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On decidability of concept satisfiability in Description Logic with product semantics

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Abstract

The aim of the present paper is to prove that concept validity and positive satisfiability with an empty ontology in the Fuzzy Description Logic $\mathfrak{I}\mathcal{ALE}$, under standard product semantics and with respect to quasi-witnessed models, are decidable. In our framework we are not considering reasoning tasks over ontologies. The proof of our result consists in reducing the problem to a finitary consequence problem in propositional product logic with Monteiro-Baaz delta operator, which is known to be decidable. Product FDL and first order logic are known not to enjoy the finite model property, so we cannot restrict to finite interpretations. Thus, in order to obtain our result, we need to codify infinite interpretations using a finite number of propositional formulas. Such result was conjectured in [10], but the proof given was subsequently found incorrect. In the present work an improved reduction algorithm is proposed and a proof of the same result is provided.

1. Introduction

Obtaining methods and tools suited to give a high-level description of the world and to implement intelligent systems plays a key role in the area of Knowledge Representation and Reasoning. Classical Description Logics (DLs) are knowledge representation languages particularly suited to specify formal ontologies, which have been studied extensively over the last three decades. A comprehensive reference manual on the field is [2]. In real applications, the knowledge used is usually imperfect and it has to address situations of uncertainty and vagueness. From a real world viewpoint, it is easy to find domains where concepts like “patient with a high fever” and “person living near a pollution source” have to be considered. One way to treat the *vagueness* aspect is to model DL concepts as fuzzy sets and roles as binary fuzzy relations.

Fuzzy Sets and Fuzzy Logic were born to deal with the problem of approximate reasoning [25,26]. In recent times, formal logic systems have been developed for such semantics, and the logics based on triangular norms (t-norms) have become the central paradigm of fuzzy logic. The development of this field of research, called Mathematical

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Fuzzy Logic (MFL for short), is intimately linked to [15], which shows the connection of fuzzy logic systems with many-valued residuated lattices based on continuous t-norms. There are three basic continuous t-norms: minimum, Łukasiewicz and product t-norm. For each one of these three t-norms a propositional and a first order logical system have been studied in the literature. A reference text about fuzzy logics is [14]. In MFL a distinction between general (that is, a semantics based on all Gödel, Łukasiewicz or product algebras, respectively) and standard semantics (that is, a semantics restricted to just minimum, Łukasiewicz or product t-norms) is done. In the propositional case, standard completeness results have been proved and, thus, the logic defined by general and standard semantics coincide. This is also the case for first order Gödel Logic, but not for first order Łukasiewicz and product logics, where it has been proved that general and standard semantics give different logics. Traditionally, the semantics adopted for FDL systems is the one based on those t-norms, hence, in this paper we deal with logics given by the standard semantics and, when not specified otherwise, by "semantics", we will mean "standard semantics".

The first paper on FDLs is [24], but FDLs become to be more studied since [22]. In [16], Hájek introduced the fuzzy version of DL, as a fragment of its first order fuzzy logic with the standard semantics. The main result in [16] says that, if the semantics is based on the Łukasiewicz t-norm, then, for the language $\mathcal{ALC}$, concept satisfiability and validity are decidable problems. This is proved using a reduction of such problems to a decidable problem in the propositional fuzzy logic given by the corresponding t-norm; the idea behind this reduction is the fact that finite interpretations can be codified using a finite number of propositional formulas. This is possible in the case of Łukasiewicz t-norm, and the proof of this fact is based on the notion of *witnessed model*. In the case of Łukasiewicz t-norm it is well known [17] that satisfiable first order formulas (in particular DL concepts), coincide with satisfiable first order formulas in witnessed models. Hájek also proves that the result about decidability is true for the logic of any continuous t-norm, restricted to witnessed models, but only for Łukasiewicz satisfiability coincides with satisfiability w.r.t. witnessed (finite) models. After the seminal paper by Hájek, there have been more works on FDL based on the framework of Mathematical Fuzzy Logic, a comprehensive account can be found in [6]. As proved in [16], in product t-norm, as well as in Gödel t-norm, there exist satisfiable FDL concepts and first order formulas that are not satisfiable in any witnessed interpretation. This means, in particular, that reasoning tasks, based on product or minimum t-norm do not enjoy the finite model property, at least with respect to Kripke-like structures (see [21] for a proof of decidability of Gödel modal logic, based on other methods). The lack of this property makes more challenging a proof of decidability, since it makes necessary to cope with infinite models. In [20] it was proved that product first order logic with general semantics is complete with respect to structures that were called *closed models* in [20] and that were subsequently called *quasi-witnessed models* in [10] and [9]. In the Appendix of [10], it is proved that this result can be restricted to product first order logic with standard semantics for the validity and positive satisfiability problems, but not for the 1-satisfiability problem, that is still an open problem. Despite the fact that completeness with respect to quasi-witnessed structures does not assure finite model property, nevertheless it reduces the complexity in the shape of the models that decide satisfiability in first order product logic. Thus, quasi-witnessed model property allows to try to prove decidability of an $\mathcal{ALC}$-like language based on product t-norm, by managing well-behaved infinite structures.

Several results were obtained in the literature about Fuzzy Description Logics in the last twenty years. Decidability of different reasoning tasks in a quite expressive DL language, based on Gödel t-norm with witnessed interpretations, with a non-empty ontology was proved in [7]. This result implies decidability of concept satisfiability with empty ontologies for the same language, as well as for the fragment we are interested in. In [16], decidability of concept satisfiability in $\mathcal{ALC}$, based on Łukasiewicz t-norm with respect to witnessed interpretations and with an empty ontology was proved, while in [12] it was proved that ontology consistency in the same language is undecidable in general. The same undecidability result for $\mathfrak{I}\mathcal{ALE}$, based on product t-norm (this is exactly the DL language studied in the present paper) with respect to witnessed models is proved in [3] and [4]. In [3] is also proved that ontology consistency in $\mathfrak{I}\mathcal{ALE}$, based on product t-norm with respect to quasi-witnessed models is undecidable. In [8] it is proved that ontology consistency of a very expressive language, based on product t-norm and with respect to witnessed models can be linearly reduced to consistency of a crisp ontology. This result does not contradict the result in [3] and [4] because in such works the expressivity allowed in the ontology is greater.

In [10] a proof that concept positive satisfiability with empty knowledge bases is a decidable problem for language $\mathfrak{I}\mathcal{ALE}$, based on product t-norm, is proposed. Later on, the proof has been found incorrect while trying to implement the reduction algorithm in [1]. In the present work an improved reduction algorithm is proposed and an alternative proof of the same result is provided. Indeed, the aim of the present paper is to prove that positive concept satisfiability with an empty ontology in language $\mathfrak{I}\mathcal{ALE}$, under product semantics and with respect to quasi-witnessed models is decidable. We point out that in our framework we are not considering reasoning tasks over ontologies. The proof of our result follows the same pattern as Hájek's one, reducing the problem to a consequence problem in the propositional fuzzy logic (of product t-norm this time), but in this occasion we cannot restrict to finite interpretations. Thus, in order to deal with the product case, we need to codify infinite interpretations using a finite number of propositional formulas. Moreover, in the present work, we propose a behavioral analysis of the algorithm that determines a worst-case lower bound of complexity for our algorithm, as well as for the one in [16]. Such bound partially answers a problem that was left open by Hájek in his work.

The paper is organized as follows. In the Preliminary section (Section 2) we provide the basic definitions. In Section 3 it is proved that concept satisfiability w.r.t. quasi-witnessed models can be restricted to what we call *canonical interpretations*. In Section 4 the reduction algorithm is defined, soundness and completeness of such algorithm are proved and an analysis of the worst case complexity of the rough algorithm is undertaken. In Section 5 we give the decidability results, obtained as consequences of the reduction algorithm, and some remarks about the 1-satisfiability open problem are provided. Finally, in Section 6 we conclude with some comments about the applicability of our results to other t-norms and to Modal Logic.

2. Preliminaries

In this preliminary section we will give basic definitions and known results about the logic of product t-norm, product description logic and about witnessed and quasi-witnessed models. The interested reader can find further details in [6].

2.1. The logic of product t-norm

Product t-norm $*$ is one of the basic continuous t-norms and it is defined as the product of real numbers in [0, 1]. It is a binary operation, that is commutative, associative, monotone and having 1 as neutral element. As a continuous t-norm, it has a residuum $\Rightarrow$, satisfying the residuation condition:

$$a * b \leq c \iff a \leq b \Rightarrow c$$

and defined as:

$$a \Rightarrow b = \begin{cases} 1, & \text{if } a \leq b, \\ \frac{b}{a}, & \text{otherwise.} \end{cases}$$

Moreover we consider the following operations on [0, 1]:

$$\begin{aligned} \neg a &:= a \Rightarrow 0 && \text{(negation)} \\ a \Leftrightarrow b &:= \min\{a \Rightarrow b, b \Rightarrow a\} && \text{(equivalence degree).} \end{aligned}$$

These operations are defined by:

$$\neg a := \begin{cases} 1, & \text{if } a = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$a \Leftrightarrow b := \frac{\min\{a,b\}}{\max\{a,b\}}.$$

Moreover, from the two basic operation $*$ and $\Rightarrow$ we can obtain min and max operations in [0, 1], since the following equalities hold:

$$\min\{a, b\} = a * (a \Rightarrow b)$$

$$\max\{a, b\} = \min\{(a \Rightarrow b) \Rightarrow b, (b \Rightarrow a) \Rightarrow a\}$$

We will denote by $[0, 1]_\Pi$ the algebra on the real unit interval,

$$[0, 1]_\Pi := \langle [0, 1], \min, \max, *, \Rightarrow, \neg, 1, 0 \rangle$$

Another operation on [0, 1] that we will consider in the proof of decidability is the *Monteiro-Baaz delta operator* $\triangle$, *defined in [5] as:*

$$\triangle a := \begin{cases} 1, & \text{if } a = 1 \\ 0, & \text{otherwise.} \end{cases}$$

The propositional logic of product t-norm with standard semantics, Π_{St} is the logic whose language is defined from a set of propositional variables using the basic connectives $\odot, \rightarrow, \bot$ (plus $\wedge, \vee, \neg$ and $\equiv$ as definable connectives). Its semantics is based on evaluations of the propositional variables in [0, 1] interpreting the logical connectives as $*, \Rightarrow$ and 0 respectively. The consequence relation $\models_{\Pi_{St}}$ is defined as usual as:

$$\Gamma \models_{\Pi_{St}} \varphi \text{ iff for each evaluation } e \text{ such that } e(\gamma) = 1 \text{ for all } \gamma \in \Gamma, \text{ we have that } e(\varphi) = 1.$$

The first order logic of product t-norm, $\Pi\forall_{St}$ is the logic whose language is defined from sets of individual variables and predicate symbols (and, possibly empty, sets of function and constant symbols) using the same connectives as in the propositional case, plus the two quantifiers $\forall$ and $\exists$. Its semantics[1] is defined in the usual way for first order logic, by means of interpretations of the individual variables as elements from a given domain, where the n-ary predicate symbols are interpreted as n-ary fuzzy relations in the domain with values in $[0, 1]_\Pi$. The consequence relation $\models_{\Pi\forall_{St}}$ is defined as usual as:

$$\Gamma \models_{\Pi\forall_{St}} \varphi \text{ iff for each interpretation of individual variables and predicate symbols such that all } \gamma \in \Gamma \text{ have value 1, then } \varphi \text{ has value 1 as well.}$$

The logics here defined are not the propositional (resp. first order) product logic Π (resp. $\Pi\forall$) defined in the general framework of Mathematical Fuzzy Logic, since here we have restricted the general semantics (based on any product algebra) to the standard semantics based on $[0, 1]_\Pi$. The logics Π_{St} and $\Pi\forall_{St}$ are known as propositional product logic (resp. first order) with *standard semantics*. In the propositional case product logic Π and product logic with standard semantics Π_{St} coincide since product logic is standard complete (see [18]), but this is not the case for first order logics. In fact, first order Product logic $\Pi\forall$ is recursively enumerable (it has a finite axiomatization) while first order product logic $\Pi\forall_{St}$ is not arithmetic (see [15]).

