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Strict core fuzzy logics and quasi-witnessed models

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Abstract In this paper we prove strong completeness of axiomatic extensions of first-order strict core fuzzy logics with the so-called quasi-witnessed axioms with respect to quasi-witnessed models. As a consequence we obtain strong completeness of Product Predicate Logic with respect to quasi-witnessed models, already proven by M.C. Laskowski and S. Malekpour in [19]. Finally we study similar problems for expansions with Δ, define Δ-quasi-witnessed axioms and prove that any axiomatic extension of a first-order strict core fuzzy logic, expanded with Δ, and Δ-quasi-witnessed axioms are complete with respect to Δ-quasi-witnessed models.

Keywords Foundations of fuzzy logic · Mathematical fuzzy logic · First-order monoidal t-norm based logic · First-order Product Logic · Witnessed and quasi-witnessed models

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1 Introduction

Fuzzy Logics (both propositional and first-order) as many-valued residuated logics were defined by Petr Hájek in his celebrated book [12]. He defined, on the one hand, propositional fuzzy logics as extensions of the Basic Fuzzy Logic BL and, on the other hand, their algebraic counterpart, the variety of BL-algebras. Moreover he proved that

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BL and all its axiomatic extensions are complete with respect to evaluations over the BL-chains belonging to the corresponding variety. The fact that for each axiomatic extension of BL there is a corresponding subvariety of BL-algebras is a consequence of the fact that BL and its extensions are logics algebraizable in the sense of Blok and Pigozzi (see [10]). Special interest have the results in Hájek [13] and in Cignoli et al. [4], where it is proved that BL is the logic of continuous t-norms and their residua. Well known axiomatic extensions of BL are Łukasiewicz, Gödel and Product Logics (denoted as Ł, G and Π respectively). In his book, Hájek also defined the predicate logic corresponding to BL and its axiomatic extensions (denoted adding $\forall$ after the name of the propositional logic). Moreover he defined their semantics as first-order safe structures taking values on BL-chains of the corresponding variety and proved their completeness with respect to these models. Taking into account that a t-norm has residuum if and only if it is left-continuous, Esteva and Godo in [8] defined both propositional and first-order MTL (for Monoidal t-norm based Logic) whose propositional logic is proved to be the logic of left-continuous t-norms in Jenei and Montagna [18]. In Esteva and Godo [8] it is also defined their algebraic counterpart, the variety of MTL-algebras. The first-order versions of MTL and its axiomatic extensions are proved to be complete with respect to first-order structures evaluated over MTL-chains belonging to the corresponding variety. In recent times first-order Fuzzy Logic has been deeply studied. Recall that generalizing the classical case, the value of a universally (existentially) quantified formula is defined as the infimum (supremum) of the values of the results of replacing the quantified variable by the interpretation of a term of the language in a first-order model. Notice that in the context of Classical Logic, as well as every finitely valued logic, infima and suprema turn out to be minima and maxima, respectively. However, when we move to infinitely valued logics, this is not the case, the infimum or supremum of a set of values C may be an element $c \notin C$, i.e., a quantified formula may have no *witness*. Following these ideas, Hájek introduced in [15,16] the notion of *witnessed model*, i.e., a model in which each quantified formula has a witness and proved that this is an important property because it implies a limited form of finite model property for certain fragments of predicate fuzzy logic (see [14]). Moreover, Cintula and Hájek introduce in [17] the so-called witnessed axioms that, added to any first-order core fuzzy logic, give a logic complete with respect to witnessed models. Subsequently they prove that these axioms are derivable in Łukasiewicz first-order Logic, showing that Ł$\forall$ is complete with respect to witnessed models (we will say that Ł$\forall$ has the witnessed model property), but also that neither Gödel, nor Product first-order Logic share this property because witnessed axioms are not theorems of these logics. In fact no other first-order logic of a continuous t-norm enjoys this property, since it is related to continuity of the truth functions, a property that only Łukasiewicz logic has. Nevertheless, in Laskowski and Malekpour [19] it is proved that Product Predicate Logic enjoys a weaker property, what we call *quasi-witnessed model property*. Quasi-witnessed models[1] are models in

[1] These models are called "closed models" in Laskowski and Malekpour [19] but we decided, after some discussions with colleagues, to use the more informative name of "quasi-witnessed models". We take into account the fact that the name "closed" is used in mathematics and logic in different contexts with different meanings and could induce some confusion.

which, whenever the value of a universally quantified formula is strictly greater than 0, then it has a witness, while existentially quantified formulas are always witnessed.

In this paper we introduce both the so-called *strict core* fuzzy logics and *quasi-witnessed* axioms (generalizations of the witnessed axioms of Hájek–Cintula to cope with quasi-witnessed models) and prove, following the style of [17] that, if we add quasi-witnessed axioms to any first-order strict core fuzzy logic, the resulting logic enjoys the quasi-witnessed model property. From this result, the one in Laskowski and Malekpour [19] about the completeness of Product first-order Logic with respect to quasi-witnessed models, will follow as a corollary. Moreover, we prove that quasi-witnessed axioms are theorems in no logic of a continuous t-norm but Product and Łukasiewicz predicate logics. Finally we study the expansion of first-order strict core fuzzy logics by Δ operator. We give the so-called Δ*-quasi-witnessed axioms* and prove that adding these axioms to any strict Δ-core fuzzy logic, we obtain a first-order fuzzy logic which is complete with respect to quasi-witnessed models.

2 Preliminaries

2.1 Propositional logic

The logic MTL has been defined in Esteva and Godo [8] and has, as primitive binary connectives, a strong conjunction $\odot$, a weak conjunction $\wedge$ and an implication $\rightarrow$ and, as primitive 0-ary connective, the constant symbol $\bot$. This logic has been axiomatized with the following set of axioms:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,
(A2) $(\varphi \odot \psi) \rightarrow \varphi$,
(A3) $(\varphi \odot \psi) \rightarrow (\psi \odot \varphi)$,
(A4) $\varphi \odot (\varphi \rightarrow \psi) \rightarrow (\varphi \wedge \psi)$,
(A5) $(\varphi \wedge \psi) \rightarrow \varphi$,
(A6) $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$,
(A7a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \odot \psi) \rightarrow \chi)$,
(A7b) $((\varphi \odot \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$,
(A8) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$,
(A9) $\bot \rightarrow \varphi$.

And its unique rule of inference is Modus Ponens (MP).

From the primitive connectives it is possible to define more, in particular:

$$\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$$
$$\varphi \equiv \psi := (\varphi \rightarrow \psi) \odot (\psi \rightarrow \varphi)$$
$$\neg\varphi := \varphi \rightarrow \bot$$
$$\top := \bot \rightarrow \bot$$

The logic SMTL[2] is defined in the literature as the axiomatic extension of MTL by the axiom:

(S) $\varphi \wedge \neg\varphi \to \bot$ (strictness)

In this paper we are going to deal with other important axiomatic extensions of MTL. The logic BL is the axiomatic extension of MTL by the following axiom,

(D) $\varphi \wedge \psi \to \varphi \odot (\varphi \to \psi)$ (divisibility)

The logic SBL is the axiomatic extension of BL by axiom (S), or, equivalently, it is the axiomatic extension of SMTL by axiom (D).

