On Nilpotent Minimum logics defined by lattice filters and their paraconsistent non-falsity preserving companions

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Abstract

Nilpotent Mininum logic (NML) is a substructural algebraizable logic that is a distinguished member of the family of systems of Mathematical Fuzzy logic, and at the same time it is the axiomatic extension of Nelson and Markov's Constructive logic with strong negation with the prelinearity axiom. In this paper our main aim is to characterise and axiomatise paraconsistent variants of NML and its extensions defined by (sets of) logical matrices over linearly ordered NM-algebra with lattice filters as designated values, with special emphasis on those that only exclude the falsum truth-value, called non-falsity preserving logics. We also consider turning these non-falsity preserving logics into Logics of Formal Inconsistency by expanding them with a consistency operator, and we axiomatise them as well. Finally, we provide a full description of the logics defined by finite products of matrices over finite NM-chains.

1 Introduction

Mathematical fuzzy logic (MFL) is a discipline of mathematical logic that aims at studying systems of fuzzy logic in narrow sense (see the classical book [32] and the handbook [12]), i.e. systems of many-valued (truth-funcional) logics intented to reason with vague or gradual properties or predicates, where truth-values are interpreted as degrees of truth. In this sense, MFL can be seen as a degree-based approach to vagueness [45].

It should be observed that, because of its nature, vague reasoning has to deal with *gaps* (undetermination of truth) and *gluts* (overdetermination of truth). Hence, given a proposition P, a gap indicates that neither P nor its negation are true, whereas a glut represents that both P and its negation are true. The latter suggests that a fuzzy negation should be paraconsistent, that is, tolerant to contradictions. However, in general MFL adopts the (full) *truth-preserving* notion of consequence relation, usual in algebraic logic. Under this perspective, a formula is a consequence of set of premises if, for every algebraic evaluation that interprets the premises as (fully) true, it also interprets the conclusion as (fully) true. Within this paradigm, most (if not all) fuzzy logics associated to well-studied algebraic structures such as Łukasiewicz and Gödel logics *are not* paraconsistent: no contradictory theory can be (fully) satisfied, hence it is always logically trivial.

Besides this feature, the truth-preserving paradigm has also been criticized since it neglects, in some sense, the many degrees of truth available in the semantical structures: after all, only the maximum value 1 (absolute truth) is relevant for the consequence relation. In [50] it was proposed the notion of *degree-preserving* consequence relation, in which a formula follows from a given set of premises if, for all algebraic evaluations, the truth-degree of the conclusion (under such evaluation) is not lower than those of the premises, see also [27, 6] for further investigations on this weaker notion of logical consequence. It can be argued that this approach to consequence relation is more coherent with the commitment of many-valued logics to truth-degree semantics. Indeed, under this definition, each truth-value (seen as a degree of truth) plays an equally important role in the corresponding notion of consequence (for a discussion on this topic see [26]).

Other than the degree-preserving logic associated to a class of algebraic structures, it is interesting to consider (families of) *lattice filters* as sets of designated values. As particular cases, taking simply the filter $\{1\}$ corresponds to the truth-preserving paradigm, while the degree-preserving consequence relation is the logic associated to the family of all the lattice filters [6]. The lattice filters approach produces an ample class of intermediate logics between the truth and the degree preserving consequence relations, some of them being paraconsistent. We analysed in [15] some intermediate systems for Łukasiewicz logics, while the intermediate Gödel logics with an involution were discussed in [17].

The aim of this paper is the development of a similar study for intermediate logics defined by lattice filters in the case of the Nilpotent Minimum logic (NML). Nipotent Minimum logic is a substructural logic at the crossroad of two different non-classical logic traditions. On the one hand, NML is a distinguished member of the family of formal systems of mathematical fuzzy logic, introduced by two of the authors of this paper in [23] as a particular extension of the Monoidal t-norm based Logic MTL, a very general logic whose equivalent algebraic semantics is the variety of prelinear (commutative, bounded, integral) residuated lattices, also known as MTL-algebras. This variety is generated by the subclass of algebras with domain the real unit interval $[0, 1]$ and defined by left-continuous t-norms[1], see [34]. In fact, the logic NM was originally defined in [23] as the axiomatic extension of MTL by the following two axioms, requiring the negation to be involutive and the weak nilpotent minimum condition:

$$\text{(INV)} \quad \neg\neg\varphi \to \varphi$$
$$\text{(WNM)} \quad (\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi).$$

NML is an algebraizable logic, as all the axiomatic extensions of MTL, and the corresponding variety of NM-algebras is generated by a single algebra on real unit interval $[0,1]$, called *standard NM-algebra*, see Section 2.

On the other hand, NML can also be considered as deriving from the well-known *Constructive logic with strong negation* introduced independently by Nelson [38] and Markov [36], also known as *Nelson logic* or even as the logic N3, as a result of the observation by Rasiowa [42] about the non-constructive property of intuitionistic negation, namely that the derivability of the formula $\neg(\varphi \wedge \psi)$ in an intuitionistic logic does not imply that at least one of the formulas $\neg\varphi$, $\neg\psi$ is derivable. Although *Nelson algebras*, the algebraic semantics of Nelson logic developed by Rasiowa [42, 43], were not originally presented as a subclass of residuated lattices, Spinks and Veroff proved in [44, 46] that Nelson logic is indeed a substructural logic by showing that Nelson algebras are termwise equivalent to certain involutive, bounded, commutative and integral residuated lattices, called *Nelson (residuated) lattices*, see also [7]. In the latter paper, the authors also show that prelinear Nelson lattices are nothing but NM-algebras, or in other words, the NM logic can also be obtained as the axiomatic extension of Nelson logic with the prelinearity axiom

$$\text{(Lin)} \quad (\varphi \to \psi) \vee (\psi \to \varphi).$$

The NM logic together with all their axiomatic and finitary extensions has been exhaustively studied by Gispert in [29, 30]. They are all explosive, as any (full) truth-preserving substructural logic with respect to its residual negation $\neg\varphi = \varphi \to 0$.

In this paper our main aim is to characterise and axiomatise paraconsistent variants of NML and extensions defined by (sets of) logical matrices over linearly ordered NM-algebras with lattice filters as sets of designated values, with special emphasis on those whose lattice filters that only exclude the *falsum* truth-value, that will be called *non-falsity preserving.* Moreover, the introduction of consistency operators (in the sense of the paraconsistent logics known as *logics of formal inconsistency*, see [11, 10]) over the real unit interval $[0,1]$ with the non-zero designated values will also be considered, along the lines of the study we developed in [14] in the framework of Monoidal t-norm based fuzzy logic (MTL).

The approach followed in this paper is related to the one developed in [16] for the case of finite-valued Łukasiewicz logics and the one in [17] for the case of the logic $G_{\sim}$, i.e. Gödel logic expanded with an involutive negation, already

[1] A t-norm $*$ is a binary operation in $[0,1]$ which is commutative, associative, non-decreasing and having 1 as neutral element and 0 as absorbent elements.

mentioned above. Actually, NML is interpretable in $G_\sim$, and the n-valued NM logics NML_n are interpretable in the n-valued Łukasiewicz logics $Ł_n$, since for instance Baaz-Monteiro's projection operator Δ is definable both in $G_\sim$ (by letting $\Delta\varphi := \neg{\sim}\varphi$) and in $Ł_n$ for each n, while it is neither definable in NML nor in NML_n. Also in a related approach, more recently, Esteva et al. [21] have considered the paraconsistent degree-preserving logics of distributive involutive residuated lattices expanded with a consistency operator $\circ$ in order to get logics of formal inconsistency (LFIs) in the sense of [18, 11], and in particular the cases of the subvarieties of Nelson lattices and of NM-algebras are explored.

More specifically, the outline of this paper is as follows. After this introduction, in Section 2 we provide the needed logic preliminaries about NML itself and the variety of NM-algebras, as well as the basic definitions and notations of logics defined by a given NM-algebra with a lattice filter. In Section 3 we focus on the logics defined by matrices over the standard NM-algebra $[\mathbf{0}, \mathbf{1}]_{\mathbf{NM}}$ and with a lattice filters F of this algebra and we show that they basically lead to only four different logics: the truth-preserving logic NML when $F = \{1\}$, the well-known 3-valued Łukasiewicz logic $Ł_3$ when $F = (1/2, 1]$, the also well-known 3-valued paraconsistent logic J_3 when $F = [1/2, 1]$, and the non-falsity preserving companion of NML, nf-NML, when $F = (0, 1]$. We present general axiomatisations and completeness results. In Section 4, we study the expansion of the paraconsistent *nf*-NML with a consistency operator $\circ$. Section 5 generalises the results of Sections 3 and 4 to the case of logics defined by matrices over general NM-chains with lattice filters. Section 6 is devoted to the full study and characterisation of the logics defined by matrices over finite products of finite NM-chains, and moreover, among them, the maximal paraconsistent ones are identified. Finally, we conclude in Section 7 with some final remarks and prospects for future research.

2 Preliminaries: NM logic and some of its sublogics defined by matrices with lattice filters

The *nilpotent minimum logic*, NML for short, was firstly introduced by Esteva and Godo in [23] in order to formalize the logic of the nilpotent minimum t-norm, defined by Fodor in [25] as an example of an involutive left continuous t-norm which is not continuous.[2]

The language of NML consists of countably many propositional variables $p_1, p_2, \ldots$, binary connectives $\wedge, *, \to$, and the truth constant $\bot$. Formulas, which will be denoted by lower case greek letters $\varphi, \psi, \chi, \ldots$, are recursively defined from propositional variables, connectives and truth-constant as usual. Further definable connectives and constants are as follows: $\neg\varphi$ stands for $\varphi \to \bot$, $\varphi \vee \psi$ stands for $\neg(\neg\varphi \wedge \neg\psi)$, and $\top$ stands for $\neg\bot$.

[2] Actually, Pei showed later in [41] that NML and NM-algebras are equivalent to Wang's $\mathcal{L}^*$ logic and R_0-algebras, respectively [48, 49].

NML is obtained from the monoidal t-norm logic MTL introduced also in [23], by adding the involutive condition axiom

$$\text{(INV)} \quad \neg\neg\varphi \to \varphi$$

and the (weak) nilpotent minimum condition axiom

$$\text{(WNM)} \quad (\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi).$$

It is worth observing that NML enjoys the following form of deduction theorem: $\Gamma \cup \{\varphi\} \vdash_{NM} \psi$ iff $\Gamma \vdash_{NM} \varphi \to (\varphi \to \psi)$. It is well known that NML is algebraizable and the class $\mathbb{NM}$ of all nilpotent minimum algebras is its equivalent algebraic quasivariety semantics [23].

A *nilpotent minimum algebra* (*NM-algebra*) $\mathbf{A} = \langle A, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$, is an involutive MTL-algebra (i.e. a bounded, commutative, integral, involutive, prelinear residuated lattice) that satisfies the following equation

$$\text{(WNM)} \quad (x * y \to \mathbf{0}) \vee (x \wedge y \to x * y) \approx \mathbf{1}.$$

We say that an NM-algebra is an NM-chain provided that its underlying lattice order (defined as $x \leq y$ if $x \to y = \mathbf{1}$) is total. Since the class $\mathbb{NM}$ of all NM-algebras is a proper subvariety of MTL-algebras it inherits the subdirect representation of MTL-algebras, and thus each NM-algebra is representable as a subdirect product of NM-chains (see [23, Proposition 3]).

NM-chains can be easily characterised. Namely, given a bounded totally ordered set $(A, \leq)$, with upper bound $\mathbf{1}$ and lower bound $\mathbf{0}$, equipped with an involutive negation $\neg$ dually order preserving, denoting by $\wedge$ and $\vee$ the meet and join in $(A, \leq)$, and defining for every $a, b \in A$,

$$a * b = \begin{cases} \mathbf{0}, & \text{if } b \leq \neg a \\ a \wedge b, & \text{otherwise} \end{cases} \quad \text{and} \quad a \to b = \begin{cases} \mathbf{1}, & \text{if } a \leq b \\ \neg a \vee b, & \text{otherwise} \end{cases},$$

it follows that $\mathbf{A} = \langle A, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ is an NM-chain. And moreover, every NM-chain is of this form.

From the standard completeness theorem for NM in [23], it follows that the variety of NM-algebras $\mathbb{NM}$ is generated by the *canonical standard* NM-chain

$$[\mathbf{0}, \mathbf{1}]_{\mathbf{NM}} = \langle [0,1], *, \to, \wedge, \vee, 0, 1\rangle$$

where the above operations boil down to:

$$a * b = \begin{cases} \mathbf{0}, & \text{if } b \leq 1 - a \\ \min\{a, b\}, & \text{otherwise} \end{cases}, \quad a \to b = \begin{cases} \mathbf{1}, & \text{if } a \leq b \\ \max\{1 - a, b\} & \text{otherwise} \end{cases}.$$

As for finite NM-chains, we define the *canonical* $(2n{+}1)$- and $2n$-element NM-chains respectively as follows:

$$\mathbf{NM}_{2n+1} = \langle [-n, n] \cap \mathbb{Z}, *, \to, \wedge, \vee, -n, n\rangle, \text{ for every } n \geq 0, \text{ and}$$
$$\mathbf{NM}_{2n} = \langle NM_{2n+1} \smallsetminus \{0\}, *, \to, \wedge, \vee, -n, n\rangle, \text{ for every } n > 0.$$

Notice that $\mathbf{NM}_1$ is the trivial algebra, $\mathbf{NM}_2$ the 2-element Boolean algebra, and $\mathbf{NM}_3$ the 3-element MV-algebra. Furthermore, every numerable NM-chain is embeddable into $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$. For the finite NM-chain $\mathbf{NM}_n$, sometimes we will also use the set $\{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ as the universe of $\mathbf{NM}_n$ as a subalgebra of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$.

From the above it follows that:

(1) All the NM-algebras over $[0,1]$ are isomorphic, since all the involutive order-reversing mappings $n : [0,1] \to [0,1]$ are in turn isomorphic due to a result by Trillas [47].

(2) Also, up to isomorphism, for each $n \in \mathbb{N} \setminus \{0\}$, there is only one NM-chain $\mathbf{NM}_n$ with exactly n elements.

Given an NM-algebra $\mathbf{A}$, we recall that $a \in A$ is a *negation fixpoint* (or just *fixpoint*, for short) if, and only if, $\neg a = a$. Any NM-algebra has at most one fixpoint [33]. Clearly, both the algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ and the algebras $\mathbf{NM}_{2n+1}$, for any n, have fixpoint, while the algebras $\mathbf{NM}_{2n}$ have not. It is easy to see that if $\mathbf{A}$ is an NM-chain with a negation fixpoint $a \in A$ then $A \setminus \{a\}$ is the universe of a NM-subalgebra of $\mathbf{A}$, which we denote by $\mathbf{A}^-$. Notice that $\mathbf{NM}_{2n} = \mathbf{NM}^-_{2n+1}$.

Notation: Given an NM-algebra $\mathbf{A}$ and a lattice filter $F \subseteq A$,[3] the pair $\mathcal{M} = \langle \mathbf{A}, F \rangle$ is called a *logical matrix* and induces a logic, denoted by $\models_{\mathcal{M}}$, that is defined as follows: for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_{\mathcal{M}} \varphi \text{ if, for any } \mathbf{A}\text{-evaluation } e,\ e(\psi) \in F \text{ for all } \psi \in \Gamma \text{ implies } e(\varphi) \in F.$$

The lattice filter F plays the role of set of *designated values* for the logic $\models_{\mathcal{M}}$. Given a set of matrices $\mathcal{K} = \{\mathcal{M}_i\}_{i \in I}$, the logic induced by $\mathcal{K}$ is the intersection of the family of logics $\{\models_{\mathcal{M}_i}\}_{i \in I}$. Moreover, a matrix $\mathcal{M}' = \langle \mathbf{B}, G \rangle$ is a *submatrix* of another matrix $\mathcal{M} = \langle \mathbf{A}, F \rangle$ if $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $G = F \cap B$, and in that case, $\models_{\mathcal{M}} \subseteq \models_{\mathcal{M}'}$.[4] Finally, we recall that the logic $\models_{\mathcal{M}}$ is called *explosive* when from a pair of contradictory formulas everything follows, i.e. for every φ, ψ it holds that $\{\varphi, \neg\varphi\} \models_{\mathcal{M}} \psi$. Otherwise, the logic $\models_{\mathcal{M}}$ is called *paraconsistent*.

For any NM-chain $\mathbf{A}$ and for every $a \in A \setminus \{\mathbf{0}\}$, consider the lattice filters $F_a = \{x \in A \mid a \leq x\}$ and $F_{(a} = \{x \in A \mid a < x\}$. Then, the *finitary* logics corresponding to the matrices $\langle \mathbf{A}, F_a \rangle$ and $\langle \mathbf{A}, F_{(a} \rangle$, denoted $\vdash_a^A$ and $\vdash_{(a}^A$ respectively, are defined as follows.

Definition 1. For any finite set of formulas $\Gamma \cup \{\varphi\}$, we define:

- $\Gamma \vdash_a^A \varphi$ if, for any $\mathbf{A}$-evaluation e, if $e(\psi) \geq a$ for all $\psi \in \Gamma$, then $e(\varphi) \geq a$.
- $\Gamma \vdash_{(a}^A \varphi$ if, for any $\mathbf{A}$-evaluation e, if $e(\psi) > a$ for all $\psi \in \Gamma$, then $e(\varphi) > a$.

[3] F is a lattice filter of $\mathbf{A}$ if it is a non-empty upset of A closed by $\wedge$.

[4] For a modern algebraic treatment of logical matrices see e.g. [13].

As costumary, we extend these definitions for arbitrary sets of formulas $\Gamma \cup \{\varphi\}$ by stipulating that $\Gamma \vdash \varphi$, for $\vdash \in \{\vdash^A_a, \vdash^A_{(a}\}$, whenever there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.

Notice that if $\mathbf{A}$ is finite, then any matrix logic $\models_{\langle \mathbf{A}, F\rangle}$ is finitary and thus, in particular, $\models_{\langle \mathbf{A}, F_a\rangle}$ and $\models_{\langle \mathbf{A}, F_{(a}\rangle}$ coincide with $\vdash^A_a$ and $\vdash^A_{(a}$ respectively.

It is very easy to check that $\vdash^A_a$ is paraconsistent iff $a \leq \neg a$, while $\vdash^A_{(a}$ is paraconsistent iff $a < \neg a$.

At this point, let us recall three well-known particular cases of such logics, where with $\mathbf{NM}_3$ we denote the three element NM-chain (over the carrier $\{0, 1/2, 1\}$):

- By the standard completeness of NML, the logic of the matrix $\langle [\mathbf{0}, \mathbf{1}]_{\mathbf{NM}}, \{1\}\rangle$ coincides with $\vdash^{[0,1]}_1$ and with the logic NM itself.
- The logic of the matrix $\langle \mathbf{NM}_3, \{1\}\rangle$ coincides with the 3-valued Łukasiewicz logic $Ł_3$, since in fact the chain $\mathbf{NM}_3$ is term-equivalent to the 3-element MV-algebra $\mathbf{MV}_3$.
- The (paraconsistent) logic of the matrix $\langle \mathbf{NM}_3, \{1/2, 1\}\rangle$ coincides (up to language) with D'Ottaviano and da Costa's three-valued logic J_3 [19].

For a given a NM-chain $\mathbf{A}$, we can also consider its corresponding finitary *degree-preserving logic* $\vdash^{\leq}_A$ as defined next, following [6].

Definition 2. (c.f. [6]) For any finite set of formulas $\Gamma \cup \{\varphi\}$, we define $\Gamma \vdash^{\leq}_A \varphi$ if, for any $\mathbf{A}$-evaluation e, and for all $a \in A$, if $e(\psi) \geq a$ for all $\psi \in \Gamma$, then $e(\varphi) \geq a$. In other words, $\Gamma \vdash^{\leq}_A \varphi$ if, for any $\mathbf{A}$-evaluation e, $\inf\{e(\psi) \mid \psi \in \Gamma\} \leq e(\varphi)$.

Moreover, if $\mathbb{V}$ is a variety of NM-algebras, one can define its corresponding degree-preserving logic $\vdash^{\leq}_{\mathbb{V}}$ by stipulating $\Gamma \vdash^{\leq}_{\mathbb{V}} \varphi$ whenever $\Gamma \vdash^{\leq}_A \varphi$ for every chain $\mathbf{A} \in \mathbb{V}$. Finally, we extend the above definitions of $\vdash^{\leq}_A$ and $\vdash^{\leq}_{\mathbb{V}}$ for an arbitrary set of premises Γ as in Definition 1.

It is easy to check that $\vdash^{\leq}_A$ is indeed the intersection of all the finitary matrix logics $\langle \mathbf{A}, F_a\rangle$ for all $a \in A$, namely, $\Gamma \vdash^{\leq}_A \varphi$ iff $\Gamma \vdash^A_a \varphi$ holds for any $a \in A$. It also directly follows that $\vdash^{\leq}_A$ is paraconsistent.

As a matter of fact, the logic $\vdash^{\leq}_A$ is strongly related to the 1-preserving logic $\vdash^A_1$. Indeed, on the one hand, it holds that $\vdash^{\leq}_A \varphi$ iff $\vdash^A_1 \varphi$, so both logics share the set of valid formulas. Moreover, if for any finite set of formulas Γ we let $\Gamma^\wedge = \wedge\{\psi \mid \psi \in \Gamma\}$, we can observe that

$$\Gamma \vdash^{\leq}_A \varphi \text{ iff } \vdash^A_1 \Gamma^\wedge \to \varphi,$$

and hence, iff $\vdash^{\leq}_A \Gamma^\wedge \to \varphi$. This property can be seen as a sort of deduction theorem for $\vdash^{\leq}_A$. Furthermore, since the variety $\mathbb{NM}$ is generated by the standard NM-algebra $[\mathbf{0}, \mathbf{1}]_{\mathbf{NM}}$, it also follows that $\vdash^{\leq}_{\mathbb{NM}} = \vdash^{\leq}_{[0,1]_{NM}}$.

