On Triangular Norm Based Fuzzy Description Logics

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Abstract— *Description Logics (DLs) are knowledge representation languages useful to represent concepts and roles. Fuzzy Description Logics (FDLs) incorporate both vague concepts and vague roles modeling them as fuzzy sets and fuzzy relations respectively. In the present paper, following ideas from Hájek, we propose the use of* t*-norm based (fuzzy) logics with truth constants in the language as logics underlying the fuzzy description language. We introduce the languages* $\mathcal{ALC}_{L^*(\mathbf{S})}$ *and* $\mathcal{ALC}_{L^*_{\sim}(\mathbf{S})}$ *as an adequate syntactical counterpart of some semantic calculi given in different works dealing with FDLs.*

Keywords— Description Logics, Fuzzy Description Logics, t-Norm Based Fuzzy Logics, Truth-constants, Involutive negation.

1 Introduction

Description Logics (DLs) are knowledge representation languages (particularly suited to specify formal ontologies), which have been studied extensively over the last two decades. A full reference manual of the field is [1]. The vocabulary of DLs consists of *concepts*, which denote sets of individuals, and *roles*, which denote binary relations among individuals. From atomic concepts and roles DL systems allow, by means of *constructors*, to build complex descriptions of both concepts and roles, which are used to describe a domain through a knowledge base (KB) containing the definitions of relevant domain concepts or some hierarchical relationships among them (TBox) and a specification of properties of the domain instances (ABox). One of the main issues of DLs is the fact that the semantics is given in a Tarski-style presentation and the statements in both TBox and ABox can be identified with formulae in first-order logic or a slight extension of it, and hence we can use reasoning to obtain implicit knowledge from the explicit knowledge in the KB.

A natural generalization to cope with vague concepts and relations consists in interpret DL concepts and roles as fuzzy sets and fuzzy relations, respectively. From this point of view, it is at the end of the last decade (from 1998) when several proposals of Fuzzy Description Logics (FDLs) were introduced (e.g., the first ones by Yen [19], Tresp and Molitor [18] and Straccia [14]). However, the logic framework behind these initial works is very limited. With the aim of enriching the expressive possibilities Hájek [9] proposes to take t-norm based fuzzy logics as logics underlying FDLs. This change of view gives a wide number of choices on which a DL can be based: for every particular problem we can consider the t-norm based (fuzzy) logic that seems to be more adequate. As an example, Hájek studies the FDL associated with the description language $\mathcal{ALC}$. After this work, several researchers on FDLs have developed approaches based on the spirit of Hájek's paper, even though their work is more related to expressiveness and algorithms than in its logical base (see for instance [16, 13, 17]).

The main motivation of the present work is based on the following consideration: since the axioms of the bases of knowledge in FDLs include truth degrees (see for instance [14]), a natural choice is to include symbols for these degrees in both, the description language and, as truth constants, in the t-norm based logic where that language is interpreted.

To this goal in the present paper we propose two new families of description languages, denoted by $\mathcal{ALC}_{L^*(\mathbf{S})}$ and $\mathcal{ALC}_{L^*_{\sim}(\mathbf{S})}$ that are extensions of the language $\mathcal{ALC}$ considered by Hájek in [9]. After some introductory notions we define their semantics and describe the corresponding knowledge base (TBox and ABox) from a syntactic and semantic perspective and, taking advantage of having truth constants in the logic, we define graded notions of validity, satisfiability and subsumption. We also give some representative example and some new results for the case of $\mathcal{ALC}$ language over Gödel logic with truth constants and an involutive negation.

2 Fuzzy Logic: basics

In the last decade a family of fuzzy logics as t-norm based fuzzy logics has defined and studied (see, for instance, the monograph [8] for the main notions used in this section). They are multi-valued systems with additive conjunction and disjunction, multiplicative conjunction, implication, negation and the constant $\bar{0}$ which are interpreted in $[0,1]$ as $\min$, $\max$, a continuous t-norm $*$, its residuum $\rightarrow_*$, the negation function $n(x) = x \rightarrow_* 0$ and 0, respectively. This interpretation is mainly defined by a continuous t-norm and its residuum which justify the name t-norm based logics. Remember that the main continuous t-norms are the minimum, the product and the Lukasiewicz since all other continuous t-norm are ordinal sum of these three basic ones.

2.1 From BL to the logic of a continuous t-norm

The Basic fuzzy Logic (BL) (defined in [8]) has the following basic connectives: *multiplicative conjunction* ($\&$) *implication* ($\rightarrow$) (both binary) and *falsity* ($\bar{0}$) (nullary). It is defined by the inference rule of *Modus Ponens* and the following schemata (taking $\rightarrow$ as the least binding connective):

ISBN: 978-989-95079-6-8

(BL1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(BL2) $\varphi \& \psi \to \varphi$
(BL3) $\varphi \& \psi \to \psi \& \varphi$
(BL4) $\varphi \& (\varphi \to \psi) \to \psi \& (\psi \to \varphi)$
(BL5a) $(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
(BL5b) $(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(BL6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(BL7) $\bar{0} \to \varphi$

The usual defined connectives are introduced as follows:

$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$,
$\varphi \wedge \psi := \varphi \& (\varphi \to \psi)$, $\quad \varphi \leftrightarrow \psi := (\varphi \to \psi) \& (\psi \to \varphi)$,
$\neg \varphi := \varphi \to \bar{0}$, $\quad \bar{1} := \neg \bar{0}$.