Product logic has been defined in [11] and it can be seen as the axiomatic extension of SBL by the following axiom,

(Π) $\neg\neg\chi \to (((\varphi \odot \chi) \to (\psi \odot \chi)) \to (\varphi \to \psi))$ (simplification)

Hence Product Logic is the axiomatic extension of SMTL by axioms (D) and (Π).

Gödel logic is the axiomatic extension of BL (or either SBL or SMTL) by the following axiom:

(Id) $\varphi \to (\varphi \odot \varphi)$ (idempotence)

Finally, Łukasiewicz logic is the axiomatic extension of BL by the following axiom:

(Inv) $\neg\neg\varphi \to \varphi$ (involutive negation)

Definition 1 1. An *MTL-algebra* $\mathbf{A} = \langle \mathbf{A}, \cap, \cup, *, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ is a bounded commutative integral residuated lattice which satisfies the equation:
(PL) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$ (pre-linearity)
2. An *SMTL-algebra* $\mathbf{A} = \langle \mathbf{A}, \cap, \cup, *, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ is a MTL-algebra which satisfies the equation:
(S) $x \cap (x \Rightarrow 0) = 0$ (strictness)
3. A *BL-algebra* $\mathbf{A} = \langle \mathbf{A}, \cap, \cup, *, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ is an MTL-algebra which satisfies the equation:
(D) $x \cap y = x * (x \Rightarrow y)$ (divisibility)
4. A Π-*algebra* $\mathbf{A} = \langle \mathbf{A}, \cap, \cup, *, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ is an SMTL-algebra which satisfies the equations (D) and:
(Π) $((z \Rightarrow 0) \Rightarrow 0) \Rightarrow (((x * z) \Rightarrow (y * z)) \Rightarrow (x \Rightarrow y)) = 1$ (simplification)
5. A *Gödel-algebra* $\mathbf{A} = \langle \mathbf{A}, \cap, \cup, *, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ is a BL-algebra which satisfies the equation:
(Id) $x = x * x$ (idempotence)
6. An *MV-algebra* $\mathbf{A} = \langle \mathbf{A}, \cap, \cup, *, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ is a BL-algebra which satisfies the equation:
(Inv) $x = (x \Rightarrow 0) \Rightarrow 0$ (involutive negation)

Moreover, if any of them is linearly ordered, we say that it is an *MTL-chain* (respectively *SMTL-chain*, Π-*chain* and so on).

[2] SMTL means *strict MTL* in the sense that $(\varphi \wedge \neg\varphi) \leftrightarrow 0$ is a theorem. Algebraically this property is called "pseudo-complementation" and denoted as (PC) in some more algebraic works like [9].

All the logics defined in these preliminaries are algebraizable in the sense of Blok and Pigozzi (see [10]) and its algebraic semantics is the variety of the corresponding MTL-algebras. Moreover all of these logics are chain-complete (what is called "semi-linear" in [7]) in the sense that they are strong complete for evaluations over the chains of the corresponding variety.

A natural semantics for the MTL logic and their axiomatic extensions are the evaluations over the real unit interval, i.e. over the MTL-chains whose lattice reduct is [0, 1] with the usual order. These chains, called standard chains are related to a special kind of operation called "t-norms".

Definition 2 A t-norm is a binary operation $*$ on the real unit interval [0, 1] that is associative, commutative, non-decreasing in both arguments and having 1 as neutral (unit) element.

Left continuity of a t-norm is characterized by the existence of an unique binary operation $\Rightarrow$ satisfying for all $a, b, c \in [0, 1]$ the following condition (called *residuation*):

$$a * b \leq c \text{ if and only if } a \leq b \Rightarrow c$$

The operator $\Rightarrow$ is called the *residuum* of the t-norm $*$ and it is defined as

$$x \Rightarrow y = \max\{z \in [0, 1] \mid x * z \leq y\}$$

Using this residuum, the following result characterize standard chains.

Proposition 1 *A structure* $\langle [0, 1], \cap, \cup, *, \Rightarrow, 0, 1\rangle$ *is a standard MTL-chain if and only if* $*$ *is a left-continuous* t*-norm and* $\Rightarrow$ *is its residuum. This structure will be denoted from now on as* $[0, 1]_*$. *Moreover a standard chain satisfies divisibility* (*Hence it is a BL-chain*) *if and only if the* t*-norm is continuous.*

In Jenei and Montagna [18] it is proved that MTL are *strong standard complete* (strong complete for evaluations over the standard chains), i.e. for any set of formulas $\Gamma \cup \{\varphi\}$ and any evaluation e over a standard chain,

$$\Gamma \vdash_{MTL} \varphi \text{ iff } e(\varphi) = 1 \text{ for any evaluation } e \text{ such that } e(\gamma) = 1 \text{ for all } \gamma \in \Gamma.$$

This result is not automatically translatable to axiomatic extensions of MTL. It is easily extended to SMTL and the standard SMTL-chains but not to BL and the standard BL-chains (hence neither to its axiomatic extensions). If $\mathcal{L}$ is either BL or SBL or Łukasiewicz or Product or Gödel logic only the finite strong standard completeness results are valid, i.e. for any *finite* set of formulas $\Gamma \cup \{\varphi\}$ and any evaluation e over a standard $\mathcal{L}$-chain,

$$\Gamma \vdash_{\mathcal{L}} \varphi \text{ iff } e(\varphi) = 1 \text{ for any evaluation } e \text{ such that } e(\gamma) = 1 \text{ for all } \gamma \in \Gamma,$$

Table 1 The three main continuous t-norms

$*$	Minimum (Gödel)	Product (of real numbers)	Łukasiewicz
$x * y$	$\min(x, y)$	$x \cdot y$	$\max(0, x + y - 1)$
$x \to_* y$	$\begin{cases} 1, & \text{if} x \leq y \\ y, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if} x \leq y \\ y/x, & \text{otherwise} \end{cases}$	$\min(1, 1 - x + y)$
n_*	$\begin{cases} 1, & \text{if} x = 0 \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} 1, & \text{if} x = 0 \\ 0, & \text{otherwise} \end{cases}$	$1 - x$

An interesting result for Łukasiewicz Product and Gödel logics is that the corresponding standard-chains are all isomorphic[3]. The most used representative of standard chains of these three logics (unique up to isomorphisms), are the ones defined by the so-called Łukasiewicz, product and minimum t-norms and their residua (collected in Table 1).

