Zero-probability and coherent betting: A logical point of view

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Abstract. The investigation reported in this paper aims at clarifying an important yet subtle distinction between (i) the logical objects on which measure theoretic probability can be defined, and (ii) the interpretation of the resulting values as rational degrees of belief. Our central result can be stated informally as follows. Whilst all subjective degrees of belief can be expressed in terms of a probability measure, the converse doesn't hold: probability measures can be defined over linguistic objects which do not admit of a meaningful betting interpretation. The logical framework capable of expressing this will allow us to put forward a precise formalisation of de Finetti's notion of *event* which lies at the heart of the Bayesian approach to uncertain reasoning.

1 Introduction: the epistemic structure of de Finetti's betting interpretation

De Finetti's theory of subjective probability is well-known and widely scrutinized in the literature (cf. [2]), so we will review only on those aspects which are directly relevant to our present purposes[1].

Let $\theta_1, \ldots, \theta_n$ be events of interest. De Finetti's *betting problem* is the choice that an idealised agent called *bookmaker* must make when publishing a *book*, i.e. when making an assignment $B = \{(\theta_i, \beta_i) : i = 1, \ldots, n\}$ in which each event of interest θ_i is given value $\beta_i \in [0, 1]$. Once a book has been published, a *gambler* can place bets $S_i \in \mathbb{R}$ on any event θ_i by paying $S_i\beta_i$ to the bookmaker. In return for this payment, the gambler will receive S_i, if θ_i obtains and nothing otherwise. Note that "betting on θ_i" effectively amounts, for the gambler, to choosing a real-valued S_i which determines the amount payable to the bookmaker[2].

[1] The reader who wishes to consult the originals is referred to [2, 3, 5, 6].
[2] In order to avoid potential distortions arising from the diminishing value of money, de Finetti invokes the "rigidity hypothesis" to the effect that S_i should be small.

De Finetti's construction of the betting problem proceeds by forcing the bookmaker to write *fair betting odds* for any given book B. To this end, two modelling assumptions are built into the problem, namely (i) the bookmaker must accept any number of bets on B and (ii) when betting on θ_i, gamblers can choose the sign of the stakes S_i, thereby possibly (and unilaterally) imposing a payoff swap to the bookmaker. Taken jointly, conditions (i-ii) force the bookmaker to publish books with zero-expectation, for doing otherwise may offer gamblers the possibility of making a sure profit, possibly by swapping payoffs. As the game is zero-sum, this is equivalent to forcing the bookmaker into sure loss. The Dutch Book theorem states that this possibility is avoided exactly when the bookmaker chooses betting odds which are probabilities.

This line of argument presupposes an *epistemic structure* which de Finetti mentions only in passing in his major contributions to this topic [3–5]. A more direct, albeit very informal, reference to the point appears in [6]. For reasons that will be apparent in a short while, the underlying epistemic structure of the betting problem is fundamental to understanding the notion of *event*:

> [T]he characteristic feature of what I refer to as an "event" is that the circumstances under which the event will turn out to be "verified" or "disproved" have been fixed in advance. [6] (p. 150)

This very informal characterisation echoes the characterisation de Finetti gives of random quantities –of which events are special cases. A random quantity is a "well-determined" unknown, namely one which is so formulated as "to rule out any possible disagreement on its actual value, for instance, as it might arise when a bet is placed on it." ([5], Section 2.10.4).

The epistemic structure implicit in the betting framework clearly builds on the presupposition that at the time of betting bookmakers *and* gamblers ignore the truth value of the event on which they are betting, i.e. they agree that, say $v(\theta)$ is undefined. Yet, for the bet to be meaningful, i.e. payable at all, players must also agree on the conditions which will *decide* the truth value of θ. This implies that a betting interpretation of probability is meaningful only for those sentences whose truth value is presently (at the time of betting) undecided, but which the players know that will eventually be decided. Now, there are certainly well-formed formulas escaping this restriction, so probability functions defined on them cannot have a betting interpretation.

Before introducing the logical framework that will formalise this in Section 2, let us pause for a second to appreciate why the interpretation of probability which arises in this context is clearly subjective. Whether a sentence qualifies as an *event* depends crucially on the *state of information* of the individuals involved in the betting problem. Compare this with the logical, measure-theory inspired, characterisation of probability functions which is derived under the tacit assumption that the agent's state of information is empty, that is to say the set of events includes all possible sentences. This assumption will be relaxed in our framework and indeed this will lead us to generalise the scope of the representation theorem of probability functions on sentences by introducing a refinement of the notion of probability functions which we call *bet functions* and we denote by $Bet(\cdot)$. In particular, we shall be interested in characterising sentences of SL in such a way that the resulting definitions of *facts* and *events* (Section 3.2) will give us enough structure to prove that $Bet(\cdot)$ so defined is *consistent* in the sense of de Finetti (Section 4) and to show that its extension to *inaccessible sentences* preserves consistency (Section 5). Section 6 concludes by pointing to the future work which we envisage within the framework fleshed out in this paper.