2.2. Product description logic

In this section we give basic definitions and known results about Product Description Logic (ΠDL).

2.2.1. Syntax

Let us define a *description signature* as a triple $\mathcal{D} = \langle N_I, N_A, N_R \rangle$, where N_I, N_A and N_R are pairwise disjoint sets of individual names, concept names (or *atomic concepts*), and role names (or *atomic roles*), respectively. The set of *concepts* for the description language considered in the present paper, is inductively defined by the following syntactic rules:

$C, C_1, C_2 \rightsquigarrow$	$A \mid$	(atomic concept)
	$C_1 \sqcap C_2 \mid$	(concept strong intersection)
	$C_1 \rightarrow C_2 \mid$	(concept implication)
	$\neg C \mid$	(weak complementary concept)
	$\bot \mid$	(empty concept)
	$\top \mid$	(universal concept)
	$\forall R.C \mid$	(value restriction)
	$\exists R.C$	(existential restriction).

Such description language is called $\mathfrak{I}\mathcal{ALE}$, according to the literature (see [6] for more details). As in Section 2.1, it is possible to define the following concepts constructors:

[1] Note that the ΠDL semantics defined in Section 2.2.2 is a restriction of first order semantics.

$$\begin{aligned} C \wedge D &:= C \sqcap (C \to D) \\ C \vee D &:= ((C \to D) \to D) \wedge ((D \to C) \to C) \\ C \equiv D &:= (C \to D) \wedge (D \to C) \end{aligned}$$

Since individual names are used to define neither concept satisfiability nor validity, the reasoning tasks we are interested in, we could omit them from our syntax. Nevertheless, in order to define the reduction algorithm, we will use expressions of the form $C(a)$ and $R(a, b)$, where C is a $\mathfrak{I}\mathcal{ALE}$ concept, R is a role name and $a, b \in N_I$, which, with a slight language abuse, we will call *instances*.

2.2.2. Semantics for basic concept constructors

An *interpretation* for a description signature $\mathcal{D} = \langle N_I, N_A, N_R\rangle$ is a pair $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\rangle$ consisting of a nonempty (crisp) set $\Delta^{\mathcal{I}}$ (called *domain*) and of a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$ that assigns: (i) to each concept name $A \in N_A$ a fuzzy set, that is, a function $A^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0, 1]$, (ii) to each role name $R \in N_R$ a fuzzy relation, that is, a function $R^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0, 1]$, (iii) to each individual name $a \in N_I$ an object $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. Given an interpretation $\mathcal{I}$, the interpretation $\cdot^{\mathcal{I}}$ is extended to complex concepts by assigning to every complex concept D a fuzzy set $D^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0, 1]$ inductively defined as follows:

$$D^{\mathcal{I}}(x) := \begin{cases} 0 & \text{if } D = \bot \\ 1 & \text{if } D = \top \\ A^{\mathcal{I}}(x) & \text{if } D = A \in N_A \\ C^{\mathcal{I}}(x) \Rightarrow \bot & \text{if } D = \neg C \\ C_1^{\mathcal{I}}(x) * C_2^{\mathcal{I}}(x) & \text{if } D = C_1 \sqcap C_2 \\ C_1^{\mathcal{I}}(x) \Rightarrow C_2^{\mathcal{I}}(x) & \text{if } D = C_1 \to C_2 \\ \inf\{R^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y) \mid y \in \Delta^{\mathcal{I}}\} & \text{if } D = \forall R.C \\ \sup\{R^{\mathcal{I}}(x, y) * C^{\mathcal{I}}(y) \mid y \in \Delta^{\mathcal{I}}\} & \text{if } D = \exists R.C. \end{cases}$$

2.3. Results on witnessed and quasi-witnessed models

Definition 1. ([16]) A model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\rangle$ is called *witnessed* if in the interpretation of any quantified formula the infimum (supremum) is a minimum (maximum), i.e. for every $x \in \Delta^{\mathcal{I}}$,

- For all $D = \forall R.C$ there is $z \in \Delta^{\mathcal{I}}$ such that $D^{\mathcal{I}}(x) = R^{\mathcal{I}}(x, z) \Rightarrow C^{\mathcal{I}}(z)$,
- For all $E = \exists R.C$ there is $t \in \Delta^{\mathcal{I}}$ such that $E^{\mathcal{I}}(x) = R^{\mathcal{I}}(x, t) * C^{\mathcal{I}}(t)$.

The elements z and t are called *witnesses* of the quantified concepts $D = \forall R.C$, $E = \exists R.C$, respectively.

In [16] Hájek proves that satisfiability of a concept in Łukasiewicz Description Logic (ŁDL) is a decidable problem. He uses an algorithm that reduces satisfiability of a concept in ŁDL to satisfiability of a set of propositions in propositional Łukasiewicz Logic, that is a known decidable problem. The proof uses the fact that satisfiability on ŁDL coincides with satisfiability with respect to witnessed models and equivalently, with respect to finite models. The proof of this fact is based on the continuity of the Łukasiewicz t-norm and its residuum. This implies that the result is not valid for description logics based on others continuous t-norms, because their implications are not continuous, in particular for ΠDL. Hájek himself in the cited paper, shows that concept $\neg\forall R.C \sqcap \neg\exists R.\neg C$ is satisfiable in ΠDL, but not by any finite model, i.e., by a witnessed model.

In [20], its authors introduced the so called *closed models* and they proved that first order Product Logic Π∀ is complete with respect to these models. In [9] new results on first order Product Logic and its relations with closed models are provided, but now those structures are called *quasi-witnessed models*. In this paper we will use the name quasi-witnessed models.

Definition 2. A model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\rangle$ is called *quasi-witnessed* (*qw-models* for short) if, for any $x \in \Delta^{\mathcal{I}}$, we have that:

- For all $D = \forall R.C$ either $D^{\mathcal{I}}(x) = 0$ or there is $z \in \Delta^{\mathcal{I}}$ such that $D^{\mathcal{I}}(x) = R^{\mathcal{I}}(x, z) \Rightarrow C^{\mathcal{I}}(z)$,
- For all $E = \exists R.C$ there is $t \in \Delta^{\mathcal{I}}$ such that $E^{\mathcal{I}}(x) = R^{\mathcal{I}}(x, t) * C^{\mathcal{I}}(t)$

The element z and t are called *witnesses* of the quantified formulas $D = \forall R.C$ (if it exists) and $E = \exists R.C$, respectively.

In the Appendix of [10] it is proved that validity of a formula in first order product logic with standard semantic $\Pi\forall_{St}$ coincides with validity with respect to quasi-witnessed models. The proof is based on the following consequence of a result in [10]: for every first order $[0, 1]_{\Pi}$-model there is a quasi-witnessed $[0, 1]_{\Pi}$-model which approximates as much as you want the value of a formula to the value that this formula has in the former model. Using this result it is easy to prove that positive satisfiability coincides with positive satisfiability in quasi-witnessed models. But this is not the case for 1-satisfiability. In fact it is still an open problem whether 1-satisfiability of concepts in ΠDL coincides with 1-satisfiability in quasi-witnessed models.

2.4. Reasoning tasks

In the present paper we are going to consider the following reasoning tasks:

(Val$_1$) We say that a concept C is 1-*valid* ($C \in \mathrm{Val}_1$) if $C^{\mathcal{I}} = 1$, for every product interpretation $\mathcal{I}$.
(Val$_+$) We say that a concept C is *positively valid* ($C \in \mathrm{Val}_+$) if $C^{\mathcal{I}} > 0$, for every product interpretation $\mathcal{I}$.
(Sat$_1$) We say that a concept C is 1-*satisfiable* ($C \in \mathrm{Sat}_1$) if there exists a product interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} = 1$. We will use Sat_r, with $r \in [0, 1]$, for the set of r-satisfiable concepts.
(Sat$_+$) We say that a concept C is *positively satisfiable* ($C \in \mathrm{Sat}_+$) if there exists a product interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} > 0$.
(Subs) We say that a concept D is *subsumed by* concept C ($(C, D) \in \mathrm{Subs}$) if $\inf_{x \in \Delta^{\mathcal{I}}}\{C(x) \to D(x)\} = 1$ for every product interpretation $\mathcal{I}$.

If the above reasoning tasks are restricted to qw-interpretations, we will denote the respective sets as QVal_1, QVal_+, QSat_1, QSat_+ and QSubs. In product first order logic, the sets Val_1 and Sat_+ coincide with their respective quasi-witnessed restriction, as proved in the Appendix of [10]. Section 4 is devoted to prove that the satisfiability problem of a concept in ΠDL restricted to quasi-witnessed models is decidable. In Section 5, we will use such result and the one from [10], to prove unrestricted decidability results for ΠDL. Before that, we need some preliminary results about ΠDL quasi-witnessed interpretations, that are proved in Section 3.

3. Canonical models

As we said in the introduction, we found an error in our previous algorithm (see [10]). In the present paper we define a new algorithm, introducing two new constraints (see constraints (∀3) and (∀4) in Section 4). In order to prove completeness of the reduction algorithm, we need a special type of qw-interpretations that we call *canonical interpretations* with respect to a particular concept. The completeness proof is based on the fact that we can restrict ourselves to canonical interpretations. In this section we are going to prove that, whenever a concept has a certain value in a given qw-interpretation, we can always find a canonical qw-interpretation where the concept has the same value.