From the previous results seems natural the definition of the logic of a (continuous) t-norm.

Definition 3 We say that a logic (called $\mathcal{L}(*)$) is the logic of a continuous t-norm $*$ if it is an axiomatic extension of BL which is finite strong standard complete with respect to evaluations over the standard chain $[0, 1]_*$, i.e. for any *finite* set of formulas $\Gamma \cup \{\varphi\}$ and any evaluation e over $[0, 1]_*$,

$$\Gamma \vdash_{\mathcal{L}(*)} \varphi \text{ iff } e(\varphi) = 1 \text{ for any evaluation } e \text{ such that } e(\gamma) = 1 \text{ for all } \gamma \in \Gamma.$$

All the logics considered so far enjoy two important properties we need to define the class of logics we are interested in.

Definition 4 1. We say that a logic $\mathcal{L}$ enjoys the *Local Deduction Theorem* (LDT, for short) if for each theory T and formulas φ, ψ, it holds that $T, \varphi \vdash \psi$ iff there exists a natural number n such that $T \vdash \varphi^n \to \psi$, where $\varphi^n = \varphi \odot \ldots \odot \varphi$, n times.
2. We say that a logic $\mathcal{L}$ enjoys *Invariance under Substitution* (Sub, for short) if, for every formulas φ, ψ, χ it holds that $\varphi \equiv \psi \vdash \chi(\varphi) \equiv \chi(\psi)$.

Next we recall the definition of *core fuzzy logic* given in Hájek and Cintula [17] (a family of logics that encompasses all logics considered so far) and we introduce the *strict core fuzzy logic* we will deal with in this paper.

Definition 5 1. We say that a logic $\mathcal{L}$ is a *core fuzzy logic* if it is finitary, enjoys LDT, Sub and expands MTL.
2. We say that a logic $\mathcal{L}$ is a *strict core fuzzy logic* if it is finitary, enjoys LDT, Sub and expands SMTL.

Throughout this preliminary section, we will denote by $\mathcal{L}$ any core fuzzy logic.

[3] In fact for Gödel logic there is only one standard chain while for Łukasiewicz and Product there are infinite different but isomorphic ones.

2.2 Predicate logic

In order to define what a predicate logic is, we have, previously, to define what a *predicate language* is.

Definition 6 A *predicate language* Γ is composed by a set of relation symbols $P_1, \ldots, P_n, \ldots$, each one with arity ≥ 1, a set of function symbols $f_1, \ldots, f_n, \ldots$, each one with its arity, and a set of constant symbols $c_1, \ldots, c_n, \ldots$, that are 0-ary function symbols.

Terms and *formulas* of a predicate language are defined as usual in the literature.

Following [12], given a propositional residuated logic $\mathcal{L}$, we define the first-order logic associated with $\mathcal{L}$ (denoted by $\mathcal{L}\forall$), as follows:

Definition 7 $\mathcal{L}\forall$ is the first-order logic such that:

1. its language is composed by a predicate language Γ and a set of logical symbols obtained by adding, to the set of logical symbols of $\mathcal{L}$, the two "classical" quantifiers $\forall$ and $\exists$ and,
2. it is axiomatized by means of the following set of axiom schemata:
   - (P) the axioms resulting from the axioms of $\mathcal{L}$ after the substitution of propositional variables by formulas of the new predicate language.
   - ($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ.
   - ($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ.
   - ($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi(x))$, where x is not free in χ.
   - ($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi(x) \to \chi)$, where x is not free in χ.
   - ($\forall$3) $(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi(x))$, where x is not free in χ.
3. its rules of inference are Modus Ponens (MP) and generalization (G): From φ infer $(\forall x)\varphi(x)$.

The following definitions are required to prove the main results given in Section 3. They are typical within the framework of Classical first-order Logic. Their presentation in our context follows the generalization, due to [17], necessary to adapt them to a many-valued framework.

Definition 8 We say that a theory T' in a predicate language Γ' is an *expansion* of a theory T in a predicate language Γ, if $\Gamma \subseteq \Gamma'$ and each formula provable in T is provable in T'. We say that T' is a *conservative expansion* of T if T' is an expansion of T and each formula in the language of T, provable in T', is provable in T.

Definition 9 A theory T is *linear* if, for each pair of sentences φ, ψ, we have $T \vdash \varphi \to \psi$ or $T \vdash \psi \to \varphi$.

Definition 10 Let Γ and Γ' be predicate languages such that $\Gamma \subseteq \Gamma'$ and T a Γ'-theory. We say that T is $\forall$-Γ-*Henkin* if, for each Γ-sentence $\varphi = (\forall x)\psi(x)$ such that $T \nvdash \varphi$, there is a constant c in Γ' such that $T \nvdash \psi(c)$.

We say that T is $\exists$-Γ-*Henkin* if, for each Γ-sentence $\varphi = (\exists x)\psi(x)$ such that $T \vdash \varphi$, there is a constant c in Γ' such that $T \vdash \psi(c)$.

A theory is called Γ-*Henkin* if it is both $\forall$-Γ-Henkin and $\exists$-Γ-Henkin.

If $\Gamma = \Gamma'$, we say that T is $\forall$-Henkin ($\exists$-Henkin, Henkin).

From a semantic point of view first-order models are composed of a set of elements, an algebra of truth values and an assignation function.

Definition 11 A *first-order structure* for a given predicate language Γ is a pair $(\mathbf{A}, \mathbf{M})$, where $\mathbf{A}$ is an $\mathcal{L}$-chain and $\mathbf{M}=(M, (P_{\mathbf{M}})_{P\in\Gamma}, (f_{\mathbf{M}})_{f\in\Gamma}, (c_{\mathbf{M}})_{c\in\Gamma})$, where:

1. The set M, called *domain*, is a non-empty set,
2. for each predicate symbol $P \in \Gamma$ of arity n, $P_{\mathbf{M}}$ is an n-ary $\mathbf{A}$-fuzzy relation on M,
3. for each function symbol $f \in \Gamma$ of arity n, $f_{\mathbf{M}}$ is an n-ary (crisp) function on M and
4. for each constant symbol $c \in \Gamma$, $c_{\mathbf{M}}$ is an element of M.

The truth value $\|\varphi\|_v^{\mathbf{A},\mathbf{M}}$ of a predicate formula φ in a given model v is defined as follows.

