FIRST ORDER SMTL LOGIC AND QUASI-WITNESSED MODELS

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Abstract

In this paper we prove strong completeness of axiomatic extensions of First Order SMTL logic adding the so-called quasi-witnessed axioms with respect to quasi-witnessed Models. In order to achieve this result, we make use of methods that are typical of Classical Predicate Logic, and have been later generalized by P. Hájek to cope with Predicate Fuzzy Logic. At the end of the paper, we obtain, as a particular case, the result of strong completeness, already proven by M.C. Laskowski and S. Malekpour, for Product Predicate Logic with respect to quasi-witnessed Models.

Keywords: Foundations of Fuzzy Logic, Mathematical Fuzzy Logic, First Order Monoidal t-norm based logic, First Order Product Logic, witnessed and quasi-witnessed models.

1 INTRODUCTION

In recent times there has been a growing interest on First Order Fuzzy Logic due, among other facts, to its role in the construction of Description Logics (DLs) for the Semantic Web (see [1]). Since its birth in the eighties, Classical DLs enjoyed of an already existing wide amount of results in Classical First Order Logic. Also, during the last twenty years there have been attempts to develop DLs based on Fuzzy Logic, in order to cope with vague information but, in this field, the main results obtained are limited to the case of the so-called Zadeh logic[1] and, in the last time, after Hájek's paper [5], some results on FDL over Gödel and Łukasiewicz Logic (see [2]). A basic result on FDL over Łukasiewicz are based on an important property shared by Classical and Łukasiewicz First Order Logic: the so-called *witnessed model property* defined in [5]. In [4], generalizing the classical case, the value of an universally (existentially) quantified formula is the infimum (supremum) of the values of the results of replacing the quantified variable by the evaluation of a term of the language in **M**. In the context of Classical Logic, as well as every finitely valued logic, the infimum and supremum turn out to be a minimum and a maximum, respectively, but, when we move to infinitely valued logics, we can find sets of formulas which have values whose infimum (resp. supremum) is strictly different from each element of the set, i.e., the quantified formula has no *witness*. A *witnessed model* is then a model in which each quantified formula has a witness and it is an important property because Hájek proves that it implies a limited form of finite model property for Description Logics based on Łukasiewicz logic. Moreover, he introduces the so-called witnessing axioms that are satisfied in Łukasiewicz Logic and, adding them to any axiomatic extension of MTL∀, we obtain a logic complete with respect to witnessed models. The $\mathcal{ALC}$ description language over these logics extended by means of witnessing axioms, enjoys the finite model property. In particular, using such results, in [5], Hájek proves that assertion satisfiability and validity, for fuzzy $\mathcal{ALC}$ based on Łukasiewicz t-norm are decidable problems.

In [8] it is proven that Łukasiewicz First Order Logic is complete with respect to witnessed models (or, in other words, it has the witnessed model property), but also that neither Gödel, nor Product First Order Logic share this property. In fact no other first order logic of a continuous t-norm enjoys this property, since it is related to continuity of the truth functions that it is known that only Łukasiewicz logic has. However, in [9]

[1]This is a logic whose connectives are related to max, min, the negation function $\neg x = 1 - x$ and the implication function $x \to y = \max(\neg x, y)$, all of them defined over the real unit interval. Even though this is not a residuated logic, it is definable in Łukasiewicz logic.

is proven that Product Predicate Logic enjoys a weaker property, what we call *quasi-witnessed model property*. Quasi-witnessed models[2] are models in which whenever the value of an universally quantified formula is strictly greater than 0, then it has a witness, while existentially quantified formulas are always witnessed. In this paper we prove, following the style of [8] that there is an axiomatic extension of SMTL that enjoys quasi-witnessed model property[3]; the result of [9] about the completeness of Product First Order Logic with respect to quasi-witnessed models, will follow as a corollary from our main result.