2 Background

Let $L = \{p_1, \ldots, p_n\}$ be a finite set of propositional variables, and let $SL = \{\theta, \phi, \ldots\}$ be the set of sentences built as usual from L in the language of classical propositional logic. Denote by AT^L be the set of maximally elementary conjunctions of L, that is the set of sentences of the form $\alpha = p_1^{\epsilon_1} \wedge p_2^{\epsilon_2} \wedge \ldots \wedge p_n^{\epsilon_n}$, with $\epsilon_i \in \{0, 1\}$ and where $p_i^1 = p_i$ and $p_i^0 = \neg p_i$, for $i = 1, \ldots, n$.

Note that the Lindenbaum algebra[3] on SL is a finite Boolean algebra and hence it is atomic. In particular the elements of AT^L exactly correspond the atoms of the Lindenbaum algebra.

AT^L is in 1-1 correspondence with the set $\mathbb{V}$ of (classical) valuations on L. This implies that there is a unique valuation satisfying $v(\alpha) = 1$ namely $v_\alpha(p_i^{\epsilon_i}) = \epsilon_i$ for $1 \leq i \leq n$. Conversely, given a valuation $v \in \mathbb{V}$ there exists a unique atom $\alpha \in AT^L$ such that $v(\alpha) = 1$. Now let

$$M_\theta = \{\alpha \in AT^L \mid \alpha \models \theta\},$$

where $\models$ denotes the classical Tarskian consequence. Since there exists a unique valuation satisfying α, say v_α, by definition of $\models$ it must be the case that $v_\alpha(\theta) = 1$. Thus

$$M_\theta = \{\alpha \in AT^L \mid v_\alpha(\theta) = 1\}.$$

This framework is sufficient to provide a very general representation theorem for probability functions.

Theorem 1 (Paris 1994).

1. *Let P be a probability function on SL.*[4] *Then the values of P are completely determined by the values it takes on $AT^L = \{\alpha_1, \ldots, \alpha_J\}$, as fixed by the vector*

$$\langle P(\alpha_1), P(\alpha_2), \ldots, P(\alpha_J)\rangle \in \mathbb{D}^L = \{\boldsymbol{a} \in \mathbb{R}^J \mid \boldsymbol{a} \geq 0, \sum_{i=1}^{J} a_i = 1\}.$$

[3] Recall that the Lindenbaum algebra over L is the quotient set $SL/\equiv$, where $\equiv$ is the logical equivalence relation (defined as $\theta_1 \equiv \theta_2$ iff $\models \theta \leftrightarrow \theta_2$), with the operations induced by the classical conjunction, disjunction and negation connectives.

[4] $P : SL \to [0, 1]$ is a probability function on sentences if (i) $P(\top) = 1$, (ii) $P(\theta_1 \vee \theta_2) = P(\theta_1) + P(\theta_2)$ if $\models \neg(\theta_1 \wedge \theta_2)$, and (iii) $P(\theta_1) = P(\theta_2)$ if $\models \theta_1 \leftrightarrow \theta_2$.

2. *Conversely, fix* $\boldsymbol{a} = \langle a_1, \ldots, a_J \rangle \in \mathbb{D}^L$ *and let* $P' : SL \to [0,1]$ *be defined by*

$$P'(\theta) = \sum_{i:\alpha_i \in M_\theta} a_i. \tag{1}$$

Then P' *is a probability function.*

In words, Theorem 1 shows that every probability function arises from distributing the unit mass of probability across the $J = 2^n$ atoms of the Lindenbaum algebra generated by $L = \{p_1, \ldots, p_n\}$.