Łukasiewicz, Product and Gödel Logics can be obtained as axiomatic extensions of BL with the following axioms: $\neg\neg\varphi \to \varphi$ for Łukasiewicz; $(\varphi \wedge \neg\varphi) \to \bar{0}$ and $\neg\neg\chi \to (((\varphi \& \chi) \to (\psi \& \chi)) \to (\varphi \to \psi))$ for Product; and $\varphi \to \varphi \& \varphi$ for Gödel.

An *evaluation of propositional variables* is a mapping e assigning to each variable p a *truth value* $e(p) \in [0,1]$. Given a continuous t-norm $*$, the evaluation e is extended inductively to a mapping of all formulas into the so-called *standard algebra* $[0,1]_* = \langle [0,1], *, \to_*, \max, \min, 0, 1 \rangle$ defined on [0,1] by the t-norm and its residuum in the following way: $e(\varphi \& \psi) = e(\varphi) * e(\psi)$; $e(\varphi \to \psi) = e(\varphi) \to_* e(\psi)$; $e(\bar{0}) = 0$. In [3] it is proved that the formal system BL is sound and complete w.r.t. the standard algebras defined in $[0,1]$ by continuous t-norms and their residua, i.e., a formula φ is provable in BL if and only if it is a common tautology of all standard algebras defined by a continuous t-norm and its residuum. Consequently we say that BL is the logic of continuous t-norms and their residua. It is also well known that Łukasiewicz (Product, Gödel) Logic is the logic of the t-norm of Łukasiewicz (Product, Gödel) and its residuum in the sense that a formula is provable in each logic if and only if it is a tautology over the standard algebra defined by the corresponding t-norm and its residuum.

In [6] the logic L^* of each continuous t-norm $*$ and its residuum is proved to be finitely axiomatizable as extension of BL. Moreover it is also given an algorithm to find a finite set of axioms characterizing each logic L^*. When the negation $\neg\varphi := \varphi \to \bar{0}$ defined in L^* is not involutive, a new logic $L^*_{\sim}$, expanding L^* by adding an involutive negation, can be considered. This negation, denoted by $\sim$, could be introduced, as is done in the context of intuitionistic logic (see [11]) or in the context of Gödel or Product logics (cf.[5]) by adding the axioms

(~ 1) $\sim\sim \varphi \to \varphi$
(~ 2) $\sim (\varphi \vee \psi) \leftrightarrow (\sim \varphi \wedge \sim \psi)$
(~ 3) $\neg\varphi \to \sim \varphi$

Notice that having an involutive negation in the logic enriches the representational power of the logical language in a non-trivial way because a multiplicative (or strong) disjunction $\varphi \veebar \psi$ is definable now (by duality) as $\sim (\sim \varphi \& \sim \psi)$, being the associated truth function $\oplus$ defined as $x \oplus y := n(n(x) * n(y))$, where n is the truth function of $\sim$ and, on the other hand, a contrapositive implication $\varphi \hookrightarrow \psi$ is definable as $\sim \varphi \veebar \psi$, with truth function $\hookrightarrow_\oplus$ defined as $x \hookrightarrow_\oplus y = n(x) \oplus y$.

2.2 The logics $BL\forall$ and $L^*\forall$

The language of the *basic fuzzy predicate logic* $BL\forall$ consists of a set of *predicate symbols* $\mathcal{P} = \{P, Q, \ldots\}$, each together with its *arity* $n \geq 1$ and a set of *constant symbols* $\mathcal{C} = \{c, d, \ldots\}$. The logical symbols are *variable symbols* $x, y, \ldots$, *connectives* $\&, \to, \bar{0}$ and *quantifiers* $\forall, \exists$. Other connectives $(\vee, \wedge, \neg, \leftrightarrow, \bar{1})$ are defined as in BL. *Terms* are constant symbols and variable symbols. An *atomic formula* is an expression of the form $P(t_1, \ldots, t_n)$, where P is a predicate letter of arity n and $t_1, \ldots, t_n$ are terms. The set of predicate formulas is built from atomic formulas in the usual way.

A *fuzzy interpretation* for our language is a tuple $\mathbf{M} = \langle M, (r_P)_{P \in \mathcal{P}}, (m_c)_{c \in \mathcal{C}} \rangle$, where M is a set, for each n-ary predicate symbol P, r_P is a fuzzy n-ary relation $M^n \to [0,1]$; and for each constant symbol c, m_c is an element of M.

Given a continuous t-norm $*$, an $\mathbf{M}$-*evaluation* of the variables assigns to each variable x an element $v(x)$ of M. From M and v we define the *truth value of a term* t in the following way: $\|t\|_{\mathbf{M},v} = v(t)$ when t is a variable, and $\|t\|_{\mathbf{M},v} = m_c$ when t is a constant c. The *truth value of a formula* φ *for an evaluation* v, denoted by $\|\varphi\|^*_{\mathbf{M},v}$, is a value in $[0,1]$ defined inductively as follows:

$\|\varphi\|_{\mathbf{M},v}$	if φ is an atomic formula,
0	if $\varphi = \bar{0}$,
$\|\alpha\|^*_{\mathbf{M},v} * \|\beta\|^*_{\mathbf{M},v}$	if $\varphi = \alpha \& \beta$,
$\|\alpha\|^*_{\mathbf{M},v} \to_* \|\beta\|^*_{\mathbf{M},v}$	if $\varphi = \alpha \to \beta$,
$\inf\{\|\alpha\|^*_{\mathbf{M},v} : v \equiv_x v'\}$	if $\varphi = (\forall x)\alpha$,
$\sup\{\|\alpha\|^*_{\mathbf{M},v} : v \equiv_x v'\}$	if $\varphi = (\exists x)\alpha$,

where $v \equiv_x v'$ means that for all variables $y \neq x$, $v(y) = v'(y)$.