2 PRELIMINARIES

The logic SMTL is defined in the literature as the axiomatic extension of the Monoidal t-norm Logic MTL by the axiom:

(PC) $\varphi \wedge \neg\varphi \to \bot$

At the semantic level it is the logic of left-continuous t-norms whose associated negation is the so called *Gödel negation*, i.e. the negation whose truth function, in the standard semantics, is defined by:

$$n(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x = 0 \end{cases}$$

In this paper we deal with an axiomatic extension of predicate SMTL, denoted by SMTL$\forall^{qw}$, which we define below; following [4], we previously define first order SMTL logic, (denoted by SMTL$\forall$).

Definition 2.1 SMTL$\forall$ *is the expansion of propositional* SMTL *adding the two "classical" quantifiers* $\forall$ *and* $\exists$ *and the following set of axiom schemata:*

(P) the axioms resulting from the axioms of SMTL *after the substitution of propositional variables by formulas of the new predicate language.*

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, *where* t *is substitutable for* x *in* φ.

($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, *where* t *is substitutable for* x *in* φ.

($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi(x))$, *where* x *is not free in* χ.

($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi(x) \to \chi)$, *where* x *is not free in* χ.

($\forall$3) $(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi(x))$, *where* x *is not free in* χ.

and which has, as rules of inference, Modus Ponens (MP) and generalization (G): From φ *infer* $(\forall x)\varphi(x)$.

Definition 2.2 *Let* $\mathcal{L}\forall$ *be an axiomatic extension of* SMTL$\forall$. *We denote by* $\mathcal{L}\forall^{qw}$ *the axiomatic extension of* $\mathcal{L}\forall$ *by the following axiom schemata called, from now on, "quasi-witnessed axioms":*

(C$\exists$) $(\exists y)((\exists x)\varphi(x) \to \varphi(y))$,

(ΠC$\forall$) $\neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

These quasi-witnessed axioms are a modification of witnessed axioms. The first one, $(C\exists)$, is a witnessed axiom and the second one says that the witnessed axiom $(C\forall)$ $(\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$ is valid only when $(\forall x)\varphi(x)$ is different from 0, i.e., when $\neg\neg(\forall x)\varphi(x) = 1$.

Definition 2.3 *The logic* $\Pi\forall$ *(resp.* $\Pi\forall^{qw}$*) is the axiomatic extension of* SMTL$\forall$ *(resp.* SMTL$\forall^{qw}$*) by the following axioms*[4]*:*

(C) $\varphi \wedge \psi \leftrightarrow \varphi \odot (\varphi \to \psi)$,

(Π) $\neg\neg\chi \to (((\varphi \odot \chi) \to (\psi \odot \chi)) \to (\varphi \to \psi))$.

The following definitions are required to prove the main results given in Section 3. They are typical within the framework of Classical First Order Logic. Their presentation in our context, slightly different from the classical one, follows the generalization, due to [8], and necessary to adapt them to a many-valued framework.

Definition 2.4 *We say that a theory* T' *in a predicate language* Γ' *is an* expansion *of a theory* T *in a predicate language* Γ, *if* $\Gamma \subseteq \Gamma'$ *and, each formula provable in* T *is provable in* T'. *We say that* T' *is a* conservative expansion *of* T *if* T' *is an expansion of* T *and each formula in the language of* T, *provable in* T', *is provable in* T.

Definition 2.5 *A theory* T *is* linear *if, for each pair of sentences* φ, ψ, *we have* $T \vdash \varphi \to \psi$ *or* $T \vdash \psi \to \varphi$.

[2] This models are called closed models in [9] but we decided, after some discussions with colleagues, to use the more informative name of quasi-witnessed models. We take into account the fact that the name closed is used in mathematics and logic in many different context with different meanings and could induce some confusions.

[3] In what follows we implicitly maintain that we are considering always *safe* models: a safe model, as defined in [4], is a model in which, for each first order formula of a given language, the value (the required infima and suprema) is always defined.

[4] Alternatively, $\Pi\forall$ (resp. $\Pi\forall^{qw}$) can be defined as the expansion of the well known Propositional Product Logic Π by classical quantifiers and axioms in Definition 2.2 (resp. 2.1).