It has been shown in [6] that in the case the logic $\vdash_1^A$ has a complete axiomatisation with Modus Ponens as the only inference rule, then the logic $\vdash_A^{\leq}$ admits a complete axiomatisation as well, having as axioms the axioms of $\vdash_1^A$ and as inference rules the rule of adjunction:

$$\text{(Adj)} \quad \frac{\varphi, \quad \psi}{\varphi \wedge \psi},$$

and the following restricted form of the Modus Ponens rule

$$\text{(r-MP)} \quad \frac{\varphi, \quad \varphi \to \psi}{\psi}, \quad \text{if } \vdash_1^A \varphi \to \psi.$$

If the logic $\vdash_1^A$ has additional inference rules

$$(\mathrm{R}_i) \quad \frac{\Gamma_i}{\varphi}$$

for $i \in I$, then [20, Proposition 1] shows that $\vdash_A^{\leq}$ is axiomatised with the above axioms and rules together with the following restricted forms of the rules (R_i):

$$(\text{r-R}_i) \quad \frac{\Gamma_i}{\varphi}, \quad \text{if } \vdash_1^A \Gamma_i.$$

Finally, let us consider the subalgebra $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$ of the standard NM-algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$, where $[0,1]^-_{NM} = [0,1] \setminus \{1/2\}$. We recall that the logic defined by the matrix $\langle [0,1]^-_{NM}, \{1\}\rangle$ can be syntactically characterised as the axiomatic extension of NML with the following axiom [29]:

$$(BP) \quad \neg(\neg\varphi^2)^2 \leftrightarrow (\neg(\neg\varphi)^2)^2,$$

where φ^2 is a shorthand for $\varphi * \varphi$. We will call this axiomatic extension NML$^-$ and its corresponding variety of algebras $\mathbb{NM}^-$, which is generated by the algebra $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$. Actually, in the frame of NML, the above axiom can be simplified and equivalently expressed as

$$(BP) \quad \neg((\varphi \leftrightarrow \neg\varphi)^2).$$

We will assume this form when referring to the axiom (BP) in the rest of the paper. Note that the axiom (BP) is not only valid in $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$ but also in any NM-chain without fixpoint. Even more, a NM-chain validates (BP) if, and only if, the chain has no negation fixpoint [29].

3 Logics defined by matrices over $[0,1]_{\mathbf{NM}}$: completeness results

In this section we pay attention to the the matrix logics $\vdash_a^A$ and $\vdash_{(a}^A$ introduced in the last section in the particular case $\mathbf{A} = [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$, that is, to the logics

$\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_a\rangle$ for any $a \in (0,1]$ and $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_{(a}\rangle$ for any $a \in [0,1)$. Among all these logics we will show that there are only four different logics, two explosive and two paraconsistent.

For the sake of a simpler notation, in what follows we will omit the superscript A and will simply write $\vdash_a$, $\vdash_{(a}$ without danger of confusion. Moreover, we will also use the notation $\vdash_{\leq}$ instead of $\vdash^{\leq}_{[0,1]_{NM}}$.

Proposition 1. *For any $a \in [0,1]$, the logics $\vdash_a$ and $\vdash_{(a}$ respectively defined by the matrices $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_a\rangle$ and $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_{(a}\rangle$ satisfy the following properties:*

1. *$\vdash_a$, $\vdash_{(a}$ and $\vdash_1$ are the same logic for all $a \in (1/2,1)$,*
2. *$\vdash_a$, $\vdash_{(a}$ and $\vdash_{(0}$ are the same logic for all $a \in (0,1/2)$,*
3. *$\vdash_{(1/2}$ and $\vdash_{L_3}$ are the same logic,*
4. *$\vdash_{1/2}$ and $\vdash_{J_3}$ are the same logic,*
5. *$\vdash_1 \subsetneq \vdash_{(1/2}$,*
6. *$\vdash_{(0} \subsetneq \vdash_{1/2}$, yet $\vdash_{(0} \varphi$ iff $\vdash_{1/2} \varphi$,*
7. *$\vdash_{(1/2}$ and $\vdash_{1/2}$ are not comparable,*
8. *$\vdash_1$ and $\vdash_{1/2}$ are not comparable,*
9. *$\vdash_{(1/2}$ and $\vdash_{(0}$ are not comparable,*
10. *$\vdash_1$ and $\vdash_{(0}$ are not comparable.*

Proof. Property 1: Let $a \in (1/2,1)$. Assume $\{\varphi_i \mid i \in I\} \nvdash_1 \psi$, then there is an evaluation e such that $e(\varphi_i) = 1$ and $e(\psi) \neq 1$. Then the map $h : [0,1] \to [0,1]$ such that

$$h(x) = \begin{cases} 1, & \text{if } x = 1; \\ (2a-1)x + 1 - a, & \text{if } 0 < x < 1; \\ 0, & \text{if } x = 0. \end{cases}$$

is a homomorphism and $h \circ e$ is an evaluation such that $h \circ e(\varphi_i) = 1 > a$ and $h \circ e(\psi) < a$. Thus $\{\varphi_i \mid i \in I\} \nvdash_a \psi$ and $\{\varphi_i \mid i \in I\} \nvdash_{(a} \psi$.

If $\{\varphi_i \mid i \in I\} \nvdash_a \psi$, then there is an evaluation e such that $e(\varphi_i) \geq a$ and $e(\psi) = d < a$. Then the map $g : [0,1] \to [0,1]$ such that

$$g(x) = \begin{cases} 1, & \text{if } x \geq a; \\ x, & \text{if } 1 - a < x < a; \\ 0, & \text{if } x \leq 1 - a. \end{cases}$$

is a homomorphism and $g \circ e$ is an evaluation such that $g \circ e(\varphi_i) = 1$ and $g \circ e(\psi) < 1$. Thus $\{\varphi_i \mid i \in I\} \nvdash_1 \psi$.

If $\{\varphi_i \mid i \in I\} \nvdash_{(a} \psi$, then using the map $f : [0,1] \to [0,1]$ such that

$$f(x) = \begin{cases} 1, & \text{if } x > a; \\ x, & \text{if } 1 - a \leq x \leq a; \\ 0, & \text{if } x < 1 - a. \end{cases}$$

it follows that $\{\varphi_i \mid i \in I\} \not\vdash_1 \psi$.

Property 2 is proved analogously as Property 1 with the homomorphisms

$$h'(x) = \begin{cases} 1, & \text{if } x = 1; \\ (1-2a)x + a, & \text{if } 0 < x < 1; \\ 0, & \text{if } x = 0. \end{cases} \qquad g'(x) = \begin{cases} 1, & \text{if } x > 1-a; \\ x, & \text{if } a \leq x \leq 1-a; \\ 0, & \text{if } x < a. \end{cases}$$

$$\text{and} \qquad f'(x) = \begin{cases} 1, & \text{if } x \geq 1-a; \\ x, & \text{if } a < x < 1-a; \\ 0, & \text{if } x \leq a. \end{cases}$$

Property 3. Recall that by the completeness theorem of the 3-valued Łukasiewicz logic $Ł_3$ is complete with respect the matrix logic $\langle \mathbf{MV_3}, \{1\}\rangle = \langle \mathbf{NM_3}, \{1\}\rangle$. Since $\langle \mathbf{NM_3}, \{1\}\rangle$ is a submatrix of $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_{1/2}\rangle$ then $\{\varphi_i \mid i \in I\} \vdash_{1/2} \psi$ implies $\{\varphi_i \mid i \in I\} \vdash_{Ł_3} \psi$. Moreover since the map $h : [0,1] \to \{0, 1/2, 1\}$ defined by

$$h(x) = \begin{cases} 1, & \text{if } x > 1/2; \\ 1/2, & \text{if } x = 1/2; \\ 0, & \text{if } x < 1/2, \end{cases}$$

is a homorphism such that $h(F_{(1/2)} = \{1\}$, then $\{\varphi_i \mid i \in I\} \vdash_{Ł_3} \psi$ implies $\{\varphi_i \mid i \in I\} \vdash_{1/2} \psi$.

Property 4: Recall that J_3 is the logic of the matrix $\mathcal{J}_3 = \langle \mathbf{MV_3}, \{1/2, 1\}\rangle = \langle \mathbf{NM_3}, \{1/2, 1\}\rangle$. Since $\langle \mathbf{NM_3}, \{1/2, 1\}\rangle$ is a submatrix of $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_{1/2}\rangle$ then $\{\varphi_i \mid i \in I\} \vdash_{1/2} \psi$ implies $\{\varphi_i \mid i \in I\} \vdash_{J_3} \psi$. Moreover since the map $h : [0,1] \to \{0, 1/2, 1\}$ defined by

$$h(x) = \begin{cases} 1, & \text{if } x > 1/2; \\ 1/2, & \text{if } x = 1/2; \\ 0, & \text{if } x < 1/2, \end{cases}$$

is an onto homorphism such that $h(F_{1/2}) = \{1/2, 1\}$, then $\{\varphi_i \mid i \in I\} \vdash_{J_3} \psi$ implies $\{\varphi_i \mid i \in I\} \vdash_{1/2} \psi$.

Property 5 is a consequence of Property 3, since $Ł_3$ is a proper axiomatic extension of NML.

Property 6: Since $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_{1/2}\rangle$ coincides with the logic J_3 (Property 4), and the matrix $\mathcal{J}_3$ is a submatrix of $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_{(0}\rangle$, then $\vdash_{(0} \subseteq \vdash_{1/2}$. To prove that they do not define the same logic, it is enough to see that $p, q \vdash_{1/2} (\neg p \to p) * q$, while $p, q \not\vdash_{(0} (\neg p \to p) * q$.

Clearly, if $\vdash_{1/2} \varphi$ then $\vdash_{(0} \varphi$. For the converse direction, suppose $\not\vdash_{1/2} \varphi$. Then there exists e such that $e(\varphi) < 1/2$. Consider again the homomorphism

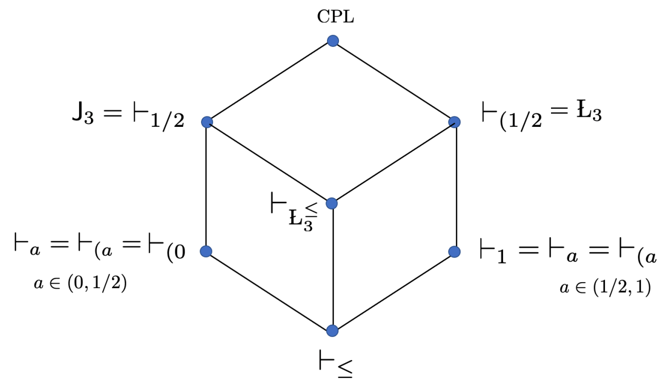

Figure 1: The lattice of the four different logics in Proposition 1 and their relation to classical propositional logic CPL and to the degree-preserving companions of $Ł_3$ and NML.

$h : [0, 1] \to \{0, 1/2, 1\}$ defined by

$$h(x) = \begin{cases} 1, & \text{if } x > 1/2; \\ 1/2, & \text{if } x = 1/2; \\ 0, & \text{if } x < 1/2, \end{cases}$$

Then the evaluation $e' = h \circ e$ is such that $e'(\varphi) = 0$. Hence $\not\vdash_{(0} \varphi$.

Finally, by Properties 5 and 6, Properties 7 to 10 hold as well because $\varphi, \neg\varphi \vdash_1 \bot$ while $\varphi, \neg\varphi \not\vdash_{\mathsf{J}_3} \bot$, and $\vdash_{(0} \varphi \vee \neg\varphi$ while $\not\vdash_{Ł_3} \varphi \vee \neg\varphi$. □

In the previous section, we have observed that the degree-preserving companion $\vdash^{\leq}_{\mathbb{NM}}$ of the logic NM coincides with $\vdash^{\leq}_{[0,1]_{NM}}$, that we simply denote now $\vdash_{\leq}$, that in turn coincides with the intersection of the logics $\vdash_a$, for all $a \in (0, 1]$, that is, $\vdash^{\leq}_{\mathbb{NM}} = \bigcap_{a>0} \vdash_a$. Now, as a consequence of Proposition 1, this intersection can be significantly simplified.

Lemma 1. $\vdash_{\leq} = \vdash_1 \cap \vdash_{(0}$.

In Figure 1 there is a graphical representation of the different logics involved above, where CPL denotes classical propositional logic.

Next lemma is a key observation that, thanks to the involutivity of the NM negation, tightly relates both logics $\vdash_1$ and $\vdash_{(0}$ through the negation connective.

Lemma 2. *For every formula φ,*

$$\psi \vdash_{(0} \varphi \textit{ if, and only if, } \neg\varphi \vdash_1 \neg\psi.$$

In particular, $\vdash_{(0} \varphi$ if, and only if, $\vdash_1 \neg(\neg\varphi)^2$.

Proof. By definition, $\psi \vdash_1 \varphi$ iff for every $[\mathbf{0},\mathbf{1}]$-evaluation e, if $e(\psi) = 1$ then $e(\varphi) = 1$; that is, if $e(\varphi) < 1$ then $e(\psi) < 1$, for all e; that is, $e(\neg\varphi) > 0$ then $e(\neg\psi) > 0$, for all e; iff $\neg\varphi \vdash_{(0} \neg\psi$.

Therefore, $\top \vdash_{(0} \varphi$ iff $\neg\varphi \vdash_1 \bot$, and by the deduction theorem for NML, this holds iff $\vdash_1 (\neg\varphi)^2 \to \bot$, that is, $\vdash_1 \neg(\neg\varphi)^2$. □

Corollary 1. *For every formulas $\psi_1, \ldots, \psi_n, \varphi$,*

$$\psi_1, \ldots, \psi_n \vdash_{(0} \varphi \textit{ if, and only if, } \vdash_1 (\psi_1 \wedge \ldots \wedge \psi_n) \to \neg(\neg\varphi)^2.$$

Proof. The following chain of equivalences hold: $\psi_1, \ldots, \psi_n \vdash_{(0} \varphi$ iff $\psi_1 \wedge \ldots \wedge \psi_n \vdash_{(0} \varphi$ iff $\neg\varphi \vdash_1 \neg(\psi_1 \wedge \ldots \wedge \psi_n)$ iff $\vdash_1 (\neg\varphi)^2 \to \neg(\psi_1 \wedge \ldots \wedge \psi_n)$ iff $\vdash_1 (\psi_1 \wedge \ldots \wedge \psi_n) \to \neg(\neg\varphi)^2$. □

Now we introduce two new inference rules. Consider the following two *restricted* inference rules, which are intended for the logic axiomatising $\vdash_{(0}$:

- (r-MP^2): From φ and $\varphi \to \neg(\neg\psi)^2$ derive ψ, whenever $\vdash_{\mathrm{NML}} \varphi \to \neg(\neg\psi)^2$,
- (r-MP): From φ and $\varphi \to \psi$ derive ψ, whenever $\vdash_{NML} \varphi \to \psi$

Note that both inference rules involve conditions on the derivability of formulas in the logic NM.

Proposition 2. *The rule (r-MP^2) is sound for $\vdash_{(0}$ and the Restricted Modus Ponens rule (r-MP) is derivable from (r-MP^2).*

Proof. As for the soundness of (r-MP^2), let e be an $[\mathbf{0},\mathbf{1}]$-evaluation, and assume $e(\varphi) > 0$ and that $e(\varphi \to \neg(\neg\psi)^2) = 1$, where the latter clearly holds iff $e(\varphi) \leq e(\neg(\neg\psi)^2)$. Therefore $0 < e(\neg(\neg\psi)^2)$. Hence $e((\neg\psi)^2) < 1$. Now suppose $e(\psi) = 0$, then it would be $e((\neg\psi)^2) = 1$, contradiction. Therefore it has to be $e(\psi) > 0$.

The derivability of (r-MP) follows from the fact that $\psi \to \neg(\neg\psi)^2$ is a theorem of NML. Therefore, from φ and $\vdash_{NML} \varphi \to \psi$, we also have $\vdash_{NML} \varphi \to \neg(\neg\psi)^2$, and by applying (r-MP^2), we finally get ψ. □

Definition 3. The *non-falsity preserving companion* of NML, denoted nf-NML, is the logic defined by the following axioms and rules:

- Axioms: those of NML
- Rules: Adjunction and (r-MP^2) .

Next theorem proves that nf-NML defined above syntactically captures the logic of the matrix $\langle[\mathbf{0},\mathbf{1}], F_{(0}\rangle$.

Theorem 1. nf-NML *is a sound and complete axiomatisation of* $\vdash_{(\mathbf{0}}$.

Proof. Soundness follows from the fact that the Adjunction and (r-MP^2) rules are sound as proved above.

As for completeness, suppose $\psi_1, \ldots, \psi_n \vdash_{(0} \varphi$ (semantically). This is equivalent to $\vdash_1 (\psi_1 \wedge \ldots \wedge \psi_n) \to \neg(\neg\varphi)^2$. By completeness of NML, there is a proof $\langle \Pi_1, \ldots \Pi_r \rangle$, where $\Pi_r = (\psi_1 \wedge \ldots \wedge \psi_n) \to \neg(\neg\varphi)^2$ and where each Π_i is either an axiom of NML, or has been obtained from previous Π_k, Π_j $(k, j < r)$ and the application of Modus ponens rule. Note that all the Π_i's are theorems of NML. Then, in order to get a proof of ψ in nf-NM we only need to do the following:

(i) add a previous step $\Pi_0 = \psi_1 \wedge \ldots \wedge \psi_n$ that is obtained from the premises by the Adjunction rule,

(ii) add a final step $\Pi_{r+1} = \psi$ that is obtained from Π_0 and Π_r by application of the (r-MP^2) rule.

Therefore, the sequence $\Pi_0, \Pi_1, \ldots \Pi_r, \Pi_{r+1}$ is a proof of ψ in the logic nf-NM with the proviso that the applications of the modus ponens in the proof $\Pi_1, \ldots \Pi_r$ have to be considered as applications of the Restricted Modus ponens rule (r-MP), which we know it is derivable. □

It is interesting to observe that, although the axioms of NML and of nf-NM are the same, the set of theorems of nf-NM is larger than that of NML. It is clear that the excluded-middle axiom $\varphi \vee \neg\varphi$ is not a theorem of NML, but is a theorem of nf-NM. Indeed, $\varphi \vee \neg\varphi$ follows from the application of rule (r-MP^2) by taking $\varphi := \top$ and $\psi := \varphi \vee \neg\varphi$, since $\vdash_{NM} \neg(\neg(\varphi \vee \neg\varphi))^2$.

As a consequence of the above completeness theorem, for the different logics appearing in Proposition 1 and their intersections we have the axiomatisations given in Table 1. In this table we use (V3) to refer to the following axiom

$$(V3) \quad (\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_3) \vee (\varphi_3 \to \varphi_4)$$

that forces the chains of the corresponding variety to be of cardinal less or equal to 3.

Logics	Matrix	Axioms	Inference Rules
NML: $\vdash_1$	$\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, \{1\} \rangle$	NM	MP
$Ł_3$	$\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, (1/2, 1] \rangle$	NM + (V3)	MP
J_3	$\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, [1/2, 1] \rangle$	NM + (V3)	Adj, $MP^*_{Ł_3}$: $\frac{\varphi, \quad \vdash_{Ł_3} \varphi \to \neg(\neg\psi)^2}{\psi}$
nf-NML: $\vdash_{(0}$	$\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, (0, 1] \rangle$	NM	Adj, MP^* : $\frac{\varphi, \quad \vdash_{\mathbf{1}} \varphi \to \neg(\neg\psi)^2}{\psi}$
$Ł_3 \cap J_3 = Ł_3^{\leq}$		NM + (V3)	Adj, r-MP : $\frac{\varphi, \quad \vdash_{Ł_3} \varphi \to \psi}{\psi}$
$\vdash_1 \cap \vdash_{(0} = \vdash_{\leq}$		NM	Adj, r-MP : $\frac{\varphi, \quad \vdash_{\mathbf{1}} \varphi \to \psi}{\psi}$

Table 1: Axiomatisations of the logics appearing in Proposition 1 defined by matrices over $[0, 1]_{NM}$ by a lattice filter.

4 Expanding the paraconsistent logic nf-NM with a consistency operator $\circ$

As mentioned in the Introduction, paraconsistent logics and fuzzy logics are conceptually related, although not all the systems of MFL are paraconsistent. In the case of NML, paraconsistency can be obtained by replacing the truth-preserving consequence relation by the degree-preserving one, or by consequence relations defined by matrices with suitable lattice filters as designated values.

Let us briefly recall some notions from paraconsistency. A logic has an *explosive* negation $\neg$ when any formula can be derived from a contradiction $\{\varphi, \neg\varphi\}$.[5] A logic L is paraconsistent (w.r.t. $\neg$) when $\neg$ is a non explosive negation, meaning that there are ($\neg$-)contradictory but non-trivial theories in L. Among the plethora of paraconsistent logics proposed in the literature, one of the best behaved families of paraconsistent logics are the so-called *Logics of Formal Inconsistency* (in short LFIs, see for instance [11] and [10]). The idea behind LFIs is that explosiveness can be locally recovered by means of a (primitive or defined) unary connective $\circ$, in the following sense: in spite of having formulas φ and ψ such that ψ does not follow from $\{\varphi, \neg\varphi\}$ (given that $\neg$ is a paraconsistent negation), the set $\{\varphi, \neg\varphi, \circ\varphi\}$ is always logically trivial (or explosive). Within this context, the connective $\circ$ is called a *consistency* (or *recovery*, or *classicality*) operator. LFIs generalize the well-known hierarchy of da Costa's paraconsistent logics C_n introduced in 1963, in which the calculus C_n at level n has a defined consistency operator $\circ_n$ which 'tolerates' n degrees of contradiction (see [18]).