Our goal is to refine this result by isolating a class of sentences on which, we argue, there should be no distribution of epistemically significant mass. More specifically, we aim at building a framework in which those probabilities which bear a meaning as *betting quotients* can be formally distinguished from those which do not. Central to achieving this will be a rigorous definition of de Finetti's notion of *event*, which will be distinguished from the related notion of *fact*. Under certain conditions, all sentences in SL will either be events or facts. Under more general conditions a third class of inaccessible sentences will feature in SL. The central result of this paper can be intuitively phrased as establishing that *probabilities which are defined on sentences which are not events can only be given trivial values*. Trivial, as we will shortly see, means one of two things. Either a sentence can (coherently) be given only its truth value (and this characterises betting on facts), or it should be given 0. This means that the "uncertainty mass" is really concentrated only on events, for which we provide a formal definition.

3 Formal preliminaries: information frames, facts and events

In what follows, we denote subsets of SL by capital Greek letters $\Gamma, \Delta, \ldots$, and the classical Tarskian consequence is denoted by either $\models$ or Cn depending on whether its relational or operational definition is more suited to the specific to the context. Recall that a (total, classical) valuation is a function $v : L \to \{0,1\}$ which extends uniquely to the sentences in SL. A total valuation represents a "fully informed" epistemic state since it allows agents to assign a truth-value (either 1 or 0) to any sentence of SL. However, an epistemic state determined by a set Γ of sentences (the ones known to be true), will permit an assignment of truth-values 1 or 0 only to some subset of sentences. In fact, each Γ uniquely determines a three-valued map on SL, $e_\Gamma : SL \to \{0, 1, u\}$, defined as

$$e_\Gamma(\theta) = \begin{cases} 1 & \text{if } \theta \in Cn(\Gamma), \\ 0 & \text{if } \neg\theta \in Cn(\Gamma), \\ u & \text{otherwise.} \end{cases} \tag{2}$$

where the new value u reads as *unknown*.

Notice that partial evaluations are not truth-functional. Note also that, if $\Gamma \subseteq \Gamma'$ then $Cn(\Gamma) \subseteq Cn(\Gamma')$. From now on, we will say that a mapping

$e : SL \to \{0,1,u\}$ is a *partial evaluation* whenever there exists $\Gamma \subseteq SL$ such that $e = e_\Gamma$.

Given two partial valuations e, e', we say that e' extends e, written $e \subseteq e'$, when the class of formulas which e sends into $\{0,1\}$ is included into that one which e' sends into $\{0,1\}$. Note that if $e = e_\Gamma$ and $e' = e_{\Gamma'}$ then

$$e \subseteq e' \Leftrightarrow \Gamma \subseteq \Gamma'. \tag{3}$$

By a *theory* we mean a deductively closed subset of SL. So, Γ is a theory if and only if $Cn(\Gamma) = \Gamma$. We denote the set of theories on L by $\mathbf{T}$. Let us finally recall that a theory $\Gamma \in \mathbf{T}$ is *maximally consistent* iff for every $\theta \in SL$, either $\Gamma \models \theta$, or $\Gamma \models \neg\theta$. Note also that for any maximally consistent $\Gamma \in \mathbf{T}$, there exists a (total) valuation $v \in \mathbb{V}$ such that for all $\theta \in SL$, $e_\Gamma(\theta) = v(\theta)$.

Definition 1 (Determined sentences). *We say that* $\Gamma \subseteq SL$ determines $\theta \in SL$, *written* $\Gamma \succ \theta$ *if and only if,* $\forall p_i \in Var(\theta)$, $e_\Gamma(p_i) \in \{0,1\}$.

Definition 2 (Decided sentences). *We say that* $\Gamma \subseteq SL$ decides $\theta \in SL$, *written* $\Gamma \rhd \theta$ *if and only if* $e_\Gamma(\theta) \in \{0,1\}$.

It is clear that for all $\Gamma \subseteq SL$ and $\theta \in SL$, if $\Gamma \succ \theta$ then $\Gamma \rhd \theta$ as well. Furthermore, as remarked above, if $\Gamma \in \mathbf{T}$ is maximally consistent, then $\Gamma \succ \theta \Leftrightarrow \Gamma \rhd \theta$. The following are immediate consequences of the above definitions.