Definition 2.6 *Let Γ and Γ' be predicate languages such that $\Gamma \subseteq \Gamma'$ and T a Γ'-theory. We say that T is* $\forall$-Γ-Henkin *if, for each Γ-sentence $\varphi = (\forall x)\psi(x)$ such that $T \nvdash \varphi$, there is a constant c in Γ' such that $T \nvdash \psi(c)$.*
We say that T is $\exists$ -Γ-Henkin *if, for each Γ-sentence $\varphi = (\exists x)\psi(x)$ such that $T \vdash \varphi$, there is a constant c in Γ' such that $T \vdash \psi(c)$.*
A theory is called Γ-Henkin *if it is both $\forall$-Γ-Henkin and $\exists$ -Γ-Henkin.*
If $\Gamma = \Gamma'$, we say that T is $\forall$-Henkin ($\exists$-Henkin, Henkin).

From a semantic point of view, before defining what a first order model is, we need a definition of SMTL-algebra.

Definition 2.7 *An* SMTL-*algebra* $\mathbf{A} = \langle \mathbf{A}, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ *is a bounded commutative integral residuated lattice which satisfies the following equations:*

1. $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$,
2. $x \cap (x \Rightarrow 0) = 0$.

Moreover, if it is linearly ordered, we say that it is a SMTL-*chain.*
For each axiomatic extension $\mathcal{L}$ of SMTL, *a $\mathcal{L}$-algebra is a* SMTL-*algebra satisfying the equations corresponding to added axioms.*

Let Γ be a predicate language without function symbols and $\mathbf{A}$ a SMTL-chain. An $\mathbf{A}$-structure for a given predicate language Γ is a structure $\mathbf{M}{=}(M, (P_{\mathbf{M}})_{P\in\Gamma}, (c_{\mathbf{M}})_{c\in\Gamma})$, where $M \neq \emptyset$, each $P_{\mathbf{M}}$ is an n-ary $\mathbf{A}$-fuzzy relation on M and each $c_{\mathbf{M}}$ is an element of M. The truth value $\|\varphi\|^{\mathbf{A},\mathbf{M}}$ of a predicate formula φ is defined in the usual way and so are the concepts of *satisfiability* and *validity*. In [8], we find the following useful definitions and result, which we report without proof. In what follows, we will denote by $\mathbf{A}$ any SMTL-chain.

Definition 2.8 *Let $(\mathbf{A_1}, \mathbf{M_1})$ and $(\mathbf{A_2}, \mathbf{M_2})$ be models in the languages Γ_1 and Γ_2 respectively and let $\Gamma_1 \subseteq \Gamma_2$. We say that a pair (f, g) is an* elementary embedding *if:*

1. *the mapping f is an injection of M_1 into M_2,*
2. *the mapping g is an embedding of $\mathbf{A_1}$ into $\mathbf{A_2}$,*
3. *for each Γ_1-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M_1$, it holds that $g(\|\varphi(a_1, \ldots, a_n)\|^{(\mathbf{A_1},\mathbf{M_1})}) = \|\varphi(f(a_1), \ldots, f(a_n))\|^{(\mathbf{A_2},\mathbf{M_2})}$.*

Definition 2.9 *Let T be a linear Henkin theory, then the* canonical model *of T is the structure $(\boldsymbol{Lind}_T, \boldsymbol{CM}(T))$, where $\boldsymbol{Lind}_T$ is the Lindenbaum algebra of theory T, the domain of $\boldsymbol{CM}(T)$ consists of the object constants $c_{\mathbf{CM(T)}} = c$ and, for every predicate n-ary symbol $P \in \Gamma$, $P_{\mathbf{CM(T)}}(t_1, \ldots, t_n) = [P(t_1, \ldots, t_n)]_T$.*

From here on, for simplicity, we will write $\mathbf{CM}(T)$ to denote $(\mathbf{Lind}_T, \mathbf{CM}(T))$.