The non-falsity preserving logic nf-NM introduced in the previous section is a paraconsistent logic, but it is not an LFI, that is, it can be shown that a consistent operator in the above sense is not definable (see Proposition 3 below). Therefore, in this section we study the expansion nf-NM with a proper consistency operator $\circ$ so that the resulting logic is an LFI.

We start with some basic definitions and algebraic considerations about the $\circ$ operators before going into more details in the rest of the section.

4.1 Preliminary definitions and some algebraic considerations

Let us first recall the definition of Logics of Formal Inconsistency.

Definition 4. ([11, 10]) Let L be a logic defined in a language containing a negation $\neg$ and a unary operator $\circ$, and whose deduction system is denoted by $\vdash$. L is a Logic of Formal Inconsistency (with respect to $\neg$ and $\circ$) if the following conditions hold:

(i) $\varphi, \neg\varphi \nvdash \psi$, for some formulas φ, ψ, i.e. L is not explosive w.r.t. $\neg$;

[5] By the way, it may be observed that explosiveness is a basic feature of negation in many logics.

(ii) $\circ\varphi, \varphi \nvdash \psi$, for some formulas φ, ψ;

(iii) $\circ\varphi, \neg\varphi \nvdash \psi$, for some formulas φ, ψ; and

(iv) $\varphi, \neg\varphi, \circ\varphi \vdash \bot$, for every formula φ.

Condition (i) states that for L to be an LFI, it must be first a paraconsistent logic with respect to the negation $\neg$, namely: not every theory containing $\{\varphi, \neg\varphi\}$ is logically trivial. In addition, conditions (ii)-(iv) describe the properties a consistency operator should satisfy, namely: not every theory containing $\{\circ\varphi, \varphi\}$ is logically trivial; analogously, not every theory containing $\{\circ\varphi, \neg\varphi\}$ is logically trivial; however, any theory containing $\{\circ\varphi, \varphi, \neg\varphi\}$ is logically trivial. This means that, in an LFI, trivialization always occurs when the three formulas $\circ\varphi$, φ and $\neg\varphi$ are placed together in a theory, but the presence of only two of them does not guarantee logical trivialization.

A consistency operator in a LFI logic can be primitive or it can be defined from other connectives of the language. For instance, in the well-known system C_1 of da Costa, consistency is defined by the formula $\circ\varphi = \neg(\varphi \wedge \neg\varphi)$, while in C_n (for $n \geq 2$) a formula $\circ_n\varphi$ obtained by iterating $\circ\varphi$ in a suitable way expresses consistency (see [18]). Another example of an LFI logic is the case of the degree-preserving companion of Gödel logic with an involutive negation $G_\sim$ (see [17]), where the Baaz-Monteiro Δ operator is definable ($\Delta\varphi = \neg \sim \varphi$), and consistency is defined by the formula $\circ\varphi = \Delta(\varphi \vee \neg\varphi)$. In fact, it is known [23] that the logic $G_\sim$ is equivalent to the expansion of the NM logic with Δ, NM_Δ, and hence a consistency operator in the degree-preserving companion of NM_Δ keeps being definable as $\circ\varphi = \Delta(\varphi \vee \neg\varphi)$.

Let us now turn our attention to $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$. Observe that, among the logics depicted in Fig. 1 defined by matrices $\langle[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F\rangle$, where F is a lattice filter, the only paraconsistent ones are J_3, nf-NM $= \vdash_{(\mathbf{0}}$, $Ł_3^{\leq} = \mathsf{J}_3 \cap Ł_3$ and $NM^{\leq} = NM \cap \mathsf{nf}\text{-}NM$. Moreover, the following result can be obtained.

Proposition 3. *The logic nf-NM $= \vdash_{(\mathbf{0}}$, defined by the matrix $\mathcal{M} = \langle[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, (0,1]\rangle$, is not an LFI.*

Proof. Assume $\circ$ is definable in NML in such a way that $\models_{\mathcal{M}}$ is an LFI, hence the conditions (ii)-(iv) in the definition above are satisfied. Since the 2-element Boolean algebra $\mathbf{2}$ over $\{0,1\}$ is a subalgebra of any NM-chain, if $\circ$ were definable in NML (by a unary term), the only consistency operator that could be definable would be the one where $\circ(0) = \circ(1) = 1$, since this is the only compatible possibility when restricting $\circ$ to $\mathbf{2}$. Thus $\circ$ satisfies conditions (ii) and (iii). On the other hand, if we want condition (iv) be satisfied, this implies that, for any $x \in [0,1]$ the following condition has to be satisfied:

$$x > 0, \neg x > 0 \text{ implies } \circ(x) = 0. \tag{1}$$

Now, consider the NM-homomorphism $h : [\mathbf{0},\mathbf{1}]_{\mathbf{NM}} \to \mathbf{NM_3}$, where $\mathbf{NM_3}$ is the NM-subalgebra of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ on the set $\{0, 1/2, 1\}$, defined as $h(x) = 1$ if $x > 1/2$, $h(1/2) = 1/2$ and $h(x) = 0$ if $x < 1/2$. Then, since $\circ(x)$ is a term defined over the algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$, it should be that $h(\circ(x)) = \circ(h(x))$ for all $x \in [0,1]$. However, take x such that $1/2 \geq x > 0$, then $\neg x \geq 1/2 > 0$ and thus, by condition (1), $\circ(x) = 0$. Then, by definition of h and since $\circ(x) = 0$, we have $h(\circ(x)) = h(0) = 0$. But this is in contradiction with the fact that $\circ(h(x)) = \circ(0) = 1$. □

Observe that, as a consequence of this proposition, the degree-preserving logic $\vdash_{\leq} = \vdash_1 \cap \vdash_{(0}$ is not an LFI either. On the other hand, as expected, the cases of the logic $\mathsf{J}_3 = \vdash_{1/2}$ and the logic $Ł_3^{\leq} = \mathsf{J}_3 \cap Ł_3$ do not fall in the scope of the proposition. In fact, the term $\circ(x) = x^2 \vee (\neg x)^2$ defines a consistency operator in J_3 and in $Ł_3^{\leq}$, and hence they are LFIs.

Nonetheless, similarly to what has been done in the case of fuzzy logics preserving degrees of truth in [14], we can expand the above paraconsistent but not LFI logics with a suitable consistency operator $\circ$ such that they become an LFI. Actually, as announced, in this section we focus on the case of the logic nf-NM, and our task will be then to study its expansion with a new unary connective $\circ$ so that the resulting logic is an LFI. We will denote by $\mathcal{L}_\circ$ the expansion of the language of NML with $\circ$.

From a semantical point of view, consider a given unary operator $\circ : [0,1] \to [0,1]$,[6] and let us consider the following matrices:

$$\mathcal{M}_\circ^1 = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}^\circ}, \{1\} \rangle \text{ and } \mathcal{M}_\circ^0 = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}^\circ}, (0,1] \rangle,$$

where the algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}^\circ}$ is the expansion of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ with $\circ$. We start by considering the most general semantical conditions on $\circ$ guaranteeing that the logic $\models_{\mathcal{M}_\circ^0}$ is an LFI, in other words, requiring that the following conditions are satisfied:

- $\circ\varphi, \varphi, \neg\varphi \models_{\mathcal{M}_\circ^0} \bot$
- $\varphi, \circ\varphi \not\models_{\mathcal{M}_\circ^0} \bot$
- $\neg\varphi, \circ\varphi \not\models_{\mathcal{M}_\circ^0} \bot$

It immediately follows that these conditions are satisfied if, and only if, in the algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}^\circ}$ the following conditions are in turn satisfied:

- For all $x \in [0,1]$, $x \wedge \neg x \wedge \circ x = 0$,
- There exists $x \in [0,1]$, such that $x \wedge \circ x \neq 0$,
- There exists $x \in [0,1]$, such that $\circ x \wedge \neg x \neq 0$.

It is readily seen that requiring these three conditions amount to require the next three constraints on $\circ$:

(C0) $\circ x = 0$ for all $x \in (0,1)$,

[6] Without danger of confusion, we will use the same symbol $\circ$ to denote the connective and a generic unary operation on the unit real interval $[0,1]$.

(1-NZ) $\circ 1 > 0$,

(0-NZ) $\circ 0 > 0$.

We will call an operator basic when satisfying these conditions.

Definition 5. A unary operator $\circ : [0,1] \to [0,1]$ that satisfies conditions (C0), (1-NZ) and (0-NZ) will be called a *basic* consistency operator.

From conditions (1-NZ) and (0-NZ) above, it is clear that the value of $\circ(0)$ or $\circ(1)$ can be either

– equal to 1,

– a stricly positive element (SP), i.e. strictly greater than $\frac{1}{2}$ and strictly smaller than 1,

– equal to $\frac{1}{2}$, or

– a strictly negative element (SN), i.e. strictly smaller than $\frac{1}{2}$ and strictly greater than 0.

In fact, one cannot distinguish in $[0,1]_{NM}$ the case $\circ(0) = a$ from the case $\circ(0) = b$ if both a and b are SP or SN, because there exist an isomorphism f of $[0,1]_{NM}$ such that $f(a) = b$.

Moreover, it is easy to characterise the above cases by equations and inequations in $[0,1]_{NM}$. The proof is easy and thus it is omitted.

Proposition 4. *For* $\mathsf{b} \in \{0,1\}$*, the following conditions hold:*

[b-1] $\circ(\mathsf{b}) = 1$ *is equivalently characterised by the equation* $\neg(\circ(\mathsf{b})) = 0$*,*

[b-SP] $\circ(\mathsf{b}) \in (1/2, 1)$ *is characterised by the inequation* $(\circ(\mathsf{b}))^2 \wedge \neg(\circ(\mathsf{b})) > 0$*,*

[b-fix] $\circ(\mathsf{b}) = 1/2$ *is characterised by the inequation* $(\circ(\mathsf{b}) \leftrightarrow \neg(\circ(\mathsf{b})))^2 > 0$*,*

[b-SN] $\circ(\mathsf{b}) \in (0, 1/2)$ *is characterised by the inequation* $\circ(\mathsf{b}) \wedge (\neg(\circ(\mathsf{b})))^2 > 0$*.*

Combining these four conditions for $\mathsf{b} = 1$ and $\mathsf{b} = 0$, we obtain sixteen types of basic consistency operators $\circ$. In particular, the operator satisfying the conditions [1-1] and [0-1] is the maximal consistency operator $\circ_{max}$, i.e. the one such that $\circ_{max}(0) = \circ_{max}(1) = 1$.

Proposition 5. *Two interesting properties of consistency operators are the following:*

(i) The operator $\circ_{max}$ *and Baaz-Monteiro's projection operator*[7] Δ *are inter-definable.*

[7] Recall that the so-called Baaz-Monteiro operator Δ on the unit interval $[0,1]$ is defined as $\Delta(1) = 1$ and $\Delta(x) = 0$ for $x < 1$. From a logical point of view, it has been used in the frame of mathematical fuzzy logic as a way to specify that a proposition is fully true, so that, even if φ takes intermediate degrees of truth, $\Delta\varphi$ is Boolean, it can only take two truth-values: 1 when φ is 1-true, and 0 otherwise. In general, if L is an axiomatic extension of MTL, then the (conservative) expansion of L with Δ is axiomatised by adding to L the axioms (Δ1) $\Delta\varphi \vee \neg\Delta\varphi$, ($\Delta$2) $\Delta(\varphi \vee \psi) \to \Delta\varphi \vee \Delta\psi$, ($\Delta$3) $\Delta\varphi \to \varphi$, ($\Delta$4) $\Delta\varphi \to \Delta\Delta\varphi$, ($\Delta$5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$, together with the necessitation rule: (Δ-Nec) from φ derive $\Delta\varphi$. See [12] for details.

(ii) The maximal consistency operator $\circ_{max}$ (and the Δ operator) is definable from any of the sixteen types of consistency operators except from the one defined by the pair of conditions [0-SN] *and* [1-SN].

Proof. (i) To prove the first item we only need to check that $\Delta(x) = \circ_{max}(x) \wedge x$ and also that $\circ_{max}(x) = \Delta(x \vee \neg x)$.

(ii) The second item will be proved by cases:

- Suppose first that both $\circ(0), \circ(1) \geq 1/2$ (containing the cases defined by the nine pairs of conditions ([0-1],[1-1]), ([0-1],[1-SP]), ([0-1],[1-Fix]), ([0-SP],[1-1]), ([0-SP],[1-SP]), ([0-SP],[1-Fix]), ([0-Fix],[1-1]), ([0-Fix],[1-SP]), and ([0-Fix],[1-Fix]). In such a case it is easy to check that

$$\circ_{max}(x) = \neg((\neg(\circ(x)))^2) \text{ and } \Delta(x) = \circ_{max}(x) \wedge x.$$

- Suppose that $\circ(1) \geq 1/2$ and $\circ(0) \in (0, 1/2)$, that contains the three consistency operators defined by the pairs of conditions ([0-SN],[1-1]), ([0-SN],[1-SP]), and ([0-SN],[1-Fix]). In such a case it is easy to check that

$$\Delta(x) = \neg((\neg(\circ(x))^2)^2) \text{ and } \circ_{max}(x) = \Delta(x \vee \neg x).$$

- Finally when $\circ(1) \in (0, 1/2)$ and $\circ(0) \geq 1/2$, that contains the three consistency operators defined by the following pairs of conditions ([0-1],[1-SN]), ([0-SP],[1-SN]), and ([0-Fix],[1-SP]). In such a case it is easy to check that

$$\Delta(x) = \neg((\neg(\circ(\neg x))^2)^2) \text{ and } \circ_{max}(x) = \Delta(x \vee \neg x).$$

- For the remaining case, the one determined by the pair ([0-SN],[1-SN]), the conjecture is that it is not possible to define the Δ and $\circ_{max}$ operators.

□

Remark 1. It is clear that the converse of the previous results does not hold in the sense that if we add $\circ_{max}$ to the algebra $[\mathbf{0}, \mathbf{1}]_{\mathbf{NM}}$, it is not possible to recover the previous consistency operators, of course with the exception of $\circ_{max}$ itself, because Δ and $\circ_{max}$ are crisp operators (i.e. they only take values 0 or 1) and the operations of the algebra $[\mathbf{0}, \mathbf{1}]_{\mathbf{NM}}$ are classical when restricted to $\{0, 1\}$.

4.2 The maximal consistent operator and related logics: Approach 1

In this subsection we will formally define and axiomatise the expansion of the logic nf-NM with the maximal consistency operator $\circ_{max}$, i.e. the basic consistency operator $\circ$ further satisfying:

[1-1] $\circ 1 = 1$

[0-1] $\circ 0 = 1$

As already noted before, the crucial observation is that, in this case, $\circ_{max}$ and the Baaz-Monteiro operator Δ are interdefinable: $\Delta(x) = \circ_{max}(x) \wedge x$, and $\circ_{max}(x) = \Delta(x \vee \neg x)$.

We start by axiomatising first the logic $\models_{\mathcal{M}_\circ^{\max}}$ defined by the logical matrix $\mathcal{M}_\circ^{\max} = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{NM}_\circ^{\max}}, \{1\} \rangle$. It is worth noting that this logic *is not* paraconsistent, and so in particular *it is not* an LFI. However, its underlying algebra $[\mathbf{0}, \mathbf{1}]_{\mathbf{NM}_\circ^{\max}}$ was designed to be able to define an LFI when a suitable filter of designated values is considered.[8]

Definition 6. $\mathrm{NML}_\circ^{\max}$ is the logic defined by the following axioms and rules:

- Axioms of NML
- Consistency Axioms:
  (C0) $\neg(\circ\varphi \wedge \varphi \wedge \neg\varphi)$
  ($\top$-1) $\circ\top$
  ($\bot$-1) $\circ\bot$
- Modus ponens: (MP)
- Congruence rule:
  (Cong) $\dfrac{(\varphi \leftrightarrow \psi) \vee \chi}{(\circ\varphi \leftrightarrow \circ\psi) \vee \chi}$.

Observe that axiom (C0) can be equivalently replaced by the axiom

(C0') $(\varphi \wedge \neg\varphi \wedge \circ\varphi) \to \psi$,

which is characteristic of the LFIs. Moreover, it is easy to check that the following three inference rules

$$\frac{\varphi}{\circ\varphi}, \quad \frac{\neg\varphi}{\circ\varphi}, \quad \frac{\varphi}{\Delta\varphi}$$

are derivable in $\mathrm{NML}_\circ^{\max}$ from the axioms ($\top$-1) and ($\bot$-1) and the rule (Cong). Moreover, one can also check that the formula $\circ\varphi \vee \neg\circ\varphi$, stating that $\circ$ is a Boolean operator, can be proved to be a theorem of the logic as well: by applying the (Cong) rule to the axiom (C0), equivalently expressed as $\varphi \vee \neg\varphi \vee \neg\circ\varphi$, one gets $\circ\varphi \vee \neg\varphi \vee \neg\circ\varphi$, and by applying the derived rule $\neg\varphi/\circ\varphi$, one gets $\circ\varphi \vee \circ\varphi \vee \neg\circ\varphi$, which is equivalent to $\circ\varphi \vee \neg\circ\varphi$. Finally, note that from there, one can prove that $\circ\circ\varphi$ is a theorem of the logic as well.

[8]This is the approach to paraconsistency frequently adopted in the realm of MFL, see for instance [14].

Remark 2. As already mentioned above, the consistency operator $\circ_{max}$ and the operator Δ are interdefinable, and thus it follows that an alternative equivalent axiomatisation of $\mathrm{NML}_\circ^{\max}$ (where $\circ_{max}$ is primitive and Δ definable), could be given by the logic NML_Δ, the expansion of NML with Δ [23], where Δ is primitive and $\circ_{max}$ is definable. Nevertheless, the above axiomatisation of $\mathrm{NML}_\circ^{\max}$ will be more useful for our purposes of axiomatising all the other types of consistency operators which allow the definition of the Δ operator, see the last part of this subsection.

Proposition 6. *$\mathrm{NML}_\circ^{\max}$ is a sound and complete axiomatisation of $\models_{\mathcal{M}_\circ^{\max}}$.*

Proof. Soundness is easy as it reduces to check that, in the matrix $\mathcal{M}_\circ^{\max} = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}_\circ^{\max}}, \{1\}\rangle$, the above three consistency axioms (C0), ($\top$-1) and ($\bot$-1) are valid and the (Cong) rule preserves truth, and it is immediate that the $\circ_{max}$ satisfies the corresponding equations and quasi-equation. As for completeness, note that $\mathrm{NML}_\circ^{\max}$ is an expansion of NM with axioms plus the (Cong) rule, which is closed by disjunctions. So, by results in [12], the corresponding variety of $\mathrm{NM}_\circ^{\max}$-algebras keeps being prelinear. Therefore, $\mathrm{NML}_\circ^{\max}$ is complete with respect to the class of $\mathrm{NM}_\circ^{\max}$-chains. Hence, if $\Gamma \not\vdash \varphi$, there is an evaluation e on a $\mathrm{NM}_\circ^{\max}$-chain $\mathbf{A}$ such that $e(\psi) = 1$ for all $\psi \in \Gamma$ and $e(\varphi) < 1$. Consider the $\mathrm{NM}_\circ^{\max}$-subchain $\mathbf{A}'$ generated by the set of elements $\{e(\psi) \mid \psi \in \Gamma \cup \{\varphi\}\}$, which is countable. Now, from the strong standard completeness of NML, we know that every countable NM-chain embeds into the standard NM-chain $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$, and it is very easy to check that every such embedding easily extends to an embedding h from a countable $\mathrm{NM}_\circ^{\max}$-chain into the standard algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}_\circ^{\max}}$. Therefore, we can always find an evaluation e' on $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}_\circ^{\max}}$ such that $e'(\psi) = 1$ for all $\psi \in \Gamma$ and $e'(\varphi) < 1$, and hence $\Gamma \not\models_{\mathcal{M}_\circ^{\max}} \varphi$. ☐

It is worth noticing that, from this completeness result, it follows that the set of axioms for the Δ operator (defined above as $\Delta\varphi := \circ\varphi \wedge \varphi$), as proposed e.g. in [32] to syntactically characterising it, are provable in $\mathrm{NML}_\circ^{\max}$, since they are obviously valid formulas in $\mathcal{M}_\circ^{\max} = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}_\circ^{\max}}, \{1\}\rangle$.

Now we move from the matrix $\mathcal{M}_\circ^{\max} = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}_\circ^{\max}}, \{1\}\rangle$ defined by the filter $F = \{1\}$ to the matrix $\mathcal{M}_\circ^{\max 0} = \langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}_\circ^{\max}}, (0,1]\rangle$ defined by the filter $F = (0,1]$, and consider its corresponding paraconsistent logic $\models_{\mathcal{M}_\circ^{\max 0}}$.

Note that the logic $\models_{\mathcal{M}_\circ^{\max 0}}$ can be described in terms of the logic $\models_{\mathcal{M}_\circ^{\max}}$ by using the Δ connective. Namely, it holds that

$$\begin{array}{lll} \{\varphi_1, \ldots, \varphi_n\} \models_{\mathcal{M}_\circ^{\max 0}} \psi & \text{iff} & \{\nabla\varphi_1, \ldots, \nabla\varphi_n\} \models_{\mathcal{M}_\circ^{\max}} \nabla\psi, \\ & \text{iff} & \models_{\mathcal{M}_\circ^{\max}} (\nabla\varphi_1 \wedge \ldots \wedge \nabla\varphi_n) \to \nabla\psi \\ & \text{iff} & \models_{\mathcal{M}_\circ^{\max}} \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \nabla\psi, \end{array}$$

where $\nabla = \neg\Delta\neg$. Indeed, by definition of the Δ operator, for any evaluation e it holds that $e(\nabla\varphi) = 1$ if $e(\varphi) > 0$ and $e(\nabla\varphi) = 0$ otherwise.