Proposition 1. *For all* $\Gamma \subseteq SL$, *and for all* $\theta, \varphi \in SL$, *the following hold:*

1. $\Gamma \succ \theta$ *iff* $\Gamma \succ \neg\theta$; $\Gamma \rhd \theta$ *iff* $\Gamma \rhd \neg\theta$.
2. *If* $\Gamma \rhd \theta$, *and* $\Gamma \rhd \varphi$, *then* $\Gamma \rhd \theta \circ \varphi$ *for all* $\circ \in \{\wedge, \vee, \to\}$.
3. *If* $\Gamma \rhd \theta$, $\Gamma \not\rhd \varphi$, *and* $e_\Gamma(\theta) = 0$ *then* $\Gamma \not\succ \theta \circ \varphi$ *for every* $\circ \in \{\wedge, \vee, \to\}$, *but* $\Gamma \rhd \theta \wedge \varphi$ *and* $\Gamma \rhd \theta \to \varphi$, *and in particular* $e_\Gamma(\theta \wedge \varphi) = 0$, $e_\Gamma(\theta \to \varphi) = 1$.
4. *If* $\Gamma \rhd \theta$, $\Gamma \not\rhd \varphi$, *and* $e_\Gamma(\theta) = 1$ *then* $\Gamma \not\succ \theta \circ \varphi$ *for every* $\circ \in \{\wedge, \vee, \to\}$, *but* $\Gamma \rhd \theta \vee \varphi$, $\Gamma \rhd \varphi \to \theta$ *and* $\Gamma \rhd \theta \to \varphi$, *and in particular* $e_\Gamma(\theta \vee \varphi) = e_\Gamma(\varphi \to \theta) = 1$.

3.1 Information frames

Definition 3 (Information frame). *An* information frame $\mathcal{F}$ *is a pair* $\langle W, R\rangle$ *where* W *is a non-empty subset of partial valuations defined as in Equation (2) and* R *is a binary transitive relation on* W.

Remark 1. Since each partial valuation is uniquely determined by a $\Gamma \subseteq SL$, we can freely use $w_1, w_2, \ldots$ to denote either subsets of SL or their associated partial valuations, depending on which interpretation suits best the specific context. As a consequence of Equation (3) the inclusion $w \subseteq w'$ is always defined.

We interpret $w_i \in W$ as an agent's *state of information*, i.e. the sentences (equivalently, the partial valuation) which capture all and only the information available to an agent who finds itself in state w_i. Under this interpretation the relation R models the agent's *possible transitions* among information states. For reasons that will soon be apparent, we always require R to be transitive. As more structure is needed further restrictions on R will be considered.

Definition 4. *Let $\mathcal{F} = \langle W, R \rangle$ be an information frame. We say that $\mathcal{F}$ is*

- Monotone *if $(w, w') \in R$ implies $w \subseteq w'$.*
- Complete *if $w \subseteq w'$ implies $(w, w') \in R$.*

Under our interpretation, monotonicity captures the idea that agents can only learn new information, but never "unlearn" the old one. In addition, monotonicity implies that the dynamics of information is stable in the sense that once a formula is either determined or decided at state w (i.e. it is given a binary truth-value), this remains fixed at any information state reachable from w. Hence if $w \succ \phi$, then there cannot exist $(w, w') \in R$ such that $w' \not\succeq \phi$. Completeness ensures that the agent will learn all the possible consistent refinements to its current information state. So, if $(w, w') \notin R$, there exists θ such that $w' \succ \theta$ and $w \succ \neg\theta$. Finally, note that if $\mathcal{F}$ is monotonic and complete then obviously R coincides with set-inclusion among states (equivalently, sets of sentences).

3.2 Facts and events

The following definition captures the differences among facts, events and inaccessible sentences in a monotone information frame.

Definition 5. *Let $\langle W, R \rangle$ be a monotone information frame, let $w \in W$, and let $\theta \in SL$. We say that θ is a w-*fact *if $w \rhd \theta$.*

On the other hand, if $w \not\rhd \theta$, we say that θ is:

- *a w-*event *if for every (total) valuation V extending w there exists w' with $(w, w') \in R$ such that $w' \rhd \theta$ and $w'(\theta) = V(\theta)$.*
- *w-*inaccessible *if for every (total) valuation V and every world w' such that $w'(\theta) = V(\theta)$, $(w, w') \notin R$.*

We shall respectively denote by $\mathcal{F}(w)$, $\mathcal{E}(w)$ and $\mathcal{I}(w)$ the class of w-facts, w-events, and w-inaccessible sentences, for some information frame $\langle W, R \rangle$ and some $w \in W$.

The following proposition sums up some key properties of the sets $\mathcal{F}(w)$, $\mathcal{E}(w)$ and $\mathcal{I}(w)$.