Definition 2.10 *Let $(\boldsymbol{A},\boldsymbol{M})$ be a structure and $Alg((\boldsymbol{A},\boldsymbol{M}))$ be the subalgebra of $\boldsymbol{A}$ whose domain is the set $\{\|\varphi\|_v^{\mathbf{A},\mathbf{M}} \mid \varphi, v\}$ of truth degrees of formulas under all $\boldsymbol{M}$-evaluation v of variables. Call $(\boldsymbol{A},\boldsymbol{M})$* exhaustive *if $\boldsymbol{A} = Alg((\boldsymbol{A},\boldsymbol{M}))$.*

The following lemma is a direct consequence of Lemma 4 of [8] and we will not prove it here.

Lemma 2.11 *Let T_1, T_2, T be SMTL$\forall$-theories. If T_2 is a conservative expansion of T_1, then, for each exhaustive model $(\boldsymbol{A},\boldsymbol{M})$ of T_1, there exists a linear Henkin theory T extending T_2 such that $(\boldsymbol{A},\boldsymbol{M})$ can be elementarily embedded into $\boldsymbol{CM}(T)$.*

3 COMPLETENESS WITH RESPECT TO QUASI-WITNESSED MODELS

In this section we will state and prove the main result of this paper, i.e., that if we add axioms C$\exists$ and ΠC$\forall$ to any predicate fuzzy logic extending SMTL$\forall$, we obtain a logic that is complete with respect to quasi-witnessed models. Within this section, we will write $\mathcal{L}\forall$ to denote any axiomatic extencion SMTL$\forall$.

Definition 3.1 *Let Γ be a predicate language, and $(\mathbf{A}, \mathbf{M})$ a model, then we say that $(\mathbf{A}, \mathbf{M})$ is* quasi-witnessed *if, for every Γ-formula $\varphi(x, y_1, \ldots, y_n)$:*

1. *For each tuple $c_1, \ldots, c_n$ of elements in M there exist $a \in M$ such that $\|(\exists x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = \|\varphi(a, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})}$.*
2. *For each tuple $c_1, \ldots, c_n$ of elements in M either $\|(\forall x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = 0$, or there exists $b \in M$ such that $\|(\forall x)\varphi(x, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})} = \|\varphi(b, c_1, \ldots, c_n)\|^{(\mathbf{A},\mathbf{M})}$.*

Lemma 3.2 *If a model $(\mathbf{A}, \mathbf{M})$ of $\mathcal{L}\forall$ is quasi-witnessed, then $(\mathbf{A}, \mathbf{M})$ satisfies (C$\exists$) and ($\Pi C\forall$).*

Proof Let $(\mathbf{A},\mathbf{M})$ be a quasi-witnessed model of $\mathcal{L}\forall$, then:

1. Since, by the first condition of Definition 3.1, there exists $a \in M$ such

that $\|\varphi(a)\|^{(\mathbf{A},\mathbf{M})} = \|(\exists x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$, then $(\mathbf{A},\mathbf{M}) \models (\exists x)\varphi(x) \to \varphi(a)$. So, by axiom $(\exists 1)$ and (MP), $(\mathbf{A},\mathbf{M}) \models (\exists y)((\exists x)\varphi(x) \to \varphi(y))$.

2. By the second condition of Definition 3.1, there exists $b \in M$ such that either $\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})} = \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$, or $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$. If $\|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$, then, $\|\neg\neg(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})} = 0$ and, trivially, $(\mathbf{A},\mathbf{M}) \models \neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$. If, on the other hand, $\|\varphi(b)\|^{(\mathbf{A},\mathbf{M})} = \|(\forall x)\varphi(x)\|^{(\mathbf{A},\mathbf{M})}$, then $(\mathbf{A},\mathbf{M}) \models \varphi(b) \to (\forall x)\varphi(x)$, and, by axiom $(\exists 1)$ and (MP), $(\mathbf{A},\mathbf{M}) \models (\exists y)(\varphi(y) \to (\forall x)\varphi(x))$. So, $(\mathbf{A},\mathbf{M}) \models \neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$.