Now we introduce an axiomatic system for the the logic $\models_{\mathcal{M}_\circ^{\max 0}}$.

Definition 7. nf-$\mathrm{NML}_{\circ_{\max}}$ is the logic defined by the following axioms and rules:

- Axioms of $\mathrm{NML}_\circ^{\max}$
- Rule of Adjunction: (Adj) $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- Restricted Modus Ponens: (r-MP) $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$, if $\vdash_{\mathrm{NML}_\circ} \varphi \to \psi$
- Restricted Congruence: (r-Cong) $\dfrac{(\varphi \leftrightarrow \psi) \vee \chi}{(\circ\varphi \leftrightarrow \circ\psi) \vee \chi}$, if $\vdash_{\mathrm{NML}_\circ} (\varphi \leftrightarrow \psi) \vee \chi$
- Reversed (r-∇Nec) $\dfrac{\nabla\varphi}{\varphi}$.

Observe that the rule of necessitation for ∇:

$$(\nabla\text{Nec}) \quad \frac{\varphi}{\nabla\varphi},$$

which is the reverse of (r-∇Nec), is derivable. Indeed, it is a direct consequence of fact that, by definition, $\nabla\varphi = \neg\Delta\neg\varphi = \neg\circ\neg\varphi \vee \varphi$. On the other hand, from (r-∇Nec) it easily follows that the rule

$$\frac{\neg\circ\neg\varphi}{\varphi},$$

is also derivable since clearly $\neg\circ\neg\varphi \to \neg\circ\neg\varphi \vee \varphi$ is a theorem of $\mathrm{NML}_\circ^{\max}$.

Theorem 2. nf-*$NML_{\circ_{\max}}$* *is a sound and complete axiomatisation of* $\models_{\mathcal{M}_\circ^{\max 0}}$.

Proof. Suppose $\{\varphi_1, \ldots, \varphi_n\} \models_{\mathcal{M}_\circ^{\max 0}} \psi$. Then, as observed above, this holds iff $\models_{\mathcal{M}_\circ^{\max}} \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \nabla\psi$, and by completeness, iff $\vdash_{NM_\circ^{\max}} \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \nabla\psi$. Therefore, in $NM_\circ^{\max}$ there is a proof

$$\Pi_1, \ldots, \Pi_r = \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \nabla\psi,$$

where each Π_i (with $1 \leq i < r$) is either an axiom of $\mathrm{NML}_\circ$, it has been obtained from a previous Π_k by the (Cong) rule, or has been obtained from previous Π_k, Π_j $(k, j < r)$ by the application of Modus ponens rule. Then, in order to get a proof of φ from $\psi_1, \ldots, \psi_n$ in $\mathrm{NML}_\circ^0$ we only need do the following:

(i) add two previous steps Π_0^1 and Π_0^2, where
   - $\Pi_0^1 = \varphi_1 \wedge \ldots \wedge \varphi_n$, obtained from the premises by the (Adj) rule,[9]
   - $\Pi_0^2 = \nabla(\varphi_1 \wedge \ldots \wedge \varphi_n)$, obtained from Π_0^1 by the (∇Nec) rule

(ii) add two final steps Π_{r+1} and Π_{r+2}, where
   - $\Pi_{r+1} = \neg\Delta\neg\psi$, obtained by the application of the (r-MP) rule to Π_0 and Π_r, and
   - $\Pi_{r+2} = \psi$, obtained by applying the rule (r-ΔNec) to Π_{r+1}.

[9] To be precise, it would be necessary to also add n further steps, one for each of the n premises $\{\varphi_1, \ldots, \varphi_n\}$, but we skip them for simplicity.

Therefore, the sequence $\Pi_0^1, \Pi_0^2, \Pi_1, \ldots \Pi_r, \Pi_{r+1}, \Pi_{r+2}$ is a proof of ψ from $\{\varphi_1, \ldots, \varphi_n\}$ in the logic $\mathrm{NML}_\circ^0$, with the proviso that the applications of the modus ponens and the (Cong) rules in the original proof $\Pi_1, \ldots \Pi_r$ in $\mathrm{NML}_\circ$ have to be replaced now by applications of the corresponding restricted rules (r-MP) and (r-Cong). □

The same kind of approach can be used to define the logics corresponding to each type of the remaining fourteen basic consistency operators described in Proposition 4 for which the Δ operator is definable, see (ii) of Proposition 5. To do this, one has to:

(1) Replace axioms ($\top$-1) and ($\bot$-1) respectively by suitable axioms corresponding to any pair of the conditions [b-SP], [b-fix] and [b-SN] from Prop. 4, namely:

(k-SP) $(\circ(\mathsf{k}))^2 \wedge \neg\circ(\mathsf{k})$,

(k-Fix) $(\circ(\mathsf{k}) \leftrightarrow \neg\circ(\mathsf{k}))^2$,

(k-SN) $\circ(\mathsf{k}) \wedge (\neg\circ(\mathsf{k}))^2$,

one for $\mathsf{k} = \top$ and one for $\mathsf{k} = \bot$, except for the pair $\{(\top\text{-SN}), (\bot\text{-SN})\}$.

(2) Suitably change the defining abbreviation of Δ in terms of $\circ$ according to the following cases:

- for the pairs of axioms ($\top$-1, $\bot$-1), ($\top$-1,$\bot$-SP), ($\top$-1,$\bot$-Fix), ($\top$-SP,$\bot$-1), ($\top$-SP,$\bot$-SP), ($\top$-SP, $\bot$-Fix), ($\top$-Fix, $\bot$-1), ($\top$-Fix, $\bot$-SP), and ($\top$-Fix, $\bot$-Fix), define

$$\Delta\varphi := \neg((\neg\circ\varphi)^2) \wedge \varphi,$$

- for the pairs of axioms ($\bot$-SN, $\top$-1), ($\bot$-SN, $\top$-SP), and ($\bot$-SN, $\top$-Fix), define

$$\Delta\varphi := \neg((\neg(\circ\varphi)^2)^2),$$

- and for the pairs of axioms ($\bot$-1, $\top$-SN), ($\bot$-SP, $\top$-SN), and ($\bot$-Fix, $\top$-SP), define

$$\Delta\varphi := \neg((\neg(\circ\neg\varphi)^2)^2)\ .$$

4.3 The logic of basic consistency operators: Approach 2

The approach followed in the previous subsection does not work in the cases of expansions of nf-NML with a consistency operator $\circ$ where Δ is not definable. This is the case for instance of expansions with a basic consistency operators or with an operator satisfying the axioms ($\top$-SN) and ($\bot$-SN). In this subsection we explore an alternative approach.

We start by considering the expansion of the logic NM with a new connective $\circ$ requiring to satisfy the following axiom

(C0) $\neg(\circ\varphi \wedge \varphi \wedge \neg\varphi)$,

and the following inference rule

$$\text{(Cong)}\ \frac{(\varphi \leftrightarrow \psi) \vee \chi}{(\circ\varphi \leftrightarrow \circ\psi) \vee \chi}.$$

Call this logic $\overline{\mathrm{NM}}_\circ$. Since the rule (Cong) is closed by disjunction, it is readily seen that $\overline{\mathrm{NM}}_\circ$ is sound and complete w.r.t. the class of matrices

$$\mathcal{C}_{qcons} = \{\langle [\mathbf{0}, \mathbf{1}]_{\mathbf{NM}^\circ}, \{1\}\rangle : \circ \text{ satisfies condition (C0)}\}.$$

Observe that operators on $[0, 1]$ satisfying condition (C0) can be called *quasi-consistency operators* since they can verify $\circ(0) = 0$ and $\circ(1) = 0$.

Next we turn to the corresponding paraconsistent logic whose semantics is given by the class of matrices

$$\mathcal{C}^0_{qcons} = \{\langle [\mathbf{0}, \mathbf{1}]_{\mathbf{NM}^\circ}, (0, 1]\rangle : \circ \text{ satisfies condition (C0)}\}.$$

and introduce the following definition of the non-falsity preserving companion of $\overline{\mathrm{NM}}_\circ$.

Definition 8. We define the logic nf-$\overline{\mathrm{NM}}_\circ$ by the following axioms and rules:

- Axioms of $\overline{\mathrm{NM}}_\circ$
- Rule of Adjunction: (Adj) $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
- Reverse Modus Ponens: (MP^r) $\dfrac{\neg\psi \vee \chi}{\neg\varphi \ \vee\ \neg(\varphi \to \psi)\ \vee\ \chi}$
- Restricted Modus Ponens: (r-MP$_{\overline{\mathrm{NM}}_\circ}$) $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$,
  if $\vdash_{\overline{\mathrm{NM}}_\circ} \varphi$ or $\vdash_{\overline{\mathrm{NM}}_\circ} \varphi \to \psi$
- Reverse Congruence rule: (Cong^r) $\dfrac{\neg((\circ\varphi \leftrightarrow \circ\psi) \vee \chi) \vee \delta}{\neg((\varphi \leftrightarrow \psi) \vee \chi) \vee \delta}$.

In this logic, the following inference rule, which is a restricted form of modus ponens for the material implication, is derivable:

$$\text{(Contr)}\ \frac{\varphi, \quad \psi \vee \neg\varphi}{\psi}, \quad \text{if } \vdash_{\overline{\mathrm{NM}}_\circ} \varphi$$

Indeed, assume $\vdash_{\overline{\mathrm{NM}}_\circ} \varphi$. Then $\vdash_{\overline{\mathrm{NM}}_\circ} \neg\varphi \to \bot$, and hence $\vdash_{\overline{\mathrm{NM}}_\circ} \neg\varphi \to \psi$ as well, and since $\vdash_{\overline{\mathrm{NM}}_\circ} \psi \to \psi$, it follows that $\vdash_{\overline{\mathrm{NM}}_\circ} \psi \vee \neg\varphi \to \psi$. Finally, applying the (r-MP) rule to $\psi \vee \neg\varphi$ and the theorem $\psi \vee \neg\varphi \to \psi$, we get ψ.

It is straightforward to check that that the logic nf-$\overline{\mathrm{NM}}_\circ$ is sound w.r.t. the class of matrices $\mathcal{C}^0_{qcons}$. Only notice that, on the one hand, if a rule φ/ψ is sound for a matrix $\mathcal{M} = \langle [\mathbf{0}, \mathbf{1}]_{\mathrm{NM}_\circ}, \{1\}\rangle \in \mathcal{C}_{qcons}$ then the rule $\neg\psi \vee \chi/\neg\varphi \vee \chi$

is automatically sound for the matrix $\mathcal{M}' = \langle [\mathbf{0},\mathbf{1}]_{\mathrm{NM}_\circ}, (0,1] \rangle \in \mathcal{C}^0_{qcons}$. On the other hand, regarding the rule (r-MP), notice that in the case $e(\varphi) = 1$ and $e(\varphi \to \psi) > 0$, then $e(\psi) = e(\varphi \to \psi) > 0$ as well.

In order to show the logic nf-$\overline{\mathrm{NM}}_\circ$ is complete, we first prove the following proposition, relating proofs in $\overline{\mathrm{NM}}_\circ$ and in nf-$\overline{\mathrm{NM}}_\circ$.

Proposition 7. *If $\psi \vdash_{\overline{NM}_\circ} \varphi$ then in* nf-$\overline{\mathrm{NM}}_\circ$ *there is a proof of $\neg\psi$ from $\neg\varphi$.*

Proof. Suppose $\psi \vdash_{\overline{\mathrm{NM}}_\circ} \varphi$, then there is a proof $\langle \Pi_1, \dots \Pi_r \rangle$, where $\Pi_1 = \psi$, $\Pi_r = \psi$ and where each Π_i (with $1 < i \leq r$) either:

- is an axiom of $\overline{\mathrm{NM}}_\circ$,
- has been obtained from previous Π_k, Π_j $(k, j < r)$ by the application of the Modus ponens rule (MP), or
- has been obtained from a previous Π_k $(k < i)$, by the application of the (Cong) rule.

We show next that we can build a proof for $\neg\psi$ from $\neg\varphi$ in nf-$\overline{\mathrm{NM}}_\circ$. We define:

(1) $\Sigma_1 = \neg\Pi_r = \neg\varphi$,

(2) For each $i = 1, ..., r-1$ we do the following: by the iterative construction below, Σ_i will be of the form $\Sigma_i = \Sigma^* \vee \neg\Pi_{r-i+1}$ (in the case $i = 1$ we take $\Sigma^* = \bot$). Then we define:

- If Π_{r-i+1} is an axiom or theorem of $\overline{\mathrm{NM}}_\circ$, then $\Sigma_{i+1} = \Sigma_i$.

- If $\Pi_{r-i+1} = \Psi$ has been obtained from previous $\Pi_k = \Phi, \Pi_j = \Phi \to \Psi$ $(k, j < r)$ by the application of Modus ponens rule, then $\Sigma_{i+1} = \Sigma^* \vee \neg\Pi_k \vee \neg\Pi_j$ is obtained from Σ_i by application of (MP^r).

- If $\Pi_{r-i+1} = (\circ\varphi \leftrightarrow \circ\psi) \vee \chi$ has been obtained from a previous $\Pi_k = (\varphi \leftrightarrow \psi) \vee \chi$ by the application of (Cong) rule, then $\Sigma_{i+1} = \Sigma^* \vee \neg\Pi_k$ is obtained from Σ_i by application of (Cong^r).

(3) By construction, Σ_r is of the form $\neg\Pi_1 \vee \bigvee_{i=1,n} \neg\Pi_{k_i}$, where for each k_i, Π_{k_i} is an axiom or theorem of $\overline{\mathrm{NM}}_\circ$. Therefore, $\neg\Pi_1 \vee \bigvee_{i=1,n} \neg\Pi_{k_i} \to \neg\Pi_1$ is a theorem of $\overline{\mathrm{NM}}_\circ$ as well. So we define $\Sigma_{r+1} = \Sigma_r \to \Sigma_1$,[10] and thus by using the restricted Modus Ponens rule (r-MP$_{\overline{\mathrm{NM}}_\circ}$) on Σ_r and Σ_{r+1} and theorem we finally get $\Sigma_{r+2} = \neg\Pi_1 = \neg\psi$

As a consequence, after removing possible duplicates in the sequence $\langle \Sigma_1, ..., \Sigma_r, \Sigma_{r+1}, \Sigma_{r+2} \rangle$, we get a proof of $\neg\psi$ from $\neg\varphi$ in nf-$\overline{\mathrm{NM}}_\circ$. □

Example 1. Consider the derivation $\{\varphi \wedge \circ\psi \wedge (\varphi \to (\psi \leftrightarrow \chi))\} \vdash_{\overline{\mathrm{NML}}_\circ} \circ\chi$. A possible proof is the following sequence:

[10] Actually, to be formally accurate we should replace the proof step Σ_{r+1} itself by a whole proof of this theorem in $\overline{\mathrm{NML}}_\circ$, but for the sake of simplicity we leave it as it is.

$\Pi_1 = \varphi \wedge \circ\psi \wedge (\varphi \to (\psi \leftrightarrow \chi))$, premise
$\Pi_2 = (\varphi \wedge \circ\psi \wedge (\varphi \to (\psi \leftrightarrow \chi))) \to \varphi$, axiom
$\Pi_3 = \varphi$, since $\Pi_3 = MP(\Pi_1, \Pi_2)$
$\Pi_4 = (\varphi \wedge \circ\psi \wedge (\varphi \to (\psi \leftrightarrow \chi))) \to \circ\psi$, axiom
$\Pi_5 = \circ\psi$, since $\Pi_5 = MP(\Pi_1, \Pi_4)$
$\Pi_6 = (\varphi \wedge \circ\psi \wedge (\varphi \to (\psi \leftrightarrow \chi))) \to (\varphi \to (\psi \leftrightarrow \chi))$, axiom
$\Pi_7 = \varphi \to (\psi \leftrightarrow \chi)$, since $\Pi_7 = MP(\Pi_1, \Pi_6)$
$\Pi_8 = \psi \leftrightarrow \chi$, since $\Pi_8 = MP(\Pi_3, \Pi_7)$
$\Pi_9 = \circ\psi \to \circ\chi$, since $\Pi_9 = Cong(\Pi_8)$
$\Pi_{10} = \circ\chi$; since $\Pi_{10} = MP(\Pi_5, \Pi_9)$

Now, according to the procedure defined in the proof of the above proposition, we obtain the following sequence of proof steps in nf-$\overline{\text{NM}}_\circ$:

$\Sigma_1 = \neg\Pi_{10}$
$\Sigma_2 = \bot \vee \neg\Pi_5 \vee \neg\Pi_9$, since $MP^r(\neg\Pi_{10}) = \neg\Pi_5 \vee \neg\Pi_9$
$\Sigma_3 = \bot \vee \neg\Pi_5 \vee \neg\Pi_8$, $\neg\Pi_8 = Cong^r(\neg\Pi_9)$
$\Sigma_4 = \bot \vee \neg\Pi_5 \vee \neg\Pi_3 \vee \neg\Pi_7$, since $MP^r(\neg\Pi_8) = \neg\Pi_3 \vee \neg\Pi_7$
$\Sigma_5 = \bot \vee \neg\Pi_5 \vee \neg\Pi_3 \vee \neg\Pi_1 \vee \neg\Pi_6$, since $MP^r(\neg\Pi_7) = \neg\Pi_1 \vee \neg\Pi_6$
$\Sigma_6 = \Sigma_5$, since Π_6 is an axiom
$\Sigma_7 = \bot \vee \neg\Pi_1 \vee \neg\Pi_4 \vee \neg\Pi_3 \vee \neg\Pi_1 \vee \neg\Pi_6$, since $MP^r(\neg\Pi_5) = \neg\Pi_1 \vee \neg\Pi_4$
$\Sigma_8 = \Sigma_7$, since Π_4 is an axiom
$\Sigma_9 = \bot \vee \neg\Pi_1 \vee \neg\Pi_4 \vee \neg\Pi_1 \vee \neg\Pi_2 \vee \neg\Pi_1 \vee \neg\Pi_6$, since $MP^r(\neg\Pi_3) = \neg\Pi_1 \vee \neg\Pi_2$
$\Sigma_{10} = \Sigma_9$, since Π_2 is an axiom
$\Sigma_{11} = \Sigma_{10} \to \Sigma_1$, since $\Sigma_{10} \to \Sigma_1$ is a theorem of $\overline{\text{NM}}_\circ$
$\Sigma_{12} = \neg\Pi_1$, since r-MP$_{\overline{\text{NM}}_\circ}(\Sigma_{10}, \Sigma_{11})$

Therefore, after removing duplicate steps, we have that

$$\langle \Sigma_1, \ldots, \Sigma_5, \Sigma_7, \Sigma_9, \Sigma_{11}, \Sigma_{12} \rangle$$

is a proof of $\neg(\varphi \wedge \circ\psi \wedge (\varphi \to (\psi \leftrightarrow \chi)))$ in nf-$\overline{\text{NM}}_\circ$ from $\neg\circ\chi$.

Theorem 3. *The finitary* nf-$\overline{\text{NM}}_\circ$ *is sound and complete w.r.t. to the class of matrices* $\mathcal{C}^0_{qcons}$*.*

Proof. Suppose $\{\psi_1, ..., \psi_n\} \models_{\mathcal{M}} \varphi$ for every $\mathcal{M} \in \mathcal{C}^0_{qcons}$. This is equivalent to $\neg\varphi \models_{\mathcal{M}'} \neg(\psi_1 \wedge ... \wedge \psi_n)$ for every $\mathcal{M}' \in \mathcal{C}_{cons}$. By completeness of $\overline{\text{NM}}_\circ$, there is a proof $\langle \Pi_1, \ldots \Pi_r \rangle$, where $\Pi_1 = \neg\varphi$, $\Pi_r = \neg\psi_1 \vee ... \vee \neg\psi_n$. Now, by the above Proposition 7, there is also a proof of $\neg\neg\varphi$ from $\neg\neg(\psi_1 \wedge ... \wedge \psi_n)$ in $\overline{\text{NM}}^0_\circ$. Then, if $\Pi_1, ..., \Pi_r$, with $\Pi_1 = \neg\neg(\psi_1 \wedge ... \wedge \psi_n)$ and $\Pi_r = \neg\neg\varphi$, is a proof of $\neg\neg\varphi$ from $\neg\neg(\psi_1 \wedge ... \wedge \psi_n)$, to get a proof of φ from $\{\psi_1, ..., \psi_n\}$ it is enough to add:

- a previous step: $\Pi_0 = \psi_1 \wedge ... \wedge \psi_n$, obtained by $n-1$ applications of the Adjunction rule (Adj) to the premises Γ.[11] Then Π_1 is obtained by applying the (r-MP) rule to Π_0 and the theorem $\psi_1 \wedge ... \wedge \psi_n \to \neg\neg(\psi_1 \wedge ... \wedge \psi_n)$.

[11] The same comment in the proof of Prop. 2 applies here.

- a final step: $\Pi_{r+1} = \varphi$, obtained by applying the (r-MP$_{\overline{\mathrm{NM}}_\circ}$) rule to Π_r and the theorem $\neg\neg\varphi \to \varphi$. □

At this point we emphasize that the logic above introduced nf-$\overline{\mathrm{NM}}_\circ$ is indeed paraconsistent but it is not an LFI, since the operator $\circ$ is not guaranteed to be a consistency operator, i.e. it is only required to satisfy axiom (C0) but neither axiom ($\top$-1) nor ($\bot$-1). This is why we finally introduce the non-falsity preserving LFI logic nf-$\mathrm{NM}_\circ$.

