Strong Planning in the Logics of Communication and Change

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Abstract. In this contribution we study how to adapt Backward Plan search to the Logics of Communication and Change (LCC). These are dynamic epistemic logics with common knowledge modeling the way in which announcements, sensing and world-changing actions modify the beliefs of agents or the world itself. The proposed LCC planning system greatly expands the social complexity of scenarios involving cognitive agents that can be solved. For example, goals or plans may consist of a certain distribution of beliefs and ignorance among agents. Our results include: soundness and completeness of backward planning (breadth first search), both for deterministic and strong non-deterministic planning.

1 Introduction

Practical rationality or decision-making is a key component of autonomous agents, like humans, and correspondingly has been studied at large. This research has been conducted from several fields: game theory, planning, decision theory, etc. each focusing on a different aspect (strategic decision-making, propositional means-ends analysis, and uncertainty, respectively).

While the different models are well-understood, they were (understandably) designed with a considerably low level of expressivity at the object language. For instance, game-theory does not represent the logical structure underlying the states, actions and goals; planning [5], on the other hand, represents part of it with atomic facts and negation, but it traditionally disregards other existing agents. All this contrasts with the area of logic, where logics for multi-agent systems (with increasing expressivity) have been characterized.

Specially relevant to the topic of cognitive agents are the notions of belief, action, goal, norm, and so on. The first two elements are the target of dynamic epistemic logics DEL [3], [15], [16], a recent family of logics which allow us to reason about agents' communications, observations and the usual world-changing actions. We focus on the so-called Logics of Communication and Change (LCC) [13], which generalize many previously known DEL logics, and hence include a rich variety of epistemic actions (in the DEL literature) and ontic actions (from the tradition on planning). Briefly, LCC logics are dynamic epistemic logics with common knowledge, ontic actions and several types of communicative actions (truthful or lying, public or private announcements).

Less consensus exists about representing and reasoning with motivational attitudes like goals, desires or intentions. On the one hand, logics in the BDI tradition (belief-desire-intention) [12] make them explicit in the language, e.g. one can express *agent a*

has goal φ; in the planning tradition, though, one only makes explicit their propositional content φ (what makes φ a goal is just its membership to the set of goals). Here we adopt the second (and less expressive) representation of goals.

In the present contribution, we describe a system for planning that accepts arbitrary epistemic formulas (e.g. common knowledge) as goals or state descriptions, and with ontic/epistemic actions given by Kripke-like action models. The language of LCC logics (used to this end) is further extended with action composition $\otimes$ and choice $\cup$, in order to study planning with non-deterministic actions. In this sense, we slightly generalize on previous results in [9] and [10], by dropping a technical restriction on the precondition of non-deterministic actions, and proposing slightly different plan structures. In summary, we define a breadth first search (BFS) algorithm for strong planning in the extended LCC logics. This search method is proved to be sound and complete: its outputs are (logically) successful plans and if such a successful plan exists, the algorithm terminates with some such solution. Finally, this algorithm easily extends to optimal plan search when each action is assigned some cost for its execution.

Motivating example. Our aim, then, is to endow LCC logic based agents with planning capacities for this logic, so they can achieve their goals in scenarios where other agents have similar cognitive and acting abilities. In particular, LCC planning seems necessary for an agent whose goals consist in (or depend on) a certain distribution of knowledge and ignorance among agents. To illustrate the kind of rational behavior an LCC planner can exhibit, consider the following example:

Example 1. Agent a placed a bet with agent b that the next coin toss would be heads (h). Agent a knows she can toss the coin and detect its outcome, or flip the coin, without agent b knowing about it. Given a sensing action that tells a whether h holds or not, a successful plan seems to be: toss the coin; if sense that h, then show h to b; otherwise flip the coin and show h.

2 Related Work

Among logics for action guidance, the family of BDI [12] and related logics for intention are possibly the more popular. While these logics usually allow for considerable expressivity w.r.t. motivational attitudes (and their interaction with beliefs), they are not completely understood at a syntactic level. In fact, the use of planning methods has been suggested for an implementation of a BDI architecture. In particular, [4] suggest the use of LCC planning for the corresponding fragment of BDI logic. In this work [4] (see also [8]), the authors study LCC forward planning based on the semantics of update models; the BFS search algorithm is shown to be complete for LCC forward planning and in addition this problem (LCC forward planning) is shown to be semi-decidable in the general multi-agent case. An extension for (single-agent) conditional plan search in AND/OR-graphs can be found in [1]. The present work addresses the multi-agent case using instead a backward search approach (in OR-graphs). The motivation for this lies in the nature of communicative actions: while forward search is based on actions that are *executable*, backward search focuses on actions that are *relevant to the current* *goals*. This makes a difference in LCC since many actions will exist which are everywhere executable, so forward planning will typically face the state explosion problem. Another work along the same lines is [2] (and related papers) where regression methods are introduced for the fragment of LCC without common knowledge. Regression can also be used as a (non-incremental) planning algorithm for LCC.

3 Preliminaries: The Logics of Communication and Change

Logics for agents with epistemic and communicative abilities have been developed in the recent years, ranging from epistemic logic [7] (for individual, group or common belief or knowledge), to logics of announcements [3], [15] (public or private, honest or dishonest), and finally to incorporating ontic actions (i.e. world-changing actions) [16]. All this has been unified within the single framework of Logics of Communication and Change [13], or LCC logics, formally a dynamic extension of epistemic logic using action models. This work proposes a general (translation-based) method that provides a complete axiomatization of an LCC logic from the specification of its particular action model. Since LCC logics are built by adding dynamic action models U on top of $\mathrm{E{\cdot}PDL}$ (propositional dynamic logic PDL under an epistemic reading), we recall PDL first.

3.1 Epistemic PDL

Propositional dynamic logic [6] is a modal logic for reasoning about programs, with modalities $[\pi]$ (and $\langle\pi\rangle$) expressing *after executing program π it is necessarily (resp. possibly) the case that*. Using a semantics for programs π based on relations R_π (between the internal states of a machine running the program), the PDL programs π are built from basic actions a and the program constructors of composition $a; b$ (*do a then b*), choice $a \cup b$ (*either do a or b*), test $?\varphi$ (*test φ, and proceed if true or terminate*) and iteration a^* (*do a; repeat*) (the Kleene-star for the reflexive transitive closure). It was later suggested [13] that the dynamic modalities of PDL naturally admit an epistemic interpretation as well, called $\mathrm{E{\cdot}PDL}$, if we read the basic "program" $[a]$ as the modality for agent a's knowledge or belief; that is, $[a]\varphi$ reads: *a knows φ*, or *a believes φ*; and $\langle a\rangle$ reads: *a considers it possible that φ*. Note that epistemic PDL does not distinguish between knowledge and belief, as usually understood by the S5 and KD45 modal logics, respectively. And thus, at the abstract level of PDL we will indistinctly refer to $[a]$ as knowledge or belief. Within a particular model, though, we can properly refer to one or the other depending on the semantic properties, e.g. whether $[a]\varphi \to \varphi$ holds, etc.

Definition 1. *The language of* $\mathrm{E{\cdot}PDL}$*, denoted by* $\mathcal{L}_{\mathrm{E\cdot PDL}}$*, for a given sets of atoms* $p \in \mathsf{Var}$ *and agents* $a \in \mathsf{Ag}$ *consists of the following formulas* φ *and programs* π*:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid [\pi]\varphi \qquad \pi ::= a \mid ?\varphi \mid \pi_1; \pi_2 \mid \pi_1 \cup \pi_2 \mid \pi^*$$

The symbols $\bot, \vee, \leftrightarrow$ and $\langle\pi\rangle$ are defined from the above as usual. Under the epistemic reading, the PDL program constructors allow us to model, among others,

$[a; b]$	*agent a believes that b believes that*	(nested belief)
$[B]$, or $[a \cup b]$	*agents in $B = \{a, b\}$ believe that*	(group belief)
$[B^*]$, or $[(a \cup b)^*]$	*it is common knowledge among B that*	(comm. knowl.)

An E·PDL model $M = (W, \langle R_a \rangle_{a \in \mathsf{Ag}}, V)$ does, as usual, contain a set of worlds W, a relation R_a in W for each agent a, and an evaluation $V : \mathsf{Var} \to \mathcal{P}(W)$).

Definition 2. *The semantics of* E·PDL *consists of models* $M = (W, \langle R_a \rangle_{a \in \mathsf{Ag}}, V)$*, containing: a set of worlds* W*, a relation* R_a *in* W *for each agent* a*, and an evaluation* $V : \mathsf{Var} \to \mathcal{P}(W)$*. This map* V *extends to a map* $[\![\varphi]\!]^M$ *for each formula* φ *in* $\mathcal{L}_{\mathrm{E \cdot PDL}}$*:*

$$\begin{array}{rlrl} [\![\top]\!]^M &= W & [\![a]\!]^M &= R(a) \\ [\![p]\!]^M &= V(p) & [\![?\varphi]\!]^M &= \mathsf{Id}_{[\![\varphi]\!]} \\ [\![\neg\varphi]\!]^M &= W \setminus [\![\varphi]\!]^M & [\![\pi_1;\pi_2]\!]^M &= [\![\pi_1]\!]^M \circ [\![\pi_2]\!]^M \\ [\![\varphi_1 \wedge \varphi_2]\!]^M &= [\![\varphi_1]\!]^M \cap [\![\varphi_2]\!]^M & [\![\pi_1 \cup \pi_2]\!]^M &= [\![\pi_1]\!]^M \cup [\![\pi_2]\!]^M \\ & & [\![\pi^*]\!]^M &= ([\![\pi]\!]^M)^* \end{array}$$

$$[\![[\pi]\varphi]\!]^M = \{w \in W \mid \forall v((w,v) \in [\![\pi]\!]^M \Rightarrow v \in [\![\varphi]\!]^M\}$$

where $\circ$ *and* * *are the composition and reflexive transitive closure of relations.*

Notice in particular that $[\![?\bot]\!]^M = \varnothing$ and $[\![?\top]\!]^M = \mathsf{Id}_W$ (the identity relation on W). We recall the axioms/rules of E·PDL that provide a sound and complete axiomatization:

$$\begin{array}{rl} (K) & \vdash\ [\pi](\varphi \to \psi) \to ([\pi]\varphi \to [\pi]\psi) \\ (test) & \vdash\ [?\varphi_1]\varphi_2 \leftrightarrow (\varphi_1 \to \varphi_2) \\ (sequence) & \vdash\ [\pi_1;\pi_2]\varphi \leftrightarrow [\pi_1][\pi_2]\varphi \\ (choice) & \vdash\ [\pi_1 \cup \pi_2]\varphi \leftrightarrow [\pi_1]\varphi \wedge [\pi_2]\varphi \\ (mix) & \vdash\ [\pi^*]\varphi \leftrightarrow \varphi \wedge [\pi][\pi^*]\varphi, \text{ and} \\ (induction) & \vdash\ \varphi \wedge [\pi^*](\varphi \to [\pi]\varphi)) \to [\pi^*]\varphi. \\ (Modus\ ponens) & \text{From} \vdash \varphi_1 \text{ and} \vdash \varphi_1 \to \varphi_2, \text{ infer } \varphi_2, \\ (Necessitation) & \text{From} \vdash \varphi, \text{ infer} \vdash [\pi]\varphi. \end{array}$$

3.2 Action models U, e

An LCC logic will add to an E·PDL language a set of modalities $[\mathsf{U}, \mathsf{e}]$ for each pointed action model U, e with distinguished (actual) action e. These new operators $[\mathsf{U}, \mathsf{e}]$ read *after each execution of action* e *it is the case that*. An action model is a tuple $\mathsf{U} = (\mathsf{E}, \mathsf{R}, \mathsf{pre}, \mathsf{post})$ containing

- $\mathsf{E} = \{\mathsf{e}_0, \ldots, \mathsf{e}_{n-1}\}$, a set of actions
- $\mathsf{R} : \mathsf{Ag} \to (\mathsf{E} \times \mathsf{E})$, a map assigning a relation R_a to each agent $a \in \mathsf{Ag}$
- $\mathsf{pre} : \mathsf{E} \to \mathcal{L}_{\mathrm{PDL}}$, a map assigning a precondition $\mathsf{pre}(\mathsf{e})$ to each action e
- $\mathsf{post} : \mathsf{E} \times \mathsf{Var} \to \mathcal{L}_{\mathrm{PDL}}$, a map assigning a post-condition $\mathsf{post}(\mathsf{e})(p)$, or $p^{\mathsf{post}(\mathsf{e})}$, to each $\mathsf{e} \in \mathsf{E}$ and $p \in \mathsf{Var}$

Let us fix the above enumeration $\mathsf{e}_0, \ldots, \mathsf{e}_{n-1}$ which will be used throughout the paper, unless stated otherwise. During plan search, in particular, when we refine a plan with some new action, the different alternatives will be considered according to this ordering: the refinement with e_0 will be considered before the refinement with e_1, and so on.

Definition 3. *The language of the* LCC*-logic for an action model* U *extends the formulas of* E·PDL *(for the same set of variables* Var *and agents* Ag*) with modalities for pointed action models* U, e*, giving the following sets of formulas* φ *and programs* π*:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid [\pi]\varphi \mid [\mathsf{U}, \mathsf{e}]\varphi \qquad \pi ::= a \mid ?\varphi \mid \pi_1; \pi_2 \mid \pi_1 \cup \pi_2 \mid \pi^*$$

The new modalities $[\mathsf{U}, \mathsf{e}]\varphi$ represent *"after the execution of* e*,* φ *will hold"*. The semantics of LCC computes $M, w \models [\mathsf{U}, \mathsf{e}]p$ in terms of the product update of M, w and U, e. This product update is (again) an E·PDL pointed model $M \circ \mathsf{U}, (w, \mathsf{e})$, with

$$M \circ \mathsf{U} = (W', \langle R'_a \rangle_{a \in \mathsf{Ag}}, V') \qquad \text{where}$$

- the set W' consists of those worlds (w, e) such that $M, w \models \mathsf{pre}(\mathsf{e})$ (so executing e will lead to the corresponding state (w, e).)
- the relation $(w, \mathsf{e})R'_a(v, \mathsf{f})$ holds iff both $wR_a v$ and $\mathsf{e}\mathsf{R}_a\mathsf{f}$ hold; and
- the valuations are $V'(p) = \{(w, \mathsf{e}) \in W' \mid M, w \models \mathsf{post}(\mathsf{e})(p)\}$, (the truth-value of p after e depends on that of $\mathsf{post}(\mathsf{e})(p)$ before the execution)

An updated model $(W', \langle R'_a \rangle_{a \in \mathsf{Ag}}, V')$ will be denoted $(W^{M \circ \mathsf{U}}, \langle R_a^{M \circ \mathsf{U}} \rangle_{a \in \mathsf{Ag}}, V^{M \circ \mathsf{U}})$.

Example 2. Several types of announcement (that φ by agent a) can be expressed. As purely epistemic actions, they are assigned the trivial post-condition $\mathsf{post}(\cdot)(p) = p$.

- a (successful) *truthful* announcement to sub-group $X \subseteq \mathsf{Ag}$, denoted $[\mathsf{U}, \varphi!^a_X]$, with

$$\mathsf{pre}(\varphi!^a_X) = \varphi \quad \text{and} \quad \mathsf{R}_b(\varphi!^a_X, \mathsf{e}) \Leftrightarrow \begin{cases} \mathsf{e} = \varphi!^a_X & \text{if } b \in X \cup \{a\} \\ \mathsf{e} \in \{\varphi!^a_X, \mathsf{skip}\} & \text{if } b \notin X \cup \{a\} \end{cases}$$

- a (successful) *lying* announcement to X, denoted $\mathsf{U}, \varphi\dagger^a_X$, is defined by the same accessibility relation but with precondition $\mathsf{pre}(\varphi\dagger^a_X) = \neg\varphi$.

(Here skip is the null action defined $\mathsf{pre}(\mathsf{skip}) = \top$, and $\mathsf{post}(\mathsf{skip})(p) = p$.)

From here on we assume that post-conditions $\mathsf{post}(\mathsf{e})(p)$ are restricted to the elements $\{p, \top, \bot\}$, rather than $\mathsf{post}(\mathsf{e})(p)$ being an arbitrary formula. This restriction was studied in [16] for logics similar to LCC, with epistemic modalities for agents $[a]$ and group common knowledge $[B^*]$ for $B \subseteq \mathsf{Ag}$. The authors show that the logic resulting after this restriction on post-conditions is as expressive as the original where post-conditions are arbitrary formulas.

Later, we recover this expressivity by introducing a non-deterministic choice operator for actions. Let us remark that choice is more general than arbitrary post-conditions φ, since it can model the toss of a coin without describing which conditions φ would result in the coin landing heads.

This restriction makes the truth-value of p after e to be either of the following:

if $\mathsf{post}(\mathsf{e})(p) = \dots$	then the truth-value of p after e is ...
$\top$	true (since $\top$ is always true, hence true before e)
p	its truth-value before the execution of e
$\bot$	false (since $\bot$ is always false)

3.3 Logics of Communication and Change

The PDL semantics $[\![\cdot]\!]$ for E·PDL-formulas extends to a semantics for LCC by adding:

$$[\![[\mathsf{U},\mathsf{e}]\varphi]\!]^M = \{w \in W \mid \text{ if } M, w \models \mathsf{pre}(\mathsf{e}) \text{ then } (w,\mathsf{e}) \in [\![\varphi]\!]^{M\circ\mathsf{U}}\}.$$

In [13], the authors define program transformers $T^{\mathsf{U}}_{ij}(\pi)$ that provide a mapping between E·PDL programs (see Def. 4). Given any combination of actions in a model U the transformers provide a complete set of reduction axioms, reducing LCC to E·PDL. In a sketch, the U,e-modalities are pushed inside the formula, up to the case $[\mathsf{U},\mathsf{e}]p$.