As for witnessed models and Ł$\forall^w$, the converse of the last lemma does not hold: to show this fact, consider the model $([0,1]_\Pi, (N, r_P))$, where $r_P(n) = \frac{1}{m} + \frac{1}{n+2}$, for a fixed but arbitrary positive integer $m > 1$. Such model, when we take $\varphi(x) = P(x)$, validates axiom $(\Pi C\forall)$, but it is not quasi-witnessed because, on the one hand, $\|(\forall x)\varphi(x)\|^{([0,1]_\Pi,(N,r_P))} = \frac{1}{m} > 0$ and on the other hand, for each $n \in N$, $\|\varphi(n)\|^{([0,1]_\Pi,(N,r_P))} > \frac{1}{m} = \|(\forall x)\varphi(x)\|^{([0,1]_\Pi,(N,r_P))}$. So, it does not respect condition 2 of Definition 3.1. However, as in [8], it is possible to prove the following result.

Lemma 3.3 *Let Γ be a predicate language, and $(\mathbf{A},\mathbf{M})$ an exhaustive model of a Γ-theory T. Then $(\mathbf{A},\mathbf{M})$ is a $\mathcal{L}\forall^{qw}$-model of T iff it can be elementarily embedded into a quasi-witnessed model of T.*

Proof ($\Rightarrow$) Let $(\mathbf{A},\mathbf{M})$ be an exhaustive $\mathcal{L}\forall^{\mathrm{qw}}$-model of T. By Lemma 2.11, there is a linear Henkin theory T' extending T, such that $(\mathbf{A},\mathbf{M})$ can be elementarily embedded into $\mathbf{CM}(T')$. Hence $\mathbf{CM}(T')$ is a $\mathcal{L}\forall^{\mathrm{qw}}$-model of T and we have to show that $\mathbf{CM}(T')$ is quasi-witnessed.

Due to the construction of the canonical model, each element of the domain of $\mathbf{CM}(T')$ is a constant. Let $\varphi(x)$ be a formula with one free variable, then:

1. Suppose, on the one hand, that $\|(\forall x)\varphi(x)\|^{\mathbf{CM(T')}} > 0$, then $\|\neg\neg(\forall x)\varphi(x)\|^{\mathbf{CM(T')}} = 1$.
   Since, by axiom $(\Pi C\forall)$, we have that $T' \vdash \neg\neg(\forall x)\varphi(x) \to ((\exists y)(\varphi(y) \to (\forall x)\varphi(x)))$, then, by (MP), $T' \vdash (\exists y)(\varphi(y) \to (\forall x)\varphi(x))$. Since T' is $\exists$-Henkin, then there exists some c such that $T' \vdash \varphi(c) \to (\forall x)\varphi(x)$. So, by axiom $(\forall 1)$, we obtain that $\|\varphi(c)\|^{\mathbf{CM(T')}} = \|(\forall x)\varphi(x)\|^{\mathbf{CM(T')}}$.
2. Suppose, on the other hand, that, for every constant $c \in \Gamma$, $\|\varphi(c)\|^{\mathbf{CM(T')}} \neq \|(\forall x)\varphi(x)\|^{\mathbf{CM(T')}}$, then we have necessarily that $\|(\forall x)\varphi(x)\|^{\mathbf{CM(T')}} = 0$, because, otherwise, as we have shown before, we have that $\|\varphi(c)\|^{\mathbf{CM(T')}} = \|(\forall x)\varphi(x)\|^{\mathbf{CM(T')}}$, contradicting the supposition.

The proof of the other condition is similar to Hájek's proof of Lemma 5 in [8] and we will not repeat it here.

($\Leftarrow$) Suppose now that $(\mathbf{A},\mathbf{M})$ can be elementarily embedded into a quasi-witnessed model of T, hence, by Lemma 3.2, is a $\mathcal{L}\forall^{\mathrm{qw}}$-model of T.