Definition 9. We define the logic nf-$\mathrm{NM}_\circ$ as the axiomatic extension of the logic nf-$\overline{\mathrm{NM}}_\circ$ with the axioms:

($\top$-1) $\circ\top$,

($\bot$-1) $\circ\bot$.

Then, as a corollary of the above theorem, it follows that $\mathrm{NM}^0_\circ$ is in fact a logic complete w.r.t. the class of matrices over $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ defined by basic consistency operators and filter $F = (0, 1]$.

Corollary 2. *The logic* nf-$\mathrm{NM}_\circ$ *is sound and complete w.r.t. the class of matrices* $\mathcal{C}^0_{cons} = \{\langle[\mathbf{0},\mathbf{1}]_{\mathbf{NM}_\circ}, (0, 1]\rangle : \circ$ *satisfies conditions* (C0), ($\top$-1) *and* ($\bot$-1)$\}$.

Proof. Observe that $\Gamma \vdash_{\mathrm{NM}^0_\circ} \varphi$ iff $\Gamma \cup \{(\top\text{-}1), (\bot\text{-}1)\} \vdash_{\overline{\mathrm{NM}}^0_\circ} \varphi$, and by completeness of nf-$\overline{\mathrm{NM}}_\circ$, iff $\Gamma \cup \{(\top\text{-}1), (\bot\text{-}1)\} \models_{\mathcal{M}} \varphi$ for every $\mathcal{M} \in \mathcal{C}^0_{qcons}$, that is, iff $\Gamma \models_{\mathcal{M}} \varphi$ for every $\mathcal{M} \in \mathcal{C}^0_{cons}$. As for the latter equivalence, note that, for any evaluation e on $[0, 1]_{NM_\circ}$, it holds that $e(\circ\top) > 0$ and $e(\circ\bot) > 0$ iff $\circ(1) > 0$ and $\circ(0) > 0$. □

Actually, the same kind of proof applies to show completeness of any axiomatic extension of $\mathrm{NM}^0_\circ$ with any pair of the axioms

(k-SP) $(\circ(\mathsf{k}))^2 \wedge \neg(\circ(\mathsf{k}))$

(k-fix) $(\circ(\mathsf{k}) \leftrightarrow \neg(\circ(\mathsf{k})))^2$

(k-SN) $\circ(\mathsf{k}) \wedge (\neg(\circ(\mathsf{k})))^2$,

one for $\mathsf{k} = \top$ and one for $\mathsf{k} = \bot$.

We end with two remarks about the approach followed in this section.

Remark 3. The approach followed in this subsection does not go through to show completeness for instance for the logic of the maximal consistency operator $\circ_{max}$, since the conditions $\circ(1) = \circ(0) = 1$ cannot be expressed by adding two axioms to nf-$\mathrm{NM}_\circ$, but rather by adding the following two inference rules:

$$\frac{\neg\circ\top}{\bot}, \qquad \frac{\neg\circ\bot}{\bot}.$$

5 Logics of matrices over NM-chains by lattice filters

In this section we are going to show that most of the results we have obtained in Section 3 can be extended to arbitrary NM-chains. As a matter of illustrative example we first consider the particular case of the logics defined over the NM-chain $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$, which is the fix-point less subalgebra of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ whose the universe is $[0,1] \smallsetminus \{1/2\}$, and then in the second part we consider the logics defined by matrices on general NM-chains and lattice filters.

5.1 Logics of matrices over $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$

First, as a matter of illustrative example, we recall the NM-chain $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$ which is the subalgebra of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ where the universe is $[0,1] \smallsetminus \{1/2\}$. Since $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$ is a subalgebra of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$, for every $a \in [0,1]$, the logic $\langle [\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}, F_a \smallsetminus \{1/2\}\rangle$ is a proper extension of the logic $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_a\rangle$ because the rule $(p \leftrightarrow \neg p)^2/\bot$ holds in $\langle [\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}, F_a \smallsetminus \{1/2\}\rangle$ but not in $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, F_a\rangle$. Using similar arguments as in the proof of Proposition 1 we obtain the following result:

Proposition 8. *For any $a \in [0,1]$, let $\vdash^-_a$ and $\vdash^-_{(a}$ be the finitary logics respectively determined by the matrices $\langle [\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}, F_a \smallsetminus \{1/2\}\rangle$ and $\langle [\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}, F_{(a} \smallsetminus \{1/2\}\rangle$. Then the following results hold:*

1. *$\vdash^-_a$, $\vdash^-_{(a}$ and $\vdash^-_1$ are the same logic for all $a \in (1/2,1)$,*
2. *$\vdash^-_a$, $\vdash^-_{(a}$ and $\vdash^-_{(0}$ are the same logic for all $a \in (0,1/2)$,*
3. *$\vdash^-_{(1/2}$, $\vdash^-_{1/2}$ and $\vdash_{CPL}$ are the same logic,*
4. *CPL is a proper extension of $\vdash^-_1$ and $\vdash^-_{(0}$,*
5. *$\vdash^-_1$ and $\vdash^-_{(0}$ are not comparable.*

Definition 10. The degree-preserving companion of the logic $\vdash^-_{NM}$ is defined as the intersection of the logics $\vdash^-_a$, for all $a \in (0,1]$, that is, $\vdash^-_{\leq} = \bigcap_{a>0} \vdash^-_a$.

Similarly to Lemma 1, Proposition 8 allows $\vdash^-_{\leq}$ to be expressed in a very simple way.

Lemma 3. $\vdash^-_{\leq} = \vdash^-_1 \cap \vdash^-_{(0}$.

In Figure 2 there is a graphical representation of the lattice of the logics appearing in Proposition 8, which in fact involves only four different logics.

Notice that the same arguments used in the proofs of Lemma 2 and Corollary 1 allow us to prove that for all formulas $\psi_1, \ldots, \psi_n, \varphi$,

$$\psi_1, \ldots, \psi_n \vdash^-_{(0} \varphi \text{ if, and only if, } \vdash^-_1 \psi_1 \wedge \ldots \wedge \psi_n \to \neg(\neg\varphi)^2.$$

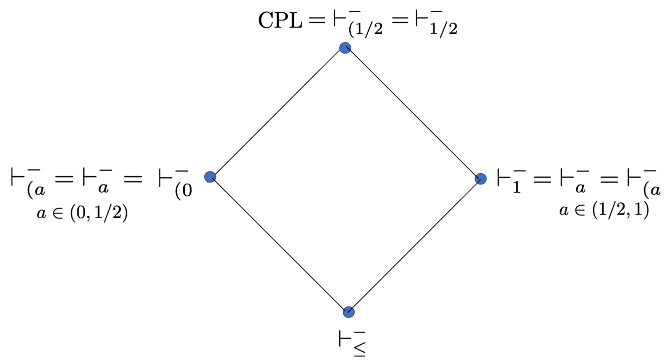

Figure 2: The lattice of the different logics in Proposition 8 and their relation to $\vdash^-_{\leq}$.

We recall that the logic NM^-, the axiomatic extension of NML with the axiom (BP), axiomatises $\vdash^-_{1}$. Thus, now we define the non-falsity preserving companion of NM^- with the following axioms and rules:

- Axioms: those of NML plus (BP), that is, those of NML^-
- Rules: Adjunction and (r-$\mathrm{MP}^2_{NM^-}$)

where the rule (r-$\mathrm{MP}^2_{NM^-}$) is similar to (r-MP^2) but restricted to theorems of NM^-, that is, the rule such that from φ and $\varphi \to \neg(\neg\psi)^2$ derives ψ, whenever $\vdash^-_1 \varphi \to \neg(\neg\psi)^2$

Finally, analogously to Theorem 1, we have the following completeness result for the logic nf-NM^-.

Theorem 4. nf-*NM^- is a sound and complete axiomatisation of* $\vdash^-_{(\mathbf{0}}$.

Now, for the different logics appearing in Proposition 8 we have the axiomatisations given in Table 2.

Logics	Matrix	Axioms	Inference Rules
NM^-: $\vdash^-_{\mathbf{1}}$	$\langle [\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}, \{1\}\rangle$	NM + (BP)	MP
nf-NM^-: $\vdash^-_{(\mathbf{0}}$	$\langle [\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}, (0,1]\rangle$	NM + (BP)	Adj, r-$\mathrm{MP}^2_{\mathrm{NM}^-}$: $\frac{\varphi, \quad \vdash^-_{\mathbf{1}} \varphi \to \neg(\neg\psi)^2}{\psi}$
CPL	$\langle [\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}, (1/2,1]\rangle$	NM + (EM)	MP
$\vdash^-_{\mathbf{1}} \cap \vdash^-_{(\mathbf{0}} = \vdash^-_{\leq}$		NM + (BP)	Adj, r-$\mathrm{MP}_{\mathrm{NM}^-}$: $\frac{\varphi, \quad \vdash^-_{\mathbf{1}} \varphi \to \psi}{\psi}$

Table 2: Axiomatisations of the logics appearing in Proposition 8 defined by matrices over $[0,1]^-_{NM}$ by a lattice filter.

In this table we use (EM) to refer to the excluded-middle axiom

$$(EM) \quad \varphi \vee \neg\varphi$$

and (BP) to refer to the axiom

$$(BP) \quad \neg((\varphi \leftrightarrow \neg\varphi)^2)$$

that is satisfied by a NM-chain only if does not contain a negation fixpoint, see [29, Theorem 2], where the equivalent expression of this axiom mentioned at the end of Section 2 is used.

Finally, the lattice of the logics appearing in Table 1 and Table 2 is depicted in Figure 3.

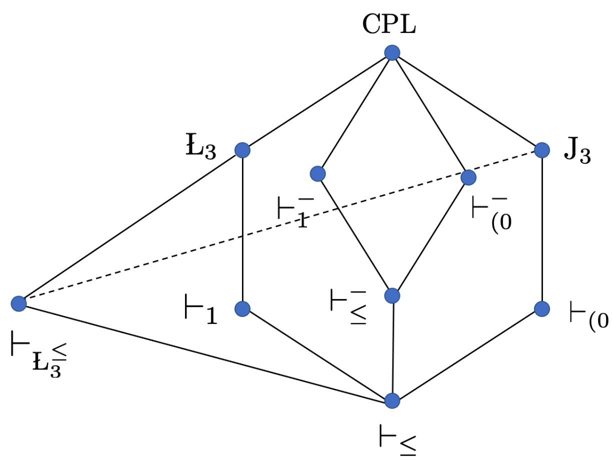

Figure 3: The lattice of the different logics in Table 1 and Table 2

Remark 4. (Adding a consistency operator to the logic nf-NM$^-$) The question of expanding the paraconsistent non-falsity preserving logic nf-NM$^-$ over the NM-algebra $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$ has a parallel development as the one studied in Section 4 for the case of the logic over the standard NM-algebra $[0,1]_{NM}$ with small modifications. In fact, one has to consider consistency operators $\circ : [0,1]^- \to [0,1]^-$ satisfying the same conditions as the ones in Section 4 once we exclude the value $1/2$ from both the domain and the image of $\circ$, a fact that restricts the number of types of consistency operators from sixteen to nine.

Anyway, if we pay our attention to the maximum consistency operator $\circ_{max}$ on $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$, similarly to what we did in Section 4.2, we can define the logic nf-NML$^-_{\circ_{\max}}$ exactly as in Definition 7 for the logic nf-NML$_{\circ_{\max}}$, only with the proviso of adding the axiom (BP) to the axioms of NML$^{\max}_{\circ}$. The same proof of Theorem 2 yields now completeness of LFI logic NML$^{\max}_{\circ}$ with respect to the intended semantics given by the matrix $\langle [\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}^{\max}_{\circ}}, F_{(0}\rangle$. Analogous results can be obtained for the logics expanded with the seven remaining types of consistency operators where the Δ is definable as well.

Finally, note that the whole approach developed in Section 4.3 for the case where Δ is not definable from $\circ$ (i.e. when both $\circ(1), \circ(0) < 1/2$), can be fully reproduced here for the case of logics over $[\mathbf{0},\mathbf{1}]^-_{\mathbf{NM}}$.

5.2 Logics defined by matrices on general NM-chains

Now, we consider logics defined by matrices on general NM-chains $\mathbf{A}$ and lattice filters and first show that, using a similar argument as in the proof of Proposition 1, they also reduce in fact to matrix logics with filters $F_{\mathbf{1}}$ or $F_{(\mathbf{0}}$, where in the following we will write $\mathbf{1}$ and $\mathbf{0}$ to denote the top and bottom element of $\mathbf{A}$ respectively. For any $F \subseteq A$, as usual we denote by F^c the complement of F on A and by $\neg F$ the set $\{\neg a : a \in F\}$. Since the negation in $\mathbf{A}$ is involutive, we recall that for every proper lattice filter F and any proper lattice ideal I, $\neg F$ and F^c are proper ideals while $\neg I$ and I^c are proper filters. Moreover $F = (F^c)^c = \neg(\neg F)$ and $I = (I^c)^c = \neg(\neg I)$. In the following, we will say that two matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ are *equivalent* when the induced logics $\models_{\mathcal{M}_1}$ and $\models_{\mathcal{M}_2}$ are the same.

A first result about logics defined by matrices over a NM-chain and a lattice filter is that, from pragmatic point of view, we can restrict ourselves to consider only matrices either with $\{\mathbf{1}\}$ or with $(\mathbf{0}$ as lattice filters.

Proposition 9. *Let* $\mathbf{A}$ *be an NM-chain and let* F *be a proper lattice filter on* A*. Then there exists a NM-chain* $\mathbf{B}$ *such that the matrix* $\langle \mathbf{A}, F\rangle$ *is equivalent either to the matrix* $\langle \mathbf{B}, F_{\mathbf{1}}\rangle$ *or to the matrix* $\langle \mathbf{B}, F_{(\mathbf{0}}\rangle$.

In more detail, by letting $A^+ = \{a \in A : \neg a < a\}$*, the following conditions hold:*

1. *If* $F \subseteq A^+$*, then* $\langle \mathbf{A}, F\rangle$ *is equivalent to* $\langle \mathbf{B}, F_{\mathbf{1}}\rangle$*, where*

$$B = \{\mathbf{0}\} \cup [A \setminus (F \cup \neg F)] \cup \{\mathbf{1}\}.$$

2. *If* $F \not\subseteq A^+$*, then* $\langle \mathbf{A}, F\rangle$ *is equivalent to* $\langle \mathbf{B}, F_{(\mathbf{0}}\rangle$*, where*

$$B = \{\mathbf{0}\} \cup [A \setminus (F^c \cup \neg(F^c))] \cup \{\mathbf{1}\}.$$

In particular, we have:

3. *If* $\mathbf{A}$ *has negation fixpoint and* $F = A^+$*, then* $\langle \mathbf{A}, F\rangle$ *is equivalent to* $L_3 = \langle \mathbf{NM_3}, \{1\}\rangle$.

4. *If* a *is the negation fixpoint of* $\mathbf{A}$ *and* $F = A^+ \cup \{a\}$*, then* $\langle \mathbf{A}, F\rangle$ *is equivalent to* $\mathcal{J}_3 = \langle \mathbf{NM_3}, \{1/2, 1\}\rangle$.

5. *If* $\mathbf{A}$ *does not have a negation fixpoint and* $F = A^+$*, then* $(\mathbf{A}, F)$ *is equivalent to* CPL $= \langle \mathbf{NM_2}, \{1\}\rangle$.

Proof. The proof is similar in every case:

1. $\langle \mathbf{B}, F_{\mathbf{1}}\rangle$ is a submatrix and also a strong homomorphic image[12] of $\langle \mathbf{A}, F\rangle$.

2. $\langle \mathbf{B}, F_{(\mathbf{0}}\rangle$ is a submatrix and also a strong homomorphic image of $\langle \mathbf{A}, F\rangle$.

[12] A homomorphism h from $\langle \mathbf{A}, F\rangle$ to $\langle \mathbf{B}, G\rangle$ is strong if $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism such that for every $a \in A$, $a \in F$ iff $h(a) \in G$.

3. $Ł_3$ is a submatrix and also a strong homomorphic image of $\langle \mathbf{A}, F \rangle$.

4. $\mathcal{J}_3$ is a submatrix and also a strong homomorphic image of $\langle \mathbf{A}, F \rangle$.

5. $\langle \mathbf{NM}_2, F_{\mathbf{1}} \rangle$ is a submatrix and also a strong homomorphic image of $\langle \mathbf{A}, F \rangle$.

□

The logics of matrices with $\{\mathbf{1}\}$ as a filter are explosive and in the literature are usually referred as *truth-preserving* logics (understanding 1 as full truth), while the logics of matrices with $(\mathbf{0}$ as a filter are paraconsistent (except for the case CPL) and can be called as *non-falsity preserving* logics, see e.g. [3].

The truth-preserving logics defined over NM-chains have been fully studied in [29, 30]. The rest of this section is devoted in general to the non-falsity preserving logics, and in particular to the axiomatisation of the logics given by the matrices $(\mathbf{A}, F_{(\mathbf{0}})$. We start by characterising the set of tautologies of the logics $\vdash^{A}_{(\mathbf{0}}$.

Proposition 10. *Let* $\mathbf{A}$ *be a non trivial NM-chain.*

1. *If* $\mathbf{A}$ *has a fixpoint,* J_3 *is an extension of* $\vdash^{A}_{(\mathbf{0}}$*. Moreover, for every formula* φ*,* $\vdash^{A}_{(\mathbf{0}} \varphi$ *iff* $\vdash_{\mathsf{J}_3} \varphi$*. i.e.* $Taut(\langle \mathbf{A}, F_{(\mathbf{0}} \rangle) = Taut(\mathsf{J}_3)$.

2. *If* $\mathbf{A}$ *has no fixpoint, CPC is an extension of* $\vdash^{A}_{(\mathbf{0}}$*. Moreover, for every formula* φ*,* $\vdash^{A}_{(\mathbf{0}} \varphi$ *iff* $\vdash_{CPL} \varphi$*. i.e.* $Taut(\langle \mathbf{A}, F_{(\mathbf{0}} \rangle) = Taut(CPL)$.

Proof. 1. Let c be the fixpoint of A. Then $\{\mathbf{0}, c, \mathbf{1}\}$ is the subuniverse of a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{NM_3}$. Therefore, $\langle \mathbf{NM_3}, F_{(\mathbf{0}} \rangle$ is embeddable as a submatrix into $\langle \mathbf{A}, F_{(\bar{\mathbf{0}}} \rangle$, thus $\mathcal{J}_3 = \langle \mathbf{NM_3}, F_{(\mathbf{0}} \rangle$ is an extension of $\vdash^{A}_{(\mathbf{0}}$.

Assume $\nvdash^{A}_{(\mathbf{0}} \varphi$, then there is an A-evaluation e such that $e(\varphi) = \mathbf{0}$. Moreover since the map $h : A \to \{-1, 0, 1\}$ defined by

$$h(x) = \begin{cases} 1, & \text{if } x > c; \\ 0, & \text{if } x = c; \\ -1, & \text{if } x < c. \end{cases}$$

is a homorphism from $\mathbf{A}$ onto $\mathbf{NM_3}$, then $h \circ e$ is an NM_3-evaluation such that $h \circ e(\varphi) = -1$. Thus $\nvdash_{\mathsf{J}_3} \varphi$.

2. If A does not have a fixpoint, then the 2-element Boolean algebra $\mathbf{B_2}$ is not only a subalgebra of $\mathbf{A}$ but also a homomorphic image of $\mathbf{A}$ and a similar argument as in the previous item can be used to prove that $Taut(\langle \mathbf{A}, F_{(0} \rangle) = Taut(CPL)$.

□

As in the case of the matrices over $[\mathbf{0}, \mathbf{1}]_{\mathbf{NM}}$, thanks to the involutivity of the NM negation there is a tight relation among truth-preserving logics and non-falsity preserving logics defined by matrices over NM-chains.

Lemma 4. *Let* **A** *be a non trivial NM-chain. For every formula* φ*,*

$$\psi \vdash^{A}_{(\mathbf{0}} \varphi \text{ if, and only if, } \neg\varphi \vdash^{A}_{\mathbf{1}} \neg\psi.$$

In particular, $\vdash^{A}_{(\mathbf{0}} \varphi$ *if, and only if,* $\vdash^{A}_{\mathbf{1}} \neg(\neg\varphi)^2$*.*

Corollary 3. *Let* **A** *be a non trivial NM-chain. For every formulas* $\psi_1, \ldots, \psi_n, \varphi$*,*

$$\psi_1, \ldots, \psi_n \vdash^{A}_{(\mathbf{0}} \varphi \text{ if, and only if, } \vdash^{A}_{\mathbf{1}} (\psi_1 \wedge \ldots \wedge \psi_n) \to \neg(\neg\varphi)^2.$$

Finally, we can extend the axiomatisation of $\vdash_{(0}$ obtained in Theorem 1 to non-fasilty preserving logics of matrices over NM-chains..