Definition 4. *Let an action model* U *with* $\mathsf{E} = \{\mathsf{e}_0, \dots, \mathsf{e}_{n-1}\}$ *be given. The* program transformer *function* T^{U}_{ij} *is defined as follows:*

$$\begin{aligned}
T^{\mathsf{U}}_{ij}(a) &= \begin{cases} ?\mathsf{pre}(e_i); a & \text{if } \mathsf{e}_i \mathsf{R}(a) \mathsf{e}_j, \\ ?\bot & \text{otherwise} \end{cases} \\
T^{\mathsf{U}}_{ij}(?\varphi) &= \begin{cases} ?(\mathsf{pre}(\mathsf{e}_i) \wedge [\mathsf{U},\mathsf{e}_i]\varphi), & \text{if } i = j \\ ?\bot & \text{otherwise} \end{cases} \\
T^{\mathsf{U}}_{ij}(\pi_1;\pi_2) &= \textstyle\bigcup_{k=0}^{n-1} (T^{\mathsf{U}}_{ik}(\pi_1); T^{\mathsf{U}}_{kj}(\pi_2)) \\
T^{\mathsf{U}}_{ij}(\pi_1 \cup \pi_2) &= T^{\mathsf{U}}_{ij}(\pi_1) \cup T^{\mathsf{U}}_{ij}(\pi_2) \\
T^{\mathsf{U}}_{ij}(\pi^*) &= K^{\mathsf{U}}_{ijn}(\pi). \\
&\quad \text{where } K^{\mathsf{U}}_{ijn} \text{ is inductively defined as follows:} \\
K^{\mathsf{U}}_{ij0}(\pi) &= \begin{cases} ?\top \cup T^{\mathsf{U}}_{ij}(\pi) & \text{if } i = j \\ T^{\mathsf{U}}_{ij}(\pi) & \text{otherwise} \end{cases} \\
K^{\mathsf{U}}_{ij(k+1)}(\pi) &= \begin{cases} (K^{\mathsf{U}}_{kkk}(\pi))^* & \text{if } i = k = j \\ (K^{\mathsf{U}}_{kkk}(\pi))^*; K^{\mathsf{U}}_{kjk}(\pi) & \text{if } i = k \neq j \\ K^{\mathsf{U}}_{ikk}(\pi); (K^{\mathsf{U}}_{kkk}(\pi))^* & \text{if } i \neq k = j \\ K^{\mathsf{U}}_{ijk}(\pi) \cup (K^{\mathsf{U}}_{ikk}(\pi); (K^{\mathsf{U}}_{kkk}(\pi))^*; K^{\mathsf{U}}_{kjk}(\pi)) & \text{if } i \neq k \neq j \end{cases}
\end{aligned}$$

A calculus for the LCC logic of a given action model U is given by the following:

the axioms and rules for E·PDL	
$[\mathsf{U},\mathsf{e}]\top \leftrightarrow \top$	(top)
$[\mathsf{U},\mathsf{e}]p \leftrightarrow (\mathsf{pre}(\mathsf{e}) \to \mathsf{post}(e)(p))$	(atoms)
$[\mathsf{U},\mathsf{e}]\neg\varphi \leftrightarrow (\mathsf{pre}(\mathsf{e}) \to \neg[\mathsf{U},\mathsf{e}]\varphi)$	(negation)
$[\mathsf{U},\mathsf{e}](\varphi_1 \wedge \varphi_2) \leftrightarrow ([\mathsf{U},\mathsf{e}]\varphi_1 \wedge [\mathsf{U},\mathsf{e}]\varphi_2)$	(conjunction)
$[\mathsf{U},\mathsf{e}_i][\pi]\varphi \leftrightarrow \bigwedge_{j=0}^{n-1} [T^{\mathsf{U}}_{ij}(\pi)][\mathsf{U},\mathsf{e}_j]\varphi$	(E·PDL-programs)
if $\vdash \varphi$ then $\vdash [\mathsf{U},\mathsf{e}]\varphi$	(Necessitation)

The completeness for this calculus is shown by reducing LCC to E·PDL. The translation, simultaneously defined for formulas $t(\cdot)$ and programs $r(\cdot)$ is

$$\begin{array}{ll}
t(\top) & = \top \\
t(p) & = p \\
t(\neg\varphi) & = \neg t(\varphi) \\
t(\varphi_1 \wedge \varphi_2) & = t(\varphi_1) \wedge t(\varphi_2) \\
t([\pi]\varphi) & = [r(\pi)]t(\varphi) \\
t([\mathsf{U},\mathsf{e}]\top) & = \top \\
t([\mathsf{U},\mathsf{e}]p) & = t(\mathsf{pre}(\mathsf{e})) \to p^{\mathsf{post}(\mathsf{e})} \\
t([\mathsf{U},\mathsf{e}]\neg\varphi) & = t(\mathsf{pre}(\mathsf{e})) \to \neg t([\mathsf{U},\mathsf{e}]\varphi) \\
t([\mathsf{U},\mathsf{e}](\varphi_1 \wedge \varphi_2)) & = t([\mathsf{U},\mathsf{e}]\varphi) \wedge t([\mathsf{U},\mathsf{e}]\varphi_2) \\
t([\mathsf{U},\mathsf{e}_i][\pi]\varphi) & = \bigwedge_{j=0}^{n-1}[T_{ij}^{\mathsf{U}}(r(\pi))]t([\mathsf{U},\mathsf{e}_j]\varphi) \\
t([\mathsf{U},\mathsf{e}][\mathsf{U},\mathsf{e}']\varphi) & = t([\mathsf{U},\mathsf{e}]t([\mathsf{U},\mathsf{e}']\varphi))
\end{array}
\qquad
\begin{array}{ll}
r(a) & = a \\
r(B) & = B \\
r(?\varphi) & = ?t(\varphi) \\
r(\pi_1;\pi_2) & = r(\pi_1);r(\pi_2) \\
r(\pi_1 \cup \pi_2) & = r(\pi_1) \cup r(\pi_2) \\
r(\pi^*) & = (r(\pi))^*
\end{array}$$

These translation functions t and r will be part of the backward planning algorithms presented in the next sections.

Some basic properties of LCC needed in later results are stated next. Most claims in the next lemma seem to be folklore among the dynamic epistemic logic community.

Lemma 1. *The following hold for any* $\mathsf{e} \in \mathsf{E}$*:*

$$\begin{array}{rll}
\textit{(a)} & \models [\mathsf{U},\mathsf{e}]\bigvee_{k\leq n}\varphi_k \leftrightarrow \bigvee_{k\leq n}[\mathsf{U},\mathsf{e}]\varphi_k & \textit{any } \varphi \in \mathcal{L}_{\mathrm{LCC}} \\
\textit{(b)} & \models [\mathsf{U},\mathsf{e}]\varphi \leftrightarrow (\mathsf{pre}(\mathsf{e}) \to [\mathsf{U},\mathsf{e}]\varphi) & \textit{any } \varphi \in \mathcal{L}_{\mathrm{PDL}} \\
\textit{(c)} & \models [\mathsf{U},\mathsf{e}]\varphi \to (\mathsf{pre}(\mathsf{e}) \to \langle \mathsf{U},\mathsf{e}\rangle\varphi) & \textit{any } \varphi \in \mathcal{L}_{\mathrm{LCC}} \\
\textit{(c')} & \models \mathsf{pre}(\mathsf{e}) \leftrightarrow \langle \mathsf{U},\mathsf{e}\rangle\top & \\
\textit{(d)} & \models [\mathsf{U},\mathsf{e}]\theta \leftrightarrow (\mathsf{pre}(\mathsf{e}) \to \theta^{\mathsf{post}(\mathsf{e})}) & \textit{any Boolean } \theta \\
\textit{(e)} & \models \langle \mathsf{U},\mathsf{e}\rangle\varphi \leftrightarrow \langle \mathsf{U},\mathsf{e}\rangle\top \wedge [\mathsf{U},\mathsf{e}]\varphi & \textit{any } \varphi \in \mathcal{L}_{\mathrm{LCC}} \\
\textit{(f)} & \models [\mathsf{U},\mathsf{e}](\varphi \to \psi) \leftrightarrow ([\mathsf{U},\mathsf{e}]\varphi \to [\mathsf{U},\mathsf{e}]\psi) & \textit{(hence, axiom } \mathsf{K}\textit{)}
\end{array}$$

Proof. For the claim 1(a), we only show the case $n = 2$. The general case is completely analogous.

$$\begin{array}{l}
\quad M,w \models [\mathsf{U},\mathsf{e}]\varphi_1 \vee \varphi_2 \\
\text{iff } M,w \models [\mathsf{U},\mathsf{e}]\neg(\neg\varphi_1 \wedge \neg\varphi_2) \\
\text{iff } M,w \models \mathsf{pre}(\mathsf{e}) \to \neg[\mathsf{U},\mathsf{e}](\neg\varphi_1 \wedge \neg\varphi_2) \\
\text{iff } M,w \models \mathsf{pre}(\mathsf{e}) \to \neg([\mathsf{U},\mathsf{e}]\neg\varphi_1 \wedge [\mathsf{U},\mathsf{e}]\neg\varphi_2) \\
\text{iff } M,w \models \neg\mathsf{pre}(\mathsf{e}) \vee \neg[\mathsf{U},\mathsf{e}]\neg\varphi_1 \vee \neg[\mathsf{U},\mathsf{e}]\neg\varphi_2 \\
\text{iff } M,w \models \neg\mathsf{pre}(\mathsf{e}) \vee \neg[\mathsf{U},\mathsf{e}]\neg\varphi_1 \vee \neg\mathsf{pre}(\mathsf{e}) \vee \neg[\mathsf{U},\mathsf{e}]\neg\varphi_2 \\
\text{iff } M,w \models (\mathsf{pre}(\mathsf{e}) \to \neg[\mathsf{U},\mathsf{e}]\neg\varphi_1) \vee (\mathsf{pre}(\mathsf{e}) \to \neg[\mathsf{U},\mathsf{e}]\neg\varphi_2) \\
\text{iff } M,w \models [\mathsf{U},\mathsf{e}]\neg\neg\varphi_1 \vee [\mathsf{U},\mathsf{e}]\neg\neg\varphi_2) \\
\text{iff } M,w \models [\mathsf{U},\mathsf{e}]\varphi_1 \vee [\mathsf{U},\mathsf{e}]\varphi_2
\end{array}$$

1(b) This is immediate:

$$
\begin{array}{ll}
& M, w \models \mathsf{pre}(\mathsf{e}) \to [\mathsf{U}, \mathsf{e}]\varphi \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \text{ implies } M, w \models [\mathsf{U}, \mathsf{e}]\varphi \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \text{ implies } w \in [\![[U, \mathsf{e}]\varphi]\!]^M \qquad\qquad (\text{Def. } [\![\cdot]\!]) \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \text{ implies} \\
& \qquad (M, w \models \mathsf{pre}(\mathsf{e}) \text{ implies } (w, \mathsf{e}) \in [\![\varphi]\!]^{M \circ \mathsf{U}}) \ (\text{Def. 39 of [13]}) \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \text{ implies } (w, \mathsf{e}) \in [\![\varphi]\!]^{M \circ \mathsf{U}} \\
\text{iff } & w \in [\![[U, \mathsf{e}]\varphi]\!]^M \qquad\qquad (\text{Def. } [\![\cdot]\!]) \\
\text{iff } & M, w \models [\mathsf{U}, \mathsf{e}]\varphi
\end{array}
$$

1(c) Let M, w be arbitrary.

$$
\begin{array}{lllll}
& M, w \models [\mathsf{U}, \mathsf{e}]\varphi & & & \\
\text{iff } & M, w \models (\mathsf{pre}(\mathsf{e}) \to \mathsf{pre}(\mathsf{e})) \wedge [\mathsf{U}, \mathsf{e}]\varphi & & & \\
\text{iff } & M, w \models (\mathsf{pre}(\mathsf{e}) \to \mathsf{pre}(\mathsf{e})) \wedge (\mathsf{pre}(\mathsf{e}) \to [\mathsf{U}, \mathsf{e}]\varphi) & & & \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \to (\mathsf{pre}(\mathsf{e}) \wedge [\mathsf{U}, \mathsf{e}]\varphi) & & & \\
\text{iff } & M, w \not\models \mathsf{pre}(\mathsf{e}) & \text{or} & M, w \models & \mathsf{pre}(\mathsf{e}) \wedge [\mathsf{U}, \mathsf{e}]\varphi \\
\text{iff } & M, w \not\models \mathsf{pre}(\mathsf{e}) & \text{or} & M, w \models & \neg(\neg\mathsf{pre}(\mathsf{e}) \vee \neg[\mathsf{U}, \mathsf{e}]\varphi) \\
\text{iff } & M, w \not\models \mathsf{pre}(\mathsf{e}) & \text{or} & M, w \not\models & \neg\mathsf{pre}(\mathsf{e}) \vee \neg[\mathsf{U}, \mathsf{e}]\varphi \\
\text{iff } & M, w \not\models \mathsf{pre}(\mathsf{e}) & \text{or} & M, w \not\models & \mathsf{pre}(\mathsf{e}) \to \neg[\mathsf{U}, \mathsf{e}]\varphi \\
\text{iff } & M, w \not\models \mathsf{pre}(\mathsf{e}) & \text{or} & M, w \not\models & [\mathsf{U}, \mathsf{e}]\neg\varphi \\
\text{iff } & M, w \not\models \mathsf{pre}(\mathsf{e}) & \text{or} & M, w \models & \neg[\mathsf{U}, \mathsf{e}]\neg\varphi \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \to \neg[\mathsf{U}, \mathsf{e}]\neg\varphi & & & \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \to \langle \mathsf{U}, \mathsf{e} \rangle\varphi & & &
\end{array}
$$

1(c') For the particular case of $\varphi = \top$, we just add the validity $\models [\mathsf{U}, \mathsf{e}]\top$, and thus obtain from (c) that $\models \mathsf{pre}(\mathsf{e}) \to \langle \mathsf{U}, \mathsf{e} \rangle\top$. For the other direction, towards a contradiction, let

$$
\begin{array}{lll}
& M, w \models \langle \mathsf{U}, \mathsf{e} \rangle\top \wedge \neg\mathsf{pre}(\mathsf{e}) & \\
\text{so } & M, w \models \langle \mathsf{U}, \mathsf{e} \rangle\top \wedge (\mathsf{pre}(\mathsf{e}) \to [\mathsf{U}, \mathsf{e}]\bot) & \\
\text{iff } & M, w \models \langle \mathsf{U}, \mathsf{e} \rangle\top \wedge [\mathsf{U}, \mathsf{e}]\bot & (\text{Lemma 1(a)}) \\
\text{iff } & M, w \models \langle \mathsf{U}, \mathsf{e} \rangle\top \wedge \neg\langle \mathsf{U}, \mathsf{e} \rangle\neg\bot & \\
\text{iff } & M, w \models \langle \mathsf{U}, \mathsf{e} \rangle\top \wedge \neg\langle \mathsf{U}, \mathsf{e} \rangle\top & (\text{contradiction})
\end{array}
$$

Thus, for arbitrary M and $w \in W$, we have $M, w \models \mathsf{pre}(\mathsf{e}) \to \langle \mathsf{U}, \mathsf{e} \rangle\top$.

1(d) By induction. We denote by σ the postcondition of e: $\sigma = \mathsf{post}(\mathsf{e})$.
(Case p) The reduction axiom for p just gives: $\models [\mathsf{U}, \mathsf{e}]p \leftrightarrow (\mathsf{pre}(\mathsf{e}) \to p^\sigma)$. (Case $\neg\varphi$) Assume (Ind. Hyp.) that $M, w \models [\mathsf{U}, \mathsf{e}]\theta$ iff $M, w \models (\mathsf{pre}(\mathsf{e}) \to \theta^\sigma)$. Then, we have

$$
\begin{array}{lll}
& M, w \models [\mathsf{U}, \mathsf{e}]\neg\theta & \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \to \neg([\mathsf{U}, \mathsf{e}]\theta) & (\text{LCC axiom for } \neg) \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \to \neg(\mathsf{pre}(\mathsf{e}) \to \theta^\sigma) & (\text{Ind. Hyp.}) \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \to (\mathsf{pre}(\mathsf{e}) \wedge \neg\theta^\sigma) & \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \to \neg\theta^\sigma & \\
\text{iff } & M, w \models \mathsf{pre}(\mathsf{e}) \to (\neg\theta)^\sigma & .
\end{array}
$$

(Case $\theta_1 \wedge \theta_2$) Assume (Ind. Hyp.) that the claim holds for θ_1 and for θ_2. Then

$$
\begin{aligned}
&M, w \models [\mathsf{U}, \mathsf{e}]\theta_1 \wedge \theta_2 \\
\text{iff } &M, w \models [\mathsf{U}, \mathsf{e}]\theta_1 \wedge [\mathsf{U}, \mathsf{e}]\theta_2 \\
\text{iff } &M, w \models [\mathsf{U}, \mathsf{e}]\theta_1 \text{ and } M, w \models [\mathsf{U}, \mathsf{e}]\theta_2 && \text{(LCC axiom for } \wedge) \\
\text{iff } &M, w \models \mathsf{pre}(\mathsf{e}) \to \theta_1^\sigma \text{ and } M, w \models \mathsf{pre}(\mathsf{e}) \to \theta_2^\sigma && \text{(Ind. Hyp.)} \\
\text{iff } &M, w \models \mathsf{pre}(\mathsf{e}) \to (\theta_1^\sigma \wedge \theta_2^\sigma) \\
\text{iff } &M, w \models \mathsf{pre}(\mathsf{e}) \to (\theta_1 \wedge \theta_2)^\sigma.
\end{aligned}
$$

This case concludes the inductive proof for the equivalence between $[\mathsf{U}, \mathsf{e}]\theta$ and $\mathsf{pre}(\mathsf{e}) \to \theta^{\mathsf{post}(\mathsf{e})}$.