Definition 12 Let Γ be a predicate language, $\mathbf{A}$ an $\mathcal{L}$-chain and $(\mathbf{A}, \mathbf{M})$ a first-order structure, then a first-order assignation v is a homomorphism $v : Var \to M$. As usual each assignation, defined on the set of individual variables, extends univocally to a first-order assignation (that we will denote by v as well) satisfying, for every terms $t_1, \ldots, t_n$ and each n-ary function $f \in \Gamma$, that $v(f(t_1, \ldots, t_n)) = f_{\mathbf{M}}(v(t_1), \ldots, v(t_n))$. Moreover, each assignation v, defined on the set of individual variables yields a first-order model $\|\cdot\|_v^{(\mathbf{A},\mathbf{M})} : Fm_{\mathcal{L}\forall} \to \mathbf{A}$ such that:

1. for each n-tuple of terms $t_1, \ldots, t_n$ and each n-ary relation $P \in \Gamma$, it holds that $\|P(t_1, \ldots, t_n)\|_v^{(\mathbf{A},\mathbf{M})} = P_{\mathbf{M}}(v(t_1), \ldots, v(t_n)) \in A$,
2. if φ, ψ are formulas, $\star_{\mathcal{L}}$ a binary logical connective and $\star_{\mathbf{A}}$ its truth function, then $\|\varphi \star_{\mathcal{L}} \psi\|_v^{(\mathbf{A},\mathbf{M})} = \|\varphi\|_v^{(\mathbf{A},\mathbf{M})} \star_{\mathbf{A}} \|\psi\|_v^{(\mathbf{A},\mathbf{M})}$.
3. if $\varphi(x_1, \ldots, x_n)$ is a formula with n free variables and v is a first-order assignation such that $v(x_i) = a_i$ and $a_i \in M$, for $1 < i \leq n$, then we have that $\|(\forall x_1)\varphi(x_1, x_2, \ldots, x_n)\|_v^{(\mathbf{A},\mathbf{M})} = \inf_{a\in M}\{\|\varphi(a, a_2, \ldots, a_n)\|^{(\mathbf{A},\mathbf{M})}\}$,
4. if $\varphi(x_1, \ldots, x_n)$ is a formula with n free variables and v is a first-order assignation such that $v(x_i) = a_i$ and $a_i \in M$, for $1 < i \leq n$, then we have that $\|(\exists x_1)\varphi(x_1, x_2, \ldots, x_n)\|_v^{(\mathbf{A},\mathbf{M})} = \sup_{a\in M}\{\|\varphi(a, a_2, \ldots, a_n)\|^{(\mathbf{A},\mathbf{M})}\}$.

Clearly, depending on the model, the infimum and supremum of a set of values of formulas do not necessarily exist and, in this case we will say that a given quantified formula has an undefined truth value. Following [12], we will say that if, for a given model v, both infima and suprema of sets of values are defined for every formula, then v is a *safe* model. Moreover, if, for a given first-order structure $(\mathbf{A}, \mathbf{M})$, each assignation v, defined in it, is safe, we will say that $(\mathbf{A}, \mathbf{M})$ is a safe structure.

From now on and for simplicity, we will omit the name "safe" before the first-order structures, i.e., when we speak about a first-order structure $(\mathbf{A}, \mathbf{M})$, we implicitly mean a *safe* first-order structure $(\mathbf{A}, \mathbf{M})$.

The concepts of *satisfiability* and *validity* are defined in the usual way.

In Hájek and Cintula [17], we find the following useful definitions and result, which we report without proof. In what follows, we will denote by $\mathbf{A}$ any $\mathcal{L}$-chain.

Definition 13 Let $(\mathbf{A_1}, \mathbf{M_1})$ and $(\mathbf{A_2}, \mathbf{M_2})$ be structures in the languages Γ_1 and Γ_2 respectively and let $\Gamma_1 \subseteq \Gamma_2$. We say that a pair (f, g) is an *elementary embedding* if:

1. the mapping f is an injection of M_1 into M_2,
2. the mapping g is an embedding of $\mathbf{A_1}$ into $\mathbf{A_2}$,
3. for each Γ_1-formula $\varphi(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n \in M_1$, it holds that $g(\|\varphi(a_1, \ldots, a_n)\|^{(\mathbf{A_1},\mathbf{M_1})}) = \|\varphi(f(a_1), \ldots, f(a_n))\|^{(\mathbf{A_2},\mathbf{M_2})}$.

Definition 14 Let T be a theory. We define $[\varphi]_T = \{\psi \mid T \vdash \varphi \equiv \psi\}$ and $L_T = \{[\varphi]_T \mid \varphi \text{ a formula }\}$. The *Lindenbaum algebra* of the theory T ($\mathbf{Lind}_T$, in symbols) has domain L_T and operations $c_{\mathbf{Lind}_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [c(\varphi_1, \ldots, \varphi_n)]_T$, for every n-ary propositional connective c.

Definition 15 Let T be a linear Henkin theory, then the *canonical model* of T is the structure $(\mathbf{Lind_T}, \mathbf{CM(T)})$, where $\mathbf{Lind_T}$ is the Lindenbaum algebra of theory T, the domain of $\mathbf{CM(T)}$ consists of object constants $c_{\mathbf{CM}(T)} = c$ and terms built without variables. Moreover for every predicate n-ary symbol $P \in \Gamma$, $P_{\mathbf{CM}(T)}(t_1, \ldots, t_n) = [P(t_1, \ldots, t_n)]_T$.

From here on, for simplicity, we will write $\mathbf{CM(T)}$ to denote $(\mathbf{Lind_T}, \mathbf{CM(T)})$.

Definition 16 For each structure $(\mathbf{A}, \mathbf{M})$, let $\mathbf{Alg}((\mathbf{A}, \mathbf{M}))$ be the subalgebra of $\mathbf{A}$ whose domain is the set $\{\|\varphi\|_v^{\mathbf{A},\mathbf{M}} \mid \varphi, v\}$ of truth degrees of formulas under all $\mathbf{M}$-assignation v of variables. Call $(\mathbf{A}, \mathbf{M})$ *exhaustive* if $\mathbf{A} = \mathbf{Alg}((\mathbf{A}, \mathbf{M}))$.

The next lemma is a direct consequence of Lemma 4 in [17] and we will not prove it here.

Lemma 17 *Let T_1, T_2 be $\mathcal{L}\forall$-theories. If T_2 is a conservative expansion of T_1, then, for each exhaustive model $(\mathbf{A}, \mathbf{M})$ of T_1, there exists a linear Henkin $\mathcal{L}\forall$-theory T extending T_2 such that $(\mathbf{A}, \mathbf{M})$ can be elementarily embedded into $\mathbf{CM(T)}$.*

2.3 The witnessed model property

Witnessed models have been firstly defined in [14] in the following way:

Definition 18 For any structure $(\mathbf{A}, \mathbf{M})$, a formula $(\forall y)\varphi(y, x_1, \ldots, x_n)$ is $\mathbf{A}$-witnessed in $\mathbf{M}$ if, for each assignation $c_1, \ldots, c_n \in M$, to $x_1, \ldots, x_n$, there is $c \in M$ such that $\|(\forall y)\varphi(y, c_1, \ldots, c_n)\|^{\mathbf{A},\mathbf{M}} = \|\varphi(c, c_1, \ldots, c_n)\|^{\mathbf{A},\mathbf{M}}$. Similarly for $(\exists y)\varphi(y, x_1, \ldots, x_n)$. $\mathbf{M}$ is $\mathbf{A}$-witnessed if all quantified formulas are $\mathbf{A}$-witnessed in $\mathbf{M}$.