Proposition 2. *Let $\langle W, R \rangle$ be a monotone information frame, and let $w \in W$. Then the following hold:*

1. *The structure $\langle \mathcal{F}(w), \wedge, \neg, \perp \rangle$ is a Boolean algebra.*
2. *If w is a total valuation, then $SL = \mathcal{F}(w)$, while if $w = \emptyset$ is the empty valuation, then $\mathcal{F}(w) = \emptyset$.*
3. *If $\langle W, R \rangle$ is complete, then $\langle \mathcal{E}(w), \wedge, \neg, \perp \rangle$ is a Boolean algebra.*
4. *If $\langle W, R \rangle$ is complete, then for all $w \in W$, $SL = \mathcal{F}(w) \cup \mathcal{E}(w)$. Therefore, in particular, if $\langle W, R \rangle$ is complete, then $\mathcal{I}(w) = \emptyset$.*
5. *If $\mathcal{I}(w) \neq \emptyset$, then for every w' such that its corresponding valuation is total, $(w, w') \notin R$.*

It is worth noticing that in arbitrary monotone information frameworks one cannot ensure that sentences which are neither w-facts nor w-events, are w-inaccesible, so that the sets $\mathcal{F}(w)$, $\mathcal{E}(w)$, $\mathcal{I}(w)$ form a partition of SL. As we will discuss in further detail in the concluding section, it is surprisingly difficult to find natural properties on frames which ensure the rather desirable property that $SL = \mathcal{F}(w) \cup \mathcal{E}(w) \cup \mathcal{I}(w)$. When the information framework is also complete then we trivially get this condition since $\mathcal{I}(w) = \emptyset$.

4 Formalising the betting problem

Next we formalise a notion of Dutch book in our generalised framework.

Definition 6. *Let $\langle W, R\rangle$ be an information frame, and let $\Gamma = \{\theta_1, \ldots, \theta_n\}$. A* book *is any mapping $B : \Gamma \to [0, 1]$. Then we further define:*

- *for $w \in W$, the book B is said to be* w-Dutch *iff there exist $S_1, \ldots, S_n \in \mathbb{R}$ such that for every $w' \in W$ such that $w' \rhd \theta_i$ for every i, and $(w, w') \in R$,*

$$\sum_{i=1}^{n} S_i(w'(\theta_i) - B(\theta_i)) < 0;$$

- *the book B is said to be* w-coherent*, or* non-w-Dutch*, if B is not w-Dutch;*
- *the book B is said to be a* w-book*, if each formula $\theta_i \in \Gamma$ is a w-event.*

For w-books, being w-Dutch is a notion that collapses to the usual case. In fact if all the θ_i's are w-events, by definition, each possible evaluation of θ_i is accessible from w, and hence the *extra* requirement that the book be w-Dutch is redundant. On the other hand, a w-coherent w-book can be extended to more general books satisfying w-coherence, as shown by the following result.

Theorem 2. *Let (W, R) be a monotone information frame, let $w \in W$ and let $B : \theta_i \in \Gamma \mapsto \beta_i \in [0, 1]$ be a w-coherent w-book. Let φ be a sentence which is not a w-event and consider the book $B' = B \cup \{(\varphi, \alpha)\}$. Then:*

(1) B' is w-coherent iff $\alpha = w(\varphi)$, in case φ is a w-fact.
(2) B' is w-coherent iff $\alpha = 0$, in case φ is w-inaccessible.

Proof: (1). ($\Rightarrow$). Suppose, to the contrary, that $\alpha \neq w(\varphi)$, and in particular suppose that $w(\varphi) = 1$, so that $\alpha < 1$. Then, the gambler can secure a sure win by betting a positive S on φ. In this case in fact, since the information frame is monotonic by the definition of w-book, $w(\varphi) = 1$ holds in every world w' accessible from w. Thus the gambler pays $S \cdot \alpha$ in order to surely receive S in any such w'. Conversely, if $w(\varphi) = 0$, then, under the absurd hypothesis, $\alpha > 0$ and in that case it is easy to see that a sure-winning choice for the gambler consists in swapping payoffs with the bookmaker, i.e. to bet a negative amount of money on φ.

($\Leftarrow$). Let $S_1, \ldots, S_n, S$ be a system of bets on on $\theta_1, \ldots, \theta_n, \varphi$. Since B is coherent, there exists a w' accessible from w that realizes every θ_i, and such that

$$\sum_{i=1}^{n} S_i(\beta_i - w'(\theta_i)) = 0.$$

Since φ is a w-fact and w' is accessible from w, it follows that $w'(\varphi) = w(\varphi) = \alpha$. Therefore one also has

$$\left(\sum_{i=1}^{n} S_i(\beta_i - w'(\theta_i))\right) + S(\alpha - w'(\varphi)) = 0$$

and hence B' is also w-coherent.