Theorem 5. *Let* L *be a finitary extension of* nf*-NML defined by a class of matrices of type* $\langle \mathbf{A}, F_{(\mathbf{0}} \rangle$ *where each* **A** *is an NM-chain. Then there is an axiomatic extension* L' *of NML such that* L *is axiomatised as follows:*

- *Axioms: those of* L'
- *Rules: Adjunction and (r-*$MP^2_{L'}$*)*

Thus, L *is exactly* nf-L'*.*

Proof. Let $\mathbb{M}$ be a class of non-falsity preserving matrices, meaning matrices of type $\langle \mathbf{A}, F_{(\mathbf{0}} \rangle$. We denote by $\mathbb{M}^1$ the associated truth preserving class of matrices, that is, $\mathbb{M}^1 = \{\langle \mathbf{A}, F_{\mathbf{1}} \rangle : \langle \mathbf{A}, F_{(\mathbf{0}} \rangle \in \mathbb{M}\}$. By definition, the finitary logic defined by $\mathbb{M}$ is the non-falsity preserving companion of the logic defined by $\mathbb{M}^1$. It follows from the characterisation of finitary extensions of NML in [30], that the logic obtained by $\mathbb{M}^1$ is an axiomatic extension of NML, thus the same arguments used in the proof of Theorem 1 provide a proof of this theorem. □

All axiomatic extensions of NML were described in [29, Theorem 4]. The following theorem and figure summarize this result. But before let us introduce first the following notation conventions that will be used in the rest of the paper.

Notation: From now on, with an abuse of language,

- $\mathcal{N}_n$ will denote the matrix $\langle \mathbf{NM_n}, F_{\mathbf{1}} \rangle$
- $\mathcal{J}_n$ will denote the matrix $\langle \mathbf{NM_n}, F_{(\mathbf{0}} \rangle$

Theorem 6. *L is an axiomatic extension of* NML *iff there exists* $(n, m) \in \{(n, m) \in (\omega^+)^2 : n \geq m\}$ *such that L is the finitary logic defined by the following finite set of matrices* $\{\mathcal{N}_{2n}, \mathcal{N}_{2m+1}\}$*, where with an abuse of language we use* $\mathcal{N}_{2\omega} = \langle [\mathbf{0}, \mathbf{1}]^-_{\mathbf{NM}}, \{1\} \rangle$ *and* $\mathcal{N}_{2\omega+1} = \langle [\mathbf{0}, \mathbf{1}]_{\mathbf{NM}}, \{1\} \rangle$*. Moreover L is then axiomatised relative to* NML *by the axiom*

$$[BP(\varphi) \wedge S_n(\varphi_0, \ldots, \varphi_n)] \vee S_m(\varphi_0, \ldots, \varphi_m).$$

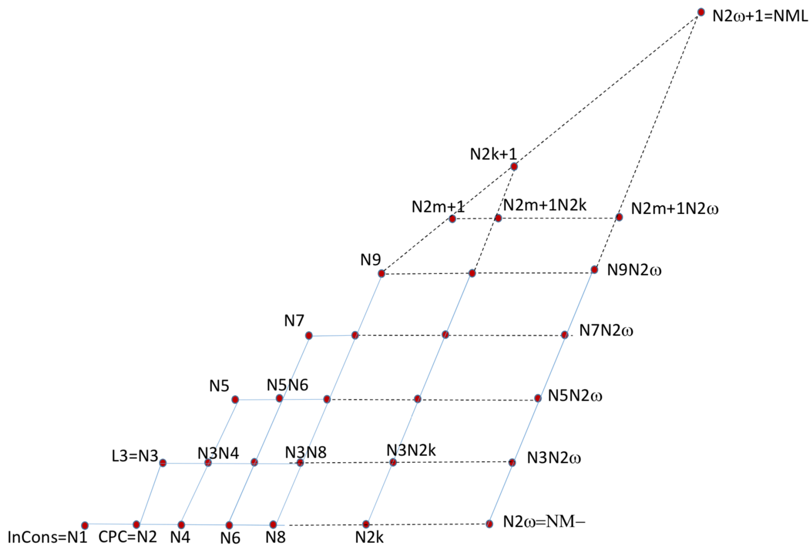

Figure 4: Lattice of axiomatic extensions of $\mathbb{NML}$.

where

$$S_n(\varphi_0, \ldots, \varphi_n) = \begin{cases} \bot, & \textit{if } n = 0; \\ \bigwedge_{i<n} ((\varphi_i \to \varphi_{i+1}) \to \varphi_{i+1}) \to \bigvee_{i<n+1} \varphi_i, & \textit{if } 0 < n < \omega; \\ \top, & \textit{if } n = \omega, \end{cases}$$

Figure 4 depicts the lattice of axiomatic extensions of NML. As a corollary of Theorem 5, we obtain the following result about the lattice of extensions of the non-falsity preserving logic nf-NML.

Corollary 4. *The lattice of finitary extensions of* nf*-NML defined by a class of matrices whose algebras are NM-chains is isomorphic to the lattice of axiomatic extensions of NML of Figure 4.*

6 Logics of matrices over finite NM-algebras

In the preceding sections we have dealt with matrix logics over NM-chains and lattice filters. However, it is clear that there are finitary extensions of NML and of nf-NML that are not complete w.r.t. matrices over finite NM-chains, e.g. the 1-preserving logic defined by the matrix $\langle \mathbf{NM_2} \times \mathbf{NM_3}, F_1 \rangle = \langle \mathbf{NM_2}, F_1 \rangle \times \langle \mathbf{NM_3}, F_1 \rangle$ or the non-falsity preserving logic defined by the matrix $\bar{J}_3 = \langle \mathbf{NM_2} \times \mathbf{NM_3}, F_{(\mathbf{0}} \times F_{(\mathbf{0}} \rangle = \mathcal{N}_2 \times \mathcal{J}_3 = \mathcal{J}_2 \times \mathcal{J}_3$, that is a proper extension of J_3, see e.g. [16].

It follows from Theorem 5 and Corollary 4 that for matrices over NM-chains, there is a one-one correspondence among truth preserving logics and non-falsity preserving logics. Moreover, it is well known (see for instance [16]) that for the case of the three element NM-chain, $Ł_3$ and J_3 are equivalent deductive systems. We are going to see in this section that this is not the case for axiomatic extensions of NML different from CPL and $Ł_3$.

Since the variety of NM-algebras is locally finite any logic defined by matrices over NM-algebras and lattice filters can be reduced to finite matrices. On the other hand, unlike e.g. the case of case of finite MV-algebras (related to Łukasiewicz logics), not every finite NM-algebra is a finite direct product of finite NM-chains. Actually, it is well-known that every finite algebra is isomorphic to a direct product of (finite) directly indecomposable algebras, but directly indecomposable NM-algebras are not necessarily linearly ordered, for instance, the subalgebra of $\mathbf{NM_4} \times \mathbf{NM_4}$ given by the following universe $\{(n, m) \in NM_4 \times NM_4 : \text{either } n, m > 0 \text{ or } n, m < 0\}$ is directly indecomposable but not a chain.

For NM-algebras we have a weaker result, in the sense that a matrix logic over a finite NM-algebra can always be seen as an extension of a logic of a product of finitely-many (finite) matrices $\mathcal{N}_{n_i}$'s and $\mathcal{J}_{m_j}$'s. Before presenting this result, notice that if F is a lattice filter of a finite NM-algebra $\mathbf{A}$, then it is a principal filter. Indeed, if $a = \bigwedge\{x \mid x \in F\}$, then $F = F_a$. In the particular case of $\mathbf{A}$ being a product of finite NM-chains $\mathbf{NM_{k_1}} \times \cdots \times \mathbf{NM_{k_j}}$ and $a = (a_1, \ldots, a_k) \in A$, then $F_a = F_{a_1} \times \ldots \times F_{a_k}$.

Lemma 5. *Let* $\mathbf{A}$ *be a finite* NM*-algebra and let* F *be a lattice filter of* A *such that* $F \neq A$*, then* $\models_{\langle \mathbf{A}, F\rangle}$ *is an extension of* $\models_{\mathcal{N}_{n_1} \times \cdots \times \mathcal{N}_{n_k} \times \mathcal{J}_{m_1} \times \cdots \times \mathcal{J}_{m_r}}$ *for some* $n_1, \ldots n_k, m_1, \ldots m_r \geq 2$*, where* $r + k > 0$.

Proof. Since $\mathbf{A}$ is a finite NM-algebra, $F \neq A$ has a minimum element $\min(F) \neq \mathbf{0}$ and $F = \{b \in A : b \geq \min(F)\}$. By the subdirect representation theorem, $\mathbf{A} \subset_{sd} \mathbf{NM_{k_1}} \times \cdots \times \mathbf{NM_{k_j}}$ for some $k_i, \ldots, k_j \geq 2$, and $\min(F) = (a_1, \ldots a_j)$ for some $(a_1, \ldots a_j) \in NM_{k_1} \times \cdots \times NM_{k_j}$. Hence $\langle \mathbf{A}, F\rangle$ is a submatrix of $\langle \mathbf{NM_{k_1}} \times \cdots \times \mathbf{NM_{k_j}}, F_{a_1} \times \cdots \times F_{a_j}\rangle$. By Proposition 9, $\langle \mathbf{NM_{k_1}} \times \cdots \times \mathbf{NM_{k_j}}, F_{a_1} \times \cdots \times F_{a_j}\rangle$ is equivalent to $\langle \mathbf{NM_{r_1}} \times \cdots \times \mathbf{NM_{r_j}}, F_{b_1} \times \cdots \times F_{b_j}\rangle$, where $r_i \leq k_i$ and the subindexes b_i's are:

$$b_i = \begin{cases} \mathbf{1}, & \text{if } a_i > 0; \\ (\mathbf{0}, & \text{if } \bot(\mathbf{NM_{k_i}}) < a_i \leq 0; \\ \mathbf{0}, & \text{if } a_i = \bot(\mathbf{NM_{k_i}}), \end{cases}$$

where $\bot(\mathbf{NM_{k_i}})$ denotes the bottom element of the chain $\mathbf{NM_{k_i}}$. Notice that

$$\langle \mathbf{NM_{r_i}}, F_{b_i}\rangle = \begin{cases} \mathcal{N}_{r_i} \text{ and } r_i \geq 2, & \text{if } b_i = \mathbf{1}; \\ \mathcal{J}_{r_i} \text{ and } r_i \geq 2, & \text{if } b_i = (\mathbf{0}; \\ \mathcal{N}_1, & \text{if } b_i = \mathbf{0}. \end{cases}$$

Finally, we can forget trivial components $\mathcal{N}_1$'s in order to obtain the desired matrix. □

All the above considerations make the task of identifying and classifying all the logics of matrices over finite NM algebras with lattice filters much more complex for instance than the case of MV-algebras. Therefore, in the first subsection we restrict ourselves to this task for logics defined by matrices over *finite products of finite NM-chains*. In a second subsection we identify which logics of matrices over finite NM-algebras are maximal paraconsistent.

Notation: In the following we will write $i = 1 \div n$ to denote "for all $i \in \{1, \ldots, n\}$".

6.1 The case of finite products of finite NM-chains

Our first main result in this section is to show that any logic of a matrix $\langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is a finite product of finite NM-chains, can be reduced to a finite set of matrices from ten different types, each one in turn being a product of at most three basic components of the form $\mathcal{N}_{n_i}$ or $\mathcal{J}_{n_j}$. This is proved in Theorem 7. Moreover, we also prove that each matrix of that set of ten different types defines a different logic. This is done in Corollary 8.

Before proving Theorem 7, we need three previous lemmas.

Lemma 6. *Let $n_1, \ldots n_k, m_1, \ldots, m_r \geq 2$ where $r + k > 0$, and consider the product matrix*

$$\mathcal{M} = \mathcal{N}_{n_1} \times \cdots \times \mathcal{N}_{n_k} \times \mathcal{J}_{m_1} \times \cdots \times \mathcal{J}_{m_r}.$$

Then, the following conditions hold:

- *The logic $\models_{\mathcal{M}}$ is an extension of NML iff either $r = 0$, or $m_i = 2$ for every $1 \leq i \leq r$.*
- *The logic $\models_{\mathcal{M}}$ is an extension of* nf*-NML iff either $k = 0$, or $n_i = 2$ for every $1 \leq i \leq k$.*

Proof.

- The right to left implication is immediate since any $\mathcal{N}_{n_i}$ is an extension of NML and $\mathcal{N}_2 = \mathcal{J}_2$. If $r \neq 0$ and there is some $m_i > 2$, then Modus Ponens does not hold in $\models_{\mathcal{M}}$, and hence $\models_{\mathcal{M}}$ is not an extension of NML
- Similarly to previous cases if $k = 0$ or $n_i = 2$ for every $1 \leq i \leq k$, then since any $\mathcal{J}_{m_i}$ is an extension of nf-NML and $\mathcal{N}_2 = \mathcal{J}_2$, $\models_{\mathcal{M}}$ is an extension of nf-NML. If $k \neq 0$ and there is some $n_i > 2$, then excluded-middle axiom (EM) does not hold in $\models_{\mathcal{M}}$. Therefore, $\models_{\mathcal{M}}$ is not an extension of nf-NML.

□

Lemma 7. *For every $n > 1$,*

- *$\mathcal{N}_{2n+1}$ is embeddable into $\mathcal{N}_{2n+1} \times \mathcal{N}_3$*
- *$\mathcal{N}_{2n+1}$ is embeddable into $\mathcal{N}_{2n+1} \times \mathcal{J}_3$*

- $\mathcal{N}_{2n}$ *is embeddable into* $\mathcal{N}_{2n} \times \mathcal{N}_2 = \mathcal{N}_{2n} \times \mathcal{J}_2$

Proof. To prove the first two items, it is easy to check that the mapping $h : NM_{2n+1} \to NM_{2n+1} \times NM_3$ defined by

$$h(a) = \begin{cases} (a, 1), & \text{if } a > \neg a \\ (a, 1/2), & \text{if } a = \neg a \\ (a, 0), & \text{if } a < \neg a \end{cases}$$

is an embedding such that $a \in F_{\mathbf{1}}$ iff $h(a) = (a, 1)$, and then $a \in F_{\mathbf{1}}$ iff $h(a) \in F_{\mathbf{1}} \times F_{\mathbf{1}}$ iff $h(a) \in F_{\mathbf{1}} \times F_{(\mathbf{0}}$.

The third item can be proved using the restriction of h to the domain NM_{2n} and codomain $NM_{2n} \times NM_2$. □

Next lemma is a technical result with a sufficient condition to embed products of matrices.

Lemma 8. *Let* $\mathcal{M}_1, \ldots, \mathcal{M}_n, \mathcal{K}_1, \ldots, \mathcal{K}_k$ *be some logical matrices. Whenever*

- *For every* $1 \leq i \leq n$*, there is* $1 \leq j \leq k$ *such that* $\mathcal{M}_i$ *is embeddable into* $\mathcal{K}_j$*, and*
- *For every* $1 \leq j \leq k$*, there is* $1 \leq i \leq n$ *such that* $\mathcal{M}_i$ *is embeddable into* $\mathcal{K}_j$*,*

then $\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ *is embeddable into* $\mathcal{L}_1 \times \cdots \times \mathcal{L}_l$ *for some integer* $l \geq n$ *where* $\{\mathcal{L}_i : 1 \leq i \leq l\} = \{\mathcal{K}_j : 1 \leq j \leq k\}$*, and thus the logic* $\models_{\mathcal{M}_1 \times \cdots \times \mathcal{M}_n}$ *is an extension of the logic* $\models_{\mathcal{K}_1 \times \cdots \times \mathcal{K}_k}$*.*

Proof. If both hypothesis hold then there exist maps $m : \{1, \ldots, n\} \to \{1, \ldots, k\}$ and $s : \{1, \ldots, k\} \to \{1, \ldots, n\}$ and embeddings $h_{i,m(i)} : \mathcal{M}_i \hookrightarrow \mathcal{K}_{m(i)}$ for every $1 \leq i \leq n$ and $g_{s(j),j} : \mathcal{M}_{s(j)} \hookrightarrow \mathcal{K}_j$ for every $1 \leq j \leq k$. Let $A = \{j_1, \ldots, j_p\} = \{1 \leq j \leq k : j \neq m(i) \text{ for all } i \leq n\}$ and let $l = n + p$. Then for every $1 \leq i \leq l$, we define

$$\mathcal{L}_i = \begin{cases} \mathcal{K}_{m(i)}, & \text{if } i \leq n; \\ \mathcal{K}_{j_r} & \text{if } i > n \text{ where } r = i - n. \end{cases}$$

It is easy to check that the map $f : \prod_{1 \leq i \leq n} \mathcal{M}_i \to \prod_{1 \leq j \leq l} \mathcal{L}_j$ defined as follows

$$f\left((a_i)_{1 \leq i \leq n}\right)(j) = \begin{cases} h_{j,m(j)}(a_j), & \text{if } j \leq n; \\ g_{s(j_r),j_r}(a_{s(j)}), & \text{if } j > n, \text{ where } r = j - n. \end{cases}$$

is a matrix embedding □

Theorem 7. *Let* $\mathbf{A}$ *be a finite product of finite NM-chains and let* $F \neq A$ *be a lattice filter on* $\mathbf{A}$*. Then the logic defined by the matrix* $\mathcal{M} = \langle \mathbf{A}, F \rangle$ *can be reduced to a finite set of the following matrices:*

1. $\mathcal{N}_n$ *for some positive integer* $n > 1$*.*

2. $\mathcal{N}_2 \times \mathcal{N}_{2m+1}$ *for some positive integer* m.

3. $\mathcal{J}_n$ *for some positive integer* $n > 1$.

4. $\mathcal{J}_n \times \mathcal{J}_k$ *for some positive integers* $n \neq k$.

5. $\mathcal{J}_{2n} \times \mathcal{J}_{2k} \times \mathcal{J}_{2l+1}$ *for some positive integers* $l < n < k$.

6. $\mathcal{J}_{2n} \times \mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1}$ *for some positive integers* $m < n$ *and* $m < l$.

7. $\mathcal{N}_{2h+1} \times \mathcal{J}_{2k}$ *for some positive integer* h, k *such that* $k > 1$.

8. $\mathcal{N}_{2h+1} \times \mathcal{J}_{2n} \times \mathcal{J}_{2k}$ *for some positive integers* h, n, k *such that* $1 < n < k$.

9. $\mathcal{N}_3 \times \mathcal{J}_{2n} \times \mathcal{J}_{2m+1}$ *for some positive integers* n, m *such that* $n > 1$.

10. $\mathcal{N}_3 \times \mathcal{J}_{2m+1}$ *for some positive integer* m.

Proof. By Lemmas 5 and 6 we may assume that $\mathcal{M}$ is:

- a finite product of $\mathcal{N}_n$'s, if $\models_{\mathcal{M}}$ is an extension of NML;
- a finite product of $\mathcal{J}_m$'s, if $\models_{\mathcal{M}}$ is an extension of nf-NML;
- a finite product of $\mathcal{N}_n$'s and $\mathcal{J}_m$'s if $\models_{\mathcal{M}}$ is neither an extension of NML, nor an extension of nf-NML.

Then we have the following cases:

- If the logic generated by $\mathcal{M}$ is an extension of NML, by Lemmas 7 and 8 and as mentioned later in Theorem 9, $\mathcal{M}$ can be reduced to

  $$\{\mathcal{N}_{2n}, \mathcal{N}_2 \times \mathcal{N}_{2m+1}, \mathcal{N}_{2k+1}, \text{ where } n \geq m \geq k \geq 0\}.$$

  Notice that these matrices are of type 1. or 2.

- If the logic generated by $\mathcal{M}$ is an extension of nf-NML, then we may assume that $\mathcal{M} = \mathcal{J}_{n_1} \times \cdots \times \mathcal{J}_{n_k}$

  If all n_i's are even numbers or all n_i's are odd numbers, let $n = \min\{n_1, \ldots n_k\}$ and let $m = \max\{n_1, \ldots n_k\}$. Then, by Lemma 8, $\mathcal{J}_n \times \mathcal{J}_m$ is a submatrix of $\mathcal{M}$ and $\mathcal{M}$ is a submatrix of $(\mathcal{J}_n \times \mathcal{J}_m)^{k-1}$. Thus both matrices define the same logic. Notice that if $n \neq m$, $\mathcal{J}_n \times \mathcal{J}_m$ is of type 4 and if $n = m$ then $\mathcal{J}_n \times \mathcal{J}_n$ can be reduced to $\mathcal{J}_n$ of type 3.

  When there are even components and odd components in $\mathcal{M}$, with an analogous argument by Lemma 8, we can reduce $\mathcal{M}$ to a product

  $$\mathcal{M}' = \mathcal{J}_{2n} \times \mathcal{J}_{2k} \times \mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1}$$

  where $2n$ is the minimum of all even components, $2k$ is the maximum of all even components, $2m+1$ is the minimum of all odd components and

$2l+1$ is the maximum of all even components. Moreover, since, by Lemma 8,

$$\mathcal{M}_1 = \mathcal{J}_{2n} \times \mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1} \text{ (type 6)}$$

and

$$\mathcal{M}_2 = \mathcal{J}_{2n} \times \mathcal{J}_{2k} \times \mathcal{J}_{2l+1} \text{ (type 5)}$$

are both submatrices of $\mathcal{M}'$, and $\mathcal{M}'$ is a submatrix of $\mathcal{M}_1 \times \mathcal{M}_2$, therefore the logic defined by $\mathcal{M}'$ is the logic defined by the set of matrices $\{\mathcal{M}_1, \mathcal{M}_2\}$. Moreover,
- If $n \leq m$, $\mathcal{M}_1$ can be reduced to $\mathcal{J}_{2n} \times \mathcal{J}_{2l+1}$.
- If $n \leq l$, then $\mathcal{M}_2$ can be reduced to

$$\mathcal{J}_{2n} \times \mathcal{J}_{2l+1} \text{ and } \mathcal{J}_{2n} \times \mathcal{J}_{2k};$$

- If $k \leq l$, then $\mathcal{M}_2$ can be reduced to $\mathcal{J}_{2n} \times \mathcal{J}_{2l+1}$;

- If the logic generated by $\langle \mathbf{A}, F \rangle$ is neither an extension of NML nor of nf-NML, then, by Lemma 5 and without loss of generality, we may assume $\mathcal{M}$ can be reduced to

$$\mathcal{N}_{n_1} \times \cdots \times \mathcal{N}_{n_k} \times \mathcal{J}_{m_1} \times \cdots \times \mathcal{J}_{m_r},$$

where $k, r \geq 1$, $n_1 > 2$, $n_i > n_{i+1} \geq 2$ for every $i = 1 \div k$, and $m_i > m_{i+1} > 2$ for every $i = 1 \div r$.

