefore, it is clear that ($\circ$1) implies ($\circ d$). □

This means in particular that all the three logics $\mathsf{dIRL}_c^{\leq}$, $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ and $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$, besides being LFIs, are LFUs as well. In other words, these logics are *strict LFIUs*, as coined in [14]. Therefore, in these logics, the operator $\circ$ is in fact a *classicality* operator in the sense of Omori in his recent paper [43].

On the other hand, the other implication ($\circ d$) implies ($\circ$1) does not hold in general, since ($\circ d$) is equivalent to $\circ(x) * (x \wedge {\sim}x) = 0$ which, in general, is strictly weaker than ($\circ$1). However, it indeed holds true if we require $\circ(x)$ be Boolean.

Lemma 5.23. *Let* $\mathbf{A}$ *be a dIRL-algebra and let* $\circ : A \to A$ *be a unary operation such that the following equation holds:*

$$\circ(x) \vee {\sim}\circ(x) = 1$$

Then ($\circ d$) *implies* ($\circ$1).

Proof. It follows by easily checking that the following equations hold in any dIRL-algebra: $x \to (x \wedge y \to x * y) = 1$ and $\neg x \to (x \wedge y \to x * y) = 1$, and hence the equation $(x \vee \neg x) \to (x \wedge y \to x * y) = 1$ holds as well. Therefore if $\circ(x) \vee \sim\circ(x) = 1$, then we have $\circ(x) \wedge y \to \circ(x) * y = 1$, for every y. In particular, taking $y = x \wedge \sim x$, we get $\circ(x) \wedge (x \wedge \sim x) \to \circ(x) * (x \wedge \sim x) = 1$, that is, $\circ(x) \wedge (x \wedge \sim x) \leq \circ(x) * (x \wedge \sim x)$, as desired. □

Therefore, being LFI turns out to be equivalent to being LFU if we replace ($\circ 1$) by ($\circ d$) in the logics $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ and $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$, but not in $(\mathsf{dIRL}_c)^{\leq}$.

6 Concluding remarks

In this paper we have been concerned with introducing Logics of Formal Inconsistency (LFIs) upon the class of substructural logics having subvarieties of distributive involutive residuated lattices as algebraic semantics. To do so, we have first introduced and studied, from an algebraic point of view, distributive involutive residuated lattices expanded with three suitable types of consistency operators. Particular attention has been paid to the subvariety of Nelson lattices. Then, the corresponding truth-preserving and degree-preserving logics have been axiomatised, the latter being paraconsistent and falling within the class of LFIs.

At this point there are several observations we deem interesting to discuss. The first one concerns with our request for the involutive lattices to be distributive. In fact, although distributivity is quite often a necessary request (see for instance the proofs of Theorem 3.8 and Theorem 5.9) it is questionable if similar results could be obtained getting rid of that property. Besides being more general, avoiding distributivity might frame the results of this paper in the setting of affine Linear Logic without exponentials of which (not necessarily distributive) involutive residuated lattices provide an algebraic semantics.

The second one is in fact an open problem. In Section 4.2 we have shown an example of dIRL$_c$-algebra (in fact a NL$_c$-algebra) that is subdirectly irreducible but not simple. It would be interesting to know whether there exists a subdirectly irreducible IMTL$_c$-algebra that is not simple as well.

The third observation concerns the status of the modus ponens rule in the logics $\mathsf{dIRL}_c^{\leq}$, $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ and $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$. By the very definition of degree-preserving logics, modus ponens is not a valid rule in any of them, only a restricted version where the implication is a theorem holds. However, in the logics $(\mathsf{dIRL}_c^{\mathsf{mB}})^{\leq}$ and $(\mathsf{dIRL}_c^{\mathsf{Bm}})^{\leq}$, due to the PDAT theorems for them (Theorems 5.16 and 5.20 resp.), it is clear that assuming the propositions involved are *consistent* guarantee the validity of the following modus ponens-like inference:

$$\{\varphi, \varphi \to \psi, \circ\varphi, \circ\psi\} \vdash \psi.$$

The situation in the logic $\mathsf{dIRL}_c^{\leq}$ is different, as even this weaker form of modus ponens is not valid. Indeed, take the dIRL$_c$-algebra **L** in depicted in Fig. 2 and an evaluation e on L such that $e(\varphi) = e$ and $e(\psi) = a$, and hence $e(\circ\varphi) = \circ(e) =$

b and $e(\circ\psi) = \circ(a) = b$. A simple computation also shows that $e(\varphi \to \psi) = c$. Then we have $e(\varphi) \wedge e(\varphi \to \psi) \wedge e(\circ\varphi) \wedge e(\circ\psi) = e \wedge c \wedge b \wedge b = b \not\leq a = e(\psi)$. Therefore,

$$\{\varphi, \varphi \to \psi, \circ\varphi, \circ\psi\} \not\vdash^{\leq}_{\mathsf{dIRL}_c} \psi.$$

Nevertheless, such a 'witnessed form' of modus ponens holds in $\mathsf{dIRL}^{\leq}_c$ if we replace the residual implication $\varphi \to \psi$ by the material implication $\neg\varphi \vee \psi$. In fact, using axiom (A1), it is not difficult to check that $\varphi \wedge (\neg\varphi \vee \psi) \wedge \circ\varphi \wedge \circ\psi$ is logically equivalent to $\varphi \wedge \psi \wedge \circ\varphi \wedge \circ\psi$, that logically implies ψ. Therefore the following pattern of inference

$$\{\varphi, \neg\varphi \vee \psi, \circ\varphi, \circ\psi\} \vdash^{\leq}_{\mathsf{dIRL}_c} \psi.$$

is indeed valid in $\mathsf{dIRL}^{\leq}_c$.

We finish with two further questions for future research. We have seen that, in the prelinear case, the quasivarieties of $\mathrm{IMTL}^{\mathsf{mB}}_c$- and $\mathrm{IMTL}^{\mathsf{Bm}}_c$-algebras are in fact varieties, since in these algebras the Baaz-Delta connective Δ is definable as $\Delta(x) = x \wedge \circ(x)$. Thus a first question to be investigated is whether the quasivarieties of dIRL_c-, $\mathrm{dIRL}^{\mathsf{mB}}_c$- and $\mathrm{dIRL}^{\mathsf{Bm}}_c$-algebras are actually varieties or, otherwise, proper quasivarieties. A second question is to study an alternative way of defining LFIs related to dIRL logics without relying on the degree-preserving companions. The idea would be to start from the work of Busaniche and Cignoli [8] on the algebraic characterization of the paraconsistent Nelson logic N4 in terms of a certain variety of non-integral involutive residuated lattices with a constant, and try to generalise it to the logic dIRL, and then adding a suitable consistency operator.

Acknowledgments

The authors are indebted to the anonymous reviewers for their remarks and suggestions that have helped to significantly improve the final layout of this paper. Esteva, Flaminio and Godo acknowledge partial support by the Spanish project PID2019-111544GB-C21. Flaminio also acknowledges partial support by the Spanish Ramón y Cajal research program RYC-2016-19799. Figallo-Orellano acknowledges the support of a post-doctoral grant 2016/21928-0 from São Paulo Research Foundation (FAPESP), Brazil.

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