Let C_0 be an $\mathfrak{I}\mathcal{ALE}$ concept, $\mathcal{I}$ be a qw-interpretation and $v \in \Delta^{\mathcal{I}}$. In order to compute the value of $C_0^{\mathcal{I}}(v)$, for each $x \in \Delta^{\mathcal{I}}$ we need to compute the value of a finite number of either universally or existentially quantified concepts, i.e., we need to compute a family $U_i = \forall R_i.C_i$ and a family $E_k = \exists R_k.F_k$ for $i = 1, 2, \ldots, l$ and $k = 1, 2, \ldots, n$. Since $\mathcal{I}$ is quasi-witnessed, each $E_k^{\mathcal{I}}(x)$ has a witness y_{E_k}, but this is not the case for universally quantified concepts. Any interpretation induces two kinds of universally quantified concepts:

- The ones satisfying $(\forall R.D)^{\mathcal{I}}(x) = R^{\mathcal{I}}(x, y) \Rightarrow D^{\mathcal{I}}(y)$, for some $y \in \Delta^{\mathcal{I}}$. In this case we say that concept $\forall R.D$ *is witnessed* in x and that y is a witness.

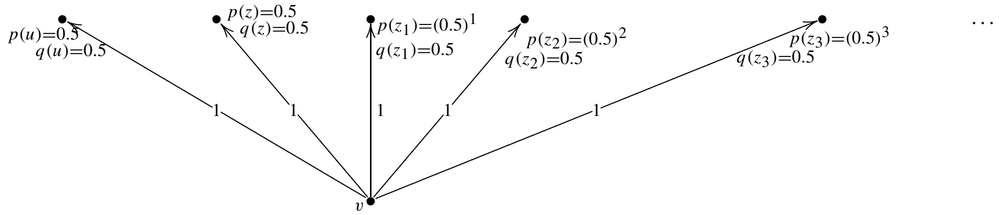

Fig. 1. The quasi-witnessed structure from Example 4.

- The ones satisfying $0 = (\forall R.C)^{\mathcal{I}}(x) = \inf_{w\in\Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,w) \Rightarrow C^{\mathcal{I}}(w)\} \neq R_i^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$ for all $y \in \Delta^{\mathcal{I}}$. In this case we say that concept $\forall R.C$ *has no witness* in x or that it is *quasi-witnessed* in x.

From now on we will denote by $qw_R(x)$ and by $w_R(x)$ the sets of quantified subconcepts of C_0 with respect to role R, that are quasi-witnessed and witnessed respectively, in $x \in \Delta^{\mathcal{I}}$. In order to simplify the proofs, we will consider the *unravelling* of interpretation $\mathcal{I}$, that we still denote by $\mathcal{I}$. Such construction is standard in the literature and the details can be found in [13] (Theorem 2.19) for the classical case and in [21] (Lemma 3.6) for the many-valued case.

Definition 3. A qw-interpretation $\mathcal{I}$ is said to be *canonical* in $v \in \Delta^{\mathcal{I}}$ with respect to a finite set $\mathcal{M}$ of quantified concepts if, for each $U = \forall R.C \in qw_R(v) \cap \mathcal{M}$, there exists $u \in \Delta^{\mathcal{I}}$, such that

$$0 < C(u) < R(v,u) \leq D(u) \tag{1}$$

for all D such that $(\forall R.D)^{\mathcal{I}}(x) > 0$.

A qw-interpretation $\mathcal{I}$ is said to be *canonical* with respect to a $\mathfrak{I}\mathcal{ALE}$ concept C_0 if it is canonical in every $v \in \Delta^{\mathcal{I}}$ with respect to the set of quantified subconcepts of C_0.

In general quasi-witnessed interpretations do not satisfy this property, as the following example shows.

Example 4. Consider the atomic concepts C and B, the role R and the quantified formulas $\forall R.B$ and $\forall R.C$. Consider the interpretation $\mathcal{I}$ in Fig. 1, such that:

1. $\Delta^{\mathcal{I}} = \{v, u, z\} \cup \{z_i \mid i \in \mathbb{N}\backslash\{0\}\}$.
2. There is a binary relation r such that:
   - $r(v,x) = 1$, if $x \neq v$,
   - $r(x,y) = 0$, otherwise.
3. There are two unary predicates p and q, that is, two fuzzy sets in $\Delta^{\mathcal{I}}$, such that:
   - $p(u) = p(z) = 0.5$,
   - $p(z_i) = (p(z))^i$,
   - $q(u) = q(z) = q(z_i) = 0.5$.

In these conditions, if we take $R^{\mathcal{I}} = r$, $C^{\mathcal{I}} = p$, and $B^{\mathcal{I}} = q$, then it is easy to check that $\mathcal{I}$ is a quasi-witnessed interpretation, since an easy computation shows that:

- $(\forall R.C)^{\mathcal{I}}(v) = 0$ with $R^{\mathcal{I}}(v,x) \Rightarrow C^{\mathcal{I}}(x) \neq 0$, for all $x \in \Delta^{\mathcal{I}}$,
- $(\forall R.B)^{\mathcal{I}}(v) = 0.5$.

Nevertheless, $\mathcal{I}$ is not canonical in v w.r.t. the set of concepts $\mathcal{M} = \{\forall R.B, \forall R.C\}$, since $R^{\mathcal{I}}(v,v) = 0$ and, for all $x \neq v$, we have $B^{\mathcal{I}}(x) = 0.5 < 1 = R^{\mathcal{I}}(v,x)$.

Now we will prove that, given a quasi-witnessed interpretation $\mathcal{I}$, we can always obtain a canonical interpretation $\mathcal{I}'$ which preserves graded satisfiability for a given concept. Previously we prove a technical lemma.

Lemma 5. *Let $\mathcal{I}$ be a quasi-witnessed interpretation, $v \in \Delta^{\mathcal{I}}$ and $\forall R.C \in qw_R(v)$. Then, for every finite set $\mathcal{M}$, such that $(\forall R.D)^{\mathcal{I}}(v) > 0$, for every $\forall R.D \in \mathcal{M}$, there exist infinitely many elements $u \in \Delta^{\mathcal{I}}$ such that*

$$0 < C^{\mathcal{I}}(u) < D^{\mathcal{I}}(u) \leq 1.$$

Proof. Since $\mathcal{M}$ is finite, let $s = \min_{D \in \mathcal{M}}\{(\forall R.D)^{\mathcal{I}}(v)\} > 0$. Since $\forall R.C \in qw_R(v)$, then there exists an infinite set $\Delta \subset \Delta^{\mathcal{I}}$ such that, for all $u \in \Delta$, we have that $0 < R^{\mathcal{I}}(v,u) \Rightarrow C^{\mathcal{I}}(u) < s \leq R^{\mathcal{I}}(v,u) \Rightarrow D^{\mathcal{I}}(u)$ for all $D \in \mathcal{M}$. Since residuum $\Rightarrow$ is non-decreasing in the second argument, for all $u \in \Delta$ and for all $D \in \mathcal{M}$ we have that $0 < C^{\mathcal{I}}(u) < D^{\mathcal{I}}(u)$. $\square$

Now we can prove the desired result.

Lemma 6. *Given a qw-interpretation $\mathcal{I}$ and a finite set $\mathcal{M}$ of quantified concepts, for every $v \in \Delta^{\mathcal{I}}$ exists a qw-interpretation $\mathcal{I}'$, that is canonical in v w.r.t. $\mathcal{M}$ such that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}'}$ and $H^{\mathcal{I}}(v) = H^{\mathcal{I}'}(v)$, for all $H \in \mathcal{M}$.*

Proof. We will prove the result just for a set $\mathcal{M}$ where only role R appears, the extension of the proof to a set $\mathcal{M}$ containing concepts with a finite number of different roles is straightforward. Let W be the set containing one witness for any concept in $\mathcal{M} \setminus qw_R(v)$. Let $\forall R.C \in \mathcal{M} \cap qw_R(v)$. By Lemma 5, there exists an infinite set $\Delta \subseteq \Delta^{\mathcal{I}}$ such that, for each $z \in \Delta$, we have that $0 < C^{\mathcal{I}}(z) < R^{\mathcal{I}}(v,z)$ and $C^{\mathcal{I}}(z) < D^{\mathcal{I}}(z)$, for all concept D within the scope of a quantified concept in $\mathcal{M}' = \{\forall R.D \in \mathcal{M} \colon (\forall R.D)^{\mathcal{I}}(v) > 0\}$. Choose $u \in \Delta \setminus W$ and define the qw-interpretation $\mathcal{I}'$ that coincides with $\mathcal{I}$ except for $R^{\mathcal{I}'}(v,u) = \min\{D^{\mathcal{I}}(u), R^{\mathcal{I}}(v,u) \colon \forall R.D \in \mathcal{M}'\}$. By definition of $\mathcal{I}'$ we have that $0 < C^{\mathcal{I}'}(u) < R^{\mathcal{I}'}(v,u) \leq D^{\mathcal{I}'}(u)$. Moreover, since $R^{\mathcal{I}'}(v,u) \leq R^{\mathcal{I}}(v,u)$, then both $R^{\mathcal{I}'}(v,u) \Rightarrow F^{\mathcal{I}'}(u) \geq R^{\mathcal{I}}(v,u) \Rightarrow F^{\mathcal{I}}(u)$ and $R^{\mathcal{I}'}(v,u) * F^{\mathcal{I}'}(u) \leq R^{\mathcal{I}}(v,u) * F^{\mathcal{I}}(u)$, for any concept F. Consequently, $H^{\mathcal{I}}(v) = H^{\mathcal{I}'}(v)$, for all $H \in \mathcal{M}$. Repeating the process for every $\forall R.C' \in \mathcal{M} \cap qw_R(v)$ we obtain a qw-interpretation that is canonical in v w.r.t. $\mathcal{M}$. Observe that for each $\forall R.C' \in \mathcal{M} \cap qw_R(v)$ we have to choose an element u', which can be different from the ones chosen for other concepts in $\mathcal{M} \cap qw_R(v)$. $\square$

The following proposition is an obvious consequence of Lemma 6.

Proposition 7. *Let C a $\mathfrak{I}\mathcal{ALE}$ concept and $r \in [0,1]$. There exists a quasi-witnessed interpretation $\mathcal{I}$ and $v \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(v) = r$ if and only if there exists a quasi-witnessed canonical interpretation $\mathcal{I}'$ for C such that $C^{\mathcal{I}'}(v) = r$.*

4. Decision procedure

In this section we define a computable reduction of the concept satisfiability problem in $\mathfrak{I}\mathcal{ALE}$ language, with respect to canonical interpretations, to a finitary entailment problem in the corresponding propositional calculus with Monteiro-Baaz $\triangle$ operator. We recall that, as we have seen in the last section, unrestricted satisfiability in qw-models coincides with satisfiability w.r.t. canonical qw-models. This allows us to restrict to canonical interpretations in the proofs of the next sections. In Definition 10, below, a procedure to obtain two distinct sets of formulas from a given $\mathfrak{I}\mathcal{ALE}$ concept C_0 is provided. These formulas are among seven fixed formula schemata. For example, one of them has the form $\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))) \vee \neg\forall R.C(d_\sigma)$, where R is a role name, C is a concept, both appearing in C_0 and d_σ, $d_{\sigma,\alpha}$ are individual names produced by the same process. Such formulas are not from any formalism that has been defined in the present work. They are not FDL concepts since most of them do not obey the rules defined in Section 2.2.1, they are rather propositional combinations of concepts and role names instantiated by individual names. They are not modal formulas, since relations between individuals can not be expressed in a modal language. They are not first order formulas, since the quantifiers that appear in them are not treated as first order quantifiers. Following the indications in [6], these expressions can easily be translated into first order formulas, but this process falls out of the scope of the present work. Nevertheless, since any instantiation of a concept or role name, appearing in these formulas, by means of individual names, can be seen as an atomic first order formula, we will use the name *atom* to denote such instances of concept names or role names by means of an individual name. Since atomic first order formulas behave like propositional variables (in the sense that they can be assigned a value in [0, 1]), a process to obtain suitable propositional formulas from these somehow intermediate expressions is defined in order to reduce $\mathfrak{I}\mathcal{ALE}$ concept satisfiability into propositional entailment in a decidable propositional calculus. In the meantime, we will generically refer to these expressions as *constraints*, since they can be seen as intermediate constraints between FDL and some other decidable calculus. Indeed, it is not necessary to manage these constraints like propositional formulas. A former version of them from [10] has been already used to reduce FDL concept satisfiability into Satisfiability Modulo Theories (SMT) constraints in [1], in order to implement the satisfiability algorithm. Since the process described in Definition 10 is deterministic, we can consider it as a proper syntax for these constraints, in the sense that it provides a way to decide whether a given expression is or not a constraint resulting from applying Definition 10 to a given $\mathfrak{I}\mathcal{ALE}$ concept. Now we introduce some notions (taken from [16]) that we will use throughout the paper.