1(e) We have the following equivalences:

$$
\begin{aligned}
&\models \langle \mathsf{U}, \mathsf{e}\rangle\varphi \leftrightarrow \neg[\mathsf{U}, \mathsf{e}]\neg\varphi && \text{(Def. } \langle \mathsf{U}, \mathsf{e}\rangle) \\
&\models \langle \mathsf{U}, \mathsf{e}\rangle\varphi \leftrightarrow \neg(\mathsf{pre}(\mathsf{e}) \to \neg[\mathsf{U}, \mathsf{e}]\varphi && \text{(Red. axiom } \neg) \\
&\models \langle \mathsf{U}, \mathsf{e}\rangle\varphi \leftrightarrow \mathsf{pre}(\mathsf{e}) \wedge \neg\neg[\mathsf{U}, \mathsf{e}]\varphi \\
&\models \langle \mathsf{U}, \mathsf{e}\rangle\varphi \leftrightarrow \mathsf{pre}(\mathsf{e}) \wedge [\mathsf{U}, \mathsf{e}]\varphi \\
&\models \langle \mathsf{U}, \mathsf{e}\rangle\varphi \leftrightarrow \langle \mathsf{U}, \mathsf{e}\rangle\top \wedge [\mathsf{U}, \mathsf{e}]\varphi && \text{(Lemma 1(c'))}
\end{aligned}
$$

1(f) Consider the following equivalences

$$
\begin{aligned}
&[\mathsf{U}, \mathsf{e}]\varphi \to \psi \\
\Leftrightarrow\ &[\mathsf{U}, \mathsf{e}]\neg\varphi \vee \psi \\
\Leftrightarrow\ &[\mathsf{U}, \mathsf{e}]\neg\varphi \vee [\mathsf{U}, \mathsf{e}]\psi && \text{Lemma 1(a)} \\
\Leftrightarrow\ &(\mathsf{pre}(\mathsf{e}) \to \neg[\mathsf{U}, \mathsf{e}]\varphi) \vee [\mathsf{U}, \mathsf{e}]\psi && \text{Axiom for } \neg \\
\Leftrightarrow\ &\neg\mathsf{pre}(\mathsf{e}) \vee \neg[\mathsf{U}, \mathsf{e}]\varphi \vee [\mathsf{U}, \mathsf{e}]\psi \\
\Leftrightarrow\ &[\mathsf{U}, \mathsf{e}]\varphi \to (\neg\mathsf{pre}(\mathsf{e}) \vee [\mathsf{U}, \mathsf{e}]\psi) \\
\Leftrightarrow\ &[\mathsf{U}, \mathsf{e}]\varphi \to (\mathsf{pre}(\mathsf{e}) \to [\mathsf{U}, \mathsf{e}]\psi) \\
\Leftrightarrow\ &[\mathsf{U}, \mathsf{e}]\varphi \to [\mathsf{U}, \mathsf{e}]\psi && \text{Lemma 1(b)}
\end{aligned}
$$

4 Backward Deterministic Planning in LCC

We proceed to introduce search algorithms for planning domains expressible in some LCC logic. In this section we study the deterministic case. The first step is to adapt the basic elements of planning systems:

- the goal and initial state are formulas of E·PDL (the static fragment of LCC).
- the set of available actions $A \subseteq \mathsf{E}$, among those in the action model U
- an available action is a pointed action model U, e where $\mathsf{e} \in A$

A deterministic plan is an executable sequence of actions in A that necessarily leads from any initial state to some goal state.

As we said, the proposed search methods for LCC planning are based on the above reduction of LCC into E·PDL. Given a (goal) formula φ for the current plan π and some action e, we want to compute the minimal conditions ψ (upon an arbitrary state) that would make φ to hold after e. After refinement of π with e, this minimal condition ψ will be the new goal replacing φ. More formally, we say $\psi \in \mathcal{L}_{\mathrm{PDL}}$ is the *weakest precondition* for a formula $[\mathsf{U}, \mathsf{e}]\varphi$, iff (in LCC)

$$\models \psi \leftrightarrow [\mathsf{U}, \mathsf{e}]\varphi.$$

This notion generalizes the definition in classical planning of open goals after refinement. Recall in classical planning, the different variables (or literals) p, q are logically independent, so the total effects of an action simply decompose into the individual effects w.r.t. each variable.

The weakest precondition for e to cause an arbitrary formula φ is the formula:

$$t([\mathsf{U}, \mathsf{e}]\varphi \wedge \langle \mathsf{U}, \mathsf{e} \rangle \top)$$

extracted from the reduction to E·PDL by way of translation using t, r. Indeed, the correctness of the translation based on t, r makes

$$\models t([\mathsf{U}, \mathsf{e}]\varphi \wedge \langle \mathsf{U}, \mathsf{e} \rangle \top) \leftrightarrow [\mathsf{U}, \mathsf{e}]\varphi \wedge \langle \mathsf{U}, \mathsf{e} \rangle \top$$

These functions t, r can then be seen as goal-transforming functions: a current goal φ is mapped into $t([\mathsf{U}, \mathsf{e}]\varphi \wedge \langle \mathsf{U}, \mathsf{e} \rangle \top)$, which becomes the new goal after we refine the plan with e.

Definition 5. *Given some* LCC *logic for an action model* U*, a* planning domain *is a triple* $\mathbb{M} = (\varphi_T, A, \varphi_G)$*, where* φ_T, φ_G *are consistent* E·PDL *formulas describing, resp., the initial and goal states; and* $A \subseteq \mathsf{E}$ *is the subset of a actions available to the agent.*

A solution *to* $\mathbb{M}$ *is a sequence* $\mathsf{f}_1, \ldots, \mathsf{f}_m \in A^{<\omega}$ *of actions in* A*, such that*

$$\models \varphi_T \to [\mathsf{U}, \mathsf{f}_1] \ldots [\mathsf{U}, \mathsf{f}_m]\varphi_G \quad \textit{and} \quad \models \varphi_T \to \langle \mathsf{U}, \mathsf{f}_1 \rangle \ldots \langle \mathsf{U}, \mathsf{f}_m \rangle \top$$

The subset $A \subseteq \mathsf{E}$ denotes those actions that are actually available to our planner-executor agent a. The reason to distinguish A from E is that some other agent $b \in \mathsf{Ag}$ might attribute our agent a some abilities which a does not actually possess, or b might fail to attribute a some of her actual abilities (and attribute her instead a decaffeinated version of some of these abilities). Thus, on the one hand, we want to distinguish the beliefs of b after an execution of some action e as depending on how b interpret this action e. On the other, we want to make explicit which abilities does our agent possess, in order to build realistic plans.

From here on, π will denote a deterministic plan, i.e. a sequence of actions e in decreasing order of execution (rather than an arbitrary epistemic PDL program as before). Plans are denoted by a pair (*action sequence*, *open goals*)

Definition 6. *Given some planning domain* $\mathbb{M} = (\varphi_T, A, \varphi_G)$*, the (initial) empty plan is the pair* $\pi_\varnothing = (\varnothing, \varphi_G)$ *and if* $\pi = (\pi, \varphi_{\mathsf{goals}(\pi)})$ *is a plan, then* $\pi(\mathsf{e}) = (\pi^\frown \langle \mathsf{e} \rangle, \varphi_{\mathsf{goals}}(\pi(\mathsf{e})))$*, defined by the goal* $\varphi_{\mathsf{goals}(\pi(\mathsf{e}))} = t([\mathsf{U}, \mathsf{e}]\varphi_{\mathsf{goals}(\pi)} \wedge \langle \mathsf{U}, \mathsf{e} \rangle \top)$*, is also a plan. A plan* π *is a* leaf *iff* $\varphi_{\mathsf{goals}(\pi(\mathsf{e}))}$ *is inconsistent, or* $\models \varphi_{\mathsf{goals}(\pi(\mathsf{e}))} \to \varphi_{\mathsf{goals}(\pi)}$*.*

Leafs are plans not worth considering, either because (a) when we add the last action refinement e, the resulting plan demands an inconsistent precondition $\varphi_{\mathsf{goals}(\pi(\mathsf{e}))}$ (and hence the plan cannot be executed) or (b) because e does not contribute to delete part of the previous goals $\varphi_{\mathsf{goals}(\pi)}$. The search space for the proposed planning algorithm (see below) is the set sequences $(\mathsf{f}_1, \ldots, \mathsf{f}_m) \in A^{<\omega}$. (These sequences are read in decreasing order of execution, i.e. as the sequence of operators $\mathsf{U}, \mathsf{f}_m, \ldots, \mathsf{U}, \mathsf{f}_1$.) Then, the planning algorithm explores just a fragment of this space, since it will not bother to generate/evaluate further refinements of leaf plans. A breadth first search (henceforth, BFS) algorithm for deterministic planning in LCC is given in Figure 1.

```
Input : 𝕄 = (φ_T, A, φ_G).
LET Plans = ⟨π_∅⟩ and π = π_∅
WHILE π does not satisfy ⊨ φ_T → φ_goals(π)
    DELETE π from Plans. SET Plans ← Plans ⌒⟨ π(e) | e ∈ A and π(e) not a leaf ⟩.
    SET π = the first element of Plans.
Output : π    (i.e. the sequence [U, e_1] . . . [U, e_k])
```

Fig. 1. BFS algorithm for backward deterministic planning in LCC.

Actions $\mathsf{e} \in \mathsf{E}$, as defined above, are deterministic, in the sense that $\models [\mathsf{U}, \mathsf{e}]\varphi \vee \psi \leftrightarrow ([\mathsf{U}, \mathsf{e}]\varphi \vee [\mathsf{U}, \mathsf{e}]\psi)$. Thus, deterministic plans consist of actions $\mathsf{e} \in \mathsf{E}$ in our current action models U. Later we will extend LCC with composition $\otimes$ and choice $\cup$ to study the non-deterministic case. There we will fully recover the expressivity of actions defined by arbitrary post-conditions $p^{\mathsf{post}(\mathsf{e})} = \varphi$ of [13], i.e. actions with conditional effects: *if* φ *then (after* e*)* p. The first contribution of this paper is the following result:[3]

Theorem 1. *BFS is sound and complete for* LCC *backward planning: the output* π *of the algorithm in Fig. 1 is a solution for* $(\varphi_T, A, \varphi_G)$*; conversely, if a solution exists, then the algorithm terminates (with a solution output).*

Proof. For Soundness, let $\pi_n = (\mathsf{f}_1, \ldots, \mathsf{f}_m)$ be the output of the algorithm. Let $\pi_k = (\mathsf{f}_k, \ldots, \mathsf{f}_n)$. We check by induction on the length of the plan that π has these two properties:

(S1) $\models \varphi_T \to [\mathsf{U}, \mathsf{f}_1] \ldots [\mathsf{U}, \mathsf{f}_k]\varphi_{\mathsf{goals}(\pi_{k+1})}$ (S2) $\models \varphi_T \to \langle \mathsf{U}, \mathsf{f}_1\rangle \ldots \langle \mathsf{U}, \mathsf{f}_k\rangle \top$

We show (S1)-(S2) by simultaneous induction on the length of the plan.

(Base Case)

(S1) The base case $\models \varphi_T \to [\mathsf{U}_1]\varphi_{\mathsf{goals}(\pi_2)}$ follows from

$$\begin{array}{ll} \models \varphi_T \to \varphi_{\mathsf{goals}(\pi_1)} & \text{(def. of output)} \\ \models \varphi_{\mathsf{goals}(\pi_1)} \to [\mathsf{U}, \mathsf{f}_1]\varphi_{\mathsf{goals}(\pi_2)} & \text{(def. of refinement).} \end{array}$$

These jointly imply our claim.

(S2) The base case $\models \varphi_T \to \langle \mathsf{U}, \mathsf{f}_1\rangle \top$, reduces to

$$\begin{array}{l} \text{(i)} \models \varphi_T \to \varphi_{\mathsf{goals}(\pi_1)}, \text{ and} \\ \text{(ii)} \models \varphi_{\mathsf{goals}(\pi_1)} \to \langle \mathsf{U}, \mathsf{e}\rangle \top. \end{array}$$

[3] Proofs for results in this paper can be found at the first author's webpage `www.iiia.csic.es/~pardo`.

But (i) holds by def. of output for π, and (ii) holds since

$$\varphi_{\mathsf{goals}(\pi_1)} = t([\mathsf{U},\mathsf{f}_1]\varphi \wedge \langle \mathsf{U},\mathsf{f}_1\rangle\top) \text{ implies } \langle \mathsf{U},\mathsf{f}_1\rangle\top.$$

(<u>Inductive Case</u>)

(S1) For the claim $\models \varphi_T \to [\mathsf{U},\mathsf{f}_1][\mathsf{U},\mathsf{f}_2]\dots[\mathsf{U},\mathsf{f}_{k+1}]\varphi_{\mathsf{goals}(\pi_{k+2})}$, consider

$$\begin{array}{lll}
(1) & \models \varphi_{\mathsf{goals}(\pi_{k+1})} \to [\mathsf{U},\mathsf{f}_{k+1}]\varphi_{\mathsf{goals}(\pi_{k+2})} & \text{(Def. 5)}, \\
(2) & \models [\mathsf{U},\mathsf{f}_k](\varphi_{\mathsf{goals}(\pi_{k+1})} \to [\mathsf{U},\mathsf{f}_{k+1}]\varphi_{\mathsf{goals}(\pi_{k+2})}) & \text{(1) + Nec.} \\
(3) & \models [\mathsf{U},\mathsf{f}_k]\varphi_{\mathsf{goals}(\pi_{k+1})} \to [\mathsf{U},\mathsf{f}_k][\mathsf{U},\mathsf{f}_{k+1}]\varphi_{\mathsf{goals}(\pi_{k+2})} & \text{(2) + } \mathsf{K} \\
(4) & \models [\mathsf{U},\mathsf{f}_{k-1}]([\mathsf{U},\mathsf{f}_k]\varphi_{\mathsf{goals}(\pi_{k+1})} \to [\mathsf{U},\mathsf{f}_k][\mathsf{U},\mathsf{f}_{k+1}]\varphi_{\mathsf{goals}(\pi_{k+2})}) & \text{(3) + Nec.} \\
\vdots & \vdots & \vdots \\
(2k+1) & \models [\mathsf{U},\mathsf{f}_1]([\mathsf{U},\mathsf{f}_2]\dots[\mathsf{U},\mathsf{f}_k]\varphi_{\mathsf{goals}(\pi_{k+1})} \to [\mathsf{U},\mathsf{f}_2]\dots[\mathsf{U},\mathsf{f}_{k+1}]\varphi_{\mathsf{goals}(\pi_{k+2})}) & \\
(2k+2) & \models [\mathsf{U},\mathsf{f}_1][\mathsf{U},\mathsf{f}_2]\dots[\mathsf{U},\mathsf{f}_k]\varphi_{\mathsf{goals}(\pi_{k+1})} \to [\mathsf{U},\mathsf{f}_1][\mathsf{U},\mathsf{f}_2]\dots[\mathsf{U},\mathsf{f}_{k+1}]\varphi_{\mathsf{goals}(\pi_{k+2})} &
\end{array}$$

Finally, combine the latter with the Ind. Hyp. (S1) for k

$$\models \varphi_T \to [\mathsf{U},\mathsf{f}_1][\mathsf{U},\mathsf{f}_2]\dots[\mathsf{U},\mathsf{f}_k]\varphi_{\mathsf{goals}(\pi_{k+1})}$$

to obtain the above claim (S1) for $k+1$.

(S2) Consider the previous proof for (S1) but replacing $[\mathsf{U},\mathsf{f}_{k+1}]\varphi_{\mathsf{goals}(\pi_{k+2})}$ by $\langle \mathsf{U},\mathsf{f}_{k+1}\rangle\top$. The result is a valid proof for claim (1) below. The proof is completed as follows:

$$\begin{array}{lll}
(1) & \models \varphi_T \to [\mathsf{U},\mathsf{f}_1]\dots[\mathsf{U},\mathsf{f}_k]\langle \mathsf{U},\mathsf{f}_{k+1}\rangle\top & \\
(2) & \models \varphi_T \to \langle \mathsf{U},\mathsf{f}_1\rangle\dots\langle \mathsf{U},\mathsf{f}_k\rangle\top & \text{(Ind. Hyp. (S2) for } k) \\
(3) & \models \varphi_T \to \langle \mathsf{U},\mathsf{f}_1\rangle\dots\langle \mathsf{U},\mathsf{f}_k\rangle\langle \mathsf{U},\mathsf{f}_{k+1}]\rangle\top & \text{(1), (2)}
\end{array}$$

The induction proof concludes with the case for m, which is itself a proof that π is a solution, so the algorithm is sound.