Theorem 3.4 *Let T be a theory and φ a formula in a given predicate language, then $T \vdash_{\mathcal{L}\forall^{qw}} \varphi$ iff $(\boldsymbol{A},\boldsymbol{M}) \models \varphi$ for every quasi-witnessed model $(\boldsymbol{A},\boldsymbol{M})$ of the theory T.*

Proof The completeness of $\mathcal{L}\forall$ with respect to all (not only quasi-witnessed) $(\mathbf{A},\mathbf{M})$-models is ensured by Theorem 5 of [8], so we will restrict ourselves to the *quasi-witnessed* part.

($\Rightarrow$) As a consequence of Theorem 5 of [8], we only have to check whether a quasi-witnessed model satisfies axioms $(C\exists)$ and $(\Pi C\forall)$, but this result has been already shown in Lemma 3.2.

($\Leftarrow$) Suppose that $T \nvdash_{\mathcal{L}\forall^{\mathrm{qw}}} \varphi$, then there exists a $\mathcal{L}\forall^{\mathrm{qw}}$-model $(\mathbf{A},\mathbf{M})$ of T, such that $(\mathbf{A},\mathbf{M}) \not\models \varphi$. Hence, by Lemma 3.3, there exists a quasi-witnessed model $(\mathbf{A}',\mathbf{M}')$ of T such that $(\mathbf{A}',\mathbf{M}') \not\models \varphi$.

4 THE CASE OF PREDICATE PRODUCT LOGIC

In this section we will show that axioms $(C\exists)$ and $(\Pi C\forall)$ are provable in $\Pi\forall$, i.e., that the logics $\Pi\forall$ and $\Pi\forall^{qw}$ are equivalent. In order to do that, let us recall that $\Pi\forall$ is complete with respect to all models over a product chain and any product chain is isomorphic to the negative cone of a linearly ordered abelian group with an added bottom (See Theorem 2.5 in [3]).

Definition 4.1 *Let $\mathbf{G} = \langle \mathbf{G}, +, -, \mathbf{0}\rangle$ be a totally ordered abelian group, then we denote by G^- the negative part of G, i.e., $G^- = \{x \in G \mid x \leq 0\}$. Moreover, we denote by $\mathfrak{P}(\mathbf{G})$ the structure $\langle G^- \cup \bot, \otimes, \Rightarrow, \bot\rangle$, where $\bot$ is an element which does not belong to G, and $\otimes$, $\Rightarrow$ are two binary operations defined as follows:*

$$x \otimes y = \begin{cases} x + y & \text{if } x, y \in G^-, \\ \bot & \text{otherwise,} \end{cases}$$

and

$$x \Rightarrow y = \begin{cases} 0 \wedge (y - x) & \textit{if } x, y \in G^-, \\ 0 & \textit{if } x = \perp, \\ \perp & \textit{if } x \in G^- \textit{ and } y = \perp. \end{cases}$$

As a consequence of Theorem 2.5 and Remark 2.2 of [3] we have the following useful result.

Proposition 4.2 *Let A be a non-trivial Π-chain*[5]. *There exists a linearly ordered abelian group* $\mathbf{G}$*, such that* $A \cong \mathfrak{P}(\mathbf{G})$*. Moreover,* $\mathbf{G}$ *is univocally determined up to isomorphism.*

Notice that the isomorphism of the above proposition maps the neutral element of the group onto the maximum element of the product chain and the added bottom $\perp$ to the minimum element of the product chain. This last result allows us to look at the theory of linearly ordered abelian groups. Moreover, let $\mathbf{G}$ be a linearly ordered abelian group and $a, \{a_i\}_{i\in\omega} \in G$: it is well known that, on the one hand, if $\{a_i\}_{i\in\omega}$ is an increasing sequence and has limit a, then $\{a - a_i\}_{i\in\omega}$ is a decreasing sequence and has limit 0. On the other hand, if $\{a_i\}_{i\in\omega}$ is a decreasing sequence and has limit a, then $\{a_i - a\}_{i\in\omega}$ is a decreasing sequence and has limit 0. So, since, by Definition 4.1, the truncated subtraction of the group is the interpretation of product implication and the constant 0 of the group is the isomorphic image of the maximum element 1 of the product chain, then, by means of Proposition 4.2, we can infer the following corollary.