Definition 8.

1. *Role depth*[2] *of the quantified subconcepts* in C (or $C(a)$) is defined inductively:
   - $rd(C) = 0$, if C is a concept name or a constant concept $\bot$ or $\top$;
   - if C is a concept, then $rd(\neg C) = rd(C)$;
   - if C and D are concepts, then $rd(C \sqcap D) = rd(C \to D) = \max(rd(C), rd(D))$;
   - if C is a concept and R a role name, then $rd(\forall R.C) = rd(\exists R.C) = rd(C) + 1$.
2. We will use the term *instance* for atoms and instantiations of complex concepts by means of individual names.
3. *Generalized atoms* are instances of quantified concepts.

Next, for every concept C_0 we are going to recursively associate two finite sets P_{C_0} and Y_{C_0} of constraints. Given a concept C_0, we construct finite sets P_{C_0} and Y_{C_0} of constraints, that are meant to be a finite codification of a, possibly infinite, interpretation for C_0. Later on, it will be proved that P_{C_0} and Y_{C_0} are the codification of an interpretation for C_0 if there is a propositional evaluation that satisfies the whole propositional translation of P_{C_0} and no proposition from a propositional translation of Y_{C_0}. The construction takes steps $0, \dots, n$ where n is the role depth of concept C_0. At each step some generalized atoms are "processed" in the sense that, at each step, (a) we add some new individual names d_σ to N_I, where σ is a sequence of generalized atoms built from quantified subconcepts of C_0, (b) we add some new constraints to P_{C_0} and Y_{C_0} and (c) we select some generalized atoms to be processed in the next step. The generalized atoms selected in step i will have role depth $\leq n - i$; after step n is completed, the algorithm stops.

Definition 9. We call *instance decomposition* the process of recursively applying the following clauses to a given instance α, until only atoms or generalized atoms remains:

1. $(C \sqcap D)(d_\sigma) := C(d_\sigma) \sqcap D(d_\sigma)$,
2. $(C \to D)(d_\sigma) := C(d_\sigma) \to D(d_\sigma)$.

Definition 10 *(Algorithm).* At step 0, we apply instance decomposition, as in Definition 9 to $C_0(d)$ and transfer the generalized atoms obtained to be further processed in step 1. For $i > 0$, step i selects the instances in formulas transferred from step $i - 1$ and processes them. We know that the generalized atoms just selected have the form $QR.C(d_\sigma)$, where $Q \in \{\forall, \exists\}$, R is a role, C a concept with role depth $\leq n - i$, d_σ is an individual name produced in the previous step. For each generalized atom α, at step i we do the following:

- If α is $\forall R.C(d_\sigma)$, then produce the new individual name $d_{\sigma,\alpha}$ and add to P_{C_0} the constraint:

$$(\forall 1) \qquad (\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))) \vee \neg\forall R.C(d_\sigma)$$

- If α is $\exists R.C(d_\sigma)$, then produce a new individual name $d_{\sigma,\alpha}$ and add to P_{C_0} the constraint:

$$(\exists 1) \qquad \exists R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \sqcap C(d_{\sigma,\alpha}))$$

[2] This is what is called *nesting degree* in [16].

We will say that $d_{\sigma,\alpha}$ is a *individual name associated to* R, d_σ and the *q-witness*[3] *of* α. Now, we consider each α of the present step and do the following:

- If α is $(\forall R.C)(d_\sigma)$ and $d_{\sigma,\beta}$ is any individual name associated to R, d_σ, then add to P_{C_0} the constraint

$$(\forall 2) \qquad \forall R.C(d_\sigma) \to (R(d_\sigma, d_{\sigma,\beta}) \to C(d_{\sigma,\beta}))$$

- If α is $(\forall R.C)(d_\sigma)$ and $d_{\sigma,\alpha}$ is its q-witness, then, for every generalized atom $(\forall R.D)(d_\sigma)$, add to P_{C_0} the constraints:

$$(\forall 3) \qquad \neg(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))) \to$$
$$(\triangle(D(d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha})) \to (\forall R.D(d_\sigma) \to \forall R.C(d_\sigma)))$$

and

$$(\forall 4) \qquad (\neg(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))) \sqcap \neg\neg\forall R.D(d_\sigma)) \to$$
$$(R(d_\sigma, d_{\sigma,\alpha}) \to D(d_{\sigma,\alpha}))$$

- If α is $(\exists R.C)(d_\sigma)$ and $d_{\sigma,\beta}$ is any individual name associated to R, d_σ, then add to P_{C_0} the constraint

$$(\exists 2) \qquad (R(d_\sigma, d_{\sigma,\beta}) \sqcap C(d_{\sigma,\beta})) \to \exists R.C(d_\sigma)$$

- If α is $(\forall R.C)(d_\sigma)$ and $d_{\sigma,\alpha}$ is its q-witness, then add to Y_{C_0} the constraint

$$(\forall 5) \qquad \neg\forall R.C(d_\sigma) \sqcap (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))$$

After these rules have been applied to every generalized atom transferred from step $i-1$, apply instance decomposition to every new instance $C(d_{\sigma,\alpha})$ produced by constraints (∀1) and (∃1) and transfer the generalized atoms obtained to be further processed in step $i+1$. If no generalized atom is produced to be further processed, the algorithm stops.

As it is said above, our aim is to reduce our problem to one in the corresponding propositional calculus. The propositional logic we need has, as variables, the set:

$$At = \{p_{R(a,b)} : R(a,b) \text{ appearing in } P_{C_0} \cup Y_{C_0}\} \cup$$
$$\{p_{C(a)} : C(a) \text{ is an atom or a generalized atom appearing in } \{C_0(d)\} \cup P_{C_0} \cup Y_{C_0}\}.$$

Every constraint is built up from instances and concept constructors respecting the syntactical rules for propositional wff. Hence, substituting concept constructors for their equivalent propositional connectives and instances for their respective propositional variables, we obtain two sets $pr(P_{C_0})$ and $pr(Y_{C_0})$ of propositional formulas with variables in At.

The next two sections are devoted to prove the following completeness result.

Proposition 11. *Let C_0 be a concept, and let P_{C_0} and Y_{C_0} be the two finite sets associated by Definition 10. For every $r \in [0, 1]$, the following statements are equivalent:*

1. *C_0 is satisfiable with truth value r in a canonical quasi-witnessed interpretation.*
2. *There is a propositional evaluation e over At such that $e(pr(C_0(d))) = r$, $e(pr(P_{C_0})) = 1$, and $e(\psi) \neq 1$ for every $\psi \in pr(Y_{C_0})$.*

From now on we will say that a propositional evaluation e is *quasi-witnessing relatively to C_0* (*quasi-witnessing*, for short) when it satisfies that $e(pr(P_{C_0})) = 1$, and $e(\psi) \neq 1$ for every $\psi \in pr(Y_{C_0})$.

[3] This q-witness is a generalization of the witness defined in [16] and indeed coincide with this notion when α is an existentially quantified generalized atom.

4.1. Soundness

The purpose of this subsection is to show the downwards implication of Proposition 11. Let us assume that, for a given concept C_0, there is a qw-interpretation $\mathcal{I}$ that is canonical in an object $v \in \Delta^{\mathcal{I}}$ w.r.t. C_0, such that $C_0^{\mathcal{I}}(v) = r$ for some $r \in [0, 1]$. Definition 12 tells us how to obtain a quasi-witnessing propositional evaluation $e_{\mathcal{I}}$ satisfying the requirements in Proposition 11.

Definition 12. Let $\mathcal{I}$ be a quasi-witnessed interpretation and v an object of the domain and C_0 a concept. Let $\mathcal{I}$ be canonical in v w.r.t. C_0. Let us consider P_{C_0}, Y_{C_0} as the sets of constraints obtained from concept C_0 by applying Definition 10. In order to define a propositional evaluation $e_{\mathcal{I}}$ on At, from $\mathcal{I}$, we extend $\mathcal{I}$ to interpret the individual names introduced in the definition of P_{C_0} and Y_{C_0}. We define such extension inductively on the length i of σ. Interpret individual d in $\mathcal{I}$ as the object v. For each $i > 0$, we assume that individual names d_σ with σ having length $i - 1$ have been interpreted in $\mathcal{I}$. In order to interpret $d_{\sigma,\alpha}$, for each generalized atom $\alpha = QR.C(d_\sigma)$, appearing in $P_{C_0} \cup Y_{C_0}$, do the following:

- If either $\alpha = \exists R.C(d_\sigma)$ or $\alpha = \forall R.C(d_\sigma) \in w_R(d_\sigma^{\mathcal{I}})$, then choose an element $u \in \Delta^{\mathcal{I}}$ witnessing α and interpret the individual name $d_{\sigma,\alpha}$ as u.
- If $\alpha = \forall R.C(d_\sigma)$ and there is no $z \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, z) \Rightarrow C^{\mathcal{I}}(z) = \inf_{w \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, w) \Rightarrow C^{\mathcal{I}}(w)\}$, then choose an element $u \in \Delta^{\mathcal{I}}$ such that

  $$0 < C^{\mathcal{I}}(u) < R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \leq D^{\mathcal{I}}(u) \leq 1,$$

  for every generalized atom $\forall R.D \in w(d_\sigma^{\mathcal{I}})$ such that $(\forall R.D)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) > 0$. Such an element u indeed exists, since $\mathcal{I}$ is canonical. Once the element u is chosen, interpret the individual name $d_{\sigma,\alpha}$ as u.

Finally, for every atom or generalized atom α, appearing in $P_{C_0} \cup Y_{C_0}$, define $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$.