For Completeness, let a solution exist for a given planning domain $(\varphi_T, A, \varphi_G)$. Let $\mathsf{e}_{i_1},\dots,\mathsf{e}_{i_m}$ be the solution in $A^{<\omega}$. Without loss of generality, we can assume this solution: (a) has minimal length and (b) it is lexicographically minimum (in the inverse order) $(i_m,\dots,i_1)$ among other solutions in $A^{<\omega}$ of the same (minimal) length $m-1$. (That is, for each other solution $(\mathsf{e}_{j_0},\dots,\mathsf{e}_{j'_m})$ we have $m' > m$, or $m' = m$ and $(i_m,\dots,i_{k+1}) = (j'_m,\dots,j_{k+1})$ and $i_k < j_k$, for some $k \leq m$). Let the sequence π encode this solution: $\pi = (\mathsf{e}_{i_m},\dots,\mathsf{e}_{i_1})$. And moreover, redefine each action e_{i_j} as f_j so π becomes $\pi = (\mathsf{f}_m,\dots,\mathsf{f}_1)$.

We proceed to show that π is indeed in the search space and that the BFS algorithm indeed terminates with this solution node π. For this, one must show that

(a) the node π is generated (i.e. each intermediate node $\pi_k = (\mathsf{f}_m,\dots,\mathsf{f}_k)$ is generated)
(b) for no other node π' with length at most that of π and with $\pi' <_{\text{lex}} \pi$, the plan π' satisfies the Terminating Condition, i.e. $\not\models \varphi_T \to \varphi_{\mathsf{goals}(\pi')}$ for each such π'

Assuming (a), claim (b) is straightforward from the above assumptions on π:assume, towards a contradiction, the contrary of (b). If some other plan π' exists with length at most that of π, with with $\pi' <_{\text{lex}} \pi$, and satisfying the Terminating Condition, then by Soundness π' is a solution, so the above assumption on π fails.

Hence it only remains to show claim (a). This is done by induction. (Base Case) That $\pi_\varnothing$ is generated is obvious by Def. 6. (Inductive Case) We must show that each refinement $\pi_k = (\mathsf{f}_m, \ldots, \mathsf{f}_{k+1}, \mathsf{f}_k)$ is generated if $\pi_{k+1} = (\mathsf{f}_m, \ldots, \mathsf{f}_{k+1})$ is. To see the inductive case, it suffices to check that the following four claims hold for each $k \leq m$:

(C1) $\models \varphi_{\mathsf{goals}(\pi_k)} \to [\mathsf{U}, \mathsf{f}_k]\varphi_{\mathsf{goals}(k+1)}$ (C2) $\models \varphi_{\mathsf{goals}(\pi_k)} \to \langle \mathsf{U}, \mathsf{f}_{k+1}\rangle\top$
(C3) $\varphi_{\mathsf{goals}(\pi_k)}$ is consistent, (C4) $\not\models \varphi_{\mathsf{goals}(\pi_k)} \to \varphi_{\mathsf{goals}(\pi_{k+1})}$

(C1) and (C2) follow from the definition of $\varphi_{\mathsf{goals}(\pi_k)}$ and the correctness of the translation defined by t, r.

For (C3), we need the next auxiliary result:

$$\varphi_{\mathsf{goals}(\pi_k)} \equiv [\mathsf{U}, \mathsf{f}_k] \ldots [\mathsf{U}, \mathsf{f}_m]\varphi_G \wedge \langle \mathsf{U}, \mathsf{f}_k\rangle \ldots \langle \mathsf{U}, \mathsf{f}_m\rangle\top$$

This is shown by induction. (Base Case m) The RHS is simply $[\mathsf{U}, \mathsf{f}_m]\varphi_G \wedge \langle \mathsf{U}, \mathsf{f}_m\rangle\top$, which is equivalent to $t([\mathsf{U}, \mathsf{f}_m]\varphi_G \wedge \langle \mathsf{U}, \mathsf{f}_m\rangle\top)$. But the latter is simply the LHS $\varphi_{\mathsf{goals}(\pi_m)}$, so we are done. (Ind. Case $k+1 \to k$.) Assume (Ind. Hyp.) that

$$\varphi_{\mathsf{goals}(\pi_{k+1})} \equiv [\mathsf{U}, \mathsf{f}_{k+1}] \ldots [\mathsf{U}, \mathsf{f}_m]\varphi_G \wedge \langle \mathsf{U}, \mathsf{f}_{k+1}\rangle \ldots \langle \mathsf{U}, \mathsf{f}_m\rangle\top$$

Then,

$$\begin{aligned}
\varphi_{\mathsf{goals}(\pi_k)} &= t([\mathsf{U}, \mathsf{f}_k]\varphi_{\mathsf{goals}(\pi_{k+1})} \wedge \langle \mathsf{U}, \mathsf{f}_k\rangle\top) & \\
&\equiv [\mathsf{U}, \mathsf{f}_k]\varphi_{\mathsf{goals}(\pi_{k+1})} \wedge \langle \mathsf{U}, \mathsf{f}_k\rangle\top) & \text{(correctness of } t\text{)} \\
&\equiv [\mathsf{U}, \mathsf{f}_k]([\mathsf{U}, \mathsf{f}_{k+1}] \ldots [\mathsf{U}, \mathsf{f}_m]\varphi_G \wedge & \\
&\qquad \wedge \langle \mathsf{U}, \mathsf{f}_{k+1}\rangle \ldots \langle \mathsf{U}, \mathsf{f}_m\rangle\top) \wedge & \\
&\quad \wedge \langle \mathsf{U}, \mathsf{f}_k\rangle\top & \text{(ind. hyp.)} \\
&\equiv [\mathsf{U}, \mathsf{f}_k][\mathsf{U}, \mathsf{f}_{k+1}] \ldots [\mathsf{U}, \mathsf{f}_m]\varphi_G \wedge & \\
&\quad \wedge [\mathsf{U}, \mathsf{f}_k]\langle \mathsf{U}, \mathsf{f}_{k+1}\rangle \ldots \langle \mathsf{U}, \mathsf{f}_m\rangle\top \wedge & \\
&\quad \wedge \langle \mathsf{U}, \mathsf{f}_k\rangle\top & \text{(Red. Axiom } \wedge\text{)} \\
&\equiv [\mathsf{U}, \mathsf{f}_k][\mathsf{U}, \mathsf{f}_{k+1}] \ldots [\mathsf{U}, \mathsf{f}_m]\varphi_G \wedge & \\
&\quad \wedge \langle \mathsf{U}, \mathsf{f}_k\rangle\langle \mathsf{U}, \mathsf{f}_{k+1}\rangle \ldots \langle \mathsf{U}, \mathsf{f}_m\rangle\top & \text{(Lemma 1(e))}
\end{aligned}$$

Now, rather than showing that $\varphi_{\mathsf{goals}(\pi_k)}$ is consistent, we show by induction that for each $1 \leq k \leq m$ there exists a model, say M, w, such that $M, w \models \varphi_T$ and also $M, w \models \langle \mathsf{U}, \mathsf{f}_1\rangle \ldots \langle \mathsf{U}, \mathsf{f}_{k-1}\rangle\varphi_{\mathsf{goals}(\pi_k)}$. From this, the claim on the consistency of $\varphi_{\mathsf{goals}(\pi_k)}$ is straightforward.

(Base Case 1) Since φ_T is consistent (by def. of planning domain), let $M, w \models \varphi_T$. Since $\pi = \pi_1$ is a solution, by def. of solution and the previous fact, we obtain that

$$\begin{aligned}
&M, w \models [\mathsf{U}, \mathsf{f}_1] \ldots [\mathsf{U}, \mathsf{f}_m]\varphi_G \wedge \langle \mathsf{U}, \mathsf{f}_1\rangle \ldots \langle \mathsf{U}, \mathsf{f}_m\rangle\top, \text{ and then} \\
&\qquad\qquad M, w \models \varphi_{\mathsf{goals}(\pi_1)} \text{ (Aux. result above)}
\end{aligned}$$

(Ind. Case $k \to k+1$) Assume (Ind. Hyp.) that $M, w \models \varphi_T$ and that

$$
\begin{array}{ll}
M, w \models \langle \mathsf{U}, \mathsf{f}_1 \rangle \ldots \langle \mathsf{U}, \mathsf{f}_{k-1} \rangle \varphi_{\mathsf{goals}(\pi_k)} & \text{(Ind. Hyp.)} \\
M, w \models \langle \mathsf{U}, \mathsf{f}_1 \rangle \ldots \langle \mathsf{U}, \mathsf{f}_{k-1} \rangle ([\mathsf{U}, \mathsf{f}_k][\mathsf{U}, \mathsf{f}_{k+1}] \ldots [\mathsf{U}, \mathsf{f}_m] \varphi_G \wedge & \\
\qquad\qquad \wedge \langle \mathsf{U}, \mathsf{f}_k \rangle \langle \mathsf{U}, \mathsf{f}_{k+1} \rangle \ldots \langle \mathsf{U}, \mathsf{f}_m \rangle \top) & \text{(Aux. result)} \\
M, w \models \langle \mathsf{U}, \mathsf{f}_1 \rangle \ldots \langle \mathsf{U}, \mathsf{f}_{k-1} \rangle ([\mathsf{U}, \mathsf{f}_k][\mathsf{U}, \mathsf{f}_{k+1}] \ldots [\mathsf{U}, \mathsf{f}_m] \varphi_G \wedge & \\
\qquad\qquad \wedge [\mathsf{U}, \mathsf{f}_k] \langle \mathsf{U}, \mathsf{f}_{k+1} \rangle \ldots \langle \mathsf{U}, \mathsf{f}_m \rangle \top \wedge & \\
\qquad\qquad \wedge \langle \mathsf{U}, \mathsf{f}_k \rangle \top) & \text{(Lemma 1(e))} \\
M, w \models \langle \mathsf{U}, \mathsf{f}_1 \rangle \ldots \langle \mathsf{U}, \mathsf{f}_{k-1} \rangle ([\mathsf{U}, \mathsf{f}_k]([\mathsf{U}, \mathsf{f}_{k+1}] \ldots [\mathsf{U}, \mathsf{f}_m] \varphi_G \wedge & \\
\qquad\qquad\qquad \wedge \langle \mathsf{U}, \mathsf{f}_{k+1} \ldots \langle \mathsf{U}, \mathsf{f}_m \rangle \top) \wedge & \\
\qquad\qquad \wedge \langle \mathsf{U}, \mathsf{f}_k \rangle \top) & \text{(Red. Axiom } \wedge \text{)} \\
M, w \models \langle \mathsf{U}, \mathsf{f}_1 \rangle \ldots \langle \mathsf{U}, \mathsf{f}_{k-1} \rangle ([\mathsf{U}, \mathsf{f}_k] \varphi_{\mathsf{goals}(\pi_{k+1})} \wedge \langle \mathsf{U}, \mathsf{f}_k \rangle \top) & \text{(Aux. result)} \\
M, w \models \langle \mathsf{U}, \mathsf{f}_1 \rangle \ldots \langle \mathsf{U}, \mathsf{f}_{k-1} \rangle \langle \mathsf{U}, \mathsf{f}_k \rangle \varphi_{\mathsf{goals}(\pi_{k+1})} & \text{(Lemma 1(e))}
\end{array}
$$

(C4) Suppose towards a contradiction that (C4) fails for π. That is, $\models \varphi_{\mathsf{goals}(\pi_k)} \to \varphi_{\mathsf{goals}(\pi_{k+1})}$, for some k. Indeed, let $(\mathsf{f}_{j_1}, \ldots, \mathsf{f}_{j_k})$ be a subsequence of $\pi^\star = (\mathsf{f}_1, \ldots, \mathsf{f}_m)$ (i.e. with $\{j_1, \ldots, j_k\} \subseteq \{1, \ldots, k-1, k+1, m\}$) such that $\not\models \varphi_{\mathsf{goals}(\pi_{j_l})} \to \varphi_{\mathsf{goals}(\pi_{j_{l+1}})}$. It only remains to show that this latter subsequence $\pi^\star$ satisfies (C1)-(C4). Conditions (C1)-(C3) are shown as above for π but adding the further facts for $\pi^\star$

$$
\varphi_{\mathsf{goals}(\pi_{j_l})} \to \varphi_{\mathsf{goals}(\pi_{j_l+1})} \to \cdots \to \varphi_{\mathsf{goals}(\pi_{j_{l+1}-1})} \to \varphi_{\mathsf{goals}(\pi_{j_{l+1}})}
$$

(C4) is obvious from the definition of $\pi^\star$. By Soundness, $\pi^\star$ is a solution, whose length is strictly less then that of π. But this contradicts the initial assumption on π. Thus, (C1)-(C4) hold for π and so we are done.

5 An Extension of LCC with Action Composition and Choice

In this section we propose an extension of LCC logic with bounded composition and choice, denoted $\mathrm{LCC}_{\cup \otimes n}$. To this end, we first expand any LCC logic with the composition of at most n actions, denoted $\otimes n$, and later we add choice $\cup$. Both operations map two actions e, f to a new action denoted, resp., $\mathsf{e} \otimes \mathsf{f}$ and $\mathsf{e} \cup \mathsf{f}$, interpreted as follows:

- $\mathsf{e} \otimes \mathsf{f}$ models an execution of e followed by an execution of f, and
- $\mathsf{e} \cup \mathsf{f}$ models non-deterministic actions: each execution of $\mathsf{e} \cup \mathsf{f}$ either instantiates as an execution of e or as an execution of f.

For the composition of actions, the resulting action models are shown equivalent to a bounded number of updates with the previous simple actions. The logic of the former action updates, denoted $\mathrm{LCC}_{\otimes n}$ reduces to the corresponding LCC logic.

Then we introduce choice $\cup$ into these models $\mathsf{U}^{\leq n}$. The semantics for non-deterministic actions $\mathsf{e} \cup \mathsf{f}$ is presented in terms of multi-pointed models (w, e) and (w, f), one for each possible realization of the former action. Again we extend the language and axioms accordingly for this logic $\mathrm{LCC}_{\cup \otimes n}$, and reduce this logic again to E·PDL. In the next section, we will study non-deterministic planning problems in terms of plan solutions expressible in this $\mathrm{LCC}_{\cup \otimes n}$ logics.

5.1 Update with the product of n actions in U^n

To define the composition of actions, we simply consider the product of an action model by itself, $\mathsf{U}_1 \otimes \cdots \otimes \mathsf{U}_k$, for each $k \leq n$. Here n denotes the maximum number of compositions allowed in the resulting logic $\mathrm{LCC}_{\otimes n}$. An obvious requirement is that these action models are defined for the same set of variables Var and agents Ag.

We define first action models of the form $\mathsf{U}^n = \mathsf{U}_1 \otimes \cdots \otimes \mathsf{U}_n$ and study them from a semantic point of view. This action model U^n just contains arbitrary products of exactly n actions: $\mathsf{f}_1 \otimes \cdots \otimes \mathsf{f}_n$.

Note that, in the next definition, the pre' functions of the product action model U^n are defined in terms of the corresponding functions pre from U, and pre' from $\mathsf{U}^2, \ldots, \mathsf{U}^{n-1}$. From here on, we let $\overrightarrow{\mathsf{f}}$ denote some sequence $\mathsf{f}_1 \otimes \cdots \otimes \mathsf{f}_k$, also written $\mathsf{f}_1, \ldots, \mathsf{f}_k$, for an appropriate k.

Definition 7. *Let* $\mathsf{U} = (\mathsf{E}, \mathsf{R}, \mathsf{pre}, \mathsf{post})$ *be an action model. We define the* product action model

$$\mathsf{U}^n \;=\; (\mathsf{E}', \mathsf{R}', \mathsf{pre}', \mathsf{post}')$$

inductively as follows:

$$\begin{aligned}
\mathsf{E}' &= \mathsf{E}^n = \{(\mathsf{f}_1, \ldots, \mathsf{f}_n) \mid \mathsf{f}_1, \ldots, \mathsf{f}_n \in \mathsf{E}\} \\
\mathsf{R}'_a &= \{\langle (\mathsf{e}, \ldots, \mathsf{e}'), (\mathsf{f}, \ldots, \mathsf{f}') \rangle \mid \mathsf{e}\mathsf{R}_a\mathsf{f} \textit{ and } \ldots \textit{ and } \mathsf{e}'\mathsf{R}_a\mathsf{f}'\} \\
\mathsf{pre}'(\mathsf{e} \otimes \mathsf{f}) &= \mathsf{pre}(\mathsf{e}) \wedge [\mathsf{U}, \mathsf{e}]\mathsf{pre}(\mathsf{f}) \qquad \textit{for the case } n = 2 \\
\mathsf{pre}'(\mathsf{f}_1 \otimes \overrightarrow{\mathsf{f}}) &= \mathsf{pre}(\mathsf{e}) \wedge [\mathsf{U}, \mathsf{e}]\mathsf{pre}(\overrightarrow{\mathsf{f}}) \\
\mathsf{post}'(\mathsf{f}_1 \otimes \cdots \otimes \mathsf{f}_n) &= \begin{cases} \mathsf{post}(\mathsf{f}_k)(p) & \textit{if } \mathsf{post}(\mathsf{f}_k)(p) \neq p = \\ & \qquad = \mathsf{post}(\mathsf{f}_{k+1})(p) = \ldots = \mathsf{post}(\mathsf{f}_n)(p) \\ \mathsf{post}(\mathsf{f}_1)(p) & \textit{if } \mathsf{post}(\mathsf{f}_1)(p) = \ldots = \mathsf{post}(\mathsf{f}_n)(p) = p \end{cases}
\end{aligned}$$

More formally, in Def. 7 we should rather define inductively (from the case $n = 2$)

$$\mathsf{pre}'(\mathsf{e} \otimes \overrightarrow{\mathsf{f}}) = \mathsf{pre}(\mathsf{e}) \wedge t([\mathsf{U}, \mathsf{e}]\mathsf{pre}'(\overrightarrow{\mathsf{f}}))$$

in order to comply with the condition upon action models: $\mathsf{pre} : \mathsf{E} \to \mathcal{L}_{\mathrm{PDL}}$. But for the sake of simplicity, we will keep the above notation. Also note that in U^n the product of actions $\mathsf{f} \otimes \cdots \otimes \mathsf{f}'$ treats p just as the latest action in this tuple satisfying $\mathsf{post}(\cdot)(p) \neq p$ (i.e. the latest action non-trivial w.r.t. p). Finally, observe that some combinations $\mathsf{e} \otimes \mathsf{f}$ in the product action model will never be applicable, e.g. when $\models [\mathsf{U}, \mathsf{e}]\neg\mathsf{pre}(\mathsf{f})$. For the purpose of planning, one can forget about the existence of these actions in the resulting model $\mathsf{U} \otimes \mathsf{U}$.