Corollary 4.3 *Let* $\mathbf{A}$ *be a product chain and* $a, \{a_i\}_{i\in\omega} \in A$*, then:*

1. *if $\{a_i\}_{i\in\omega}$ is an increasing sequence with limit a, then $\{a \Rightarrow a_i\}_{i\in\omega}$ is an increasing sequence with limit* 1*,*

2. *if $\{a_i\}_{i\in\omega}$ is a decreasing sequence with limit a, then $\{a_i \Rightarrow a\}_{i\in\omega}$ is an increasing sequence with limit* 1*.*

With the help of the last corollary, we can prove the main result of this section.

Lemma 4.4 *The quasi-witnessed axioms ($C\exists$) and ($\Pi C\forall$) are theorems of* $\Pi\forall$.

[5] A Π-chain is a SMTL-chain which satisfies the equations:

1. $x \cap y = x * (x \Rightarrow y)$,

2. $((z \Rightarrow 0) \Rightarrow 0) \Rightarrow ((x * z) \Rightarrow (y * z) \Rightarrow (x \Rightarrow y))01.$

Proof We will show it semantically. Since $\Pi\forall$ is complete w.r.t. models over linearly ordered product algebras, we have to prove that the quasi-witnessed axioms are tautologies for these models. Let $\mathbf{A}$ be a product chain and let $(\mathbf{A}, \mathbf{M})$ be a model of $\Pi\forall$, then:

($C\exists$) Since $\|(\exists y)((\exists x)\varphi(x) \rightarrow \varphi(y))\|^{(\mathbf{A},\mathbf{M})} = sup_y\{sup_x\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} \Rightarrow \|\varphi(y)\|^{(\mathbf{A},\mathbf{M})}\}$ and variables x and y range over the same values, then, by Corollary 4.3, $\|(\exists y)((\exists x)\varphi(x) \rightarrow \varphi(y))\|^{(\mathbf{A},\mathbf{M})} = 1$. So, axiom ($C\exists$) is a theorem of $\Pi\forall$.

($\Pi C\forall$) We know that $\|\neg\neg(\forall x)\varphi(x) \rightarrow ((\exists y)(\varphi(y) \rightarrow (\forall x)\varphi(x)))\|^{(\mathbf{A},\mathbf{M})} = \neg\neg inf_x\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} \Rightarrow sup_y\{\|\varphi(y)\|^{(\mathbf{A},\mathbf{M})} \Rightarrow inf_x\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\}\}$. If $inf_x\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} = 0$, the result is obvious. Otherwise (being a Gödel negation) $\neg\neg inf_x\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\} = 1$ and, therefore, the value of the whole formula will be equal to 1 iff $\|(\exists y)(\varphi(y) \rightarrow (\forall x)\varphi(x))\|^{(\mathbf{A},\mathbf{M})} = sup_y\{\|\varphi(y)\|^{(\mathbf{A},\mathbf{M})} \Rightarrow inf_x\{\|\varphi(x)\|^{(\mathbf{A},\mathbf{M})}\}\} = 1$, but this is a direct consequence of Corollary 4.3. So, axiom ($\Pi C\forall$) is a theorem of $\Pi\forall$.

This last result, together with Theorem 3.4, is an alternative way to prove the result of Laskowski and Malekpour in [9].

Corollary 4.5 *Let T be a theory and φ a formula in a given predicate language, then $T \vdash_{\Pi\forall} \varphi$ iff $(\mathbf{A}, \mathbf{M}) \models \varphi$ for every quasi-witnessed model $(\mathbf{A}, \mathbf{M})$ of the theory T.*

Next we adapt and generalize the result in [7]. Actually we can show that there is no other logic of a continuous t-norm that is complete with respect to quasi-witnessed models, but Product or Łukasiewicz. Denote by $\mathcal{L}(*)$ the propositional logic of a continuous t-norm $*$ (complete with respect to valuations over the standard chain $[0,1]_*$)[6] and by $\mathcal{L}(*)\forall$ its first order expansion.