The *truth value of a formula* φ is defined by $\|\varphi\|^*_{\mathbf{M}} := \inf\{\|\varphi\|^*_{\mathbf{M},v} : v$ is an $\mathbf{M}$-evaluation$\}$. A formula φ is a $*$-*tautology* if $\|\varphi\|^*_{\mathbf{M}} = 1$ for every fuzzy interpretation $\mathbf{M}$; φ is a *standard tautology* if it is a $*$-*tautology* for each continuous t-norm $*$. The following standard tautologies are taken as axioms of the basic fuzzy predicate logic $BL\forall$ (see [8]): *a)* the axioms of BL; *b)* the following axioms on quantifiers:

($\forall 1$) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$),
($\exists 1$) $\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$),
($\forall 2$) $(\forall x)(\varphi \to \psi) \to (\varphi \to (\forall x)\psi)$ (x not free in φ),
($\exists 2$) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to \psi)$ (x not free in ψ),
($\forall 3$) $(\forall x)(\varphi \vee \psi) \to (\forall x)\varphi \vee \psi$ (x not free in ψ).

Deduction rules are (as in classical logic) *Modus Ponens* and *Generalization*. Notions of *proof*, *provability*, *theory*, etc., are defined in the usual way. Let $\mathcal{C}$ be an axiomatic extension of BL. $\mathcal{C}\forall$ is obtained by taking the axioms and rules of $BL\forall$ plus the axioms characterizing $\mathcal{C}$. Thus, given a continuous t-norm $*$, the predicate logic $L^*\forall$ is the logic obtained from L^* by adding to its axiomatization the schemas for quantifiers and the rule of generalization.

2.3 Adding truth constants to the language

T-norm based logics are infinite-valued logics. However, the advantage of being a many-valued logic is not used in current approaches since the semantic deduction of formulas do not take into account the *intermediate or partial truth degrees*. That is to say, current approaches use a truth-preserving con-

ISBN: 978-989-95079-6-8

sequence relation in the same way as in the classical logic, i.e. deduce true formulas (having value 1) from sets of true formulas. An elegant way to take advantage from being many-valued is to introduce truth constants into the language, as it is done by Pavelka in [12] and more recently in [8, 6, 4]. The approach considered in this paper is based in these ideas.

Given a continuous t-norm $*$, its residuum $\rightarrow_*$ and its corresponding logic L^*, let $\mathbf{S} = \langle S, *, \rightarrow_*, \max, \min, 0, 1\rangle$ be a *countable* (i.e., finite or enumerable) *subalgebra* of the corresponding standard algebra $[0,1]_*$. The expansion of L^* adding into the language a truth constant $\bar{r}$ for each $r \in S$, denoted by $L^*(\mathbf{S})$, is defined as follows:

i) the language of $L^*(\mathbf{S})$ is the one of L^* plus a truth constant $\bar{r}$ for each $r \in S$,

ii) the axioms and rules of $L^*(\mathbf{S})$ are those of L^* plus the *book-keeping* axioms: for each $r, s \in S \setminus \{0,1\}$, $\bar{r}\&\bar{s} \leftrightarrow \overline{r * s}$ and $(\bar{r} \rightarrow \bar{s}) \leftrightarrow \overline{r \rightarrow_* s}$.

Completeness results for propositional logic $L^*(\mathbf{S})$ when $*$ is a continuous t-norm has been fully studied in [4].

2.4 Defining the logics $L^*(\mathbf{S})\forall$ and $L^*_\sim(\mathbf{S})\forall$

When the negation associated to the continuous t-norm $*$ is not involutive, the logic $L^*_\sim(\mathbf{S})$ can be defined in a similar way although in this case $\mathbf{S}$ has to be a countable subalgebra of the algebra obtained by adding the truth function of the involutive negation $n(x) := 1 - x$ to the operations of $[0,1]_*$. Moreover we need to add the book-keeping axioms for the involutive negation: $\sim \bar{r} \leftrightarrow \overline{n(r)}$. The corresponding predicate logics $L^*(\mathbf{S})\forall$ and $L^*_\sim(\mathbf{S})\forall$ are respectively obtained from $L^*\forall$ and $L^*_\sim\forall$ by expanding the language with a truth constant $\bar{r}$ for every $r \in S$ and by adding the book keeping axioms. The truth value of the formula $\bar{r}$ is given by $\|\bar{r}\|^*_{\mathbf{M}} = r$. The logics $L^*(\mathbf{S})\forall$ and $L^*_\sim(\mathbf{S})\forall$ will be the basis of our proposal for the description languages presented in the next section.