As said above, within the framework of classical predicate logic, where the first-order structures are evaluated on a two element chain, there is no need of making a difference between witnessed and non witnessed models, because every model is indeed witnessed, and the same holds for every finite-valued logic. The need of speaking about witnessed models arises when we move to infinite-valued logics, since we can meet sets of truth values whose infima (resp. suprema) is not an element of the set. Later on, in [17], Hájek and Cintula consider the following couple of axioms (called witnessed axioms) already given by Baaz in [1]:

(C∃) $(\exists y)((\exists x)\varphi(x) \to \varphi(y))$,
(C∀) $((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

They prove that each first-order core fuzzy logic $\mathcal{L}\forall$, extended with this couple of axioms (denoted $\mathcal{L}\forall^w$), is complete with respect to the witnessed models evaluated over $\mathcal{L}$-chains. Moreover, in [15] it is proved that Łukasiewicz predicate logic is the only logic of a continuous t-norm equivalent with its witnessed axiomatic extension, i.e., $(C\exists)$ and $(C\forall)$ are theorems of Łukasiewicz predicate Logic. As a consequence of this fact Łukasiewicz is the only logic of a continuous t-norm which is complete with respect to witnessed models, i.e. it satisfies the *witnessed model property*.

3 Completeness with respect to quasi-witnessed models

In this section we will give the definitions of quasi-witnessed axioms and quasi-witnessed models, which are a generalization of witnessed axioms and models. We stress that in this paper the starting point are strict core fuzzy logics, because the result is related with the behavior of Gödel negation. Subsequently we will state and prove the main result of this paper, i.e., that if we add quasi-witnessed axioms to any predicate strict core fuzzy logic, we obtain a logic that is complete with respect to quasi-witnessed models. In what follows $\mathcal{L}$ will denote a strict core fuzzy logic.

Definition 19 Let Γ be a predicate language and $(\mathbf{A}, \mathbf{M})$ a first-order structure, then we say that a Γ-formula $\varphi(x, y_1, \ldots, y_n)$ is **A**-*quasi-witnessed* in **M** if:

1. For each tuple $c_1, \ldots, c_n$ of elements in M there exists an element $a \in M$ such that $\|(\exists x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = \|\varphi(a, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})}$.
2. For each tuple $c_1, \ldots, c_n$ of elements in M either $\|(\forall x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = 0$, or there exists an element $b \in M$ such that $\|(\forall x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = \|\varphi(b, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})}$.

We say that a first-order structure $(\mathbf{A}, \mathbf{M})$ is quasi-witnessed if for each formula and for every assignation v of the variables on **M** the formula is quasi-witnessed.

Definition 20 Let $\mathcal{L}\forall$ be any strict core first-order logic, we denote by $\mathcal{L}\forall^{qw}$ the axiomatic extension of $\mathcal{L}\forall$ by the following axiom schemata called, from now on, "quasi-witnessed axioms":

(C∃) $(\exists y)((\exists x)\varphi(x) \to \varphi(y))$,
(ΠC∀) $\neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

These quasi-witnessed axioms are a modification of the witnessed axioms given above. The first one, $(C\exists)$, is a witnessed axiom and the second one says that the witnessed axiom $(C\forall)(\exists y)(\varphi(y) \to (\forall x)\varphi(x))$ is valid in a structure $(\mathbf{A}, \mathbf{M})$ only when the truth value of $(\forall x)\varphi(x)$ is different from 0, i.e., when $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 1$.

Next lemma proves the soundness of quasi-witnessed axioms with respect to the above defined quasi-witnessed models.

Lemma 21 *If an $\mathcal{L}\forall$-structure $(\mathbf{A}, \mathbf{M})$ is quasi-witnessed, then it satisfies $(C\exists)$ and $(\Pi C\forall)$.*

Proof Let (**A**,**M**) be a quasi-witnessed $\mathcal{L}\forall$-structure and $\varphi(x)$ a Γ formula with one free variable, then:

1. Since, by the first condition of Definition 19, there exists an element $a \in M$ such that $\|\varphi(a)\|^{(\mathbf{A},\mathbf{M})} = \|(\exists x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$, then $(\mathbf{A},\mathbf{M}) \models (\exists x)\varphi(x) \to \varphi(a)$. So, by axiom ($\exists$1) and (MP), $(\mathbf{A},\mathbf{M}) \models (\exists y)((\exists x)\varphi(x) \to \varphi(y))$.
2. By the second condition of Definition 19, there exists $b \in M$ such that either $\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})} = \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$, or $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$. If $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$, then, $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$ and, trivially we have $(\mathbf{A},\mathbf{M}) \models \neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$. If, on the other hand, $\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})} = \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$, then $(\mathbf{A},\mathbf{M}) \models \varphi(b) \to (\forall x)\varphi(x)$, and, by axiom ($\exists$1) and (MP), $(\mathbf{A},\mathbf{M}) \models (\exists y)(\varphi(y) \to (\forall x)\varphi(x))$. So, $(\mathbf{A},\mathbf{M}) \models \neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

As for witnessed models, the converse of the last lemma does not hold as we will see in Example 1.

However, as in Hájek and Cintula [17], it is possible to prove the next result.

Lemma 22 *Let Γ be a predicate language, and* $(\mathbf{A},\mathbf{M})$ *an exhaustive model of a Γ-theory T. Then* $(\mathbf{A},\mathbf{M})$ *is an $\mathcal{L}\forall^{qw}$-model of T iff it can be elementarily embedded into a quasi-witnessed model of T.*

Proof ($\Rightarrow$) Let (**A**,**M**) be an exhaustive $\mathcal{L}\forall^{\mathrm{qw}}$-model of T. By Lemma 17, there is a linear Henkin theory T' extending T, such that (**A**,**M**) can be elementarily embedded into $\mathbf{CM}(T')$. Hence $\mathbf{CM}(T')$ is an $\mathcal{L}\forall^{\mathrm{qw}}$-model of T and we have to show that $\mathbf{CM}(T')$ is quasi-witnessed.