If there is some $1 \leq i \leq k$ such that n_i is an even positive integer, then by Lemmas 8 and 7, $\mathcal{M}$ can be reduced to

$$\{\mathcal{N}_{n_i} : \ 1 \leq i \leq k \text{ and } n_i \text{ is even}\} \cup$$
$$\{\mathcal{N}_2 \times \mathcal{N}_{n_i} : \ 1 \leq i \leq k \text{ and } n_i \text{ is odd}\} \cup$$
$$\left\{\mathcal{J}_2 \times \mathcal{J}_{m_j} : \ 1 \leq j \leq r\right\}.$$

Notice that in this case all matrices are of type 1., 2. and 4. A major reduction can be obtained just by taking

$$\{\mathcal{N}_n, \mathcal{N}_2 \times \mathcal{N}_m, \mathcal{J}_2 \times \mathcal{J}_s, \mathcal{J}_2 \times \mathcal{J}_l\},$$

where $n = \max\{n_i : 1 \leq k \text{ and } n_i \text{ is even}\}$, $m = \max\{n_i : 1 \leq i \leq k \text{ and } n_i \text{ is odd}\}$, $s = \max\{m_i : 1 \leq i \leq r \text{ and } m_i \text{ is even}\}$ and $l = \max\{m_i : 1 \leq i \leq r \text{ and } m_i \text{ is odd}\}$.

If for every $1 \leq i \leq k$, n_i is an odd positive integer, let h be the positive integer such that $2h+1 = max\{n_1, \ldots, n_k\}$. Let m, n and k be defined as follows whenever they exist:
- $2m+1 = \max\{m_j : 1 \leq j \leq r \text{ and } m_j \text{ is odd }\}$,
- $2n = \min\{m_j : 1 \leq j \leq r \text{ and } m_j \text{ is even }\}$, and
- $2k = \max\{m_j : 1 \leq j \leq r \text{ and } m_j \text{ is even }\}$.

Then $\mathcal{M}$ can be reduced to:

– If m, n, k exist and $n < k$,

$$\mathcal{N}_{2h+1} \times \mathcal{J}_{2n} \times \mathcal{J}_{2k} \text{ (type 8.)} \quad \text{and} \quad \mathcal{N}_3 \times \mathcal{J}_{2n} \times \mathcal{J}_{2m+1} \text{ (type 9.)}$$

– If m, n, k exist and $n = k$,

$$\mathcal{N}_{2h+1} \times \mathcal{J}_{2n} \text{ (type 7.)} \quad \text{and} \quad \mathcal{N}_3 \times \mathcal{J}_{2n} \times \mathcal{J}_{2m+1}$$

– If m exists and n, k do not exist.

$$\mathcal{N}_{2h+1} \quad \textit{and} \quad \mathcal{N}_3 \times \mathcal{J}_{2m+1} \text{ (type 10.)}$$

– If m does not exists and n, k exist and $n < k$,

$$\mathcal{N}_{2h+1} \times \mathcal{J}_{2n} \times \mathcal{J}_{2k}$$

– If m does not exists and n, k exist and $n = k$,

$$\mathcal{N}_{2h+1} \times \mathcal{J}_{2n}$$

□

For the particular case of extensions of nf-NML, i.e. when $\mathcal{M}$ is of the form $\mathcal{M} = \mathcal{J}_{n_1} \times \cdots \times \mathcal{J}_{n_k}$, we have the following corollary.

Corollary 5. *Let* $\mathbf{A}$ *be a finite product of finite NM-chains and let* $F \neq A$ *be a lattice filter on* $\mathbf{A}$ *such that* $\langle \mathbf{A}, F \rangle$ *is an extension of* nf-*NML. Then the logic defined by the matrix* $\mathcal{M} = \langle \mathbf{A}, F \rangle$ *can be reduced to a finite set of the following matrices:*

1. $\mathcal{J}_n$ *for some positive integer* $n > 1$.
2. $\mathcal{J}_n \times \mathcal{J}_k$ *for some positive integers* $n \neq k$.
3. $\mathcal{J}_{2n} \times \mathcal{J}_{2k} \times \mathcal{J}_{2l+1}$ *for some positive integers* $l < n < k$.
4. $\mathcal{J}_{2n} \times \mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1}$ *for some positive integers* $m < n$ *and* $m < l$.

Our next task is to show that each of the ten different types of matrices identified in the above theorem defines in fact a different logic, so all of them are non-equivalent matrices. In the following we consider a *generic matrix* $\mathcal{M}$ with k *explosive* components and r *paraconsistent* components,

$$\mathcal{M} = \langle \mathbf{M}, F \rangle = \mathcal{N}_{n_1} \times \cdots \times \mathcal{N}_{n_k} \times \mathcal{J}_{m_1} \times \cdots \times \mathcal{J}_{m_r}$$

where $k + r > 0$, $n_i \geq 2$ for every $i = 1 \div k$ and $m_i > 2$ for every $i = 1 \div r$, and a number of axioms and rules, together with the conditions $\mathcal{M}$ must satisfy for the corresponding logic to validate them, that will eventually allow us univocally determine each one of the above ten types of matrices.

Notation conventions: In the following we will use the following notation conventions regarding the matrix $\mathcal{M}$:

(i) Abusing the language we will say that a component of $\mathcal{M}$ is *even* (resp. *odd*) when the NM-chain of the component has an even (resp. odd) number of elements.

(ii) Also, we will say that a rule or an axiom is valid in $\mathcal{M}$ when it is valid in the corresponding logic $\models_{\mathcal{M}}$.

(iii) For every $i = 1 \div k$, we will let s_i be such that $n_i = 2s_i$ or $n_i = 2s_i + 1$. For every $i = 1 \div r$, we will let r_i be such that $m_i = 2r_i$ or $n_i = 2r_i + 1$.

Next we consider the following axiom and rule:

- Axiom BP: Recall that the axiom

$$(BP) \;\; \neg\left((\varphi \leftrightarrow \neg\varphi)^2\right)$$

axiomatises $\vdash_1^-$ as an axiomatic extension of NML.

- Rule ∃-EVEN: We introduce the following rule that characterises when $\mathcal{M}$ has some even component:

$$(\exists\text{-EVEN}) \qquad (\varphi \leftrightarrow \neg\varphi)^2 \Big/ \bot$$

Indeed, the following result shows that (BP) and ∃-EVEN characterise matrices with all or some even components respectively.

Proposition 11.

1. *(BP) is valid in $\mathcal{M}$ iff all the components in $\mathcal{M}$ are even.*

2. *The rule ∃-EVEN is valid in $\mathcal{M}$ iff there is an even component in $\mathcal{M}$.*

Proof. 1. Assume $a \in M$.
If all components of M are even, then for every $1 \leq i \leq k + r$, $a(i) \neq 0$, recall that 0 is the fixpoint of the i-th component of M. Thus $\neg\left((a \leftrightarrow \neg a)^2\right) = \mathbf{1} \in F$. Hence (BP) is valid in $\mathcal{M}$.
If there is some odd component $1 \leq i \leq k + r$ in M, let $b \in M$ be an element such that $b(i) = 0$, then

$$\neg\left((b(i) \leftrightarrow \neg b(i))^2\right) = \begin{cases} -s_i, & \text{if } i \leq k; \\ -r_i, & \text{if } i > k. \end{cases}$$

Thus (BP) is not valid in $\mathcal{M}$.

2. Let $a \in M$. If there is a component $1 \leq i \leq n$ such that n_i is even then $(a(i) \leftrightarrow \neg a(i))^2 = -s_i$, thus $(a \leftrightarrow \neg a)^2 \notin F$ and the rule ∃-EVEN holds. Analogously for the case where there is $1 \leq j \leq r$ such that m_j is even. If all components in M are odd, then M has negation fixpoint $c = (0, \ldots, 0)$ and $(c \leftrightarrow \neg c)^2 = (s_1, \ldots, s_k, r_1, \ldots, r_r) = \mathbf{1} \in F$. Thus the rule ∃-EVEN fails in $\mathcal{M}$.

□

We continue introducing some additional axioms and rules that are needed for our task.

- Axiom Vn: Recall the axiom $(V3)$ $(\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_3) \vee (\varphi_3 \to \varphi_4)$, that axiomatises Ł3 relative to NML. We consider its generalization for any number $n > 0$:

$$(Vn) \quad \bigvee_{1 \le i \le n} (\varphi_i \to \varphi_{i+1})$$

- Rule $\mathbf{MIN0_n}$:

$$(\mathrm{MIN0}_n) \quad \bigwedge_{1 \le i \le n} \neg(\varphi_i \to \varphi_{i+1}) \Big/ \bot$$

- Rule $\mathbf{MAX0_n}$:

$$(\mathrm{MAX0}_n) \quad \varphi_1^2, \ldots, \varphi_n^2, \varphi_1 \to \psi, \ldots, \varphi_n \to \psi \Big/ \left(\bigvee_{1 \le i \le n-1} (\varphi_i \to \varphi_{i+1}) * \neg\varphi_{i+1} \right) \vee \psi$$

- Rule $\mathbf{MIN0_n^{even}}$: Consider next the following rule

$$(\mathrm{MIN0}_n^{even}) \quad \left(\bigvee_{1 \le i \le 2n+1} (\varphi_i \leftrightarrow \neg\varphi_i)^2 \right) \vee \left(\bigwedge_{1 \le i \le 2n} \neg(\varphi_i \to \varphi_{i+1}) \right) \Big/ \bot$$

- Rule $\mathbf{MAX0_n^{odd}}$:

$$(\mathrm{MAX0_n^{odd}}) \quad \frac{\chi, \chi \to \gamma, \neg\varphi_1 \to \psi, \ldots, \neg\varphi_n \to \psi, \neg\varphi_{n+1} \to \psi, \left(\neg(\varphi_1^2)\right)^2, \ldots, \left(\neg(\varphi_{n+1}^2)\right)^2}{(\neg\varphi_1)^2 \vee \left(\bigvee_{1 \le i \le n} (\varphi_i \to \varphi_{i+1}) * \varphi_i\right) \vee \psi \vee \gamma}$$

- Axiom $\varphi_{\mathbf{n}}^{\mathbf{MAX1^{odd}}}$:

$$(\varphi_{\mathrm{n}}^{\mathrm{MAX1^{odd}}}) \quad \neg\left((\varphi_1 \leftrightarrow \neg\varphi_1)^2\right) \vee \bigvee_{1 \le i \le n+1} (\varphi_{i+1} \to \varphi_i)$$

Now, for every matrix $\mathcal{M} = \langle \mathbf{M}, F \rangle = \mathcal{N}_{n_1} \times \cdots \times \mathcal{N}_{n_k} \times \mathcal{J}_{m_1} \times \cdots \times \mathcal{J}_{m_r}$, we introduce the following definitions that will be used in the next proposition:

$\mathrm{min1}(\mathcal{M}) = \min\{t : \mathcal{N}_t$ is one of the components of $\mathcal{M}\}$,
$\mathrm{min0}(\mathcal{M}) = \min\{t : \mathcal{J}_t$ is one of the components of $\mathcal{M}\}$,
$\mathrm{max0}(\mathcal{M}) = \max\{t : \mathcal{J}_{2t}$ or $\mathcal{J}_{2t+1}$ is one of the components of $\mathcal{M}\}$,
$\mathrm{min0}^e(\mathcal{M}) = \min\{t : \mathcal{J}_{2t}$ is one of the components of $\mathcal{M}\}$,
$\mathrm{max0}^o(\mathcal{M}) = \max\{t > 0 : \mathcal{J}_{2t+1}$ is one of the components of $\mathcal{M}\}$,
$\mathrm{max1}^o(\mathcal{M}) = \max\{t > 0 : \mathcal{N}_{2t+1}$ is one of the components of $\mathcal{M}\}$

Then the following characterisation results can be shown to hold.

Proposition 12. *The following characterisations of the validity of the axioms and rules considered above hold:*

- *For every $n > 1$,*
  - *(Vn) is a tautology of $\mathcal{M}$ iff either $k = 0$ or $\min1(\mathcal{M}) \leq n$.*
  - *The rule $\mathrm{MIN0_n}$ is valid in $\mathcal{M}$ iff $k > 0$ or $\min0(\mathcal{M}) \leq n$.*
- *And for every $n > 0$,*
  - *The rule $\mathrm{MAX0_n}$ is valid in $\mathcal{M}$ iff either $r = 0$ or $\max0(\mathcal{M}) \leq n$.*
  - *The rule $\mathrm{MIN0_n^{even}}$ is valid in $\mathcal{M}$ iff either there is $1 \leq i \leq k$ such that n_i is even, or $\min0^e(\mathcal{M})$ exists and $\min0^e(\mathcal{M}) \leq n$.*
  - *The rule $\mathrm{MAX0_n^{odd}}$ is valid in $\mathcal{M}$ iff either $r = 0$, or every m_j is even, or $\max0^o(\mathcal{M}) \leq n$.*
  - *$(\varphi_n^{MAX1^{odd}})$ is a tautology of $\mathcal{M}$ iff either $k = 0$, or every n_i is even, or $\max1^o(\mathcal{M}) \leq n$.*

Proof. For the sake of the simplicity and in order to ease the reading of this paper we only show the proof of first item. The proofs of the remaining items are similar, although a little bit longer.

- Let $a_1, \dots, a_{n+1} \in M$. Since $n > 1$, notice that for every $j = 1 \div r$

  $$\bigvee_{1 \leq i \leq n} (a_i(k+j) \to a_{i+1}(k+j)) \neq -r_j.$$

  If $k = 0$, by the previous remark (Vn) is a tautology. Then assume $k > 0$. If $n_j \leq n$, for every component $j = 1 \div k$ there is $1 \leq i \leq n$ such that $a_i(j) \leq a_{i+1}(j)$. Therefore $\bigvee_{1 \leq i \leq n} (a_i(j) \to a_{i+1}(j)) = s_j$ and (Vn) is a tautology of $\mathcal{M}$.
  If, without loss of generality, we assume that $n_1 > n$. Then, there exist $c_1 > c_2 > \cdots > c_{n+1} \in NM_{n_1}$. Thus, there exist $a_1, \dots, a_{n+1} \in M$ such that $a_i(1) = c_i$ and $\bigvee_{1 \leq i \leq n} (a_i(1) \to a_{i+1}(1)) \neq s_1$. Hence, (Vn) is not tautology of $\mathcal{M}$.

□

With the previous propositions, we can eventually prove that the ten types of matrices identified in Theorem 7 cannot be reduced any further, in the sense that they all define different logics.

Theorem 8. *Two different matrices of types described in Theorem 7 define different logics.*

Proof. To begin with, notice that Modus Ponens allows us to characterise NML-extensions while the Excluded-Middle axiom characterises nf-NML-extensions:

type	EM	MP
1	NO	YES
2	NO	YES
3	YES	NO
4	YES	NO
5	YES	NO
6	YES	NO
7	NO	NO
8	NO	NO
9	NO	NO
10	NO	NO

For matrices of type 1 and 2, thus NML-extensions, the axioms (BP) and (Vs) together with the rule ∃-EVEN are enough to distinguish them:[13]

type	matrix	BP	∃-even	Vs
1	$\mathcal{N}_{2n}$	YES	YES	$2n \leq s$
1	$\mathcal{N}_{2m+1}$	NO	NO	$2m+1 \leq s$
2	$\mathcal{N}_2 \times \mathcal{N}_{2m+1}$	NO	YES	$2m+1 \leq s$

For matrices of type 3, 4, 5 and 6, thus nf-NM extensions, the axiom (BP) and the rules ∃-EVEN, MIN0_s, MAX0_s, MIN0_s^{even} and MAX0_s^{odd} are enough to distinguish them:

	matrix	BP	∃-even	$\mathbf{MIN0}_s$
3	$\mathcal{J}_{2n}$	YES	YES	$2n \leq s$
3	$\mathcal{J}_{2m+1}$	NO	NO	$2m+1 \leq s$
4	$\mathcal{J}_{2n} \times \mathcal{J}_{2k}$	YES	YES	$2n \leq s$
4	$\mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1}$	NO	NO	$2m+1 \leq s$
4	$\mathcal{J}_{2n} \times \mathcal{J}_{2l+1}$	NO	YES	$\min\{2n, 2l+1\} \leq s$
5	$\mathcal{J}_{2n} \times \mathcal{J}_{2k} \times \mathcal{J}_{2l+1}$	NO	YES	$2l+1 \leq s$
6	$\mathcal{J}_{2n} \times \mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1}$	NO	YES	$2m+1 \leq s$

	matrix	$\mathbf{MAX0}_s$	$\mathbf{MIN0}_s^{even}$	$\mathbf{MAX0}_s^{odd}$
3	$\mathcal{J}_{2n}$	$n \leq s$	$n \leq s$	YES
3	$\mathcal{J}_{2m+1}$	$m \leq s$	NO	$m \leq s$
4	$\mathcal{J}_{2n} \times \mathcal{J}_{2k}$	$k \leq s$	$n \leq s$	YES
4	$\mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1}$	$l \leq s$	NO	$l \leq s$
4	$\mathcal{J}_{2n} \times \mathcal{J}_{2l+1}$	$\max\{n, l\} \leq s$	$n \leq s$	$l \leq s$
5	$\mathcal{J}_{2n} \times \mathcal{J}_{2k} \times \mathcal{J}_{2l+1}$	$k \leq s$	$n \leq s$	$l \leq s$
6	$\mathcal{J}_{2n} \times \mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1}$	$\max\{n, l\} \leq s$	$n \leq s$	$l \leq s$

[13]In the following tables, an inequality in a column, with header an axiom or a rule, stands for the condition under which the matrix in the same row validates the axiom or the rule. For instance, in the next table the matrix $\mathcal{N}_{2n}$ satisfies the axiom (Vs) whenever $2n \leq s$.

For matrices of type 7, 8, 9 and 10, the axiom $\varphi_s^{MAX1^{odd}}$ and the rules ∃-EVEN, $\mathrm{MIN0}_s$, $\mathrm{MAX0}_s$, $\mathrm{MIN0}_s^{even}$ and $\mathrm{MAX0}_s^{odd}$ are enough to distinguish them:

	matrix	$\varphi_{\mathbf{s}}^{\mathbf{MAX1^{odd}}}$	**∃-even**	$\mathbf{MIN0}_s$
7	$\mathcal{N}_{2h+1} \times \mathcal{J}_{2k}$	$h \leq s$	YES	$2k \leq s$
8	$\mathcal{N}_{2h+1} \times \mathcal{J}_{2n} \times \mathcal{J}_{2k}$	$h \leq s$	YES	$2n \leq s$
9	$\mathcal{N}_3 \times \mathcal{J}_{2n} \times \mathcal{J}_{2m+1}$	YES	YES	$\min\{2n, 2m+1\} \leq s$
10	$\mathcal{N}_3 \times \mathcal{J}_{2m+1}$	YES	NO	$2m+1 \leq s$

	matrix	$\mathbf{MAX0}_s$	$\mathbf{MIN0}_s^{even}$	$\mathbf{MAX0}_s^{odd}$
7	$\mathcal{N}_{2h+1} \times \mathcal{J}_{2k}$	$k \leq s$	$k \leq s$	YES
8	$\mathcal{N}_{2h+1} \times \mathcal{J}_{2n} \times \mathcal{J}_{2k}$	$k \leq s$	$n \leq s$	YES
9	$\mathcal{N}_3 \times \mathcal{J}_{2n} \times \mathcal{J}_{2m+1}$	$\max\{n, m\} \leq s$	$2n \leq s$	$m \leq s$
10	$\mathcal{N}_3 \times \mathcal{J}_{2m+1}$	$m \leq s$	NO	$m \leq s$

□

Next, we are in position to prove that, in general, axiomatic extensions of NML and its non-falsity preserving companions are not equivalent as deductive systems in the sense of Blok and Pigozzi [4]. We first recall that all finitary extensions of NML are described in [30]. The following theorem and figure summarize this result.

Theorem 9. *(c.f. [30, Theorem 3.7]) L is a finitary extension of* NML *iff there exists* $(n, m, k) \in \{(n, m, k) \in (\omega^+)^3 : n \geq m \geq k\}$ *such that L is the finitary logic defined by the following finite set of matrices* $\{\mathcal{N}_{2n}, \mathcal{N}_{2k+1}, \mathcal{N}_2 \times \mathcal{N}_{2m+1}\}$. *Moreover,*

- *if* $k = m$, *then L is the axiomatic extension* $\mathrm{NML}_{2n,2m+1}$ *defined by* $\{\mathcal{N}_{2n}, \mathcal{N}_{2k+1}\}$.
- *if* $k \neq m$, *then L is axiomatised relative to* $\mathrm{NML}_{2n,2m+1}$ *by the rule*

$$\frac{\varphi \leftrightarrow \neg\varphi}{S_k(\psi_0, \ldots, \psi_k)}.$$

Figure 5 depicts the dual lattice of finitary extensions of NML.