3 The description languages $\mathcal{ALC}_{L^*(\mathbf{S})}$ and $\mathcal{ALC}_{L^*_\sim(\mathbf{S})}$

Similarly as first-order logic is the counterpart for interpreting the classic description language $\mathcal{ALC}$, the logics $L^*(\mathbf{S})\forall$ and $L^*_\sim(\mathbf{S})\forall$ will be the counterpart for interpreting the description languages $\mathcal{ALC}_{L^*(\mathbf{S})}$ and $\mathcal{ALC}_{L^*_\sim(\mathbf{S})}$ we will define in this section. In these languages, take an special role the so-called evaluated formulas. An *evaluated formula* is a formula of one of the types $\bar{r} \rightarrow \varphi$, $\varphi \rightarrow \bar{r}$ where φ is a formula without new truth constants (i.e., different from $\bar{0}$ and $\bar{1}$). In this setting $\bar{r} \leftrightarrow \varphi$ is definable as $(\bar{r} \rightarrow \varphi)\&(\varphi \rightarrow \bar{r})$. The name of evaluated formulas cames from the fact that $e(\bar{r} \rightarrow \varphi) = 1$ (resp. $e(\varphi \rightarrow \bar{r}) = 1$) if and only if $e(\varphi) \geq r$ (resp. $e(\varphi) \leq r$). Thus, evaluated formulas correspond to the type of formulas (under the notation $\langle\varphi, \succcurlyeq r\rangle$ and $\langle\varphi, \preccurlyeq r\rangle$) used for the knowledge bases in papers on FDLs (cf.[14, 15, 17]). Next we define the description languages $\mathcal{ALC}_{L^*(\mathbf{S})}$ and $\mathcal{ALC}_{L^*_\sim(\mathbf{S})}$ from a syntactic and semantic perspective. Then we introduce the notions of *TBox* and *ABox* for that languages.

Syntax. In the languages of description we start from *atomic concepts* and *atomic roles*. Complex descriptions are built inductively with constructors of concepts. We will use the letters A for atomic concepts, R for atomic roles and both C and D for descriptions of concepts. Using the connectives $\bar{0}, \&, \rightarrow$ (falsity, conjunction, implication), the quantifiers $\forall, \exists$ and the point . as an auxiliary symbol, the description of concepts in classic $\mathcal{ALC}$ can be built using the following syntactical rules

$$C, D \quad \rightsquigarrow A \mid \bar{0} \mid C\&D \mid C \rightarrow D \mid \forall R.C \mid \exists R.C$$

Given a continuous t-norm $*$ and a countable subalgebra $\mathbf{S}$ of the corresponding standard algebra $[0,1]_*$, let us consider the logic $L^*(\mathbf{S})$. We define $\mathcal{ALC}_{L^*(\mathbf{S})}$ by adding to $\mathcal{ALC}$, for every $r \in S$, a nullary connective $\bar{r}$ and the rule $C \quad \rightsquigarrow \bar{r}$.

The language $\mathcal{ALC}_{L^*_\sim(\mathbf{S})}$ is defined by adding to $\mathcal{ALC}_{L^*(\mathbf{S})}$ the connective $\sim$ and the syntactic rule $C \quad \rightsquigarrow \sim C$.

Following [8], the notions of *instance of a concept* and *instance of a role* allow us to read the formulas of both languages as formulas of the corresponding predicate fuzzy logic. For each term t (variable or constant). The *instance* $D(t)$ of a concept D is defined as follows:

$$\begin{array}{ll} A(t), & \text{if } D \text{ is an atomic concept } A, \\ \bar{0}, & \text{if } D = \bar{0}, \\ \sim C(t), & \text{if } D = \sim C, \\ C_1(t) \circ C_2(t), & \text{if } D = C_1 \circ C_2, \text{ where } \circ \in \{\&, \rightarrow\}, \end{array}$$

and, if y is a variable not occurring in $C(t)$,

$$\begin{array}{ll} (\forall y)(R(t,y) \rightarrow C(y)), & \text{if } D = \forall R.C, \\ (\exists y)(R(t,y) \& C(y)), & \text{if } D = \exists R.C, \end{array}$$

where, given two terms t_1 and t_2, $R(t_1, t_2)$ is an atomic formula corresponding to the atomic role R. We will refer to the expressions of the form $R(t_1, t_2)$ as *instances of the atomic role* R.

Semantics. According to semantics for $L^*(\mathbf{S})\forall$ and $L^*_\sim(\mathbf{S})\forall$ a fuzzy interpretation $\mathbf{M}$ associates a fuzzy set $A^{\mathbf{M}}$ to each atomic concept A and a fuzzy binary relation $R^{\mathbf{M}}$ to each atomic role R, and the truth value for complex descriptions is given as follows:

$$\begin{array}{rcl} \bar{0}^{\mathbf{M}}(a) & = & 0 \\ \bar{r}^{\mathbf{M}}(a) & = & r, \text{ for every } r \in S \\ (C\&D)^{\mathbf{M}}(a) & = & C^{\mathbf{M}}(a) * D^{\mathbf{M}}(a) \\ (C \rightarrow D)^{\mathbf{M}}(a) & = & C^{\mathbf{M}}(a) \rightarrow_* D^{\mathbf{M}}(a) \\ (\forall R.C)^{\mathbf{M}}(a) & = & \inf\{R^{\mathbf{M}}(a,b) \rightarrow_* C^{\mathbf{M}}(b) : b \in M\} \\ (\exists R.C)^{\mathbf{M}}(a) & = & \sup\{R^{\mathbf{M}}(a,b) * C^{\mathbf{M}}(b) : b \in M\} \end{array}$$