Due to the construction of the canonical model, each element of the domain of $\mathbf{CM}(T')$ is a constant. Let $\varphi(x)$ be a formula with one free variable and suppose that $\|(\forall x)\varphi(x)\|^{\mathbf{CM}(T')} > 0$, then we have that $\|\neg\neg(\forall x)\varphi(x)\|^{\mathbf{CM}(T')} = 1$. Hence $T' \vdash \neg\neg(\forall x)\varphi(x)$. By axiom ($\Pi C\forall$), we have that $T' \vdash \neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$, then, by (MP), $T' \vdash (\exists y)(\varphi(y) \to (\forall x)\varphi(x))$. Since T' is $\exists$-Henkin, then there exists some c such that $T' \vdash \varphi(c) \to (\forall x)\varphi(x)$. So, by axiom ($\forall$1), we obtain that $\|\varphi(c)\|^{\mathbf{CM}(T')} = \|(\forall x)\varphi(x)\|^{\mathbf{CM}(T')}$. The proof of the other condition is similar to Hájek's and Cintula's proof of Lemma 5 in Hájek and Cintula [17] and we will not repeat it here.

($\Leftarrow$) Suppose now that $(\mathbf{A},\mathbf{M})$ can be elementarily embedded into a quasi-witnessed model of T, hence, $(\mathbf{A},\mathbf{M})$ is an $\mathcal{L}\forall$-model of T. By Lemma 21, we have that $(\mathbf{A},\mathbf{M})$ is an $\mathcal{L}\forall$-model of $T \cup \{(C\exists), (\Pi C\forall)\}$, which is equivalent to say that $(\mathbf{A},\mathbf{M})$ is an $\mathcal{L}\forall^{\mathrm{qw}}$ model of T.

Theorem 23 *Let T be a theory and φ a formula in a given predicate language, then $T \vdash_{\mathcal{L}\forall^{qw}} \varphi$ iff* $(\mathbf{A},\mathbf{M}) \models \varphi$ *for every quasi-witnessed model* $(\mathbf{A},\mathbf{M})$ *of the theory T.*

Proof The completeness of $\mathcal{L}\forall$ with respect to all (not only quasi-witnessed) $(\mathbf{A},\mathbf{M})$-models is ensured by Theorem 5 of Hájek and Cintula [17], so we will restrict ourselves to the *quasi-witnessed* part.

(⇒) As a consequence of Theorem 5 of Hájek and Cintula [17], we only have to check whether a quasi-witnessed model satisfies axioms ($C\exists$) and ($\Pi C\forall$), but this result has been already shown in Lemma 21.

(⇐) Suppose that $T \nvdash_{\mathcal{L}\forall^{\mathrm{qw}}} \varphi$, then there exists an $\mathcal{L}\forall^{\mathrm{qw}}$-model $(\mathbf{A}, \mathbf{M})$ of T, such that $(\mathbf{A}, \mathbf{M}) \nvDash \varphi$. Hence, by Lemma 22, there exists a quasi-witnessed model $(\mathbf{A}', \mathbf{M}')$ of T such that $(\mathbf{A}', \mathbf{M}') \nvDash \varphi$.

4 The case of predicate Product Logic

In this section we will show that the axioms ($C\exists$) and ($\Pi C\forall$) are provable in $\Pi\forall$, i.e., that the logics $\Pi\forall$ and $\Pi\forall^{qw}$ are equivalent. In order to do that, let us recall that $\Pi\forall$ is complete with respect to all models over a product chain and any product chain is isomorphic to the negative cone of a linearly ordered abelian group with an added bottom (See Theorem 2.5 in [5]).

Definition 24 Let $\mathbf{G} = \langle \mathbf{G}, +, -, \mathbf{0} \rangle$ be a totally ordered abelian group, then we denote by G^- the negative part of G, i.e., $G^- = \{x \in G \mid x \leq 0\}$. Moreover, we denote by $\mathfrak{P}(\mathbf{G})$ the structure $\langle G^- \cup \{\bot\}, \otimes, \Rightarrow, \bot \rangle$, where $\bot$ is an element which does not belong to G, and $\otimes, \Rightarrow$ are two binary operations defined as follows:

$$x \otimes y = \begin{cases} x + y \text{ if } x, y \in G^-, \\ \bot \text{ otherwise,} \end{cases}$$

and

$$x \Rightarrow y = \begin{cases} 0 \wedge (y - x) \text{ if } x, y \in G^-, \\ 0 \text{ if } x = \bot, \\ \bot \text{ if } x \in G^- \text{ and } y = \bot. \end{cases}$$

As a consequence of Theorem 2.5 and Remark 2.2 of [5] we have the following useful result.

Proposition 25 *Let A be a non-trivial Π-chain. There exists a linearly ordered abelian group* $\mathbf{G}$*, such that $A \cong \mathfrak{P}(\mathbf{G})$. Moreover,* $\mathbf{G}$ *is univocally determined up to isomorphism.*

Notice that the isomorphism of the above proposition maps the neutral element of the group onto the maximum element of the product chain and the added bottom $\bot$ to the minimum element of the product chain.

Let $\mathbf{G}$ be a linearly ordered abelian group and $a, \{a_i\}_{i\in\omega} \in G$: it is well known that, on the one hand, if $\{a_i\}_{i\in\omega}$ is an increasing sequence and has limit a, then $\{a - a_i\}_{i\in\omega}$ is a decreasing sequence and has limit 0. On the other hand, if $\{a_i\}_{i\in\omega}$ is a decreasing sequence and has limit a, then $\{a_i - a\}_{i\in\omega}$ is a decreasing sequence and has limit 0. So, since, by Definition 24, the truncated subtraction of the group is the interpretation of product implication and the constant 0 of the group is the isomorphic image of the maximum element 1 of the product chain, then, by means of Proposition 25, we can infer the following corollary.

Corollary 26 *Let* $\mathbf{A}$ *be a product chain and* $a, \{a_i\}_{i\in\omega} \in A$*, then if* $\{a_i\}_{i\in\omega}$ *is either an increasing or decreasing sequence with limit* a*, then* $\{a \Rightarrow a_i\}_{i\in\omega}$ *is an increasing sequence with limit* 1.

With the help of the last corollary, we can prove the main result of this section.

Lemma 27 *The quasi-witnessed axioms* $(C\exists)$ *and* $(\Pi C\forall)$ *are theorems of* $\Pi\forall$.

Proof We will show it semantically. Since $\Pi\forall$ is complete w.r.t. models over linearly ordered product algebras, we have to prove that the quasi-witnessed axioms are tautologies for these models. Let $\mathbf{A}$ be a product chain, then:

- $(C\exists)$ Since $\|(\exists y)((\exists x)\varphi(x) \to \varphi(y))\|^{(\mathbf{A},\mathbf{M})} = \sup_{y\in M}\{\sup_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} \Rightarrow \|\varphi(y)\|^{(\mathbf{A},\mathbf{M})}\}$ and variables x and y range over the same values, then, by Corollary 26, $\|(\exists y)((\exists x)\varphi(x) \to \varphi(y))\|^{(\mathbf{A},\mathbf{M})} = 1$. So, axiom $(C\exists)$ is a theorem of $\Pi\forall$.
- $(\Pi C\forall)$ We know by definition, that $\|\neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))\|^{(\mathbf{A},\mathbf{M})} = \neg\neg\inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} \Rightarrow \sup_{y\in M}\{\|\varphi(y)\|^{(\mathbf{A},\mathbf{M})} \Rightarrow \inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\}\}$. If $\inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} = 0$, the result is obvious. Otherwise (being a Gödel negation) $\neg\neg\inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} = 1$ and, therefore, the value of the whole formula will be equal to 1 iff $\|(\exists y)(\varphi(y) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})} = \sup_{y\in M}\{\|\varphi(y)\|^{(\mathbf{A},\mathbf{M})} \Rightarrow \inf_{x\in M}\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\}\} = 1$, but this is a direct consequence of Corollary 26. So, axiom $(\Pi C\forall)$ is a theorem of $\Pi\forall$.