It can be seen by direct inspection that the so-called product action model U^n is indeed an action model, provided U is. Moreover, the update of an $\mathrm{E{\cdot}PDL}$ model M by a product action model, say $\mathsf{U} \otimes \mathsf{U}$, reduces to a sequence of updates with the simpler action model, e.g. $(M \circ \mathsf{U}) \circ \mathsf{U}$. With more detail, updating a state w with an action $\mathsf{e} \otimes \mathsf{f}$ is semantically equivalent to updating w with e first, and then updating again with f. We first check this is the case for $\mathsf{U}^2 = \mathsf{U} \otimes \mathsf{U}$.

Lemma 2. *We have the following isomorphism*

$$M \circ (\mathsf{U} \otimes \mathsf{U}) \cong (M \circ \mathsf{U}) \circ \mathsf{U}.$$

Proof. Let $f : W \times (\mathsf{E} \times \mathsf{E}) \to (W \times \mathsf{E}) \times \mathsf{E}$ be the function defined by

$$f : (w, (\mathsf{e}, \mathsf{e}')) \longmapsto ((w, \mathsf{e}), \mathsf{e}')$$

(<u>States</u>: $W^{M \circ \mathsf{U}^2} \cong W^{(M \circ \mathsf{U}) \circ \mathsf{U}}$)

Let us check that f is a bijection between $W^{M \circ \mathsf{U}^2}$ and $W^{(M \circ \mathsf{U}) \circ \mathsf{U}}$:

$(w, \mathsf{e} \otimes \mathsf{f}) \in M \circ (\mathsf{E} \otimes \mathsf{E})$

iff $M, w \models \mathsf{pre}'(\mathsf{e} \otimes \mathsf{f})$

iff $M, w \models \mathsf{pre}(\mathsf{e}) \wedge [\mathsf{U}, \mathsf{e}]\mathsf{pre}(\mathsf{f})$

iff $M, w \models \mathsf{pre}(\mathsf{e})$ and $(M, w \models \mathsf{pre}(\mathsf{e})$ implies $M \circ \mathsf{U}, (w, \mathsf{e}) \models \mathsf{pre}(\mathsf{f}))$

iff $M, w \models \mathsf{pre}(\mathsf{e})$ and $M \circ \mathsf{U}, (w, \mathsf{e}) \models \mathsf{pre}(\mathsf{f})$

iff $(w, \mathsf{e}) \in W^{M \circ \mathsf{U}}$ and $((w, \mathsf{e}), \mathsf{f}) \in W^{(M \circ \mathsf{U}) \circ \mathsf{U}}$

iff $((w, \mathsf{e}), \mathsf{f}) \in W^{(M \circ \mathsf{U}) \circ \mathsf{U}}$

(<u>Accessibility relations</u>: $R_a^{M \circ (\mathsf{U} \otimes \mathsf{U})} \cong R_a^{(M \circ \mathsf{U}) \circ \mathsf{U}}$)

$(w_1, (\mathsf{e}, \mathsf{f})) R_a^{M \circ (\mathsf{U} \otimes \mathsf{U})} (w_2, (\mathsf{e}', \mathsf{f}'))$

iff $w_1 R_a^M w_2$ and $(\mathsf{e}, \mathsf{f}) \mathsf{R}'_a (\mathsf{e}', \mathsf{f}')$ (by Def. $R^{M \circ (\mathsf{U} \otimes \mathsf{U})}$)

iff $w_1 R_a^M w_2$ and ($\mathsf{e} \mathsf{R}_a \mathsf{e}'$ and $\mathsf{f} \mathsf{R}_a \mathsf{f}'$) (by Def. R')

iff ($w_1 R_a^M w_2$ and $\mathsf{e} \mathsf{R}_a \mathsf{e}'$) and $\mathsf{f} \mathsf{R}_a \mathsf{f}'$ (re-bracketing)

iff $(w_1, \mathsf{e}) R_a^{M \circ \mathsf{U}} (w_2, \mathsf{e}')$ and $\mathsf{f} \mathsf{R}_a \mathsf{f}'$ (by Def. $R^{M \circ \mathsf{U}}$)

iff $((w_1, \mathsf{e}), \mathsf{f}) R_a^{(M \circ \mathsf{U}) \circ \mathsf{U}} ((w_2, \mathsf{e}'), \mathsf{f}')$ (by Def. $R^{(M \circ \mathsf{U}) \circ \mathsf{U}'}$)

(<u>Valuations</u>: $V^{M \circ (\mathsf{U} \otimes \mathsf{U})}(p) \cong V^{(M \circ \mathsf{U}) \circ \mathsf{U}}(p)$)

$((w, \mathsf{e}), \mathsf{f}) \in V^{M \circ (\mathsf{U} \otimes \mathsf{U})}(p)$

iff $((w, \mathsf{e}), \mathsf{f}) \in (W \times \mathsf{E}) \times \mathsf{E}$ and $M, w \models \mathsf{pre}(\mathsf{e})$ and $M \circ \mathsf{U}, (w, \mathsf{e}) \models \mathsf{pre}(\mathsf{f})$
and $(M \circ \mathsf{U}) \circ \mathsf{U}, ((w, \mathsf{e}), \mathsf{f}) \models p$

iff $((w, \mathsf{e}), \mathsf{f}) \in (W \times \mathsf{E}) \times \mathsf{E}$ and $M, w \models \mathsf{pre}(\mathsf{e}) \wedge [\mathsf{U}, \mathsf{e}]\mathsf{pre}(\mathsf{f})$
and $(M \circ \mathsf{U}), (w, \mathsf{e}) \models p^{\mathsf{post}(\mathsf{f})}$

iff $((w, \mathsf{e}), \mathsf{f}) \in (W \times \mathsf{E}) \times \mathsf{E}$ and $M, w \models \mathsf{pre}(\mathsf{e} \otimes \mathsf{f})$

and $(M \circ \mathsf{U}), (w, \mathsf{e}) \models \begin{cases} p^{\mathsf{post}(\mathsf{f})} & \text{if } \mathsf{post}(\mathsf{f})(p) \neq p \\ p & \text{if } \mathsf{post}(\mathsf{f})(p) = p \end{cases}$

iff $((w, \mathsf{e}), \mathsf{f}) \in (W \times \mathsf{E}) \times \mathsf{E}$ and $M, w \models \mathsf{pre}(\mathsf{e} \otimes \mathsf{f})$

and $\begin{cases} M, w \models p^{\mathsf{post}(\mathsf{f})} & \text{if } \mathsf{post}(\mathsf{f})(p) \neq p \\ (M \circ \mathsf{U}), (w, \mathsf{e}) \models p & \text{if } \mathsf{post}(\mathsf{f})(p) = p \end{cases}$

iff $(w, (\mathsf{e} \otimes \mathsf{f})) \in W \times (\mathsf{E} \times \mathsf{E})$ and $M, w \models \mathsf{pre}(\mathsf{e} \otimes \mathsf{f})$ and $M, w \models p^{\mathsf{post}(\mathsf{e} \otimes \mathsf{f})}$

iff $(w, (\mathsf{e} \otimes \mathsf{f})) \in W \times (\mathsf{E} \times \mathsf{E})$ and $M, w \models [\mathsf{U}^2, \mathsf{e} \otimes \mathsf{f}]p$

iff $(w, (\mathsf{e} \otimes \mathsf{f})) \in V^{M \circ (\mathsf{U} \otimes \mathsf{U})}(p)$

This isomorphism extends to the valuations of arbitrary formulas and programs.

Corollary 1. *For each formula φ in the language of $\mathsf{U} \otimes \mathsf{U}$:*

$$(w, (\mathsf{e}, \mathsf{f})) \in [\![\varphi]\!]^{M \circ \mathsf{U}^2} \Leftrightarrow ((w, \mathsf{e}), \mathsf{f}) \in [\![\varphi]\!]^{(M \circ \mathsf{U}) \circ \mathsf{U}}$$

Proof. As mentioned above, the language of LCC logic for U^2 does not contain the formula $\mathsf{pre}(\mathsf{e} \otimes \mathsf{f})$, this formula being in the language for the action model U. Still, we can make use of the translation function t from [13] to obtain a formula equivalent to it, in the language of E·PDL (included in the language for U^2).

The proof is by simultaneous induction on programs and formulas. The basic cases were just considered in Lemma 2, i.e. $[\![p]\!]^{(\cdot)} = V^{(\cdot)}(p)$ and $[\![a]\!]^{(\cdot)} = R_a^{(\cdot)}$. From here on, the proof tacitly relies upon the established bijection between $W^{M \circ \mathsf{U}^2}$ and $W^{(M \circ \mathsf{U})\mathsf{U}}$ also from Lemma 2.

For programs, the correspondence for $[\![?\varphi]\!]$ is immediate from the Ind. Hyp. on $[\![\varphi]\!]$. The same occurs for $[\![\pi_1; \pi_2]\!]$ and $[\![\pi_1 \cup \pi_2]\!]$ from the Ind. Hyp. on $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$. Finally, the case $[\![\pi^*]\!]$ is also straightforward from the Ind. Hyp. on $[\![\pi]\!]$.

The same holds for formulas, the identity between $M \circ \mathsf{U}^2$ and $(M \circ \mathsf{U}) \circ \mathsf{U}$ of the corresponding sets $[\![\top]\!]^{(\cdot)}$, $[\![\neg\varphi]\!]^{(\cdot)}$ and $[\![\varphi_1 \wedge \varphi_2]\!]^{(\cdot)}$ is clear using the Ind. Hyp., resp., on none, on $[\![\varphi]\!]^{(\cdot)}$ and on $[\![\varphi_1]\!]^{(\cdot)}$ and $[\![\varphi_2]\!]^{(\cdot)}$. The case for $[\![[\pi]\varphi]\!]^{(\cdot)}$, making use of the Ind. Hyp. for $[\![\pi]\!]^{(\cdot)}$ and $[\![\varphi]\!]^{(\cdot)}$, is as follows:

$$\begin{aligned}
&[\![[\pi]\varphi]\!]^{M \circ \mathsf{U}^2} \\
&= \{(w, (\mathsf{e}, \mathsf{f})) \in W^{M \circ \mathsf{U}^2} \mid \forall (v, (\mathsf{e}', \mathsf{f}')) \in W^{M \circ \mathsf{U}^2} \\
&\qquad \text{if } ((w, (\mathsf{e}, \mathsf{f})), (v, (\mathsf{e}', \mathsf{f}'))) \in [\![\pi]\!]^{M \circ \mathsf{U}^2} \text{ then } (v, (\mathsf{e}', \mathsf{f}')) \in [\![\varphi]\!]^{M \circ \mathsf{U}^2}\} \\
&= \{((w, \mathsf{e}), \mathsf{f}) \in W^{(M \circ \mathsf{U}) \circ \mathsf{U}} \mid \forall ((v, \mathsf{e}'), \mathsf{f}') \in W^{(M \circ \mathsf{U}) \circ \mathsf{U}} \\
&\quad \text{if } (((w, \mathsf{e}), \mathsf{f}), ((v, \mathsf{e}'), \mathsf{f}')) \in [\![\pi]\!]^{(M \circ \mathsf{U}) \circ \mathsf{U}} \text{ then } ((v, \mathsf{e}'), \mathsf{f}') \in [\![\varphi]\!]^{(M \circ \mathsf{U}) \circ \mathsf{U}}\} \\
&= [\![[\pi]\varphi]\!]^{(M \circ \mathsf{U}) \circ \mathsf{U}}
\end{aligned}$$

Finally, consider the case of $[\mathsf{U}^2, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi$ formulas.

$$
\begin{aligned}
&[\![[\mathsf{U},\mathsf{f}_1\otimes\mathsf{f}_2]\varphi]\!]^{M\circ\mathsf{U}^2}\\
&= \{(w,(\mathsf{e}_1,\mathsf{e}_2))\in W^{M\circ\mathsf{U}^2} \mid M\circ\mathsf{U}^2,(w,(\mathsf{e}_1,\mathsf{e}_2))\models \mathsf{pre}(\mathsf{f}_1\otimes\mathsf{f}_2) \text{ implies}\\
&\qquad\qquad ((w,(\mathsf{e}_1,\mathsf{e}_2)),(\mathsf{f}_1,\mathsf{f}_2))\in[\varphi]^{(M\circ\mathsf{U}^2)\circ\mathsf{U}^2}\}\\
&= \{(w,(\mathsf{e}_1,\mathsf{e}_2))\in W^{M\circ\mathsf{U}^2} \mid M\circ\mathsf{U}^2,(w,(\mathsf{e}_1,\mathsf{e}_2))\models \mathsf{pre}(\mathsf{f}_1\otimes\mathsf{f}_2) \text{ implies}\\
&\qquad\qquad (((w,(\mathsf{e}_1,\mathsf{e}_2)),\mathsf{f}_1),\mathsf{f}_2)\in[\varphi]^{((M\circ\mathsf{U}^2)\circ\mathsf{U})\circ\mathsf{U}}\}\\
&= \{((w,\mathsf{e}_1),\mathsf{e}_2)\in W^{(M\circ\mathsf{U})\circ\mathsf{U}} \mid (M\circ\mathsf{U})\circ\mathsf{U},((w,\mathsf{e}_1),\mathsf{e}_2)\models \mathsf{pre}(\mathsf{f}_1)\wedge[\mathsf{U},\mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)\\
&\qquad\qquad \text{implies } (((w,(\mathsf{e}_1,\mathsf{e}_2)),\mathsf{f}_1),\mathsf{f}_2)\in[\varphi]^{(((M\circ\mathsf{U})\circ\mathsf{U})\circ\mathsf{U})\circ\mathsf{U}}\}\\
&= \{((w,\mathsf{e}_1),\mathsf{e}_2)\in W^{(M\circ\mathsf{U})\circ\mathsf{U}} \mid (M\circ\mathsf{U})\circ\mathsf{U},((w,\mathsf{e}_1),\mathsf{e}_2)\models \mathsf{pre}(\mathsf{f}_1) \text{ and}\\
&\qquad\qquad ((M\circ\mathsf{U})\circ\mathsf{U})\circ\mathsf{U},(((w,\mathsf{e}_1),\mathsf{e}_2),\mathsf{f}_1)\models \mathsf{pre}(\mathsf{f}_2)\\
&\qquad\qquad \text{imply } (((w,(\mathsf{e}_1,\mathsf{e}_2)),\mathsf{f}_1),\mathsf{f}_2)\in[\varphi]^{(((M\circ\mathsf{U})\circ\mathsf{U})\circ\mathsf{U})\circ\mathsf{U}}\}\\
&= \{((w,\mathsf{e}_1),\mathsf{e}_2)\in W^{(M\circ\mathsf{U})\circ\mathsf{U}} \mid (M\circ\mathsf{U})\circ\mathsf{U},((w,\mathsf{e}_1),\mathsf{e}_2)\models \mathsf{pre}(\mathsf{f}_1) \text{ implies}\\
&\qquad\qquad (\text{ if } ((M\circ\mathsf{U})\circ\mathsf{U})\circ\mathsf{U},(((w,\mathsf{e}_1),\mathsf{e}_2),\mathsf{f}_1)\models \mathsf{pre}(\mathsf{f}_2)\\
&\qquad\qquad \text{then } (((w,(\mathsf{e}_1,\mathsf{e}_2)),\mathsf{f}_1),\mathsf{f}_2)\in[\varphi]^{(((M\circ\mathsf{U})\circ\mathsf{U})\circ\mathsf{U})\circ\mathsf{U}}\ )\ \}\\
&= \{((w,\mathsf{e}_1),\mathsf{e}_2)\in W^{(M\circ\mathsf{U})\circ\mathsf{U}} \mid (M\circ\mathsf{U})\circ\mathsf{U},((w,\mathsf{e}_1),\mathsf{e}_2)\models \mathsf{pre}(\mathsf{f}_1) \text{ implies}\\
&\qquad\qquad ((M\circ\mathsf{U})\circ\mathsf{U})\circ\mathsf{U},(((w,\mathsf{e}_1),\mathsf{e}_2),\mathsf{f}_1)\models[\mathsf{U},\mathsf{f}_2]\varphi\}\\
&= [\![[\mathsf{U},\mathsf{f}_1][\mathsf{U},\mathsf{f}_2]\varphi]\!]^{(M\circ\mathsf{U})\circ\mathsf{U}}
\end{aligned}
$$

Also, note that the proof of Lemma 2 does not depend upon the assumption that the two action models are the same. More generally, we have the following result for different action models U,U'.