In the case of complex descriptions in $\mathcal{ALC}_{L^*_\sim(\mathbf{S})}$ we must to add the following:

$$(\sim C)^{\mathbf{M}}(a) \quad = \quad 1 - C^{\mathbf{M}}(a)$$

Fuzzy TBox and Fuzzy ABox. Now we define the notions of $TBox$ and $ABox$ for $\mathcal{ALC}_{L^*(\mathbf{S})}$ and $\mathcal{ALC}_{L^*_\sim(\mathbf{S})}$. In these definitions we use the following graded notion of inclusion between fuzzy sets: *degree*$(C \subseteq D) = \inf_x(C(x) \rightarrow_* D(x))$. Of course this degree is 1 if and only if $C(x) \leq D(x)$ for all x and 0 if the support[1] of the two fuzzy sets are disjoint. Having the truth constants in the language allows us to associate sentences like "*degree* $(C \subseteq D) \leq r$" with formulas such as $(\forall x)(C(x) \rightarrow D(x)) \rightarrow \bar{r}$.

A *fuzzy concept inclusion axiom* is a sentence of one of the following forms:

- $\bar{r} \rightarrow (\forall x)(C(x) \rightarrow D(x))$

[1] The *support* of a fuzzy set is the cardinal of the set of elements which membership degree is greater than 0.

ISBN: 978-989-95079-6-8

Table 1: The graded notation for fuzzy KB.

Evaluated Formula	FDL Graded Notation
$\bar{r} \to (\forall x)(C(x) \to D(x))$	$\langle C \sqsubseteq D, \succcurlyeq \bar{r}\rangle$
$(\forall x)(C(x) \to D(x)) \to \bar{r}$	$\langle C \sqsubseteq D, \preccurlyeq \bar{r}\rangle$
$\bar{r} \leftrightarrow (\forall x)(C(x) \to D(x))$	$\langle C \sqsubseteq D, \approx \bar{r}\rangle$
$\bar{r} \to C(a)$	$\langle a : C, \succcurlyeq \bar{r}\rangle$
$C(a) \to \bar{r}$	$\langle a : C, \preccurlyeq \bar{r}\rangle$
$\bar{r} \leftrightarrow C(a)$	$\langle a : C, \approx \bar{r}\rangle$
$\bar{r} \to R(a,b)$	$\langle (a,b) : R, \succcurlyeq \bar{r}\rangle$
$R(a,b) \to \bar{r}$	$\langle (a,b) : R, \preccurlyeq \bar{r}\rangle$
$\bar{r} \leftrightarrow R(a,b)$	$\langle (a,b) : R, \approx \bar{r}\rangle$

- $(\forall x)(C(x) \to D(x)) \to \bar{r}$
- $\bar{r} \leftrightarrow (\forall x)(C(x) \to D(x))$

A *fuzzy assertion axiom* is a sentence of one of the following forms:

- $\bar{r} \to C(a)$ or $\bar{r} \to R(a,b)$
- $C(a) \to \bar{r}$ or $R(a,b) \to \bar{r}$
- $C(a) \leftrightarrow \bar{r}$ or $R(a,b) \leftrightarrow \bar{r}$

Now a *fuzzy TBox* is defined as a finite set of *fuzzy concept inclusion axioms* while a *fuzzy ABox* is defined as a finite set of *fuzzy assertion axioms*. A *fuzzy KB* is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A}\rangle$, where the first component is a *fuzzy TBox* and the second one is a *fuzzy ABox*.

Notice that all the axioms of the *fuzzy KB* are evaluated formulas. Thus, the syntactic notion of *fuzzy KB* according to our approach, both the $TBox$ and the $ABox$ can be seen as theories of the logic $L^*(\mathbf{S})\forall$ (or $L^*_\sim(\mathbf{S})\forall$).

In Tab. 3 we present an alternative graded notation for fuzzy inclusion and fuzzy assertions that we will use in the example. This notation is similar to the one used in some papers of FDLs (see for instance [17]). Moreover this notation is according to the semantical interpretation in the sense that, for instance,

$$\langle \sigma, \succcurlyeq \bar{r}\rangle^{\mathbf{M}} = r \to_* \sigma^{\mathbf{M}}$$

and so

$$\mathbf{M} \models \langle \sigma, \succcurlyeq \bar{r}\rangle \quad \text{iff} \quad r \to_* \sigma^{\mathbf{M}} = 1 \quad \text{iff} \quad \sigma^{\mathbf{M}} \geq r$$

It is interesting to remark that in $\mathcal{ALC}_{L^*_\sim(\mathbf{S})\forall}$ the involutive negation allows us to define graded expressions like, for instance, $\langle a : C, \succ \bar{r}\rangle$ as $\sim \langle a : C, \preccurlyeq \bar{r}\rangle$ which corresponds to the formula $\sim (C(a) \to \bar{r})$.