Next Lemma 13 and Proposition 14, prove that the propositional evaluations obtained in this way are quasi-witnessing.

Lemma 13. *Let $\mathcal{I}$ be a quasi-witnessed interpretation and C_0 a concept. Then, the propositional evaluation $e_{\mathcal{I}}$ is quasi-witnessing relatively to C_0.*

Proof. We will show the result considering, case by case, the propositions we can find in $pr(P_{C_0})$ and $pr(Y_{C_0})$ from Definitions 10 and 9.

1. Consider constraint ($\forall$1) and let $d_{\sigma,\alpha}^{\mathcal{I}} = u$, according to Definition 12. Then we have two cases:
   - If $(\forall R.C(d_\sigma))^{\mathcal{I}} = 0$, then $(\neg\forall R.C(d_\sigma))^{\mathcal{I}} = 1$.
   - If $(\forall R.C(d_\sigma))^{\mathcal{I}} > 0$, then, by definition of u as witness of $(\forall R.C(d_\sigma))^{\mathcal{I}}$, we have that $(\forall R.C(d_\sigma))^{\mathcal{I}} = R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u)$.

   Therefore in both cases we have that $e_{\mathcal{I}}(pr((\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))) \vee \neg\forall R.C(d_\sigma))) = 1$.
2. Consider constraint ($\exists$1) and let $d_{\sigma,\beta}^{\mathcal{I}} = u$, according to Definition 12. Since u is a witness of $(\exists R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}})$, we have that $(\exists R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) * C^{\mathcal{I}}(u)$ and therefore $e_{\mathcal{I}}(pr(\exists R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \sqcap C(d_{\sigma,\alpha})))) = 1$.
3. Consider constraint ($\forall$2) and let $d_{\sigma,\beta}$ be an individual name associated to R, d_σ. Since $(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = \inf_{w \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, w) \Rightarrow C^{\mathcal{I}}(w)\} \leq R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,\beta}^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(d_{\sigma,\beta}^{\mathcal{I}})$ then it is straightforward that, $e_{\mathcal{I}}(pr(\forall R.C(d_\sigma) \to (R(d_\sigma, d_{\sigma,\beta}) \to C(d_{\sigma,\beta})))) = 1$.
4. Consider constraint ($\forall$3) and let $d_{\sigma,\alpha}^{\mathcal{I}} = u$, according to Definition 12. Then:
   - If $(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u)$ then $e_{\mathcal{I}}(pr(\neg(\forall R.C(d_\sigma) \equiv (R(d_\sigma, u) \to C(u))))) = 0$ and, therefore, the whole constraint takes value 1.
   - If $(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) < R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u)$ then $e_{\mathcal{I}}(pr(\neg(\forall R.C(d_\sigma) \equiv (R(d_\sigma, u) \to C(u))))) = 1$. Now we have two cases,
     - If $\triangle(D^{\mathcal{I}}(u) \Rightarrow C^{\mathcal{I}}(u)) = 0$, the whole constraint takes value 1.

– If $\triangle(D^{\mathcal{I}}(u) \Rightarrow C^{\mathcal{I}}(u)) = 1$ then $D^{\mathcal{I}}(u) \leq C^{\mathcal{I}}(u)$. Since, by Definition 12, $\mathcal{I}$ is a canonical interpretation, then, by Definition 3, we have that $\forall R.D \notin w_R(d_\sigma^{\mathcal{I}})$ and, therefore, $(\forall R.D)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = 0$. Hence, the whole constraint takes value 1.

Therefore in both cases we have $e_{\mathcal{I}}(pr(\neg(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))) \to (\triangle(D(d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha})) \to (\forall R.D(d_\sigma) \to \forall R.C(d_\sigma)))))) = 1$.

5. Consider constraint (∀4) and let $d_{\sigma,\alpha}^{\mathcal{I}} = u$, according to Definition 12. Then:
   - If either $(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \in w_R(d_\sigma^{\mathcal{I}})$ or $(\forall R.D)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = 0$, then $e_{\mathcal{I}}(pr(\neg(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))) \sqcap \neg\neg\forall R.D(d_\sigma))) = 0$ and, therefore, the whole constraint takes value 1.
   - If $(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) < R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(u)$ and $(\forall R.D)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) > 0$, since $\mathcal{I}$ is canonical, then $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \leq D^{\mathcal{I}}(u)$ and, therefore, the whole constraint takes value 1.

   Therefore in both cases we have that $e_{\mathcal{I}}(pr((\neg(\forall R.C(d_\sigma) \equiv (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))) \sqcap \neg\neg\forall R.D(d_\sigma)) \to (R(d_\sigma, d_{\sigma,\alpha}) \to D(d_{\sigma,\alpha})))) = 1$.
6. Consider constraint (∃2) and let $d_{\sigma,\beta}$ be an individual name associated to R, d_σ. Since $(\exists R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = \sup_{w \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, w) * C^{\mathcal{I}}(w)\} \geq R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, d_{\sigma,\beta}^{\mathcal{I}}) * C^{\mathcal{I}}(d_{\sigma,\beta}^{\mathcal{I}})$, then we have that $e_{\mathcal{I}}(pr((R(d_\sigma, d_{\sigma,\beta}) \to C(d_{\sigma,\beta})) \to \exists R.C(d_\sigma))) = 1$.
7. Consider constraint (∀5) and let $d_{\sigma,\alpha}^{\mathcal{I}} = u$, according to Definition 12. Then:
   - If $(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) > 0$, then $\neg(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) * (R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u))) = 0$.
   - If $(\forall R.C)^{\mathcal{I}}(d_\sigma^{\mathcal{I}}) = 0$ then we have two different cases:
     – If $(\forall R.C)^{\mathcal{I}} \in w_R((d_\sigma^{\mathcal{I}})$ then $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) = 0$.
     – If $(\forall R.C)^{\mathcal{I}} \in qw_R((d_\sigma^{\mathcal{I}})$, then, by Definition 12, we have that $C^{\mathcal{I}}(u) < R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u)$ and, therefore, $R^{\mathcal{I}}(d_\sigma^{\mathcal{I}}, u) \Rightarrow C^{\mathcal{I}}(u) < 1$.

   Then in all the cases we have that $e_{\mathcal{I}}(pr(\neg\forall R.C(d_\sigma) \sqcap (R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha})))) < 1$.

Hence, for every proposition $pr(\varphi) \in pr(P_{C_0})$, it holds that $e_{\mathcal{I}}(pr(\varphi)) = 1$ and for every proposition $pr(\psi) \in pr(Y_{C_0})$, it holds that $e_{\mathcal{I}}(pr(\psi)) < 1$ and, therefore, $e_{\mathcal{I}}$ is a quasi-witnessing propositional evaluation. □

Proposition 14. *Let $\mathcal{I}$ be a quasi-witnessed interpretation and C_0 a $\mathfrak{I}\mathcal{ALE}$ concept. Then, for every instance α appearing in $P_{C_0} \cup Y_{C_0} \cup \{C_0(d)\}$, it holds that $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$.*

Proof. We will prove the Lemma by induction on the structure of α.

1. If α is either an atom or a generalized atom, it is straightforward from Definition 12.
2. If α is of the form $\delta \star \gamma$ where δ, γ are instances, $\star$ is a concept constructor and $\hat{\star}$ is the respective algebraic operation, suppose, by inductive hypothesis, that $e_{\mathcal{I}}(pr(\delta)) = \delta^{\mathcal{I}}$ and $e_{\mathcal{I}}(pr(\gamma)) = \gamma^{\mathcal{I}}$. Hence,

$$\begin{aligned} e_{\mathcal{I}}(pr(\alpha)) &= e_{\mathcal{I}}(pr(\delta \star \gamma)) = \\ &= e_{\mathcal{I}}(pr(\delta)) \,\hat{\star}\, e_{\mathcal{I}}(pr(\gamma)) = \\ &= \delta^{\mathcal{I}} \,\hat{\star}\, \gamma^{\mathcal{I}} = \\ &= (\delta \star \gamma)^{\mathcal{I}} = \alpha^{\mathcal{I}}. \end{aligned}$$

Hence, for every instance α appearing in $P_{C_0} \cup Y_{C_0} \cup \{C_0(d)\}$, it holds that $e_{\mathcal{I}}(pr(\alpha)) = \alpha^{\mathcal{I}}$. □

In particular, $e_{\mathcal{I}}(pr(C_0(d))) = C_0^{\mathcal{I}}(d^{\mathcal{I}})$, as we wanted to prove. This finishes the proof of the downwards implication of Proposition 11.

4.2. Completeness

The aim of this subsection is to prove the upwards implication of Proposition 11. Let us assume that there is a propositional evaluation that is quasi-witnessing relatively to C_0 such that $e(pr(C_0(d))) = r$ for some $r \in [0, 1]$. The goal is to obtain a ΠDL qw-interpretation $\mathcal{I}$ such that $C_0^{\mathcal{I}}(v) = r$ for some element $v \in \Delta^{\mathcal{I}}$. It will be defined in two steps. First we define the witnessed interpretation given from the quasi-witnessing propositional evaluation, analogously as done in [16]. Then, on top of this witnessed interpretation, we define the quasi-witnessed interpretation that satisfies the required conditions.

Definition 15. Let C_0 be an $\mathfrak{I}\mathcal{ALE}$ concept and let e be a quasi-witnessing propositional evaluation. Then we define a witnessed interpretation $\mathcal{I}_e^w$ as follows:

1. $\Delta^{\mathcal{I}_e^w}$ is the set of all individual names d_σ occurring in formulas of $P_{C_0} \cup Y_{C_0}$.
2. For each atomic concept A, let:
   (a) $A^{\mathcal{I}_e^w}(d_\sigma) = e(pr(A(d_\sigma)))$, if $pr(A(d_\sigma))$ occurs in $pr(P_{C_0})$,
   (b) $A^{\mathcal{I}_e^w}(d_\sigma) = 0$, otherwise.
3. For each role R let:
   (a) $R^{\mathcal{I}_e^w}(d_\sigma, d_{\sigma'}) = e(pr(R(d_\sigma, d_{\sigma'})))$, if $pr(R(d_\sigma, d_{\sigma'}))$ occurs in $pr(P_{C_0})$,
   (b) $R^{\mathcal{I}_e^w}(d_\sigma, d_{\sigma'}) = 0$, otherwise.

This definition gives us a witnessed interpretation and a finite model. As Hájek proved in [16], this kind of interpretation can not decide satisfiability in general for ΠDL concepts. We need a more complex interpretation (infinite and quasi-witnessed), defined on top of the witnessed interpretation given in Definition 15.