(2). ($\Rightarrow$). Suppose that $\alpha > 0$. By contract, the bettor is accepting to pay a positive stake $S > 0$ on φ, and this means that the he must pay $\alpha \cdot S$ to the bookmaker, thus occurring in a sure loss since φ will not be decided in any world w' accessible from w.
($\Leftarrow$). Since B is w-coherent and since by hypothesis $\alpha = 0$, B' extends B in way which is trivial in the following sense: any gambler betting strictly positive stakes $S_1, \ldots, S_n, S$ on B' will pay to the bookmaker $\sum_i S_i\alpha_i + S\alpha = \sum_i S_i\alpha_i + 0$. And since φ is w inaccessible, in every world w' accessible from w, she will receive $\sum_i S_i w'(\theta_i)$. Hence the coherence of B' follows from the coherence of B. $\square$

The following example illustrates that w-coherent w-books cannot be characterised, in general, within the standard axiomatic framework for probability.

Example 1. Let $L = \{p, q\}$ with the following intuitive interpretation:

- p reads "the electron ε has position π";
- q reads "the electron ε has energy η".

Suppose further that our agent is in a state w such that the truth value of both p and q are unknown. In the usual quantum mechanics interpretation, an agent in w may either learn the position of ε, or its energy, but not both. This gives rise to the information frame depicted in Figure 1 where we may assume the following conditions hold:

$w_1 \rhd p,\ w_1 \not\rhd q$, and $w_1(p) = 0$;	$w_2 \rhd p,\ w_2 \not\rhd q$, and $w_2(p) = 1$;
$w_3 \rhd q,\ w_3 \not\rhd p$, and $w_3(q) = 0$;	$w_4 \rhd q,\ w_4 \not\rhd p$, and $w_4(q) = 1$;
$w_5 \rhd p, q$, and $w_5(p) = w_5(q) = 1$	$w_6 \rhd p, q$, and $w_5(p) = w_5(q) = 0$.
$w_7 \rhd p, q$, and $w_7(p) = 0, w_7(q) = 1$	$w_8 \rhd p, q$, and $w_8(p) = 1, w_8(q) = 0$.

It is immediate to see that p and q are w-events, but $p \wedge q$ is not. In fact, for instance, due to the inaccessibility of w_5, the valuation v mapping p and q to 1 has no correspondence in the worlds which are accessible from w. Analogously, $\neg p \wedge q$, $p \wedge \neg q$ and $\neg p \wedge \neg q$ are not w-events either.

Each probability assignement which coherently assigns a value to $p \wedge q$ returns $P(p \wedge q) = 0$. In fact either $p \wedge q$ turns out to be realized in an accessible state

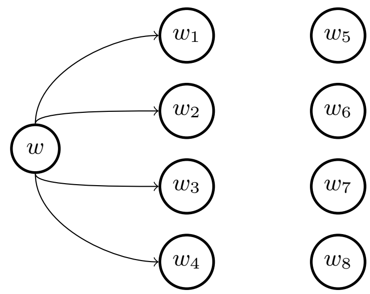

Fig. 1. Heisenberg's principle allows for the information frame to be such that states w_1, w_2, w_3, w_4 are reachable from w. Any world in which both variables are decided, namely w_5, w_6, w_7 and w_8, are not accessible from w.

(i.e. in w_1, or in w_3) in which it turns out to be false, or it turns out to be true, but in the world w_5 which is not accessible. Therefore, by an argument entirely analogous to the proof of Theorem 2, every assignment giving a positive value β to $p \wedge q$ would lead to a sure loss for the bookmaker.

Compare this with the standard measure-theoretic approach. In particular let $\mathbf{L}_2$ be the 16 element Lindenbaum algebra generated by the variables p and q with atoms $p \wedge q$, $\neg p \wedge q$, $p \wedge \neg q$, and $\neg p \wedge \neg q$. In the absence of the structure imposed by information frames, it would be very natural to assume a uniform probability distribution over the atoms of $\mathbf{L}_2$, thereby mapping $p \wedge q$ into a strictly positive value and therefore exposing the bookmaker to sure loss for the bookmaker.

5 Betting on inaccessible sentences

Example 1 illustrates that an otherwise standard probability assignment on the atoms of $\mathbf{L}_2$ may lead to sure loss because of the *inaccessibility* of w_5. The purpose of this section is to show that de Finetti's own coherence criterion fully applies when the information frame shared by the bookmaker and gamblers are *complete*, so that no sentence is inaccessible.