An example. We will use a data set composed of nine robots (Fig. 1), each one with either the same of different shape of head and body (i.e., they are homogeneous or not homogeneous respectively), they can or cannot wear a tie, they can or cannot smile, and they hold some object. Taking into account all these characteristics, robots can have different *friendliness* degree. The domain of interpretation of the robots is the set: $M_{\mathcal{R}} = \{r_i : 1 \leq i \leq 9\} \cup \{o_i : 1 \leq i \leq 9\}$, where the r_i are the robots and each o_i is the object that the robot r_i holds (e.g., the object o_4 is the flower that r_4 holds). Atomic concepts of the language are the following: Robot, Happy, Object, FriendlyObject, Homogeneous, Balloon, Flag, Flower, Sword, Ax and HasTie. There is only one atomic role: hasObject.

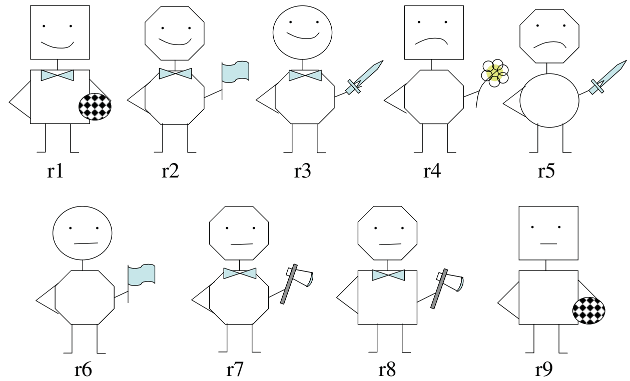

Figure 1: The 9 little robots.

The $TBox$ concerning the robots domain is the following:

$$\text{Friendly} \equiv \text{Robot}\&(\exists\text{hasObject.FriendlyObject})\&(\text{Happy} \veebar \text{Homogeneous})$$
$\langle$Robot & Object $\sqsubseteq \bar{0}, \approx \bar{1}\rangle$
$\langle \bar{1} \sqsubseteq$ Robot $\veebar$ Object$, \approx \bar{1}\rangle$
$\langle$Flower $\sqsubseteq$ FriendlyObject$, \approx \bar{1}\rangle$
$\langle$Balloon $\sqsubseteq$ FriendlyObject$, \approx \overline{0.75}\rangle$
$\langle$Flag $\sqsubseteq$ FriendlyObject$, \approx \overline{0.50}\rangle$
$\langle$Sword $\sqsubseteq$ FriendlyObject$, \approx \overline{0.25}\rangle$
$\langle$Ax $\sqsubseteq$ FriendlyObject$, \approx \bar{0}\rangle$

where $C \veebar D$ is an abbreviation of $\sim (\sim C\& \sim D)$ and $C \equiv D$ is an abbreviation for the conjunction of the formulas $\langle C \sqsubseteq D, \approx \bar{1}\rangle$ and $\langle D \sqsubseteq C, \approx \bar{1}\rangle$.

Notice that objects have different friendliness degree. For instance, a sword is a friendly object with degree 0.25 and an ax is a friendly object with degree 0 (i.e., it should be considered unfriendly in the classical case). On the other hand, the $TBox$ also contains a definition of Friendly allowing to assess the friendliness degree of a robot.

The $ABox$ containing the descriptions of the robots is the following:

For each i, $1 \leq i \leq 9$, $\langle r_i :$ Robot$, \approx \bar{1}\rangle$, $\langle (r_i, o_i) :$ hasObject$, \approx \bar{1}\rangle$
$\langle r_1 :$ Homogeneous$, \approx \bar{1}\rangle$, $\langle o_1 :$ Balloon$, \approx \bar{1}\rangle$, $\langle r_1 :$ Happy$, \approx \bar{1}\rangle$
$\langle r_2 :$ Homogeneous$, \approx \bar{1}\rangle$, $\langle o_2 :$ Flag$, \approx \bar{1}\rangle$, $\langle r_2 :$ Happy$, \approx \bar{1}\rangle$
$\langle r_3 :$ Homogeneous$, \approx \overline{0.75}\rangle$, $\langle o_3 :$ Sword$, \approx \bar{1}\rangle$, $\langle r_3 :$ Happy$, \approx \bar{1}\rangle$
$\langle r_4 :$ Homogeneous$, \approx \overline{0.50}\rangle$, $\langle o_4 :$ Flower$, \approx \bar{1}\rangle$, $\langle r_4 :$ Happy$, \approx \bar{0}\rangle$
$\langle r_5 :$ Homogeneous$, \approx \overline{0.50}\rangle$, $\langle o_5 :$ Sword$, \approx \bar{1}\rangle$, $\langle r_5 :$ Happy$, \approx \bar{0}\rangle$
$\langle r_6 :$ Homogeneous$, \approx \overline{0.75}\rangle$, $\langle o_6 :$ Flag$, \approx \bar{1}\rangle$, $\langle r_6 :$ Happy$, \approx \overline{0.50}\rangle$
$\langle r_7 :$ Homogeneous$, \approx \bar{1}\rangle$, $\langle o_7 :$ Ax$, \approx \bar{1}\rangle$, $\langle r_7 :$ Happy$, \approx \overline{0.50}\rangle$
$\langle r_8 :$ Homogeneous$, \approx \overline{0.75}\rangle$, $\langle o_8 :$ Ax$, \approx \bar{1}\rangle$, $\langle r_8 :$ Happy$, \approx \overline{0.50}\rangle$
$\langle r_9 :$ Homogeneous$, \approx \bar{1}\rangle$, $\langle o_9 :$ Balloon$, \approx \bar{1}\rangle$, $\langle r_9 :$ Happy$, \approx \overline{0.50}\rangle$

Notice that, using truth constants, we can assess different degrees of homogeneity according to the shape of both head and body. In particular, we assess, in a subjective way, that a combination of round shapes of head and body (i.e., a circle and an octagon) give a more homogeneous aspect to the robot than combining round and square shapes. Thus, robots r_6 and r_8 are considered more homogeneous than robot r_4. Similarly, robots have different form of mouth that give them different degree of happiness (i.e., robot r_1 is assessed as more happy than robots r_8 and r_4).