Definition 16. Let C_0 be an $\mathfrak{I}\mathcal{ALE}$ concept, e be a quasi-witnessing propositional evaluation. Then we define the interpretation $\mathcal{I}_e$ as the following expansion of $\mathcal{I}_e^w$:

1. The domain $\Delta^{\mathcal{I}_e}$ is obtained by adding to $\Delta^{\mathcal{I}_e^w}$ an infinite set of new individuals $\{d_\sigma^i | i \in \mathbb{N}\backslash\{0\}\}$, for each $d_\sigma \in \Delta^{\mathcal{I}_e^w} \setminus \{d\}$.
2. $A^{\mathcal{I}_e}(d_\sigma^i) = (A^{\mathcal{I}_e}(d_\sigma))^i$, for each atomic concept A.
3. For each role R:
   (a) if R appears in a universally quantified generalized atom $\forall R.C(d_\sigma)$ such that $e(pr(\forall R.C(d_\sigma))) \neq e(pr(R(d_\sigma, d_{\sigma,\alpha}) \to C(d_\alpha)))$, then:
      i. $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}^i) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}))^i$, for every $i \in \mathbb{N}\backslash\{0\}$,
      ii. $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,\alpha}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}))^j$, for every $i, j \in \mathbb{N}\backslash\{0\}$, with $i \leq j$,
      iii. $R^{\mathcal{I}_e}(d_\sigma^i, x) = 0$, otherwise.
   (b) if R appears in an existentially quantified generalized atom $\exists R.C(d_\sigma)$ or in a universally quantified generalized atom $\forall R.C(d_\sigma)$ such that $e(pr(\forall R.C(d_\sigma))) = e(pr(R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha})))$, then:
      i. $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}^i) = 0$, for every $i \in \mathbb{N}\backslash\{0\}$,
      ii. $R^{\mathcal{I}_e}(d_\sigma^i, d_{\sigma,\alpha}^j) = (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}))^j$, for every $i, j \in \mathbb{N}\backslash\{0\}$, with $i = j$,
      iii. $R^{\mathcal{I}_e}(d_\sigma^i, x) = 0$, otherwise.

Next Lemma 17 and Proposition 18, prove that the interpretations obtained applying Definition 16 are indeed qw-interpretations satisfying a given concept with the value determined by evaluation e, that is, $C_0^{\mathcal{I}_e}(d^{\mathcal{I}_e}) = e(pr(C_0(d)))$.

Lemma 17. *Let $D \in Sub(C_0)$ and e a quasi-witnessing propositional evaluation. Then, for each $i \in \mathbb{N}\backslash\{0\}$ and each sequence σ, it holds that*

$$D^{\mathcal{I}_e}(d_\sigma^i) = (D^{\mathcal{I}_e}(d_\sigma))^i.$$

Proof. The proof is by induction on the structure of D.

(P) Let D be any concept but a generalized atom:
1. If D is an atomic concept, then the result is straightforward from Definition 16.
2. Let $D = E \square F$, where E, F are concepts and $\hat{\star}$ is the respective semantics of $\square \in \{\to, \sqcap\}$. Suppose, by inductive hypothesis, that the claim holds for two concepts E, F, then:

$$(E \square F)^{\mathcal{I}_e}(d_\sigma^i) = E^{\mathcal{I}_e}(d_\sigma^i) \,\hat{\star}\, F^{\mathcal{I}_e}(d_\sigma^i)$$

$$
\begin{aligned}
&= (E^{\mathcal{I}_e}(d_\sigma))^i \,\hat{\star}\, (F^{\mathcal{I}_e}(d_\sigma))^i \\
&= (E^{\mathcal{I}_e}(d_\sigma) \,\hat{\star}\, F^{\mathcal{I}_e}(d_\sigma))^i \\
&= ((E \square F)^{\mathcal{I}_e}(d_\sigma))^i .
\end{aligned}
$$

(Q) Let $D(d_\sigma) = QR.E(d_\sigma)$ be a generalized atom with role depth equal to $k+1$ and suppose, by inductive hypothesis, that, for each generalized atom $E(d_{\sigma,\alpha})$ with role depth equal to k, the statement holds, then:

1. If $D(d_\sigma) = \exists R.E(d_\sigma)$, then, by Definition 16,

$$
\begin{aligned}
D^{\mathcal{I}_e}(d^i_\sigma) &= \sup_{w \in \Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d^i_\sigma, w) * E^{\mathcal{I}_e}(w)\} \\
&= R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,\alpha}) * E^{\mathcal{I}_e}(d^i_{\sigma,\alpha})
\end{aligned}
$$

and, by inductive hypothesis, Definition 10 and Definition 16,

$$
\begin{aligned}
R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,\alpha}) * E^{\mathcal{I}_e}(d^i_{\sigma,\alpha}) &= (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}))^i * (E^{\mathcal{I}_e}(d_{\sigma,\alpha}))^i \\
&= (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) * E^{\mathcal{I}_e}(d_{\sigma,\alpha}))^i \\
&= (D^{\mathcal{I}_e}(d_\sigma))^i .
\end{aligned}
$$

2. If $D(d_\sigma) = \forall R.E(d_\sigma)$, and $e(pr(\forall R.E(d_\sigma))) = (R(d_\sigma, d_{\sigma,\alpha}) \to E(d_{\sigma,\alpha}))$, then, by Definition 16,

$$
\begin{aligned}
D^{\mathcal{I}_e}(d^i_\sigma) &= \inf_{w \in \Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d^i_\sigma, w) \Rightarrow E^{\mathcal{I}_e}(w)\} \\
&= R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,\alpha}) \Rightarrow E^{\mathcal{I}_e}(d^i_{\sigma,\alpha})
\end{aligned}
$$

and, by inductive hypothesis, Definition 10 and Definition 16,

$$
\begin{aligned}
R^{\mathcal{I}_e}(d^i_\sigma, d^i_{\sigma,\alpha}) \Rightarrow E^{\mathcal{I}_e}(d^i_{\sigma,\alpha}) &= (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}))^i \Rightarrow (E^{\mathcal{I}_e}(d_{\sigma,\alpha}))^i \\
&= (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) \Rightarrow E^{\mathcal{I}_e}(d_{\sigma,\alpha}))^i \\
&= (D^{\mathcal{I}_e}(d_\sigma))^i .
\end{aligned}
$$

3. If $D(d_\sigma) = \forall R.E(d_\sigma)$, and $e(pr(\forall R.E(d_\sigma))) \neq (R(d_\sigma, d_{\sigma,\alpha}) \to E(d_{\sigma,\alpha}))$, then, by Definition 10, we have that $D^{\mathcal{I}_e}(d_\sigma) = 0$ and, therefore, by Definition 16,

$$
\begin{aligned}
D^{\mathcal{I}_e}(d^i_\sigma) &= \inf_{w \in \Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d^i_\sigma, w) \Rightarrow E^{\mathcal{I}_e}(w)\} \\
&= \inf_{j \in \mathbb{N}\setminus\{0\}} \{R^{\mathcal{I}_e}(d^i_\sigma, d^j_{\sigma,\alpha}) \Rightarrow E^{\mathcal{I}_e}(d^j_{\sigma,\alpha})\} \\
&= 0 = (D^{\mathcal{I}_e}(d_\sigma))^i .
\end{aligned}
$$

So, in every case we have that $D^{\mathcal{I}_e}(d^i_\sigma) = (D^{\mathcal{I}_e}(d_\sigma))^i$. □

Proposition 18. *Let e be a quasi-witnessing propositional evaluation, then, for every instance α, we have that $e(pr(\alpha)) = \alpha^{\mathcal{I}_e}$.*

Proof. The proof is by induction on the structure of α.

(P) Let α be any instance of a concept but a generalized atom.

1. If α is an atom, then it is straightforward from Definition 15.
2. Let $\alpha = (C \square D)(d_\sigma)$, where C, D are concepts and $\square \in \{\to, \sqcap\}$. Let $\star \in \{\to, \odot\}$ and $\hat{\star} \in \{\Rightarrow, *\}$. Suppose that the inductive hypothesis holds for concepts C, D, then, by Definition 9 we have that, for each concept constructor $\square$:

$$\begin{aligned}((C\square D)(d_\sigma))^{\mathcal{I}_e} &= C^{\mathcal{I}_e}(d_\sigma)\,\hat{\star}\,D^{\mathcal{I}_e}(d_\sigma)\\ &= e(pr(C(d_\sigma)))\,\hat{\star}\,e(pr(D(d_\sigma)))\\ &= e(pr(C(d_\sigma))\star pr(D(d_\sigma)))\\ &= e(pr((C\square D)(d_\sigma))).\end{aligned}$$

(Q) Let α be a generalized atom with role depth equal to $k+1$ and suppose, by inductive hypothesis, that, for each instance β with role depth $\leq k$, it holds that $e(pr(\beta)) = \beta^{\mathcal{I}_e}$.

1. If $\alpha = \exists R.C(d_\sigma)$, then, since e is quasi-witnessing we have that $e(pr(\alpha)) = e(pr(R(d_\sigma, d_{\sigma,\alpha}) \sqcap C(d_{\sigma,\alpha})))$ and, by Definition 15 and inductive hypothesis, we have that $e(pr(R(d_\sigma, d_{\sigma,\alpha}) \sqcap C(d_{\sigma,\alpha}))) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) * C^{\mathcal{I}_e}(d_{\sigma,\alpha})$. Let $v \in \Delta^{\mathcal{I}_e}$ be any individual name appearing in P_{C_0} and different from d_σ, then either v is associated to R, d_σ or not. In the first case, since e is quasi-witnessing and $e(pr(R(d_\sigma, v) \sqcap C(v))) \Rightarrow e(pr(\alpha)) = 1$, then $R^{\mathcal{I}_e}(d_\sigma, v) * C^{\mathcal{I}_e}(v) \leq e(pr(\alpha))$. In the second case, by Definition 15, we have that $R^{\mathcal{I}_e}(d_\sigma, v) * C^{\mathcal{I}_e}(v) = 0 \leq e(pr(\alpha))$. Let now $v \in \Delta^{\mathcal{I}_e}$ be any element $d^i_{\sigma,\beta}$ (obviously with $\beta \neq \alpha$, since, by Definition 16, $R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,\alpha}) = 0$), then either $R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,\beta}) = 0$ or not. The first case is straightforward. In the second case, by Definition 16 and Lemma 17, we have that:
$$\begin{aligned}R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,\beta}) * C^{\mathcal{I}_e}(d^i_\sigma) &= (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\beta}))^i * (C^{\mathcal{I}_e}(d_{\sigma,\beta}))^i\\ &= (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\beta}) * C^{\mathcal{I}_e}(d_{\sigma,\beta}))^i\\ &\leq R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\beta}) * C^{\mathcal{I}_e}(d_{\sigma,\beta})\\ &\leq R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) * C^{\mathcal{I}_e}(d_{\sigma,\alpha}).\end{aligned}$$
Hence, in each case,
$$\begin{aligned}e(pr(\alpha)) &= R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) * C^{\mathcal{I}_e}(d_{\sigma,\alpha})\\ &= \sup_{w\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d_\sigma, w) * C^{\mathcal{I}_e}(w)\}\\ &= \alpha^{\mathcal{I}_e}.\end{aligned}$$
2. Let $\alpha = \forall R.C(d_\sigma)$ and $e(pr(\alpha)) = e(pr(R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha})))$. Without loss of generality, we can suppose that $\forall R.C(d_\sigma) > 0$, since otherwise the proof is straightforward. By Definition 15 and inductive hypothesis, we have that $e(pr(R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha}))) = R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,\alpha})$. Let $v \in \Delta^{\mathcal{I}_e}$ be any individual name appearing in P_{C_0} and different from $d_{\sigma,\alpha}$. Then either v is associated to R, d_σ or not. In the first case, since e is quasi-witnessing and $e(pr(\alpha)) \Rightarrow e(pr(R(d_\sigma, v) \to C(v))) = 1$, then $e(pr(\alpha)) \leq R^{\mathcal{I}_e}(d_\sigma, v) \Rightarrow C^{\mathcal{I}_e}(v)$. In the second case, by Definition 15, since $R^{\mathcal{I}_e}(d_\sigma, v) = 0$, then we have that $R^{\mathcal{I}_e}(d_\sigma, v) \Rightarrow C^{\mathcal{I}_e}(v) = 1 \geq e(pr(\alpha))$. Let now $v \in \Delta^{\mathcal{I}_e}$ be any element $d^i_{\sigma,\beta}$ (obviously with $\beta \neq \alpha$, since, by Definition 16, $R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,\alpha}) = 0$), then either $R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,\beta}) = 0$ or not. The first case is straightforward. In the second case, by Definition 16, this means that there exists $\forall R.D \in qw(d_\sigma)$. Since e is a quasi-witnessing propositional evaluation, then constraint (∀4) is satisfied and, therefore, $R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\beta}) \leq C^{\mathcal{I}_e}(d_{\sigma,\beta})$. Hence, by Definition 16 and Lemma 17:
$$\begin{aligned}R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,\beta}) \Rightarrow C^{\mathcal{I}_e}(d^i_{\sigma,\beta}) &= (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\beta}))^i \Rightarrow (C^{\mathcal{I}_e}(d_{\sigma,\beta}))^i\\ &= (R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\beta}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,\beta}))^i\\ &= 1^i\\ &\geq R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,\alpha}).\end{aligned}$$
Hence, in each case,
$$\begin{aligned}e(pr(\alpha)) &= R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,\alpha})\\ &= \inf_{w\in\Delta^{\mathcal{I}_e}}\{R^{\mathcal{I}_e}(d_\sigma, w) \Rightarrow C^{\mathcal{I}_e}(w)\}\\ &= \alpha^{\mathcal{I}_e}.\end{aligned}$$