Next example shows that validity of quasi-witnessed axioms does not guarantee that models are quasi-witnessed (notice that last lemma ensures that all models of first-order Product Logic satisfy the quasi-witnessed axioms).

Example 1 Consider the first-order language with only one unary predicate symbol P and a model over the standard product chain $([0,1]_\Pi, (\omega, r_P))$, where $r_P(n) = \frac{1}{m} + \frac{1}{n+2}$, for a fixed but arbitrary positive integer $m > 1$. By Lemma 27, this model satisfies the quasi-witnessed axioms but it is not a quasi-witnessed model because, on the one hand, $\|(\forall x)P(x)\|^{([0,1]_\Pi,(\omega,r_P))} = \frac{1}{m} > 0$ and, on the other hand, for each $n \in N$, $\|P(n)\|^{([0,1]_\Pi,(\omega,r_P))} > \frac{1}{m} = \|(\forall x)P(x)\|^{([0,1]_\Pi,(\omega,r_P))}$. So, it does not respect condition 2 of Definition 19.

Lemma 27, together with Theorem 23, is an alternative way to prove the result in Laskowski and Malekpour [19].

Corollary 28 *Let* T *be a theory and* φ *a formula in a given predicate language, then* $T \vdash_{\Pi\forall} \varphi$ *iff* $(\mathbf{A},\mathbf{M}) \models \varphi$ *for every quasi-witnessed model* $(\mathbf{A},\mathbf{M})$ *of the theory* T.

However, we can not generalize the above result to the logic defined by an arbitrary left-continuous t-norm (even restricted to a continuous t-norm logic). In order to prove this result we adapt and generalize the result in [15]. Actually we can show that there is no other logic of a continuous t-norm that is complete with respect to quasi-witnessed models, but Product and Łukasiewicz.

Lemma 29 *Let $*$ be a continuous t-norm. If $\mathcal{L}(*)\forall$ proves both $(C\exists)$ and $(\Pi C\forall)$, then $*$ is isomorphic to either Łukasiewicz or product t-norm.*

Proof In [15] it is already proved that $(C\exists)$ is only valid in $\mathcal{L}(*)\forall$ if $*$ is isomorphic to either the Łukasiewicz or the product t-norm. Here we give a unified proof. If the standard algebra induced by a continuous t-norm $*$ is not isomorphic to either $[0,1]_Ł$ or $[0,1]_\Pi$, then it has at least one element $a \in (0,1)$ which is idempotent. Let $([0,1]_*, (\omega, r_P))$ be a model of $\mathcal{L}(*)\forall$ and $\{a_n\}_{n\in\omega}$ a sequence of elements of $[0,1]$, different from a. Let $\{a_n\}_{n\in\omega}$ be a strictly increasing sequence of elements of $[0,1]$ such that $\sup\{a_n\}_{n\in\omega} = a$. Consider the above given structure in which, for each $n \in \omega$, $\|r_P(n))\|^{([0,1]_*,(\omega,r_P))} = a_n$. In this structure, when $\varphi(x) = P(x)$, we have that $\|(\exists y)((\exists x)\varphi(x) \to \varphi(y))\|^{([0,1]_*,(\omega,r_P))} = \sup_{m\in\omega}\{\sup_{n\in\omega}\{a_n\} \Rightarrow a_m\} = \sup_{m\in\omega}\{a \Rightarrow a_m\} = \sup_{m\in\omega}\{a_m\} = a \neq 1$. So, $(C\exists)$ is not a theorem of $\mathcal{L}(*)\forall$.

It is interesting to notice that $(\Pi C\forall)$ is only valid in $\mathcal{L}(*)\forall$ if $[0,1]_*$ is isomorphic to either $[0,1]_Ł$, $[0,1]_\Pi$ or the ordinal sum of two copies of Łukasiewicz t-norms $[0,1]_Ł \oplus [0,1]_Ł$. Let $\{a_n\}_{n\in\omega}$ be a strictly decreasing sequence of elements of $[0,1]$ such that $\inf\{a_n\}_{n\in\omega} = a$, being a either the bottom of a non-Łukasiewicz component or of a Łukasiewicz component whose top element is not 1. In both cases consider the above given structure in which, for each $n \in \omega$, $\|r_P(n))\|^{([0,1]_*,(\omega,r_P))} = a_n$. In this structure, when we take $\varphi(x) = P(x)$, an easy computation shows that axiom $(\Pi C\forall)$ is not sound. Moreover it is not difficult to prove that $(\Pi C\forall)$ is valid when $*$ is isomorphic to either Łukasiewicz or the ordinal sum of two copies of Łukasiewicz t-norm.

Last Lemma allows us to prove the next general result.

Proposition 30 *Let $*$ be a continuous t-norm. Then $\mathcal{L}(*)\forall$ proves both $(C\exists)$ and $(\Pi C\forall)$ iff $[0,1]_*$ is isomorphic to either $[0,1]_Ł$ or $[0,1]_\Pi$.*

Proof One direction is proven in Corollary 28 for Product Logic and is a consequence of witnessed completeness for Łukasiewicz. The other direction is a direct consequence of Lemma 29.

5 Δ-strict fuzzy logics

In this section we deal with the expansion of a logic $\mathcal{L}$ with the new unary connective Δ (denoted $\mathcal{L}_\Delta$) and quasi-witnessed models. Logics $\mathcal{L}_\Delta$, were introduced in [12] as the expansions of $\mathcal{L}$ with the unary connective Δ, satisfying the necessitation inference rule (from φ deduce $\Delta\varphi$) and the following axioms, introduced in [1] in the framework of Gödel Logic:

$(A_\Delta 1)$ $\Delta\varphi \vee \neg\Delta\varphi$,
$(A_\Delta 2)$ $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$,
$(A_\Delta 3)$ $\Delta\varphi \to \varphi$,
$(A_\Delta 4)$ $\Delta\varphi \to \Delta\Delta\varphi$,
$(A_\Delta 5)$ $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$.

Semantically, the main feature of these expansions is that, in $\mathcal{L}_\Delta$-chains, it holds that for each formula φ and each propositional evaluation e, $e(\Delta\varphi) = 1$, if $e(\varphi) = 1$ and $e(\Delta\varphi) = 0$, if $e(\varphi) < 1$. Moreover, expansions of a logic by Δ connective enjoy the following property.