Definition 7 (Bet functions). *Let $\langle W, R\rangle$ be a monotone information frame, and $w \in W$. We say that a partial function $Bet : SL \to [0, 1]$ satisfying:*

$$Bet(\theta) = \begin{cases} w(\theta) \in \{0,1\}, & \textit{if } \theta \in \mathcal{F}(w) \\ 0, & \textit{if } \theta \in \mathcal{I}(w) \end{cases} \tag{4}$$

*is a w-*bet function *if in addition it satisfies:*

- *$Bet(\theta) = Bet(\varphi)$, for all $\theta, \varphi \in \mathcal{E}(w)$ such that $\models \theta \leftrightarrow \varphi$,*
- *for all $\theta, \varphi, (\theta \vee \varphi) \in \mathcal{E}(w) \cup \mathcal{F}(w)$ in the domain of Bet such that $\theta \models \neg\varphi$,*

$$Bet(\theta \vee \varphi) = Bet(\theta) + Bet(\varphi) \tag{5}$$

– $Bet(\theta)$ *is not defined on each* $\theta \in SL \setminus (\mathcal{E}(w) \cup \mathcal{F}(w) \cup \mathcal{I}(w))$.

The conditions in (4) capture the (obvious) formalisation of the intuitive remarks put forward at the end of Section 2 which we now generalise to possibly incomplete frames, i.e. such that for some $w \in W$, $\mathcal{I}(w) \neq \emptyset$. The condition expressed by (5) clearly captures the additivity of the betting functions.

In order to characterise inaccessible sentences, we will be working with the corresponding partial valuations and we will identify, for the sake of notational simplicity, states with (partial) valuations.

Definition 8. *Let* w, w' *be partial valuations. We say that* w *and* w' *are* incompatible *(and we will write* $w \perp w'$*) if* $\exists p$ *such that* $w \rhd p, w' \rhd p$, *and* $w(p) \neq w'(p)$.

For a fixed w and $\Gamma \subseteq SL$ let $Var(\Gamma)$ be the set of propositional variables occurring in Γ, we define $S(\Gamma, w)$ to be the set of worlds $w' \in W$ such that:

(1) $(w, w') \in R$
(2) for all $p \notin Var(\Gamma)$, $w'(p) = u$
(3) there exists a total valuation v such that $\forall \theta \in \Gamma$, $v(\theta) = w'(\theta)$

We call the set $S(\Gamma, w)$ the w-*decisive set for* Γ. The idea is that $S(\Gamma, w)$ captures the minimal set of accessible worlds from w where all sentences of Γ are decided, and no other sentences except for those that necessarily follow from Γ. States belonging to the w-decisive set for Γ are *logically independent* in the following sense: for any set of formulas Γ, and for every $w \in W$, either $S(\Gamma, w)$ is empty, or $w' \perp w''$ for each $w', w'' \in S(\Gamma, w)$, i.e., by Definition 8, $w' \cup w'' \vdash \perp$.[5]

The following easily proved proposition sums up interesting properties of w-decisive sets.

Proposition 3. *Let* $\langle W, R \rangle$ *be a monotone information frame,* $w \in W$, $\Gamma \subseteq SL$. *Then the following hold:*

1. *If* $\Gamma \cap \mathcal{I}(w) \neq \emptyset$, *then* $S(\Gamma, w) = \emptyset$;
2. *If* $\Gamma \subseteq \mathcal{E}(w)$, *then* $S(\Gamma, w) \neq \emptyset$;

Let $\langle W, R \rangle$ be an information frame, $w \in W$, and $\Gamma \subseteq \mathcal{E}(w) \cup \mathcal{F}(w) \cup \mathcal{I}(w)$. Further, let $\Gamma' = \Gamma \cap (\mathcal{E}(w) \cup \mathcal{F}(w))$. Finally, let $\pi : S(\Gamma', w) \to [0, 1]$ satisfy $\sum_{w' \in S(\Gamma', w)} \pi(w') = 1$, and define $Bet'_\pi(\cdot) : \Gamma' \subseteq SL \to [0, 1]$ by

$$Bet'_\pi(\theta) = \sum_{w' \in S(\Gamma', w)} \pi(w') \cdot w'(\theta),$$

for all $\theta \in \Gamma'$.

[5] Note that if $\mathcal{I}(w) \neq \emptyset$, w-bets cannot be characterised as distributions on AT^L. As pointed out in Section 2 above, in fact, the formulas in AT^L correspond to *total* valuations. But by Proposition 2, whenever $\mathcal{I}(w) \neq \emptyset$, each w' corresponding to a total valuation must be such that $(w, w') \notin R$.