3. If $\alpha = \forall R.C(d_\sigma)$ and $e(pr(\alpha)) \neq e(pr(R(d_\sigma, d_{\sigma,\alpha}) \to C(d_{\sigma,\alpha})))$, then, since e is a quasi-witnessing propositional evaluation, by constraint (∀1), we have that $0 = e(pr(\alpha))$ and, by Definition 15 and inductive hypothesis, $e(pr(\alpha)) \neq R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,\alpha})$. Again since e is quasi-witnessing, by constraint (∀5) and the assumption above, we have that $0 < R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,\alpha}) < 1$. Hence, by Lemma 17, we have that:

$$\begin{aligned} e(pr(\alpha)) = 0 = & \\ &= \inf_{i \in \mathbb{N}\setminus\{0\}} \{(R^{\mathcal{I}_e}(d_\sigma, d_{\sigma,\alpha}) \Rightarrow C^{\mathcal{I}_e}(d_{\sigma,\alpha}))^i\} \\ &= \inf_{i \in \mathbb{N}\setminus\{0\}} \{R^{\mathcal{I}_e}(d_\sigma, d^i_{\sigma,\alpha}) \Rightarrow C^{\mathcal{I}_e}(d^i_{\sigma,\alpha})\} \\ &= \inf_{w \in \Delta^{\mathcal{I}_e}} \{R^{\mathcal{I}_e}(d_\sigma, w) \Rightarrow C^{\mathcal{I}_e}(w)\} \\ &= \alpha^{\mathcal{I}_e}. \end{aligned}$$

In particular, we have that $e(pr(C_0(a_0))) = C_0^{\mathcal{I}_e}(a_0)$. □

This finishes the last step in the proof of Proposition 11. This result provides a reduction of the concept satisfiability problem in ΠDL with respect to qw-interpretations to the semantic consequence problem (with a finite number of hypotheses) in the corresponding product propositional calculus with Monteiro-Baaz Delta connective. In [19, Theorem 6.2.1] it is proved that such problem is decidable (indeed, NP-complete). In the next section we give a study of the computational behavior of this reduction algorithm (Section 4.3). An analysis of the consequences of such reduction for decidability in ΠDL will be given in Section 5, along with some open problems.

4.3. A worst case lower bound for the decision procedure

In the present section, we will analyze the size of the set $P_{C_0} \cup Y_{C_0}$ w.r.t. a given concept C_0. Then we prove, by means of an example, that the size of the set $P_{C_0} \cup Y_{C_0}$ can be at least exponential in the role depth of C_0, even when the size of C_0 is linear in its role depth. This means that there exists a concept for which the size of the set of constraints produced by the algorithm is exponential in the size of the concept. In this sense, our example is among the worst cases for the computational complexity of the algorithm. We do not know whether there are other concepts for which the size of the set of constraints produced by the algorithm is more than exponential in the size of the concept. Hence, our example can be taken as a worst case lower bound. We point out that such result easily applies to the algorithm introduced in [16], as well, thus solving a problem left open by P. Hájek in that work (see Remark 6(1) in [16]).

We propose a couple of analysis of the bare algorithm behavior. We consider the case when only one role name R appears in concept C_0, since, from a computational complexity point of view, this is the worst case. Indeed, if two or more role names appear in C_0, then (∀2) and (∃2) produce less constraints. As we will see, such constraints are the greater source of complexity of the bare algorithm. The first analysis focuses on the amount of constraints produced by the algorithm in Definition 10 when it processes a single generalized atom. The second analysis focuses on the cumulated effect of constraints (∀2) and (∃2) with the aim of determining the total amount of individual names, generalized atoms and constraints produced by the algorithm, w.r.t. the number of quantified concepts that appears in a given concept C_0, divided by nesting levels (see below). The behavioral analysis undertaken in the present section will put the basis for determining a lower bound for the worst case size of the set $P_{C_0} \cup Y_{C_0}$. First we need to define a couple of notions.

Definition 19. Let C and B be $\mathfrak{I}\mathcal{ALE}$ concepts, then:

- B is a *propositional component* of C if B is a subconcept of C that does not appear within the scope of a quantifier in C.
- *Nesting level of subconcepts* of C is defined inductively:
  - (0) the propositional components of C have nesting level 0,

- $(k+1)$ if $QR.B$ has nesting level k, then the propositional components of B have nesting level $k+1$.

Analysis 1. Let C_0 be an $\mathfrak{I}\mathcal{ALE}$ concept with role depth n. According to Definition 10, for every generalized atom $\alpha = QR.C(d_\sigma)$, built up from a subconcept of C_0 with nesting level i, we have that, at step i of the algorithm:

1. constraint ($\forall$1) is added to P_{C_0}, if $Q = \forall$,
2. constraint ($\forall$5) is added to Y_{C_0}, if $Q = \forall$,
3. constraint ($\exists$1) is added to P_{C_0}, if $Q = \exists$,
4. as many from either constraint ($\forall$2) or ($\exists$2) as individual names associated to R, d_σ have been produced up to that step are added to P_{C_0}, either if $Q = \forall$ or $Q = \exists$,
5. as many from constraints ($\forall$3) and ($\forall$4) as generalized atoms $\forall R.D(d_\sigma)$ have been produced up to that step are added to P_{C_0}, if $Q = \forall$.

We stress that the major complexity source of the algorithm is item 4 above. Indeed, at each step, if m is the number of individual names associated to R, d_σ and j is the number of quantified concepts that are propositional components of C, then the algorithm will produce (i) $m \cdot j - m$ new generalized atoms by means of constraints ($\forall$2) and ($\exists$2) and (ii) m new generalized atoms by means of all the other constraints. Indeed, even though up to m^2 new ($\forall$3) and ($\forall$4) constraints may be added to $P_{C_0} \cup Y_{C_0}$, in this process no new generalized atoms are produced by such constraints, but those already produced by constraint ($\forall$2). That is, at each step, at least $m \cdot j$ new generalized atoms will be produced to be processed in the next step, for each α processed in the present step. This means, obviously, that the effect of constraints ($\forall$2) and ($\exists$2) in producing new generalized atoms, to be processed in further steps, becomes cumulative since every generalized atom, produced in the present step, will trigger in the next step the production of as many constraints as above explained. In order to see how such cumulative effect works we undertake the next analysis.

Analysis 2. Let, for every $0 \le i \le n$ the symbol $|i|$ denote the number of quantified subconcepts of C_0 with nesting level i. Then:

1. At step 0 the algorithm produces 1 individual name (the root d), no constraints is added to $P_{C_0} \cup Y_{C_0}$ and $|0|$ generalized atoms atoms to be processed in step 1 are produced.
2. At step 1 the algorithm produces (i) $|0|$ new individual names (one for each quantified concept with nesting level 0), (ii) at least $|0|^2$ constraints to be added to $P_{C_0} \cup Y_{C_0}$ (the minimum $|0|^2$ happens when there are only existential restrictions at level 0, since only constraints ($\exists$1) and ($\exists$2) are added) and (iii) exactly $|0| \cdot |1|$ new generalized atoms to be processed in step 2, since there are $|0|$ individual names associated to R, d and $|1|$ quantified concepts that are propositional components of some concept within the direct scope of a quantifier of a concept of level 0.
3. At step 2 the algorithm will then process $|0| \cdot |1|$ generalized atoms. Hence, it will produce (i) $|0| \cdot |1|$ new individual names, (ii) at least $(|0| \cdot |1|)^2$ constraints to be added to $P_{C_0} \cup Y_{C_0}$ and (iii) exactly $|0| \cdot |1| \cdot |2|$ new generalized atoms to be processed in step 2, since there are $|0| \cdot |1|$ individual names associated to R, d_σ (where σ is a generalized atom produced in step 1) and $|2|$ quantified concepts that are propositional components of some concept within the direct scope of a quantifier of a concept of level 1.
4. At the end of the process, the algorithm will have produced $1 + |0| + |0| \cdot |1| + \ldots + |0| \cdot \ldots \cdot |n-1|$ new individual names, (ii) at least $|0|^2 + (|0| \cdot |1|)^2 + \ldots + (|0| \cdot \ldots \cdot |n|)^2$ constraints will have been added to $P_{C_0} \cup Y_{C_0}$ and (iii) it will have processed exactly $|0| + |0| \cdot |1| + \ldots + |0| \cdot \ldots \cdot |n|$ generalized atoms.

An intuitive consequence of Analysis 2 is that, from every quantified concept of a given nesting level i, a generalized atom with every individual name produced in step i will be produced. So, we can consider every concept of a given nesting level i as if it was within the scope of the quantifier of every concept of nesting level $i-1$. This intuition will be useful to understand the following results.