Fuzzy reasoning. Reasoning in fuzzy description logics consists on the same kind of tasks than in the classical case but

ISBN: 978-989-95079-6-8

now they depend on the chosen continuous t-norm $*$. One of the advantages of introducing truth constants in the language is the possibility to define the graded versions of the notions of $*$-satisfiability, $*$-subsumption and $*$-validity defined in [9] without modifying the semantics. Thus, given a concept C, a t-norm $*$ and a truth value $r \in S$ we introduce the following graded notions with respect to a knowledge base $\mathcal{K}$:

• C is *$*$-satisfiable in a degree greater or equal than r* iff there is a model $\mathbf{M}$ of $\mathcal{K}$ such that $\|\bar{r} \rightarrow C(a)\|^*_{\mathbf{M}} = 1$ (being a a constant).

• C is *$*$-valid in a degree greater or equal than r* in a model $\mathbf{M}$ of $\mathcal{K}$ iff $\|\bar{r} \rightarrow (\forall x)C(x)\|^*_{\mathbf{M}} = 1$.

• C is *$*$-subsumed* by D *in a degree greater or equal than* r in a model $\mathbf{M}$ of $\mathcal{K}$ iff the concept $C \rightarrow D$ is *$*$-valid in a degree greater or equal than r* in the model $\mathbf{M}$, that is, iff $\|\bar{r} \rightarrow (\forall x)(C(x) \rightarrow D(x))\|^*_{\mathbf{M}} = 1$.

On the other hand, we can analogously define the notions of lower thresholds. For instance, a concept C is *$*$-satisfiable in a degree lower or equal than r* iff $\|C(a) \rightarrow \bar{r}\|^*_{\mathbf{M}} = 1$ for some model $\mathbf{M}$ of $\mathcal{K}$. Moreover, it is also possible to define an interval where a concept is either valid, satisfiable or subsumed. For instance a concept C is *$*$-satisfiable in an interval of degrees $[r, s]$* iff $\|\bar{r} \rightarrow C(a)\|^*_{\mathbf{M}} = 1$ and $\|C(a) \rightarrow \bar{s}\|^*_{\mathbf{M}} = 1$ for some model $\mathbf{M}$. In particular, when $r = s$ the interval became a value called the *degree of satisfiability*.

Example 3.1 *The concept C below is $*$-satisfiable with degree 0.75 in the robots model using any continuous t-norm.*

$$C := \mathsf{Homogeneous}\ \&\exists\mathsf{hasObject.FriendlyObject}$$

According to the semantics, C is $*$-satisfiable with degree 0.75 if there is at least one robot such that

$C(x) = \mathsf{Homogeneous}(x) * \sup_{y \in M_{\mathcal{R}}}(\mathsf{hasObject}(x, y) * \mathsf{FriendlyObject}(y)) = 0.75$

The equality is true since the robots r_1 and r_9 have $\mathsf{Homogeneous}(x) = 1$ and both hold a ballon that, as the $TBox$ states, it is a friendly object with degree 0.75. Thus, because 1 is the unity element of any t-norm, both robots have friendliness degree 0.75.

Example 3.2 *Let us to analyze the $*$-subsumption degree of the concept* $\mathsf{Object}\ \&\ \mathsf{Sword}\ \&\ \mathsf{FriendlyObject}$ *by $\bar{0}$ with respect to the $TBox$.*

According to the definition of $*$-subsumption, we have to analyze the following formula

$$\bar{r} \leftrightarrow (\forall x)((\mathit{Object}\ \&\ \mathit{Sword}\ \&\ \mathit{FriendlyObject})(x) \rightarrow \bar{0})$$

and thus, according to the interpretation, the $*$-subsumption has degree

$$r = \inf_{x \in M_{\mathcal{R}}} \{\mathit{Object}(x) * \mathit{Sword}(x) * \mathit{FriendlyObject}(x) \rightarrow_* 0\}$$

For all t-norms: a) when x is an element that is neither an object nor a sword, then r is $0 \rightarrow_* 0 = 1$, and b) when x is a sword, $r = \mathit{FriendlyObject}(x) \rightarrow_* 0$. In such case r depends on the t-norm. For instance, taking the Łukasiewicz's t-norm, $r = \min(1, 1 - \mathit{FriendlyObject}(x) + 0) = 0.75$. Taking either the minimum or the product t-norms $r = 0.25 \rightarrow_* 0 = 0$.