The map Bet'_π is extended to a partial map Bet_π over Γ by the coherence criterion we proved in Theorem 2. Hence, for each $\theta \in \Gamma$,

$$Bet_\pi(\theta) = \begin{cases} Bet'_\pi(\theta) & \text{if } \theta \in \Gamma', \\ 0 & \text{if } \theta \in \mathcal{I}(w). \end{cases} \tag{6}$$

The following is then easily proved.

Theorem 3. *Let Γ, w and π be as above, and let Bet_π be defined by (6). Then Bet_π is a w-bet function.*

Proof: Bet_π restricted to w-events and w-facts of Γ is clearly normalised and additive in the sense of Definition 7. In addition, for $\theta \in \mathcal{F}(w)$, $w(\theta) = w'(\theta)$ for each $w' \in S(\Gamma, w)$, and hence we have: (i) if $w(\theta) = 1$, then $Bet_\pi(\theta) = \sum_{w' \in S(\Gamma,w)} \pi(w') \cdot w'(\theta) = \sum_{w' \in S(\Gamma,w)} \pi(w') = 1$; (ii) if $w(\theta) = 0$, $Bet_\pi(\theta) = \sum_{w' \in S(\Gamma,w)} \pi(w') \cdot 0 = 0$. Therefore in any case, $Bet_\pi(\theta) = w(\theta)$ for each $\theta \in \mathcal{F}(w)$, and hence $Bet_\pi(\top) = 1$ holds. □

We close the section by stating two easily proved results which illustrate how the notion of w-coherence arises from w-bets. The notion of w-coherence will be the focus of future work.

Theorem 4. *Let Γ be any set of formulas, and let $B : \Gamma \to [0, 1]$ be a book. Then the following are equivalent:*

(1) B is w-coherent,
(2) There exists a w-bet function Bet on SL extending B.
(3) There exists a probability measure P on on the Lindenbaum algebra generated by $\Gamma \cap (\mathcal{E}(w) \cup \mathcal{F}(w))$ extending B on $\Gamma \cap (\mathcal{E}(w) \cup \mathcal{F}(w))$.

Proof: We are going to sketch the proof of (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2). If B is w-coherent, then so is the book B^- obtained by restricting B to the formulas in $\Gamma' = \Gamma \cap (\mathcal{E}(w) \cup \mathcal{F}(w))$. Since Γ' does not contain w-inaccessible formulas, B^- is coherent and hence a standard argument (see for instance [8, Theorem 2]) shows that B' is coherent iff one can find a probability distribution π on $S(\Gamma', w)$. Then the map Bet_π defined through (6) satisfies (2).
(2) $\Rightarrow$ (1). Let Bet' the partial mapping on SL defined by restricting Bet on $\mathcal{E}(w)$. Then the claim easily follows from Theorem 2. □

The above theorem shows that the usual characterization of coherence can be recovered asking for the information frame to be monotone and complete.

Corollary 1. *Let $\langle W, R\rangle$ be monotone and complete, with $w \in W$. Let $\Gamma \subseteq SL$, and let $B : \Gamma \to [0, 1]$. Then the following are equivalent:*

(1) B is w-coherent,
(2) B is coherent,
(3) There exists a w-bet function Bet on SL extending B,
(4) There exists a probability P on SL extending B.

6 Conclusions and future work

We have introduced a logical framework capable of making explicit the implicit epistemic structure that lies at the very heart of the Bayesian representation of uncertainty. As a central step towards achieving this we distinguished facts, events and inaccessible sentences with the understanding that the betting framework underlying the subjective interpretation of probability demands that genuine uncertainty be expressed only on events. The ensuing logical framework leads to a significant refinement of the classical (logical) representation of probability functions recalled in Section 1. In this spirit, Theorem 3 shows that consistent subjective degrees of belief are the subset of probability values which arise from what we call betting functions.

In further work we will tackle the question at a higher level of generality, namely by showing how Theorem 1 can be in fact derived within our framework as a special case of a more general result which involves defining bet functions over suitable quotient algebras. The idea, roughly speaking, is to capture the requirement that a specific set of sentences (events) should be given all the unit mass by factoring a Lindenbaum algebra over the ideal generated by the set of w-facts, for some $w \in W$. This will provide a suitable basis for giving a pure measure-theoretic account of subjective probability with its underlying epistemic structure. One obstacle to achieving this full generality is currently represented by our unsuccessful attempts to provide natural conditions under which SL is partitioned by facts, events and inaccessible formulas.

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