The number we are interested in, from Analysis 2 is the final size of $P_{C_0} \cup Y_{C_0}$, that is at least $|0|^2 + (|0| \cdot |1|)^2 + \ldots + (|0| \cdot \ldots \cdot |n|)^2$. In the case of the algorithm in [16] it is exactly such number, since that algorithm, being thought for witnessed concepts, only considers constraints ($\forall$1), ($\forall$2), ($\exists$1) and ($\exists$2).

There are some cases of concepts that can be processed relatively fast. For example, for concepts like $QR.QR.QR.A$, where A is an atomic concept, the algorithm will add a minimum of $|0|^2 + (|0| \cdot |1|)^2 + (|0| \cdot$

$|1| \cdot |2|)^2 + (|0| \cdot |1| \cdot |2| \cdot |3|)^2 = 1^2 + (1 \cdot 1)^2 + (1 \cdot 1 \cdot 1)^2 + (1 \cdot 1 \cdot 1 \cdot 1)^2 = 4$ constraints to $P_{C_0} \cup Y_{C_0}$, if the quantifiers are all existential ones. The maximum is calculated adding 1 more constraint to $P_{C_0} \cup Y_{C_0}$ for each universal quantifier, because we have to consider constraint (∀5). Similarly, for concepts like $QR.A \sqcap QR.B \sqcap QR.C$, where A, B and C are atomic concepts, the algorithm will add from a minimum of $|0|^2 = 3^2 = 9$ constraints, to a maximum of $9 + 3 + 4 \cdot 3 = 24$ constraints to $P_{C_0} \cup Y_{C_0}$. Note that the difference between the minimum case, when only existential restrictions appear and the maximum case, when only value restrictions appear, is polynomial. Indeed, it is $3q + 2(q-1) \cdot q$, where q is the amount of quantifiers. Hence, the size of $P_{C_0} \cup Y_{C_0}$ in this case is still bounded polynomially in the size of C_0.

Nevertheless, in the worst case, the set $P_{C_0} \cup Y_{C_0}$ may reach at least exponential size in the length of C_0. For each $n \in \mathbb{N}$, let C_n be the concept $\exists R.A_1 \sqcap \exists R.(\exists R.A_2 \sqcap \exists R.(\exists R.A_3 \sqcap \exists R.(\ldots(\exists R.A_{n-1} \sqcap \exists R.A_n)\ldots)))$, where $A_1, \ldots, A_n$ are atomic concepts. As we can see, the length of C_n is linear in n, since there are $2n-1$ existential quantifiers and role names and n atomic concepts. According to Analysis 2, though, since there are 2 existential restrictions for each nesting level, but the last one, and n nesting levels, the algorithm will add exactly $|0|^2 + (|0| \cdot |1|)^2 + \ldots + (|0| \cdot \ldots \cdot |n|)^2 = 2^2 + (2 \cdot 2)^2 + \ldots + (2 \cdot \ldots \cdot 2)^2$ constraints to $P_{C_0} \cup Y_{C_0}$, where the last summand is already strictly greater than 2^n.

5. Decidability results and open problems

The algorithm provided in the previous section allows to obtain more general decidability results in ΠDL. Using Proposition 11, we can easily prove the following results, restricted to qw-interpretations.

Theorem 20. *The problems whether a concept is in* QVal_1, QSat_1, QVal_+ *and* QSat_+ *are decidable in the description language* $\mathfrak{I}\mathcal{ALE}$*, w.r.t. product semantics and empty knowledge bases.*

Proof. Let C be an $\mathfrak{I}\mathcal{ALE}$ concept. By Proposition 11, we have that:

1. $C \in \mathrm{QVal}_1$ iff $pr(C(d_0)) \vee \bigvee pr(Y_{C_0})$ is derivable, in the propositional product logic, from the set $pr(P_{C_0})$.
2. $C \in \mathrm{QSat}_1$ iff $\bigvee pr(Y_{C_0})$ is not derivable, in the propositional product logic, from the set $\{pr(C(d_0))\} \cup pr(P_{C_0})$.
3. $C \in \mathrm{QVal}_+$ iff $C \notin \mathrm{QSat}_0$ iff $\neg C \notin \mathrm{QSat}_1$
4. $C \in \mathrm{QSat}_+$ iff $\neg C \notin \mathrm{QVal}_1$ iff $\neg C \in \mathrm{QSat}_0$. □

By means of Theorem 20 it is possible to decide whether a given concept C is satisfiable with value r. First of all, we need a previous lemma.

Lemma 21. *For every* $r, s \in (0, 1)$, $\mathrm{QSat}_r = \mathrm{QSat}_s$.

Proof. This is an immediate consequence of the fact that for every $l \in \mathbb{R}^+$, the function $x \longmapsto x^l$ is an order isomorphism (that is, it preserves suprema and infima) and a homomorphism of the operations $*$ and $\Rightarrow$. □

Now, using items (1) and (3) of Theorem 20, along with Lemma 21, it is possible to prove the following result.

Corollary 22. *For any* $r \in [0, 1]$*, the set* $QSat_r$ *is decidable.*

As a consequence of Lemma 21 and Theorem 20, we have also the following corollary about subsumption. Note that, by definition of subsumption and Lemma 21, concept subsumption, based on product t-norm, is a crisp notion, in the sense that it takes just values 0 or 1.

Corollary 23. *The problem whether a pair of concepts is in* QSubs*, is decidable.*

For some of the reasoning tasks, such results can be extended to unrestricted (not only quasi-witnessed) standard interpretations. In the Appendix of [10] it is proved that first order Val_1 in standard models coincide with first order

Table 1
The landscape of Product Logic.

	ΠDL	qw-ΠDL	$\Pi\forall_{St}$	qw-$\Pi\forall_{St}$
Val_1	A	A	B	B
Val_+	C	C	D	D
Sat_1	E	F	G	H
Sat_+	I	I	J	J

QVal_1 in standard models (B in Table 1) and the same is true for its ΠDL fragment (A in Table 1). This implies the following consequences, both for first order product logic and ΠDL:

- The set Val_+ coincide with the set QVal_+. Indeed, $\varphi \in \mathrm{Val}_+$ iff $\neg\neg\varphi \in \mathrm{Val}_1$ iff $\neg\neg\varphi \in \mathrm{QVal}_1$ iff $\varphi \in \mathrm{QVal}_+$.
- The set Sat_+ coincides with the set QSat_+. Indeed, $\varphi \notin \mathrm{Sat}_+$ iff $\neg\varphi \in \mathrm{Val}_1$ iff $\neg\varphi \in \mathrm{QVal}_1$ iff $\varphi \notin \mathrm{QSat}_+$.

As a consequence of these last results and Theorem 20, we can rely on the following decidability results for ΠDL.

Theorem 24. *The problems whether a concept is in* Val_1*,* Val_+ *and* Sat_+ *and a pair of concepts is in* Subs *are decidable in the description language* $\mathfrak{I}\mathcal{ALE}$*, w.r.t. product semantics and empty knowledge bases.*

5.1. Open problems

In order to complete the decidability landscape for product $\mathfrak{I}\mathcal{ALE}$, we need a solution to the following problem, that, as far as we know, is still unsolved.

Open Problem 1. Does $\mathrm{Sat}_1 = \mathrm{QSat}_1$ (E = F) in ΠDL?

A possible answer to Open Problem 1 could be given by a positive answer (but not by a negative one) to the following three open problems.

Open Problem 2. Does $\mathrm{Sat}_1 = \mathrm{QSat}_1$ (G = H) in $\Pi\forall_{St}$?

Open Problem 3. Does $\mathrm{Sat}_1 = \mathrm{Sat}_+$ (E = I) in ΠDL?

Open Problem 4. Does $\mathrm{Sat}_1 = \mathrm{Sat}_+$ (G = J) in $\Pi\forall_{St}$?

In Table 1 we summarize the results given in the present section. Different letters denote sets known to be different, but E = F, G = H, E = I and G = J, which are the above open problems. It is also unknown whether F = I and H = J, but these last two problems have no consequences on our Open Problem 1.

Notice that Open Problems 3 and 4 have positive answer either if we restrict to finite (witnessed) models (see [16]) or if the language does not contain universal quantifiers. For the latter case the result is a consequence of double negation being a morphism for the propositional connectives and commuting with suprema. This result is not true if the language contains universal quantifiers, since double negation does not commute with infima.

On the other hand Open Problem 4 is equivalent to the following restriction of the Deduction Theorem:

$$\varphi \vDash \bot \quad \text{iff} \quad \vDash \varphi \to \bot$$

The proof is based on the following equivalences:

$$\varphi \notin \mathrm{Sat}_+ \quad \text{iff} \quad \vDash \varphi \to \bot \qquad \text{and} \qquad \varphi \notin \mathrm{Sat}_1 \quad \text{iff} \quad \varphi \vDash \bot$$

6. Further results

The present work provides a way to prove decidability of the satisfiability problem for a calculus that is complete with respect to quasi-witnessed models. Throughout the paper we refer to the FDL language $\mathfrak{I}\mathcal{ALE}$ endowed with a product semantics, because Product First Order Logic has been proved complete w.r.t. quasi-witnessed models in [20]. The same algorithm can be used for the same FDL language over different semantics whose first order calculus has been proved complete w.r.t. quasi-witnessed models, as is the case of just Product and Łukasiewicz logics and their fragments, as proved in [9]. Indeed, for the case of $\mathcal{ALC}$ language based on Łukasiewicz t-norm, in [16] it is proved that the calculus is complete with respect to witnessed models, that are a subclass of quasi-witnessed models. This result allows to use the algorithm in Definition 10 in order to decide whether a concept is satisfiable in $\mathcal{ALC}$ language based on Łukasiewicz t-norm, just considering every quantified subconcept of a given concept C_0 as witnessed.

The algorithm of the present paper can be easily adapted to decide whether a concept is satisfiable in the same FDL language over Gödel t-norm. Nevertheless, such result would be limited to quasi-witnessed models and in [9] it is proved that the first order logic of any SBL but Product Logic is not complete with respect to quasi-witnessed models. Moreover, a general result for Modal Logics over Gödel t-norm has already been provided in [21].

Unfortunately, the results of the present paper can not be generalized to FDL's based on t-norms different from Łukasiewicz, Product or Gödel since Lemma 17 fails for FDL based on such t-norms.

6.1. Application to modal logic

It is known that, in the classical case, the importance of the language and the reasoning tasks, we are working on, goes beyond the framework of DLs, since they are equivalent to the problems of local satisfiability and validity in the minimal Modal Logic (see [2]). The generalization of such relationship between DL and minimal modal logic to the fuzzy framework has been established in [11]. In this sense, the decidability results obtained in the present work, and the methods used to obtain such results can be easily translated into results and methods in the corresponding minimal Product Modal Logic (see [23]). In particular, the result in Theorem 24 can be directly translated into decidability of the validity and local positive satisfiability problems for the minimal modal logic over product Kripke models. Hence, the local positive satisfiability and the validity problems for the minimal modal logic, defined over product Kripke models, are decidable.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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