Corollary 2. *Let U,U' be action models defined on the same sets of variables Var and agents Ag. Then, $M\circ(\mathsf{U}\otimes\mathsf{U}')\cong(M\circ\mathsf{U})\circ\mathsf{U}'$. Moreover, $[\![\varphi]\!]^{M\circ(\mathsf{U}\otimes\mathsf{U}')}=[\![\varphi]\!]^{(M\circ\mathsf{U})\circ\mathsf{U}'}$, for each φ in the language of $\mathsf{U}\otimes\mathsf{U}'$.*

Before proceeding to the generalization of this lemma, we need the claim that the update with an action model U preserves isomorphisms.

Lemma 3. *If $M\cong M'$ are isomorphic epistemic models, and U is an action model, then $M\circ\mathsf{U}\cong M'\circ\mathsf{U}$.*

Proof. Let $M=(W,R,V)$ and $M'=(W',R',V')$ be isomorphic models. Let then $f: W\mapsto W'$ be a bijection satisfying $R^M_a(w,v)\Leftrightarrow R^{M'}_a(f(w),f(v))$ and $w\in V^M(p)\Leftrightarrow f(w)\in V^{M'}(p)$. Define the map $f^+: W\times\mathsf{E}\to W'\times\mathsf{E}$ simply as $f^+(w,\mathsf{e})=(f(w),\mathsf{e})$. This is clearly a bijection and moreover

$$
\begin{aligned}
& R^{M\circ\mathsf{U}}_a((w,\mathsf{e}),(v,\mathsf{f}))\Leftrightarrow R^M_a(w,v) \text{ and } R^{\mathsf{U}}_a(\mathsf{e},\mathsf{f})\\
\Leftrightarrow\ & R^{M'}_a(f(w),f(v)) \text{ and } R^{\mathsf{U}}_a(\mathsf{e},\mathsf{f})\Leftrightarrow R^{M'\circ\mathsf{U}}_a((f(w),\mathsf{e}),(f(v),\mathsf{f}))\\
\Leftrightarrow\ & R^{M'}_a(f^+(w,\mathsf{e}),f^+(v,\mathsf{f}))
\end{aligned}
$$

$$
\begin{aligned}
& (w,\mathsf{e})\in V^{M\circ\mathsf{U}}(p)\Leftrightarrow M,w\models\mathsf{pre}(\mathsf{e})\wedge p^{\mathsf{post}(\mathsf{e})}\\
\Leftrightarrow\ & M',f(w)\models\mathsf{pre}(\mathsf{e})\wedge p^{\mathsf{post}(\mathsf{e})}\Leftrightarrow(f(w),\mathsf{e})\in V^{M'\circ\mathsf{U}}(p)\\
\Leftrightarrow\ & f^+(w,\mathsf{e})\in V^{M',\circ\mathsf{U}}(p)
\end{aligned}
$$

The previous Corollary 2 for the basic case $n = 2$ extends to an arbitrary finite number $n \geq 2$ of actions $\mathsf{f}_1, \ldots, \mathsf{f}_n$. That is, it extends to updates with products of arbitrary n actions taken from a given action model U.

Corollary 3. *We have* $M \circ \mathsf{U}^n \cong (M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_n$

Proof. The proof is by induction, where the Base Case is simply Lemma 2 for $n = 2$. For the Inductive Case $n \mapsto n + 1$, redefine the action models $\mathsf{U}_1 = \ldots = \mathsf{U}_n (= \mathsf{U})$ as the corresponding n copies of U in the product action model. The Ind. Hyp. is the claim $M \circ (\mathsf{U}_1 \otimes \cdots \otimes \mathsf{U}_n) \cong (M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_n$.

$$
\begin{aligned}
&((M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_n) \circ \mathsf{U}_{n+1} \\
&\cong (M \circ (\mathsf{U}_1 \otimes \cdots \otimes \mathsf{U}_n)) \circ \mathsf{U}_{n+1} \text{ (Ind. Hyp. and Lemma 3)} \\
&\cong M \circ (\mathsf{U}_1 \otimes \cdots \otimes \mathsf{U}_n \otimes \mathsf{U}_{n+1}) \quad \text{(Corollary 2)} \\
&= M \circ \mathsf{U}^{n+1}
\end{aligned}
$$

5.2 Update with the produce of $\leq n$ actions in $\mathsf{U}^{\leq n}$

Finally, we can define the action model $\mathsf{U}^{\leq n}$ for the product of at most n actions (from a fixed action model U) in terms of the product action models $\mathsf{U}, \mathsf{U}^2, \ldots, \mathsf{U}^n$ previously defined.

Definition 8. *Let* U *be an action model and let* $\mathsf{U}_1 = \ldots = \mathsf{U}_n (= \mathsf{U})$ *be* n *different copies of* U*, denoted* $\mathsf{U}_k = (\mathsf{E}_k, \mathsf{R}_k, \mathsf{pre}_k, \mathsf{post}_k)$ *for each* $1 \leq k \leq n$*. We define* $\mathsf{U}^{\leq n} = (\mathsf{E}^{\leq n}, \mathsf{R}^{\leq n}, \mathsf{pre}^{\leq n}, \mathsf{post}^{\leq n})$ *as follows*

$$
\begin{array}{rlrl}
\mathsf{E}^{\leq n} = & \bigcup_{k \leq n} \mathsf{E}_k & \mathsf{pre}^{\leq n} = & \bigcup_{k \leq n} \mathsf{pre}_k \\
\mathsf{R}^{\leq n}(a) = & \bigcup_{k \leq n} \mathsf{R}_k(a) & \mathsf{post}^{\leq n} = & \bigcup_{k \leq n} \mathsf{post}_k
\end{array}
$$

In parallel, the sequence of at most n updates on a model M*, denoted*

$$
(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_{\leq n} = (W^{(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_{\leq n}}, R^{(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_{\leq n}}, V^{(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_{\leq n}})
$$

can be defined in a straightforward way from each product action model $(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_k$*.*

$$
\begin{aligned}
W^{(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_{\leq n}} &= \bigcup_{k \leq n} W^{(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_k} \\
R^{(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_{\leq n}}(a) &= \bigcup_{k \leq n} R^{(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_k}(a) \\
V^{(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_{\leq n}} &= \bigcup_{k \leq n} V^{(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_k}
\end{aligned}
$$

It can be observed that $\mathsf{U}^{\leq n}$ is an action model; and also that $(M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_{\leq n}$ is an E·PDL model. Moreover, we can extend Corollary 3 to the present case:

Corollary 4. *If* U *is an action model, then*

$$
M \circ \mathsf{U}^{\leq n} \cong (M \circ \mathsf{U}_1) \cdots \circ \mathsf{U}_{\leq n}
$$

Proof. Consider the mapping $(w,(\mathsf{f}_1,\ldots,\mathsf{f}_k)) \longmapsto (((w,\mathsf{f}_1),\ \ldots),\mathsf{f}_k)$.

(States $W^{M\circ\mathsf{U}^{\leq n}} \cong W^{(M\circ\mathsf{U}_1)\cdots\mathsf{U}_{\leq n}}$) We show this mapping is a bijection:

$$\begin{array}{ll} & (w,(\mathsf{f}_1,\ldots,\mathsf{f}_k)) \in W^{M\circ\mathsf{U}^{\leq n}} \\ \text{iff} & (w,(\mathsf{f}_1,\ldots,\mathsf{f}_k)) \in \bigcup_{j\leq n} W^{M\circ\mathsf{U}^j} \\ \text{iff} & (w,(\mathsf{f}_1,\ldots,\mathsf{f}_k)) \in W^{M\circ\mathsf{U}^k} \\ \text{iff} & (((w,\mathsf{f}_1),\ldots),\mathsf{f}_k) \in W^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_k} \qquad \text{(Coro. 3)} \\ \text{iff} & (((w,\mathsf{f}_1),\ldots),\mathsf{f}_k) \in \bigcup_{k\leq n} W^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_k} \\ \text{iff} & (((w,\mathsf{f}_1),\ldots),\mathsf{f}_k) \in W^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_{\leq n}} \end{array}$$

(Relations $R^{M\circ\mathsf{U}^{\leq n}} \cong R^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_{\leq n}}$) Notice all pairs in the relation $R^{M\circ\mathsf{U}^{\leq n}}$ have the same length, and similarly for $R^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_{\leq n}}$. Hence, it suffices to consider pairs of equal length among elements $(w,(\mathsf{f}_1,\ \ldots,\mathsf{f}_k)) \in W^{M\circ\mathsf{U}^{\leq n}}$ and similarly for elements $((w,\mathsf{f}_1),\ \ldots,\mathsf{f}_k) \in W^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_{\leq n}}$. The proof proceeds similarly to the case for States, now just considering equal-length pairs and replacing

$$\begin{array}{c} \cdots \in W^{M\circ\mathsf{U}^k} \text{ by } \cdots R^{M\circ\mathsf{U}^k}\cdots \\ \cdots \in W^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_k} \text{ by } \cdots R^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_k}\cdots. \end{array}$$

(Valuations $V^{M\circ\mathsf{U}^{\leq n}}(p) = V^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_{\leq n}}(p)$) The proof is also similar to the case of States, again using Corollary 3.

$$\begin{array}{ll} & (w,(\mathsf{f}_1,\ldots,\mathsf{f}_k)) \in V^{M\circ\mathsf{U}^{\leq n}}(p) \\ \text{iff} & (w,(\mathsf{f}_1,\ldots,\mathsf{f}_k)) \in \bigcup_{j\leq n} V^{M\circ\mathsf{U}^j}(p) \\ \text{iff} & ((w,\mathsf{f}_1),\ldots,\mathsf{f}_k) \in \bigcup_{j\leq n} V^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_j}(p) \\ \text{iff} & ((w,\mathsf{f}_1),\ldots,\mathsf{f}_k) \in V^{(M\circ\mathsf{U}_1)\cdots\circ\mathsf{U}_{\leq n}}(p) \end{array}$$

5.3 The logic $\mathrm{LCC}_{\otimes n}$ of the action model $\mathsf{U}^{\leq n}$

Let U be again a fixed action model and consider the corresponding product action model $\mathsf{U}^{\leq n}$. The language $\mathcal{L}_{\mathrm{LCC}_{\otimes n}}$ of the logic $\mathrm{LCC}_{\otimes n}$ for this action model $\mathsf{U}^{\leq n}$ is simply the language of LCC, but now with action modalities of the form $[\mathsf{U}^{\leq n},\mathsf{f}_1\otimes\cdots\otimes\mathsf{f}_k]$, for each $\mathsf{f}_1\otimes\cdots\otimes\mathsf{f}_k \in \mathsf{E}^{\leq n}$ in the present action model $\mathsf{U}^{\leq n}$.

The semantics of updates with pointed action model $\mathsf{U}^{\leq n},(\mathsf{f}_1,\ldots,\mathsf{f}_k)$ is also that of simple action models U. In the present case, we have

$$\begin{array}{l} M,w \models [\mathsf{U}^n,\mathsf{e}\otimes\cdots\otimes\mathsf{f}]\varphi \text{ iff } M,w\models \mathsf{pre}(\mathsf{e}\otimes\cdots\otimes\mathsf{f}) \text{ implies} \\ \qquad\qquad\qquad\qquad\qquad\qquad M\circ\mathsf{U}^n,(w,(\mathsf{e}\otimes\cdots\otimes\mathsf{f}))\models\varphi \end{array}$$

A complete axiom system for $\mathrm{LCC}_{\otimes n}$, the logic of (bounded) product action models $\mathsf{U}^{\leq n}$, is obtained by extending the previous LCC axioms and rules with reduction axioms for the new product actions $\mathsf{f}_1\otimes\cdots\otimes\mathsf{f}_k$.

These axioms suffice for the introduction of composition. They induce again a translation function t which splits product actions $[\mathsf{U}^{\leq n},\mathsf{e}\otimes\mathsf{f}]$ into a sequence of updates $[\mathsf{U}^{\leq n},\mathsf{e}][\mathsf{U}^{\leq n},\mathsf{f}]$ and proceeds as the translation for LCC for the remaining cases.

the LCC reduction axioms and rules for $[\mathsf{U},\mathsf{e}]\varphi$ formulas with $\varphi \in \mathcal{L}_{\mathrm{LCC}_{\otimes n}}$ plus $[\mathsf{U}^{\leq n},(\mathsf{f}_1,\mathsf{f}_2,\ldots,\mathsf{f}_k)]\varphi \leftrightarrow [\mathsf{U}^{\leq n},\mathsf{f}_1][\mathsf{U}^{\leq n},(\mathsf{f}_2,\ldots,\mathsf{f}_k)]\varphi$ (Product)

Fig. 2. The axioms and rules for $\mathrm{LCC}_{\otimes n}$.

Lemma 4. *The product axiom is sound:*

$$\models [\mathsf{U}^{\leq n},\mathsf{f}_1 \otimes \mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k]\varphi \;\leftrightarrow\; [\mathsf{U}^{\leq n},\mathsf{f}_1][\mathsf{U}^{\leq n},\mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_n]\varphi$$

Proof. Let M, w be an arbitrary pointed model of LCC. Using Corollary 4, we obtain the following equivalences:

$M,w \models [\mathsf{U}^{\leq n},(\mathsf{f}_1 \otimes \mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k)]\varphi$

$M,w \models \mathsf{pre}(\mathsf{f}_1 \otimes \mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k)$
implies $M \circ \mathsf{U}^{\leq n},(w,(\mathsf{f}_1,\ldots,\mathsf{f}_k)) \models \varphi$

$M,w \models \mathsf{pre}(\mathsf{f}_1) \wedge [\mathsf{U}^{\leq n},\mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k)$
implies $(M \circ \mathsf{U}) \cdots \circ \mathsf{U}_{\leq n},((w,\mathsf{f}_1),\ldots,\mathsf{f}_k) \models \varphi$

$M,w \models \mathsf{pre}(\mathsf{f}_1)$, and
$M,w \models [\mathsf{U}^{\leq n},\mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k)$
imply $(M \circ \mathsf{U}) \cdots \circ \mathsf{U}_{\leq n},((w,\mathsf{f}_1),\ldots,\mathsf{f}_k) \models \varphi$

$M,w \models \mathsf{pre}(\mathsf{f}_1)$ and
($M,w \models \mathsf{pre}(\mathsf{f}_1)$ implies $M \circ \mathsf{U}^{\leq n},(w,\mathsf{f}_1) \models \mathsf{pre}(\mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k)$)
imply $(M \circ \mathsf{U}) \cdots \circ \mathsf{U}_{\leq n},((w,\mathsf{f}_1),\ldots,\mathsf{f}_k) \models \varphi$

$M,w \models \mathsf{pre}(\mathsf{f}_1)$ and $M \circ \mathsf{U}^{\leq n},(w,\mathsf{f}_1) \models \mathsf{pre}(\mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k)$
implies $(M \circ \mathsf{U}) \cdots \circ \mathsf{U}_{\leq n},((w,\mathsf{f}_1),\ldots,\mathsf{f}_k) \models \varphi$

$M,w \models \mathsf{pre}(\mathsf{f}_1)$ and $(M \circ \mathsf{U}) \cdots \circ \mathsf{U}_{\leq n},(w,\mathsf{f}_1) \models \mathsf{pre}(\mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k)$
implies $(M \circ \mathsf{U}) \cdots \circ \mathsf{U}_{\leq n},((w,\mathsf{f}_1),\ldots,\mathsf{f}_k) \models \varphi$

$M,w \models \mathsf{pre}(\mathsf{f}_1)$ implies
(if $(M \circ \mathsf{U}) \cdots \circ \mathsf{U}_{\leq n},(w,\mathsf{f}_1) \models \mathsf{pre}(\mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k)$
then $(M \circ \mathsf{U}) \cdots \circ \mathsf{U}_{\leq n},((w,\mathsf{f}_1),\ldots,\mathsf{f}_k) \models \varphi$)

$M,w \models \mathsf{pre}(\mathsf{f}_1)$ implies $(M \circ \mathsf{U}) \cdots \circ \mathsf{U}_{\leq n},(w,\mathsf{f}_1) \models [\mathsf{U}^{\leq n},\mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k]\varphi$

$M,w \models [\mathsf{U}^{\leq n},\mathsf{f}_1][\mathsf{U}^{\leq n},\mathsf{f}_2 \otimes \cdots \otimes \mathsf{f}_k]\varphi$

As we said, we extend the previous translation $\mathcal{L}_{\mathrm{LCC}} \to \mathcal{L}_{\mathrm{E\cdot PDL}}$ into a translation $\mathcal{L}_{\mathrm{LCC}_{\otimes n}} \to \mathcal{L}_{\mathrm{E\cdot PDL}}$ with the help of an additional clause

$$t([\mathsf{U}^{\leq n},(\mathsf{f}_1,\mathsf{f}_2,\ldots,\mathsf{f}_n)]\varphi) = t([\mathsf{U}^{\leq n},\mathsf{f}_1]t([\mathsf{U}^{\leq n},(\mathsf{f}_2,\ldots,\mathsf{f}_k)]\varphi))$$

Theorem 2. *For each formula $\varphi \in \mathcal{L}_{\mathrm{LCC}_{\otimes n}}$, we have*

$$\models \varphi \Leftrightarrow \vdash \varphi$$

Proof. ($\Leftarrow$) Soundness is established by the corresponding result for LCC in [13] plus the above result for the reduction axiom for product actions. These results also establish the correctness of the extended translation function: each formula in $\mathrm{LCC}_{\otimes n}$ is logically equivalent (in LCC) to an E·PDL-formula $t(\varphi)$.

($\Rightarrow$) E·PDL is complete, and each formula in $\mathcal{L}_{\mathrm{LCC}_{\otimes n}}$ is equivalent to some $\mathcal{L}_{\mathrm{E\cdot PDL}}$ formula.

In addition, the LCC reduction axioms that would correspond to product modalities (except for the case of E·PDL-programs) are also sound.