Concerning the entailment, we say that a fuzzy assertion α is $*$-*entailed* by the knowledge base $\mathcal{K}$ if every model M of $\mathcal{K}$ also satisfies α. For instance, the reader can easily prove that the fuzzy $ABox$ of the robots' example $*$-entails the assertion $\langle \mathsf{Friendly}(r_6), \succcurlyeq 0.50\rangle$ with respect to the fuzzy $TBox$ using either Łukasiewicz or minimum t-norms. However, using the product t-norm the above $*$-entailment is not satisfied.

4 A case study: Fuzzy description logics associated to $G_\sim\forall(S)$

The first FDL systems were related to the initial Zadeh proposal for fuzzy sets operations. The logic underlying this proposals is the logic associated to the calculus over $[0, 1]$ defined by the functions $\min$, $\max$, $n(x) = 1 - x$, the Kleene-Dienes implication $x \rightarrow y = \max(1 - x, y)$, and quantifiers interpreted as in section 2.2. This is referred as *minimalistic* apparatus of fuzzy logic by Hájek in [9]. But, as already noted by Hájek, the (non residuated) implication of this logic has some no nice behaviour. For example: (a) in this logic $\varphi \rightarrow \varphi$ is not a tautology and (b) an implication $\varphi \rightarrow \psi$ is evaluated as 1 only if either φ is evaluated as 0 or ψ is evaluated as 1. The languages $\mathcal{ALC}$ based on this logic was studied by Straccia in [14]. The notion of subsumption is defined using the notion of fuzzy subsets ($C \subseteq D$ if and only if $C(x) \leq D(x)$ for all x in the univers) with no relation to implication function due to the lack of a good relation between the implication function of this logic and the order relation in $[0, 1]$. Moreover it is not known any (Hilbert style) axiomatization of this logic.

For these reasons in this short case study we propose the interest of studying the $\mathcal{ALC}$ language associated to Gödel Logic with truth constants $G(\mathbf{S})\forall$ and specially the $\mathcal{ALC}$ language associated to its expansion with an involutive negation $G_\sim(\mathbf{S})\forall$. In [7] the authors prove the canonical completenes of the logic $G(\mathbf{S})\forall$. The canonical completeness can also be proved for $G_\sim(\mathbf{S})\forall$.

Theorem 4.1 *The logic $G_\sim(\mathbf{S})\forall$ has the Canonical Finite Strong completeness, i.e., for every finite set of formulas $\Gamma \cup \{\varphi\}$,*

$$\Gamma \vdash_{G_\sim(\mathbf{S})\forall} \varphi \quad \textit{iff} \quad \Gamma \models_{[0,1]_{G_\sim(\mathbf{S})}} \varphi$$

where
$[0, 1]_{G_\sim(\mathbf{S})} = \langle [0, 1], \min, \max, \rightarrow_G, n, \{r \mid r \in S\}, 0, 1\rangle$, *with $n(x) = 1 - x$ being the truth function associated to the involutive negation.*

The proof is a simplified version of [7, Theorem 11]. As a consequence of this theorem, we have an equivalence between $*$-entailment in $\mathcal{ALC}_{G_\sim(\mathbf{S})}$ and the semantics consequence relation on $[0, 1]_{G_\sim(\mathbf{S})}$.

Notice that in $G_\sim(\mathbf{S})\forall$ we have, in addition to the connectives of this logic, the ones being the counterpart of the minimalistic logic, since an implication having as truth function the Kleene-Dienes implication is definable as $\sim \varphi \vee \psi$. Moreover the truth constants allow us to have evaluated formulas and their negation by the involutive negation that makes possible to represent by formulas the semantical expressions saying that the interpretation of a formula is greater, greater or equal, less, and less or equal than a value r.

ISBN: 978-989-95079-6-8

Thus the description language based on $G_{\sim}\forall(S)$ seems to be a good choice because both it is actually very expressive and it maintains the canonical completeness. In this setting we have two implication connectives, that semantically correspond to the residuum of the minimum plus the Kleene-Dienes implication. Of course, as in the general case of $L^{*}_{\sim}(\mathbf{S})\forall$, the degree of subsumption between concepts is defined by means of the residuated implication, i.e, the Gödel implication. The Straccia's work [14] and the Bobillo's work [2] gives algorithms for proving satisfiability and subsumption in the $\mathcal{ALC}$ description language based on the minimalistic logic and in the $\mathcal{SROIQ}(D)$ description language based on Gödel logic respectively. As future work we want to find analogous algorithms for the $\mathcal{ALC}$ description languages based on $G_{\sim}(\mathbf{S})\forall$.

5 Conclusions

This paper is a first step on the direction proposed by Hájek concerning the relationships between some proposals of FDLs and the recent developments in mathematical fuzzy logics. The main contributions of our approach is the use of truth constants in the language of description and the introduction of an involutive negation in the required cases. This allows us to recover graded notions of satisfiability, validity and subsumption that have been used in the fuzzy logic setting. This choice is oriented to search for the syntactical counterpart of the semantic calculi proposed in some works dealing with FDLs.

Acknowledgments

This work has been supported by the Spanish projects MULOG2 TIN2007-68005-C04-01/04, "Agreement Technologies" (CONSOLIDER CSD 2007-0022, INGENIO 2010), and the MCYT-FEDER Project MID-CBR (TIN2006-15140-C03-01), the CSIC grant 200750I005, and the Generalitat de Catalunya grant 2005-SGr-00093. The authors thank Félix Bou for his assistance in the improvement of this paper.

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ISBN: 978-989-95079-6-8