Definition 31 We say that a logic $\mathcal{L}$ enjoys *Delta Deduction Theorem* (ΔDT, for short) if, for each theory T and formulas φ, ψ, it holds that $T, \varphi \vdash \psi$ iff $T \vdash \Delta\varphi \to \psi$.

The last definition gives a way to define the class of logics we are interested in throughout this section.

From [6], we report the next useful definition.

Definition 32 We say that a logic $\mathcal{L}_\Delta$ is a Δ-*core fuzzy logic* if it enjoys ΔDT, *Sub* and expands MTL_Δ.

Throughout this section $\mathcal{L}_\Delta$ will denote the extension of a Δ-core fuzzy logic by the strictness axiom (S).

As in Hájek and Cintula [17], here also the failure of Lemma 17 does not allow us to prove a similar result as Theorem 23 for a logic $\mathcal{L}_\Delta\forall$. Nevertheless, it is possible, also in this context, to prove a simpler result.

Definition 33 We denote by $\mathcal{L}_\Delta\forall^{\Delta qw}$ the axiomatic extension of $\mathcal{L}_\Delta\forall$ by the following axiom schemata called, from now on, "Δ-quasi-witnessed axioms":

$$(\text{C}_\Delta\exists)\quad (\exists y)\Delta((\exists x)\varphi(x) \to \varphi(y)),$$
$$(\Pi\text{C}_\Delta\forall)\quad \neg\neg(\forall x)\varphi(x) \to ((\exists y)\Delta(\varphi(y) \to (\forall x)\varphi(x))).$$

We can prove, as in Hájek and Cintula [17], that the extension of a logic $\mathcal{L}_\Delta\forall$ by means of these axioms, is complete with respect to quasi-witnessed models, but not with respect to models that are embeddable into a quasi-witnessed model (like the extension of a strict core fuzzy logic by the usual quasi-witnessed axioms). So, it makes sense to say that these extensions are the logics of quasi-witnessed models. The main result will follow easily after a couple of simple lemmas.

Lemma 34 *Axioms* $(C_\Delta\exists)$ *and* $(\Pi C_\Delta\forall)$ *are true in every quasi-witnessed model.*

Proof Let $\mathbf{A}$ be an $\mathcal{L}_\Delta$-chain, $(\mathbf{A}, \mathbf{M})$ be a first-order quasi-witnessed structure, then:

1. Since $(\mathbf{A}, \mathbf{M})$ is a quasi-witnessed structure, then there exists $a \in M$ such that $\|\varphi(a)\|^{(\mathbf{A},\mathbf{M})} = \sup_{b\in M}\{\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})}\} = \|(\exists x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$ and, therefore, we have that $\|(\exists y)\Delta((\exists x)\varphi(x) \to \varphi(y))\|^{(\mathbf{A},\mathbf{M})} = \sup_{b\in M}\{\|\Delta((\exists x)\varphi(x) \to \varphi(b))\|^{(\mathbf{A},\mathbf{M})}\}$ $= \|\Delta((\exists x)\varphi(x) \to \varphi(a))\|^{(\mathbf{A},\mathbf{M})} = \|\Delta(1)\|^{(\mathbf{A},\mathbf{M})} = 1$.
2. Since $(\mathbf{A}, \mathbf{M})$ is a quasi-witnessed structure, then either $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$ or there exists $a \in M$ such that $\|\varphi(a)\|^{(\mathbf{A},\mathbf{M})} = \inf_{b\in M}\{\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})}\} = \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$. In the first case, trivially, $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$ and, therefore $\|\neg\neg(\forall x)\varphi(x) \to ((\exists y)\Delta(\varphi(y) \to (\forall x)\varphi(x)))\|^{(\mathbf{A},\mathbf{M})} = 1$. In the second case, by strictness, we have that $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 1$ and the axiom is then valid since $\|(\exists y)\Delta(\varphi(y) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})} = \sup_{b\in M}\{\|\Delta(\varphi(b) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})}\} = \|\Delta(\varphi(a) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})} = \|\Delta(1)\|^{(\mathbf{A},\mathbf{M})} = 1$.

Lemma 35 *Axioms* $(C_\Delta\exists)$ *and* $(\Pi C_\Delta\forall)$ *are false in every model that is not quasi-witnessed.*

Proof We will prove it only for the second axiom, the proof for the first one is almost the same. Let **A** be an $\mathcal{L}_\Delta$-chain, $(\mathbf{A}, \mathbf{M})$ a first-order structure that is not quasi-witnessed. Then there exists a formula $\varphi(x)$ such that both for each $a \in M$, $\|\varphi(a)\|^{(\mathbf{A},\mathbf{M})} \neq \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$ and $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} \neq 0$. Hence $\|(\exists y)\Delta(\varphi(y) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})} = \sup_{b\in M}\{\|\Delta(\varphi(b) \to (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})}\} = \sup_{b\in M}\{0\} = 0$.

We are, now, ready to prove the main result of this section.

Theorem 36 *Let* T *be a theory and* φ *a formula in a given predicate language, then* $T \vdash_{\mathcal{L}_\Delta\forall^{\Delta qw}} \varphi$ *iff* $(\mathbf{A}, \mathbf{M}) \models \varphi$ *for every quasi-witnessed model* $(\mathbf{A}, \mathbf{M})$ *of the theory* T.

Proof The completeness of $\mathcal{L}_\Delta\forall$ with respect to all (not only quasi-witnessed) $(\mathbf{A}, \mathbf{M})$-models is ensured by Theorem 10 of Hájek and Cintula [17], so we will restrict ourselves to the quasi-witnessed part.

- $(\Rightarrow)$ As a consequence of Theorem 10 of Hájek and Cintula [17], we only have to check whether a quasi-witnessed model satisfies axioms $(C_\Delta\exists)$ and $(\Pi C_\Delta\forall)$, but this has been proven in Lemma 34.
- $(\Leftarrow)$ Suppose that $T \nvdash_{\mathcal{L}_\Delta\forall^{\Delta\mathrm{qw}}} \varphi$, then there exists an $\mathcal{L}_\Delta\forall^{\Delta\mathrm{qw}}$-structure $(\mathbf{A}, \mathbf{M})$ of T, such that $(\mathbf{A}, \mathbf{M}) \not\models \varphi$. By Lemma 35, structure $(\mathbf{A}, \mathbf{M})$ is quasi-witnessed and, moreover, $\|\varphi\|^{(\mathbf{A},\mathbf{M})} < 1$.

Unlike quasi-witnessed axioms of previous section, Δ-quasi-witnessed axioms are not derivable in any logic of a continuous t-norm. The argument to prove this result is the same as in Lemma 35 or in Example 1.

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