Proposition 1. *Except for the* LCC *axiom on* E·PDL*-programs, the* LCC *reduction axioms are sound for product action modalities* $[\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \cdots \otimes \mathsf{f}_k]$ *are sound.*

Proof. We just show the Base Case for $n = 2$. The Ind. Case is similar. For the different reduction axioms for $[\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi$, consider the following equivalences for each case of φ.

(Axiom *top*) This is obvious: $[\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\top$ becomes $[\mathsf{U}^{\leq n}, \mathsf{f}_1][\mathsf{U}^{\leq n}, \mathsf{f}_2]\top$ which is simply $[\mathsf{U}^{\leq n}, \mathsf{f}_1]\top$ and finally $\top$, using the *top* axiom first for f_2 and then for f_1.

(Axiom *atoms*) Consider the following equivalences

$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]p$

iff $M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1][\mathsf{U}^{\leq n}, \mathsf{f}_2]p$ (*product*)

iff $M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1](\mathsf{pre}(\mathsf{f}_2) \to p^{\mathsf{post}(\mathsf{f}_2)})$ (Nec. on *atoms*)

iff if $M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)$ then $M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1]p^{\mathsf{post}(\mathsf{f}_2)}$

iff if $M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)$ then

$$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1] \begin{cases} p^{\mathsf{post}(\mathsf{f}_2)} & \text{if } \mathsf{post}(\mathsf{f}_2)(p) = p \\ p^{\mathsf{post}(\mathsf{f}_2)} & \text{if } \mathsf{post}(\mathsf{f}_2)(p) \neq p \end{cases}$$

iff if $M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)$ then

$$M, w \models \mathsf{pre}(\mathsf{f}_1) \to \begin{cases} p^{\mathsf{post}(\mathsf{f}_1)} & \text{if } \mathsf{post}(\mathsf{f}_2)(p) = p \\ p^{\mathsf{post}(\mathsf{f}_2)} & \text{if } \mathsf{post}(\mathsf{f}_2)(p) \neq p \end{cases}$$

iff if $M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)$ then
($M, w \models \mathsf{pre}(\mathsf{f}_1)$ implies $M, w \models p^{\mathsf{post}(\mathsf{f}_1 \otimes \mathsf{f}_2)}$)

iff if $M, w \models \mathsf{pre}(\mathsf{f}_1) \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)$ then $M, w \models p^{\mathsf{post}(\mathsf{f}_1 \otimes \mathsf{f}_2)}$)

iff $M, w \models \mathsf{pre}(\mathsf{f}_1 \otimes \mathsf{f}_2) \to p^{\mathsf{post}(\mathsf{f}_1 \otimes \mathsf{f}_2)}$

(Axiom *negation*)

$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\neg\varphi$

$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1][\mathsf{U}^{\leq n}, \mathsf{f}_2]\neg\varphi$

$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1](\mathsf{pre}(\mathsf{f}_2) \to \neg[\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi)$

$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1]\neg(\mathsf{pre}(\mathsf{f}_2) \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi)$

$M, w \models \mathsf{pre}(\mathsf{f}_1) \to \neg[\mathsf{U}^{\leq n}, \mathsf{f}_1](\mathsf{pre}(\mathsf{f}_2) \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi)$

$M, w \models \mathsf{pre}(\mathsf{f}_1) \to \neg([\mathsf{U}^{\leq n}, \mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)) \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_1][\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi)$

$M, w \models \mathsf{pre}(\mathsf{f}_1) \to \neg([\mathsf{U}^{\leq n}, \mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)) \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi)$
$M, w \models \mathsf{pre}(\mathsf{f}_1) \to ([\mathsf{U}^{\leq n}, \mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)) \to \neg[\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi)$
$M, w \models (\mathsf{pre}(\mathsf{f}_1) \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_1]\mathsf{pre}(\mathsf{f}_2)) \to \neg[\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi$
$M, w \models \mathsf{pre}(\mathsf{f}_1 \otimes \mathsf{f}_2) \to \neg[\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi$

(Axiom *conjunction*)

$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi_1 \wedge \varphi_2$
$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1][\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi_1 \wedge \varphi_2$
$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1]([\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi_1 \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi_2)$
$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1][\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi_1 \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_1][\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi_2)$
$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi_1 \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi_2$
$M, w \models [\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi_1 \wedge [\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi_2$

(Necessitation rule) From $\vdash \varphi$, we use (Nec) twice for $[\mathsf{U}^{\leq n}, \mathsf{f}_2]$ and, resp., $[\mathsf{U}^{\leq n}, \mathsf{f}_1]$ to obtain $\vdash [\mathsf{U}^{\leq n}, \mathsf{f}_1][\mathsf{U}^{\leq n}, \mathsf{f}_2]\varphi$. Finally, applying the reduction axiom for product, we obtain $\vdash [\mathsf{U}^{\leq n}, \mathsf{f}_1 \otimes \mathsf{f}_2]\varphi$.

In contrast to the previous section on deterministic planning, we cannot fix a priori which action model $\mathsf{U}^{\leq n}$ (and logic) are we working with, when solving a given planning domain based on U. It is only after the planning algorithm terminates with a solution, that we (a posteriori) discover for which n the action model $\mathsf{U}^{\leq n}$ (actually $\mathsf{U}^{\cup \leq n}$, see below) will suffice to check that this plan is indeed a solution. Non-deterministic solutions are more naturally expressed if we further extend the logics $\mathrm{LCC}_{\otimes n}$ with non-deterministic choice.

5.4 $\mathrm{LCC}_{\cup\otimes n}$: choice and non-deterministic actions.

In this section we extend the LCC-logics of bounded composition with the operator choice, that maps some pairs of actions e, f into a new action $\mathsf{e} \cup \mathsf{f}$. The latter expression denotes an action with indeterminate effects: an execution of $\mathsf{e} \cup \mathsf{f}$ will turn either as an execution of e or as an execution of f. It is an external agent, the environment (nature) in principle, who chooses the particular outcome after each execution of $\mathsf{e} \cup \mathsf{f}$. (This is called *demonic* non-determinism, in opposition to so-called *angelic* non-determinism where the planner agent itself selects a course of actions e rather than another one f, if both are executable.) Choice will be indistinctly represented as follows E_d, $\{\mathsf{e}, \ldots, \mathsf{f}\}$ or $\mathsf{e} \cup \ldots \cup \mathsf{f}$.

The language of $\mathrm{LCC}_{\cup\otimes n}$ adds to that of $\mathrm{LCC}_{\otimes n}$ a clause for action modalities of the form

$$[\mathsf{U}^{\leq n}, \mathsf{E}_d]\varphi$$

where $\mathsf{E}_d \subseteq \mathsf{E}^{\leq n}$ is an arbitrary (but non-empty) set of product actions $(\mathsf{f}_1 \otimes \cdots \otimes \mathsf{f}_k)$. The new actions, say,

$$\begin{aligned} \mathsf{E}_d &= \{(\mathsf{f}_1 \otimes \cdots \otimes \mathsf{f}_k), \ldots, (\mathsf{f}'_1 \otimes \cdots \otimes \mathsf{f}'_{k'})\} \quad \text{are also denoted} \\ &= (\mathsf{f}_1 \otimes \cdots \otimes \mathsf{f}_k) \cup \ldots \cup (\mathsf{f}'_1 \otimes \cdots \otimes \mathsf{f}'_{k'}). \end{aligned}$$

The presence of post-conditions in LCC actions prevents us from modeling the new non-deterministic actions, e.g. $\mathsf{e} \cup \mathsf{f}$, as full-fledged actions in the action model (as we did for product $\mathsf{e} \otimes \mathsf{f} \in \mathsf{E}^{\leq n}$). The problem is that for actions like *tossing a coin*, the post-condition for *heads*, say the variable h, will be at each execution either $\top$ or $\bot$; hence the post-condition for h is not a unique formula, and post cannot be a map.

This contrasts with the match between $\mathsf{U}^{\leq n}$ and $\mathrm{LCC}_{\otimes n}$ above, and also with the purely epistemic action models [3]. In these logics, each action operator in the language is associated an element in the action model. In this sense, even if our set of actions in the model is the same $\mathsf{E}^{\leq n}$ that we had for $\mathrm{LCC}_{\otimes n}$ logics, each constructible non-deterministic plans will be shown "equivalent" to some E_d modality. For example, the plan -informally written as- $\mathsf{e} \otimes (\mathsf{f} \cup \mathsf{f}')$ will be associated the modality $[\mathsf{U}, (\mathsf{e} \otimes \mathsf{f}) \cup (\mathsf{e} \otimes \mathsf{f}')]$.

As suggested in [13] non-deterministic actions are introduced with the help of multi-pointed semantics.

Definition 9. *Given an epistemic model M and an action model U, let $W_d \subseteq W$ and $\mathsf{E}_d = \{\mathsf{f}_1, \ldots, \mathsf{f}_k\} \subseteq \mathsf{E}$. Then M, W_d and U, E_d are multi-pointed models. We define*

$$\begin{array}{lll} M, W_d \models \varphi & \textit{iff} & M, w \models \varphi \quad \textit{for each } w \in W_d \\ M, w \models [\mathsf{U}, \mathsf{E}_d]\varphi & \textit{iff} & M \circ \mathsf{U}, \{(w, \mathsf{f}), \ldots, (w, \mathsf{f}')\} \models \varphi \\ & & \quad \textit{for each } (w, \mathsf{f}), \ldots, (w, \mathsf{f}') \in W^{M \circ \mathsf{U}} \textit{ with } \mathsf{f}, \ldots, \mathsf{f}' \in \mathsf{E}_d \end{array}$$

In other words, this semantics for $[\mathsf{U}, \mathsf{E}_d]$ modalities simply amounts to the semantics of the operators $[\mathsf{U}, \mathsf{f}]$ for each $\mathsf{f} \in \mathsf{E}_d$. That is,

$$M, w \models [\mathsf{U}, \mathsf{E}_d]\varphi \quad \text{iff} \quad \text{for each } f \in \mathsf{E}_d, \;\; M, w \models \mathsf{pre}(\mathsf{f}) \text{ implies } M \circ \mathsf{U}, (w, \mathsf{f}) \models \varphi$$

For the reasons pointed above, non-deterministic actions $\mathsf{e} \cup \mathsf{f}$ or E_d are not actions in the action model, only their components e and f are. In other words, the action model is just $\mathsf{U}^{\leq n}$. In summary, we just add the modalities $[\mathsf{U}, \mathsf{E}_d]$ and expand the semantics to the multi-pointed case, rather than expanding the action models themselves.

In [13], the additional reduction axiom listed next is suggested for non-deterministic choice. Here we add it to the previous system $\mathrm{LCC}_{\otimes n}$:

the reduction axioms and rules of $\mathrm{LCC}_{\otimes n}$	
plus	
$[\mathsf{U}, \mathsf{E}_d]\varphi \leftrightarrow \bigwedge_{\mathsf{e} \in \mathsf{E}_d} [\mathsf{U}, \mathsf{e}]\varphi$	(*choice*)

Fig. 3. The axioms and rules for $\mathrm{LCC}_{\cup \otimes n}$.

It is straightforward that the reduction axiom (*choice*) for $[\mathsf{U}, \mathsf{E}_d]\varphi$ is sound w.r.t. the semantics above. This allows us to extend once more the translation function t from $\mathrm{LCC}_{\otimes n}$ to $\mathrm{LCC}_{\cup \otimes n}$ with the clause

$$t([\mathsf{U}^{\leq n}, \mathsf{e} \cup \ldots \cup \mathsf{f}]\varphi) \;=\; t([\mathsf{U}^{\leq n}, \mathsf{e}]\varphi) \wedge \ldots \wedge t([\mathsf{U}^{\leq n}, \mathsf{f}]\varphi)$$

The resulting translation function t splits the new modalities $[\mathsf{U}, \mathsf{E}_d]$ and then proceeds as in the case of $\mathrm{LCC}_{\otimes n}$. The soundness of the axiom (*choice*) preserves the soundness of the expanded translation function, again reducing the language of $\mathrm{LCC}_{\cup\otimes n}$ to that of E·PDL and giving the next completeness result.

Corollary 5. *The logic* $\mathrm{LCC}_{\cup\otimes n}$ *is sound and complete.*

Fact 1 *The* LCC *axioms for* $[\mathsf{U}, \mathsf{e}]$ *that do not involve preconditions* $\mathsf{pre}(\cdot)$ *are also sound for* $[\mathsf{U}, \mathsf{e} \cup \mathsf{f}]$ *modalities. That is, all the* LCC *axioms except for (*atoms*) and (*partial functionality*).*

Proof. The case for (*top*) is obvious.

For (*conjunction*), we show $\models [\mathsf{U}, \mathsf{e} \cup \mathsf{e}']\varphi_1 \wedge \varphi_2 \leftrightarrow ([\mathsf{U}, \mathsf{e} \cup \mathsf{e}']\varphi_1 \wedge [\mathsf{U}, \mathsf{e} \cup \mathsf{e}']\varphi_2)$

$M, w \models [\mathsf{U}, \mathsf{e} \cup \mathsf{e}']\varphi_1 \wedge \varphi_2$
iff $M, w \models [\mathsf{U}, \mathsf{e}]\varphi_1 \wedge \varphi_2$ and $M, w \models [\mathsf{U}, \mathsf{e}']\varphi_1 \wedge \varphi_2$
iff ($M, w \models [\mathsf{U}, \mathsf{e}]\varphi_1$ and $M, w \models [\mathsf{U}, \mathsf{e}]\varphi_2$)
and ($M, w \models [\mathsf{U}, \mathsf{e}']\varphi_1$ and $M, w \models [\mathsf{U}, \mathsf{e}']\varphi_2$)
iff ($M, w \models [\mathsf{U}, \mathsf{e}]\varphi_1$ and $M, w \models [\mathsf{U}, \mathsf{e}']\varphi_1$)
and ($M, w \models [\mathsf{U}, \mathsf{e}]\varphi_2$ and $M, w \models [\mathsf{U}, \mathsf{e}']\varphi_2$)
iff $M, w \models [\mathsf{U}, \mathsf{e} \cup \mathsf{e}']\varphi_1$ and $M, w \models [\mathsf{U}, \mathsf{e} \cup \mathsf{e}']\varphi_2$
iff $M, w \models [\mathsf{U}, \mathsf{e} \cup \mathsf{e}']\varphi_1 \wedge [\mathsf{U}, \mathsf{e} \cup \mathsf{e}']\varphi_2$

For the case $[\mathsf{U}, \mathsf{E}_d][\pi]\varphi$, the reformulation of the axiom (E·PDL-*programs*) makes the resulting claim obvious.

Also notice that the executability of non-deterministic actions $\mathsf{e} \cup \mathsf{f}$ only requires that some action e or f (or both) is executable.

Lemma 5. *The following holds:* $\models \langle \mathsf{U}, \mathsf{E}_d \rangle \top \leftrightarrow \bigvee_{\mathsf{e} \in \mathsf{E}_d} \mathsf{pre}(\mathsf{e})$.

Proof. We show the simple case $\mathsf{E}_d = \{\mathsf{e}, \mathsf{f}\} \subseteq \mathsf{E}^{\leq n}$. The proof for the general case is analogous.

$$\begin{aligned}
&M, w \models \langle \mathsf{U}^{\leq n}, \mathsf{e} \cup \mathsf{f} \rangle \top \\
&M, w \models \neg[\mathsf{U}^{\leq n}, \mathsf{e} \cup \mathsf{f}]\neg\top \\
&M, w \models \neg([\mathsf{U}^{\leq n}, \mathsf{e}]\neg\top \wedge [\mathsf{U}^{\leq n}, \mathsf{f}]\neg\top) \\
&M, w \models \neg[\mathsf{U}^{\leq n}, \mathsf{e}]\neg\top \vee \neg[\mathsf{U}^{\leq n}, \mathsf{f}]\neg\top) \\
&M, w \models \langle \mathsf{U}^{\leq n}, \mathsf{e} \rangle \top \vee \langle \mathsf{U}^{\leq n}, \mathsf{f} \rangle \top \\
&M, w \models \mathsf{pre}(\mathsf{e}) \vee \mathsf{pre}(\mathsf{f})
\end{aligned}$$

6 Non-Deterministic Plans in LCC

Now we turn into non-deterministic planning, for planning domains containing actions with disjunctive effects are available to the agent, e.g.

$$\models [\mathsf{U}, \mathsf{f}_0 \cup \mathsf{f}_1]\, p \vee q, \quad \text{but with} \quad \not\models [\mathsf{U}, \mathsf{f}_0 \cup \mathsf{f}_1]p \quad \text{and} \quad \not\models [\mathsf{U}, \mathsf{f}_0 \cup \mathsf{f}_1]q$$

as given by the post-conditions postconditions $\mathsf{post}(\mathsf{f}_0)(p) = \mathsf{post}(\mathsf{f}_1)(q) = \top$, and $\mathsf{post}(\mathsf{f}_0)(q) = q$ and $\mathsf{post}(\mathsf{f}_1)(p) = p$).

In particular, we focus on strong non-deterministic planning. Recall a strong solution for a given planning domain is a plan such that all of its possible executions in the initial state lead to a goal state. Thus, ignoring preconditions, the above action $\mathsf{f}_0 \cup \mathsf{f}_1$ is a strong solution to $(\varphi_T, \{\mathsf{f}_0 \cup \mathsf{f}_1\}, \varphi_G)$, for the goal $\varphi_G = p \vee q$; and it is a weak solution when the goal is $\varphi_G = p$.)