Proposition 13. *Let L be an axiomatic extension of NML different from CPL and Ł$_3$, then L and* nf*-L are not equivalent deductive systems.*

Proof. Assume L and nf-L were equivalent. Since L is algebraizable, so is nf-L. Moreover, equivalency is inherited for every finitary extension in the following sense: every finitary extension of L defined by the set of matrices $\{\mathcal{N}_{2n}, \mathcal{N}_2 \times \mathcal{N}_{2k+1}, \mathcal{N}_{2m+1}\}$ is equivalent to the logic defined by the set of matrices $\{\mathcal{J}_{2n}, \mathcal{J}_2 \times \mathcal{J}_{2k+1}, \mathcal{J}_{2m+1}\}$. If L is an axiomatic extension of NML such that L≠CPL and L≠ Ł$_3$, then $\models_{\mathcal{N}_4}$ is an axiomatic extension of L. Thus $\models_{\mathcal{N}_4}$ and $\models_{\mathcal{J}_4}$ are

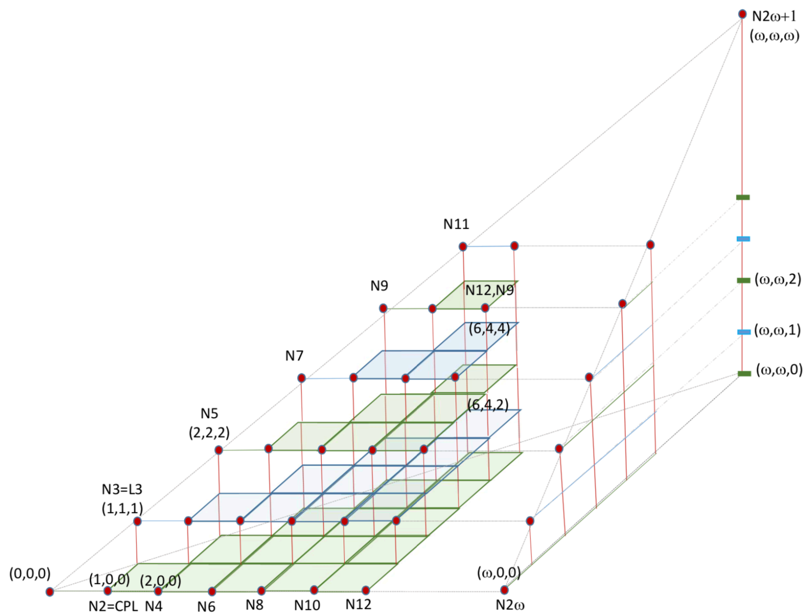

Figure 5: Lattice of finitary extensions of $\mathbb{NML}$.

equivalent and both lattices of finitary extensions are isomorphic. But, this leads to a contradiction, because the only consistent proper finitary extension of $\models_{\mathcal{N}_4}$ is CPL, as shown in Theorem 9. While, after Corollary 5 and Corollary 8, CPL and $\models_{\mathcal{N}_2 \times \mathcal{N}_4}$ are two different consistent proper finitary extensions of $\models_{\mathcal{J}_4}$. □

6.2 Maximal paraconsistent logics of finite matrices

In the previous subsection we have dealt with matrices given by a finite product of finite NM-chains, but, as already commented, not all finite NM-algebras are finite products of finite chains. As it is well known, finite NM-algebras are finite products of finite directly indecomposable NM-algebras. In this section we are going to characterise finite directly indecomposable NM-algebras and this result will help us to obtain all maximal paraconsistent logics given by matrices over finite NM-algebras.

Recall that a NM-filter of an NM-algebra $\mathbf{A}$ is any set $F \subseteq A$ such that:

- $1 \in F$.
- If $a \in F$ and $a \leq b$, then $b \in F$.
- If $a, b \in F$, then $a * b \in F$.

We say that F is proper if $0 \notin F$, and F is a prime if it is proper and for every $a, b \in A$ if $a \vee b \in F$, then $a \in F$ or $b \in F$. As usual, $Spec(\mathbf{A})$ denotes the set of prime filters of $\mathbf{A}$. Since the prelinearity condition holds for every NM-algebra, if F is an NM-filter of $\mathbf{A}$, F is prime iff $\mathbf{A}/F$ is a NM-chain.

Using Zorn's Lemma one can prove that for any proper NM-filter F there is a maximal proper NM-filter G such that $F \subseteq G$. Moreover, every maximal filter is prime. The radical of $\mathbf{A}$, denoted by $Rad(\mathbf{A})$, is the intersection of all maximal filters of $\mathbf{A}$. We define $coRad(\mathbf{A}) = \{a \in A : \neg a \in Rad(\mathbf{A})\}$. From the characterisation of the radical of MTL-algebras given in [37], we have that $Rad(\mathbf{A}) = \{a \in A : a^n > \neg a, \text{ for all } n \geq 1\}$. In the case of NM-algebras, since every NM-algebra is 3-contractive, then $Rad(\mathbf{A}) = \{a \in A : a^2 > \neg a\}$. In the case of NM-chains it can be reduced to $Rad(\mathbf{A}) = \{a \in A : a > \neg a\}$.

Definition 11. An NM-algebra $\mathbf{A}$ is *local* iff it has a unique maximal filter.

Proposition 14. *Let $\mathbf{A}$ be a local NM-algebra. Then:*

- $A = Rad(\mathbf{A}) \cup coRad(\mathbf{A})$ *if $\mathbf{A}$ does not have a negation fixpoint;*
- $A = Rad(\mathbf{A}) \cup \{c\} \cup coRad(\mathbf{A})$ *if $\mathbf{A}$ has a negation fixpoint c.*

Proof. If $\mathbf{A}$ is local then let M be its maximal filter, which coincides with $Rad(\mathbf{A})$. For any $a \in A$, let a/M denote the class of a modulo M. Since $\mathbf{A}/M$ is simple, then either $\mathbf{A}/M \cong \mathbf{NM}_2$ or $\mathbf{A}/M \cong \mathbf{NM}_3$. If $\mathbf{A}/M \cong \mathbf{NM}_2$, then $A = 1/M \cup 0/M$. Notice that $1/M = M$ and $0/M = (\neg 1)/M = \{a : \neg a \in M\}$, thus $A = Rad(\mathbf{A}) \cup coRad(\mathbf{A})$ and $\mathbf{A}$ does not have a negation fixpoint. If $\mathbf{A}/M \cong \mathbf{NM}_3$, let $a \in A$ be such that $a/M \neq 1/M = M$ and $a/M \neq 0/M$, then $a/M = (\neg a)/M$. By the subdirect representation theorem, $\mathbf{A} \subseteq_{SD} \prod_{F \in Spec(\mathbf{A})} \mathbf{A}/F$ and $a = (a/F)_{F \in Spec(\mathbf{A})}$. Since $\mathbf{A}$ is local, $a/F \subseteq a/M$ and $(\neg a)/F \subseteq (\neg a)/M$ for every prime filter of $\mathbf{A}$. Since F is prime, $\mathbf{A}/F$ is totally ordered, hence either $a/F = \neg a/F$, or $a/F > (\neg a)/F$, or $a/F < (\neg a)/F$. If $a/F > (\neg a)/F$ or $a/F < (\neg a)/F$, then either a/F or $(\neg a)/F$ belongs to $Rad(\mathbf{A}/F) = M/F$ which leads to the contradiction that either $a \in M$ or $\neg a \in M$. Thus, $a/F = \neg a/F$ for every $F \in Spec(\mathbf{A})$. Hence a is the negation fixpoint of $\mathbf{A}$ and $a/M = \{a\}$ □

Proposition 15. *Let* $\mathbf{A}$ *be an NM-algebra.* $\mathbf{A}$ *is directly indecomposable iff* $\mathbf{A}$ *is local.*

Proof. Assume $\mathbf{A}$ is indecomposable. Recall that an NM-algebra is indecomposable iff its only boolean elements are $\mathbf{0}$ and $\mathbf{1}$. Let $a \in A$, notice that $2a^2 = \neg\left(\neg a^2\right)^2$ is a boolean element, thus $2a^2 = \mathbf{1}$ or $2a^2 = \mathbf{0}$. If $2a^2 = \mathbf{1}$, then $\neg a^2 < a^2 \leq a$, thus $a \in Rad(\mathbf{A})$. If $2a^2 = \mathbf{0}$, then $a^2 = \mathbf{0}$, so a cannot belong to any proper filter. This shows that $Rad(\mathbf{A})$ is in fact a maximal filter, so $\mathbf{A}$ is local.

Assume $\mathbf{A}$ is local. Let b be a boolean element of $\mathbf{A}$, then $b/Rad(\mathbf{A})$ is also a boolean element of $\mathbf{A}/Rad(\mathbf{A})$. Since $\mathbf{A}/Rad(\mathbf{A})$ is a simple algebra, it is indecomposable, hence the class of $b/Rad(\mathbf{A})$ is either 1 or 0. Thus, either $b \in Rad(\mathbf{A})$ or $\neg b \in Rad(\mathbf{A})$, and since b is boolean, this in turn implies $b = \mathbf{1}$ or $b = \mathbf{0}$. □

Corollary 6. *Every finite NM-algebra is a finite product of finite local NM-algebras.*

Theorem 10. *The only finite matrices defining maximal paraconsistent logics are* $\mathcal{J}_3$*,* $\mathcal{J}_4$ *and* $\mathcal{J}_3 \times \mathcal{J}_4$*.*

Proof. For practical reasons, in the following proof we will identify a matrix $\mathcal{J}_i$ with its corresponding logic $\models_{\mathcal{J}_i}$.

Notice that $\mathcal{J}_3$, $\mathcal{J}_4$ and $\mathcal{J}_3 \times \mathcal{J}_4$ are not explosive. Let $\mathcal{M}$ be a paraconsistent finite matrix. By Lemmas 5 and 6 we may assume that $\mathcal{M} = \langle \mathbf{A}, F \rangle$ is a submatrix of $\mathcal{J}_{m_1} \times \cdots \times \mathcal{J}_{m_k}$ where each $m_i > 2$. By Corollary 6, $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_r$ where each $\mathbf{A}_j$ is a finite local NM-algebra. Moreover, since F is a principal lattice filter, let $a = (a_1, \ldots, a_r)$ be the generator of the filter, then $F = F_{a_1} \times \cdots \times F_{a_r}$. Since the matrix logic $\langle \mathbf{A}, F \rangle$ is not explosive then for every $j = 1 \div r$, $\langle \mathbf{A}_j, F_j \rangle$ is also not explosive.

If $\mathbf{A}_j$ has a negation fix point, then trivially $\mathbf{NM_3}$ is embeddable into $\mathbf{A}_j$ and $\mathcal{J}_3$ is a submatrix of $\langle \mathbf{A}_j, F_j \rangle$. If $\mathbf{A}_j$ does not have a negation fix point, let a be the minimum of the elements in $Rad(\mathbf{A}_j)$. If $a \neq \mathbf{1}$, then $\{\mathbf{1}, a, \neg a, \mathbf{0}\}$ is the universe of a subalgebra $\mathbf{S}$ of $\mathbf{A}_j$ isomorphic to $\mathbf{NM_4}$. Since the logic of $\langle \mathbf{A}_j, F_j \rangle$ is not explosive then $Rad(\mathbf{A}_j)$ is not trivial, $a \neq 1$ and $a, \neg a \in F_j$. Therefore $\mathcal{J}_4$ is a submatrix of $\langle \mathbf{A}_j, F_j \rangle$. Summing all up:

- If all A_j's have a negation fix point then $\mathcal{J}_3$ is an extension of the logic given by $\mathcal{M}$.
- If all A_j's do not have a negation fix point then $\mathcal{J}_4$ is an extension of the logic given by $\mathcal{M}$.
- If there is some A_j with a negation fix point and there is some A_s with no negation fix point then $\mathcal{J}_3 \times \mathcal{J}_4$ is an extension of the logic given by $\mathcal{M}$.

□

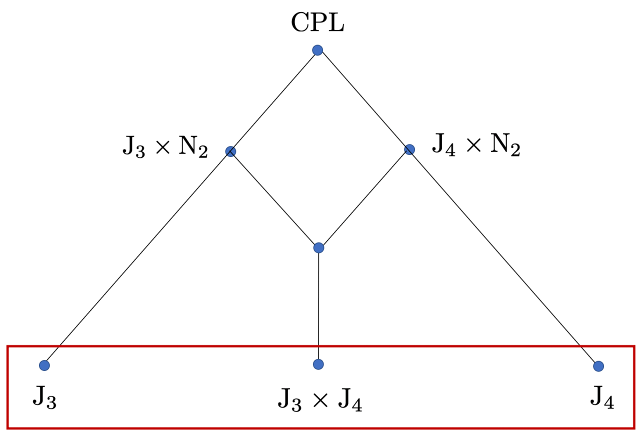

Figure 6: Maximal paraconsistent logics

It is well known (see [17]) that $\mathcal{J}_3$ is *ideal* paraconsistent[14] where the definable implication that satisfies deduction theorem (D.T.) is $\varphi \Rightarrow \psi := (\neg\varphi \to \varphi)^2 \to \psi$. However we show that both $\mathcal{J}_4$ and $\mathcal{J}_3 \times \mathcal{J}_4$ are not ideal paraconsistent.[15] Assume $\mathcal{J}_4$ is ideal paraconsistent. Then $\mathcal{J}_4$ has a definable implication $\rightharpoonup$ satisfying D.T. such that $\varphi \rightharpoonup \psi$ is classically equivalent to $\varphi \to \psi$. Then $\varphi \models_{\mathcal{J}_4} \psi$ iff $\models_{\mathcal{J}_4} \varphi \rightharpoonup \psi$. By Proposition 10, $\models_{\mathcal{J}_4} \varphi \rightharpoonup \psi$ iff $\vdash_{CPL} \varphi \rightharpoonup \psi$ iff $\varphi \vdash_{CPL} \psi$. Thus, $\mathcal{J}_4$ and CPL coincide, which is a contradiction, since the MP rule is valid in CPL but not in $\mathcal{J}_4$. For the case of $\mathcal{J}_3 \times \mathcal{J}_4$, since $\mathcal{J}_2 \times \mathcal{J}_4$ is a proper extension of $\mathcal{J}_3 \times \mathcal{J}_4$ different from CPL and the tautologies of $\mathcal{J}_2 \times \mathcal{J}_4$ are exactly the classical tautologies, $\mathcal{J}_3 \times \mathcal{J}_4$ is not maximal w.r.t. CPL. Therefore, it is not ideal paraconsistent.

[14] Recall from [2] that a propositional logic L such that it has an implication connective $\to$ for which the deduction theorem holds, and that is paraconsistent w.r.t. a negation connective $\neg$, is called *ideal* $\neg$*-paraconsistent* if: (1) there is a presentation of CPL in the same signature than L such that $\to$ and $\neg$ are interpreted as in CPL; (2) L is a sublogic of CPL; (3) L is maximal w.r.t. CPL; and (4) every proper extension of L in the same signature is not $\neg$-paraconsistent.

[15] Note that the logic defined by the matrix $\mathcal{J}_4$, denoted J_4 in Figure 6, is not the same as the logic defined by the matrix $\langle \text{Ł}_{\mathbf{4}}, F_{1/3}\rangle$ over the 4-valued MV-chain, also denoted J_4 in [16]. Actually, unlike $\mathcal{J}_4$, the logic defined by the matrix $\langle \text{Ł}_{\mathbf{4}}, F_{1/3}\rangle$ was shown in [16] to be ideal paraconsistent.

7 Conclusions and future work

In this paper we have considered logics induced by logical matrices defined on NM algebras with lattice filters, with special attention to those that are paraconsistent and preserve the non-falsity. Interestingly enough, as a first main contribution, we have shown that the logic defined by a matrix $\langle \mathbf{A}, F\rangle$, where $\mathbf{A}$ is a NM-chain and F a lattice filter of $\mathbf{A}$, can be reduced to either to a 1-preserving logic $\langle \mathbf{B}, \{\mathbf{1}\}\rangle$ for some $\mathbf{B}$ subalgebra of $\mathbf{A}$, to the well-known paraconsistent logic J_3, or to a non-falsity preserving (paraconsistent) logic $\langle \mathbf{B}, (\mathbf{0}, \mathbf{1}]\rangle$ for some $\mathbf{B}$ subalgebra of $\mathbf{A}$. Moreover, we have axiomatised the non-falsity preserving companion of the logic NM, denoted nf-NM, corresponding to the matrix $\langle [\mathbf{0}, \mathbf{1}]_{\mathbf{NM}}, (0, 1]\rangle$. A second main contribution is the study of the expansion of the paraconsistent logic nf-NM with a consistency operator so to obtain a Logic of Formal Inconsistency (LFI). Several classes of such operators and their logics have been considered and fully characterised. A final third main contribution is the full classification and characterisation of all the logics of matrices defined over finite products of finite NM-chains with lattice filters, where the presence of the $F_{(\mathbf{0}}$ filters makes the study much more complex than the case of considering only $F_{\mathbf{1}}$ filters, that was already studied in [29].

Within the class of Mathematical fuzzy logics, the lattice filter-based NM logics studied in this paper are remarkably related to those over extensions of Łukasiewicz logic Ł, over Gödel logic with involution $\mathrm{G}^{\sim}$ and over Product logic with involution $\Pi^{\sim}$, since all of them share a strong or involutive negation, although there are notable differences among them as well. Actually, Łukasiewicz and Gödel logics are, together with Product logic, the most prominent BL-logics, while NM is the most prominent logic among the extensions of the involutive MTL logic, IMTL, that is not a BL logic. At this point we would like to make the following remarks about analogies and differences among the logics nf-NM, nf-Ł, nf-$\mathrm{G}^{\sim}$ and nf-$\Pi^{\sim}$:

- Since the three-element NM-algebra, MV-algebra and $\mathrm{G}^{\sim}$-algebra are termwise equivalent, we have nf-NM_3 = nf-Ł_3 = nf-$\mathrm{G}_3^{\sim}$, which in turn coincide with the well-known d'Ottaviano and da Costa's logic J_3.

- Both nf-$\mathrm{G}^{\sim}$ and nf-$\Pi^{\sim}$ are interpretable in $\mathrm{G}^{\sim}$ and $\Pi^{\sim}$ respectively by the double negation transformation. Indeed, in both G and Π the residual negation $\neg\varphi = \varphi \to 0$ is Gödel negation, whose interpretation in $[0, 1]$ is the mapping defined by $\neg x = 1$ if $x = 0$ and $\neg x = 0$ otherwise. Then, it holds that, for L $\in \{\mathrm{G}^{\sim}, \Pi^{\sim}\}$, $\varphi \vdash^{L}_{(0} \psi$ iff $\neg\neg\varphi \vdash^{L} \neg\neg\psi$. Moreover, it is not difficult to check that if we add to the axioms and rule of L the following modified modus ponens rule

  - (mod-MP): From φ and $\neg\neg\varphi \to \neg\neg\psi$ derive ψ

  one gets a sound and complete axiomatisation of nf-L. Note that in the logics nf-$\mathrm{G}^{\sim}$ and nf-$\Pi^{\sim}$, the usual Modus Ponens rule is sound.

- The study of the lattice of matrix logics defined over finite MV-algebras and finite G$^\sim$-algebras with lattice filters is simpler than in the case of NM-logics since all finite MV-algebras or G$^\sim$-algebras are products of finite chains, this is not the case with NM-algebras. However, it is an *open problem* whether the logics defined over finite NM-algebras that are not products of NM-chains give raise to different logics.

- About the non-falsity preserving companion of Łukasiewicz logic, it is worth noticing that the technique used in Section 4.3 to prove completeness of $\mathsf{nf}\text{-}\overline{\mathrm{NM}}_\circ$ is very general, indeed it can be applied to prove completeness for the non-falsity preserving companion of any axiomatic extension of a MTL logic with an involutive negation (i.e. an extension of an IMTL logic), see e.g. [22]. In particular, the non-falsity preserving companion of the well-known Łukasiewicz logic Ł, nf-Ł, can be axiomatised by the following system:

  - Axioms of Ł
  - Rule of Adjunction: (Adj) $\dfrac{\varphi, \quad \psi}{\varphi \wedge \psi}$
  - Reverse Modus Ponens: (MP^r) $\dfrac{\neg\psi \vee \chi}{\neg\varphi \ \vee \ \neg(\varphi \to \psi) \ \vee \ \chi}$
  - Restricted Modus Ponens: (r-MP) $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$, if $\vdash_{\text{Ł}} \varphi \to \psi$

  Indeed, by applying the proof technique of Proposition 7 and Theorem 3, one readily gets that this logic is (finite strong) complete with respect to the finitary consequence relation defined by the logical matrix $\langle [\mathbf{0}, \mathbf{1}]_{\mathbf{MV}}, (0, 1] \rangle$.

  Remarkably, this logic can be seen as a more genuine paraconsistent counterpart of Ł rather than that the logic **FT** introduced by Avron in [3], since **FT** maintains the connectives $\wedge, \vee$ and $\neg$ but replaces Łukasiewicz implication by another one that validates Modus Ponens in $\langle [\mathbf{0}, \mathbf{1}]_{\mathbf{MV}}, (0, 1] \rangle$.

- Finally, about the question of whether the non-falsity preserving logics are Logics of Formal Inconsistency, there is a difference between nf-NM and nf-Ł on the one hand and nf-G$^\sim$ and nf-$\Pi^\sim$ on the other, since the former logics do not have a definable consistency operator, while in the latter logics one can define such an operator, definable in turn from the Δ operator (where $\Delta x = \neg{\sim}x$), as $\circ x = \Delta(x \vee \neg x)$. So nf-G$^\sim$ and nf-$\Pi^\sim$ are LFIs while nf-NM and nf-Ł are not.

As for future work, we envisage to extend this work in at least two lines. One aspect to further analyse is the complexity, expressive power and further properties from a paraconsistency point of view of the non-falsity preserving logics nf-L, with L being a finitary extension of NML. Another is to open the scope and study the definition and axiomatization of non-falsity preserving companions of MTL extensions in general, deepening the preliminary results [22]. We also plan to study some of the logic systems analysed here from a proof-theoretic perspective.

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