Lemma 4.6 *If a continuous t-norm $*$ is different from Product or Łukasiewicz, then there exists a model of $\mathcal{L}(*)\forall$ over $[0,1]_*$ which does not satisfy axioms ($C\exists$) and ($\Pi C\forall$).*

Proof If a continuous t-norm $*$ is different from Product or Łukasiewicz, then it has at least one element $a \in (0,1)$ which is idempotent. Let $([0,1]_*, (N, r_P))$ be a model of $\mathcal{L}(*)\forall$ and $\{a_n\}_{n\in\omega}$ a sequence of elements of $[0,1]$, different from a.

[6] By $[0,1]_* = \langle [0,1], max, min, *, \Rightarrow, \neg, 0, 1\rangle$ is denoted the linear algebra defined over the real unit interval by a left-continuous t-norm $*$ and its residuum $\Rightarrow$.

($C\exists$) Consider first order the interpretation such that $\|r_P(n))\|^{([0,1]_*,(N,r_P))} = a_n$, $\{a_n\}_{n\in\omega}$ is an increasing sequence of elements of $[0,1]$ and $sup\{a_n\}_{n\in\omega} = a \neq 1$. In this model, when we take $\varphi(x) = P(x)$, we have that $\|(\exists y)((\exists x)\varphi(x) \to \varphi(y))\|^{([0,1]_*,(N,r_P))} = sup_{m\in\omega}\{sup_{n\in\omega}\{a_n\} \Rightarrow a_m\} = sup_{m\in\omega}\{a \Rightarrow a_m\} = sup_{m\in\omega}\{a_m\} = a \neq 1$. So, ($C\exists$) is not a theorem of $\mathcal{L}(*)\forall$.

($\Pi C\forall$) Consider the first order interpretation such that $\|r_P(n))\|^{([0,1]_*,(N,r_P))} = a_n$, $\{a_n\}_{n\in\omega}$ is a decreasing sequence of elements of $[0,1]$ and $inf\{a_n\}_{n\in\omega} = a \neq 1$. In this model, when we take $\varphi(x) = P(x)$, we have that $\|\neg\neg(\forall x)\varphi(x) \to (\exists y)(\varphi(y) \to (\forall x)\varphi(x))\|^{([0,1]_*,(N,r_P))} = \neg\neg(inf_{n\in\omega}\{a_n\}) \Rightarrow sup_{m\in\omega}\{a_m \Rightarrow inf_{n\in\omega}\{a_n\}\} = 1 \Rightarrow sup_{m\in\omega}\{a_m \Rightarrow a\} = 1 \Rightarrow a = a \neq 1$. So, ($\Pi C\forall$) is not a theorem of $\mathcal{L}(*)\forall$.

This last result allows us to prove the next general result.

Proposition 4.7 *If $*$ is a continuous t-norm, $\mathcal{L}(*)\forall$ proves both ($C\exists$) and ($\Pi C\forall$) iff $*$ is either Łukasiewicz or Product.*

Proof One direction is proven in Corollary 4.5 for Product Logic and is a consequence of witnessed completeness for Łukasiewicz. The other direction is a direct consequence of Lemma 4.6.

5 CONCLUSIONS AND FUTURE WORK

In this paper we have studied quasi-witnessed models and introduced quasi-witnessed axioms, that are an extension of witnessed axioms. Finally we have showed that the extension of any SMTL$\forall$ logic with quasi-witnessed axioms is complete with respect to quasi-witnessed models. But quasi-witnessed axioms are provable in Product first order logic and then we recover the quasi-witnessed completeness proved in [9]. We also show that the only logic of a continuous t-norm that prove quasi-witnessed axioms are Łukasiewicz and Product. As future work we have in mind to study two different subjects:

1.- Is it possible to find results of decidability of Product Description Logic based on quasi-witnessed models (modifying and extending Hájek's results for witnessed models)?

2.- Is it possible to characterize the axiomatic extensions of a first order logic of a left-continuous t-norm with the witnessed axioms? And the same question for a first order logic of a strict left-continuous t-norm with the quasi-witnessed axioms?

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