Example 3. Consider the action *toss a coin*. This can be seen as a non-deterministic choice between the two deterministic actions of *toss heads* and *toss tails*. Let (resp.) toss_h and $\mathsf{toss}_{\neg h}$ denote these actions, with assigned post-conditions

$$\mathsf{post}(\mathsf{toss}_h) : h \longmapsto \top, \quad \text{and} \quad \mathsf{post}(\mathsf{toss}_{\neg h}) : h \longmapsto \bot$$

Note that the executing agent a cannot distinguish whether she executes toss_h or $\mathsf{toss}_{\neg h}$ (at least until the coin has landed and the agent proceeds to observe the result). This indistinguishability, formally given by $\mathsf{R}_a(\mathsf{toss}_h, \mathsf{toss}_{\neg h})$ and viceversa, is called *run-time* indistinguishability in [4]. Even if the agent intends the toss to result in heads (i.e. the agent intends toss_h), the action really available to a is

$$\mathsf{toss}_h \cup \mathsf{toss}_{\neg h} \qquad \text{computed as} \qquad \bigcup \{\mathsf{e} \in \mathsf{E} \mid \mathsf{R}_a(\mathsf{toss}_h, \mathsf{e})\}$$

Randomness is not essential feature to non-deterministic actions, as the next example illustrates.

Example 4. Consider for instance, the action of pressing a button on the wall, which will switch the light *on* or *off* (the latter denoting $\neg$*on*). Let the corresponding deterministic actions be denoted on and off, defined by similar post-conditions:

$$\mathsf{post}(\mathsf{on}) : \mathit{on} \longmapsto \top, \quad \text{and} \quad \mathsf{post}(\mathsf{off}) : \mathit{on} \longmapsto \bot$$

In contrast to the coin example, these two actions have different (in fact, mutually inconsistent) preconditions:

$$\mathsf{pre}(\mathsf{on}) = \mathit{off} \quad \text{and} \quad \mathsf{pre}(\mathsf{off}) = \mathit{on}$$

Suppose first our executing agent a is blind (or blind-folded), so she cannot distinguish on from off at run-time (during execution). See Figure 4 (Top). Notice that $\mathsf{on} \cup \mathsf{off}$ has a trivial precondition: $\mathit{on} \vee \neg \mathit{on}$, given by $\mathsf{pre}(\mathsf{on}) \vee \mathsf{pre}(\mathsf{off})$.

Secondly, suppose instead that the agent can see (or has been told) whether the light is initially *on*, Figure 4(Mid). She knows which of the two actions $\mathsf{on}^\star$ or $\mathsf{off}^\star$ is executable (has a true precondition), so we can model them separately as two deterministic actions.

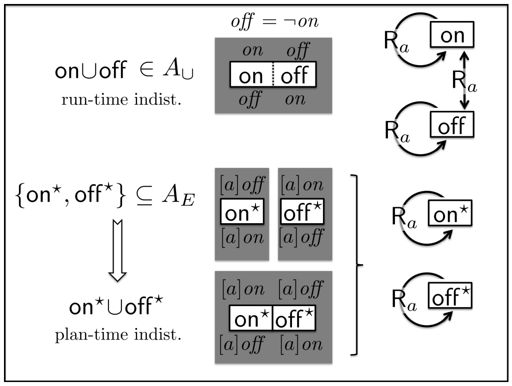

Fig. 4. (Top) A blind agent pressing the light button: $\mathsf{on} \cup \mathsf{off}$. (Mid) Switching the light on (while seeing): $\mathsf{on}^\star$. Similarly for $\mathsf{off}^\star$. (Bottom) Pressing the light button (while seeing), during the planning phase.

Along this line, the planner agent a might not know (during planning) whether she will find the light *on* or *off*, when she switches it (this being a planned action). Figure 4 (Bottom). This is called *plan-time* indistinguishability in [4], since only at execution time the agent will know whether whether she is going to turn the light on or off. This kind of actions, modeled as a choice $\mathsf{on}^\star \cup \mathsf{off}^\star$.

After this review on the effects of partial observability of states and actions, we proceed to the task of plan search. As these examples show, the previous notions of *available actions* A, *plan* and *solution* must be redefined for the present non-deterministic case. For the sake of simplicity, we will only consider the choice between two actions $\mathsf{f}_0 \cup \mathsf{f}_1$. The definitions and results in this paper can be generalized to the choice of finitely many actions $\mathsf{f}_0 \cup \mathsf{f}_1 \cup \cdots \cup \mathsf{f}_k$.

From here on, we abstract from any particular bound n upon the length of plans, so in the following we will just write the action model as U rather than as a fixed action model $\mathsf{U}^{\leq n}$. With this remark in mind, recall the set of action sequences definable in $\mathrm{LCC}_{\cup\otimes n}$ is any sequence of action modalities

$$[\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_k] \quad \text{(also written } (\mathsf{E}_0, \dots, \mathsf{E}_k))$$

Concerning the basic actions available to the agent, we have: (1) a set A_{e} of actions e from E; and (2) a set of $A_\cup$ containing pairs of actions, denoted $\mathsf{e}{\cup}\mathsf{f}$, with again $\mathsf{e} \in \mathsf{E}$ and $\mathsf{f} \in \mathsf{E}$. For an example of these basic actions, we have $\mathsf{on}^\star$in A_{E} and $\mathsf{on}^\star \cup \mathsf{off}^\star$ and $\mathsf{toss}_h \cup \mathsf{toss}_{\neg h}$ in $A_\cup$. The following definition replace the old set A from Definition 5 by the new set $A_{\mathsf{E}} \cup A_\cup$.

Definition 10. *A* non-deterministic planning domain *in* U *is a triple*

$$\mathbb{M} = (\varphi_T, A_{\mathsf{E}} \cup A_\cup, \varphi_G)$$

with $A_{\mathsf{E}} \subseteq \mathsf{E}$*, and* $A_\cup \subseteq \mathsf{E} \times \mathsf{E}$.

Not all of the above action sequences $[\mathsf{U},\mathsf{E}_1]\dots[\mathsf{U},\mathsf{E}_k]$ in the language of $\mathrm{LCC}_{\cup\otimes n}$ denote action sequences that are available to the agent according to a planning domain $\mathbb{M}$. The latter sub-class is defined next.

Definition 11. *We say* $[\mathsf{U},\mathsf{e}]$ *and* $[\mathsf{U},\mathsf{e}\cup\mathsf{f}]$ *are* $\mathbb{M}$*-sequences whenever* $\mathsf{e}\in A_{\mathsf{E}}$ *and* $\mathsf{e}\cup\mathsf{f}$ *in* $A_\cup$*. Moreover, if* $\mathsf{e}'\otimes\dots\otimes\mathsf{e}''$ *and* $\mathsf{f}'\otimes\dots\otimes\mathsf{f}''$ *are elements of* $A_{\mathsf{E}}^{<\omega}$ *and* $\mathsf{e}\cup\mathsf{f}\in A_\cup$ *satisfies* $(\mathsf{e},\mathsf{f}),(\mathsf{f},\mathsf{e})\notin\mathsf{R}_a$*, then*

$$[\mathsf{U},(\mathsf{e}\otimes\mathsf{e}'\otimes\dots\otimes\mathsf{e}'')\cup(\mathsf{f}\otimes\mathsf{f}'\otimes\dots\otimes\mathsf{f}'')]\quad \textit{is an } \mathbb{M}\textit{-sequence}$$

Finally, any finite sequence $[\mathsf{U},\mathsf{E}_k]\dots[\mathsf{U},\mathsf{E}_1]$ *of* $\mathbb{M}$*-sequences is an* $\mathbb{M}$*-sequence.*

The idea of $\mathbb{M}$-sequences is to minimally constrain (within the limits of $\mathcal{L}_{\mathrm{LCC}_{\cup\otimes n}}$) how much freedom an agent is allowed after executing a non-deterministic action $\mathsf{e}\cup\mathsf{f}$ (while preserving epistemic control):

- if the components e and f are run-time indistinguishable according to R_a, the next action after executing $\mathsf{e}\cup\mathsf{f}$ must be uniquely specified (though it can be another non-deterministic action),
- if the components e and f are run-time distinguishable, one can execute alternative (deterministic) actions, say e' or f', depending on whether the execution of $\mathsf{e}\cup\mathsf{f}$ instantiated, resp., as e or as f.

Example 5. (Cont'd) Recall the sets of available actions $A_{\mathsf{E}}=\varnothing$ and $A_\cup=\{\mathsf{toss}_h\cup\mathsf{toss}_{\neg h}\}$ from Example 3. Read the tossing action as causing the coin to land into agent a's hand. And expand these sets with a sensing action in $A_\cup$ (feeling in your hand whether the coin landed heads) and a flip (into heads) action in A_{E}:

$$\begin{array}{lcc}
\mathsf{feel}_h\cup\mathsf{feel}_{\neg h} & \mathsf{pre}(\mathsf{feel}_h)=h & \mathsf{pre}(\mathsf{feel}_{\neg h})=\neg h\\
 & \mathsf{post}(\mathsf{feel}_h)=\mathsf{id}_{\mathsf{Var}} & \mathsf{post}(\mathsf{feel}_{\neg h})=\mathsf{id}_{\mathsf{Var}}\\
 & \mathsf{R}_a(\mathsf{feel}_h,\mathsf{feel}_{\neg h}) & \mathsf{R}_a(\mathsf{feel}_{\neg h},\mathsf{feel}_h)\\
\mathsf{flip}_h & \mathsf{pre}(\mathsf{flip}_h)=\neg h & \\
 & \mathsf{post}(\mathsf{flip}_h): h\mapsto\top &
\end{array}$$

Then, the following is an $\mathbb{M}$-sequence leading to a *heads* result in any execution.

$$[\mathsf{U},\mathsf{toss}_h\cup\mathsf{toss}_{\neg h}]\;[\mathsf{U},(\mathsf{feel}_h\cup(\mathsf{feel}_{\neg h}\otimes\mathsf{flip}_h)]$$

tossing the coin, sensing it, and if tails flip it to heads

Definition 12. *We say that an* $\mathbb{M}$*-sequence* $[\mathsf{U},\mathsf{E}_1],\dots,[\mathsf{U},\mathsf{E}_r]$ *is a solution to the planning domain* $\mathbb{M}=(\varphi_T,A_{\mathsf{E}}\cup A_\cup,\varphi_G)$ *iff*

$$\begin{array}{ll}
\models\varphi_T\to[\mathsf{U},\mathsf{E}_1]\dots[\mathsf{U},\mathsf{E}_r]\varphi_G & \textit{(}\text{success}\textit{)}\\
\models\varphi_T\to\langle\mathsf{U},\mathsf{E}_1\rangle\dots\langle\mathsf{U},\mathsf{E}_r\rangle\top & \textit{(}\text{executability}\textit{)}
\end{array}$$

It can be shown that the $\mathbb{M}$-sequence from Ex. 5 is a solution for the planning domain

$$\mathbb{M}=(\;\top,\;\{\,\mathsf{toss}_h\cup\mathsf{toss}_{\neg h},\;\mathsf{feel}_h\cup\mathsf{feel}_{\neg h},\;\mathsf{flip}_h,\mathsf{skip}\,\},\;[(a\cup b)^*]h\;)$$

7 A Search Algorithm for Non-deterministic Planning in LCC

Let us then proceed to the study of search algorithms for arbitrary planning domains $\mathbb{M}$. These planning algorithms search for solutions in the space of plans, defined below. The idea is to reduce a non-deterministic plan into a sequence of pairs of deterministic plans, each pair motivated by the introduction of a non-deterministic action. These plans are a triple consisting of: (1) a (possibly empty) $\mathbb{M}$-sequence $[\mathsf{U}, \mathsf{E}_k], \ldots [\mathsf{U}, \mathsf{E}_1]$, (possibly) prefixed by an operator-like expression $[\mathsf{U}, \cdot]$ (denoting the current operator under construction); and formulas for (2) the initial state and (3) the open goals associated to the current operator or $\mathbb{M}$-sequence.

plan π = (operator + $\mathbb{M}$-sequence, init. state $\varphi_{\mathsf{init}(\pi)}$ open goals $\varphi_{\mathsf{goals}(\pi)}$)

Again we will abuse notation and refer to the $\mathbb{M}$-sequence with the label of the plan π it belongs to.

Definition 13. *Given a planning domain* $\mathbb{M} = (\varphi_T, A_{\mathsf{E}} \cup A_{\cup}, \varphi_G)$*, the* empty plan *for* $\mathbb{M}$ *is the pair* $\pi_\varnothing = (\varnothing, \varphi_G)$*. For a given plan* $\pi_k = [\mathsf{U}, \mathsf{E}_k] \ldots [\mathsf{U}, \mathsf{E}_1]$ *and its refinement with some* $\mathsf{e} \in A_{\mathsf{E}}$*, denoted* $\pi = \pi_k(\mathsf{e}) = [\mathsf{U}, \mathsf{e}]\pi_k$*, we define the refinements* $\pi(\cdot)$ *with* $\mathsf{f} \in A_{\mathsf{E}}$ *or a run-time dist. action* $\mathsf{f} \cup \mathsf{f}' \in A_{\cup}$ *as:*

$$\begin{aligned} \pi(\mathsf{f}) &= [\mathsf{U}, \mathsf{f} \otimes \mathsf{e}]\pi_k & \pi(\mathsf{f} \cup \mathsf{f}') &= [\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathbf{x})]\pi_k \\ \varphi_{\mathsf{init}(\pi(\mathsf{f}))} &= \varphi_T & \varphi_{\mathsf{init}(\pi(\mathsf{f} \cup \mathsf{f}'))} &= \text{“}[\mathsf{U}, \mathsf{f}'](\cdot)\text{”} \\ \varphi_{\mathsf{goals}(\pi(\mathsf{f}))} &= t([\mathsf{U}, \mathsf{f}]\varphi_{\mathsf{goals}(\pi)} \wedge \langle \mathsf{U}, \mathsf{f} \rangle \top) & \varphi_{\mathsf{goals}(\pi(\mathsf{f} \cup \mathsf{f}'))} &= \varphi_{\mathsf{goals}(\pi_k)} \end{aligned}$$

Given a plan π *of the form* $\pi = (\, [\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathbf{x} \otimes \mathsf{e}')]\pi_k, \text{“}[\mathsf{U}, \mathsf{f}'](\cdot)\text{”}, \varphi_{\mathsf{goals}(\pi)}\,)$*, and an action* $\mathsf{e}'' \in A_{\mathsf{E}}$ *we define the refinement* $\pi(\mathsf{e}'')$ *as*

$$\pi(\mathsf{e}'') = \begin{cases} [\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathbf{x} \otimes \mathsf{e}'' \otimes \mathsf{e})]\pi_k & \textit{if } \not\models [\mathsf{U}, \mathsf{f}'][\mathsf{U}, \mathsf{e}'' \otimes \mathsf{e}]\varphi_{\mathsf{goals}(\pi_k)} \\ [\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathsf{e}'' \otimes \mathsf{e})] & \textit{otherwise} \end{cases}$$

$$\varphi_{\mathsf{init}(\pi(\mathsf{e}''))} = \begin{cases} \varphi_{\mathsf{init}(\pi(\mathsf{e}''))} & \textit{if } \not\models [\mathsf{U}, \mathsf{f}'][\mathsf{U}, \mathsf{e}'' \otimes \mathsf{e}]\varphi_{\mathsf{goals}(\pi_k)} \\ \varphi_T & \textit{otherwise} \end{cases}$$

$$\varphi_{\mathsf{goals}(\pi(\mathsf{e}''))} = \begin{cases} t([\mathsf{U}, \mathsf{e}'']\varphi_{\mathsf{goals}(\pi)} \wedge \langle \mathsf{U}, \mathsf{e}'' \rangle \top) & \textit{if } \not\models [\mathsf{U}, \mathsf{f}'][\mathsf{U}, \mathsf{e}'' \otimes \mathsf{e}]\varphi_{\mathsf{goals}(\pi_k)} \\ t([\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathsf{e}'' \otimes \mathsf{e})]\varphi_{\mathsf{goals}(\pi_k)} & \\ \quad \wedge \langle \mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathsf{e}'' \otimes \mathsf{e}) \rangle \top) & \textit{otherwise} \end{cases}$$

Finally, if $\mathsf{f} \cup \mathsf{f}'$ *is run-time indistinguishable to the agent, i.e.* $(\mathsf{f}, \mathsf{f}'), (\mathsf{f}', \mathsf{f}) \in \mathsf{R}_a$*, we define the refinement of* π_k *with* $\mathsf{f} \cup \mathsf{f}'$ *as:*

$$\begin{aligned} \pi(\mathsf{f} \cup \mathsf{f}') &= [\mathsf{U}, \mathsf{f} \cup \mathsf{f}']\pi_k \\ \varphi_{\mathsf{init}(\pi(\mathsf{f} \cup \mathsf{f}'))} &= \varphi_T \\ \varphi_{\mathsf{goals}(\pi(\mathsf{f} \cup \mathsf{f}'))} &= t([\mathsf{U}, \mathsf{f} \cup \mathsf{f}']\varphi_{\mathsf{goals}(\pi_k)} \wedge \langle \mathsf{U}, \mathsf{f} \cup \mathsf{f}' \rangle \top) \end{aligned}$$

Given a plan π *and a refinement of it* $\pi(\cdot)$*, we say* $\pi(\cdot)$ *is a* leaf *iff either* $\varphi_{\pi(\cdot)}$ *is inconsistent or* $\models \varphi_{\mathsf{goals}(\pi(\cdot))} \to \varphi_{\mathsf{goals}(\pi)}$*. The Terminating Condition for a plan* π *is*

$$\varphi_{\mathsf{init}(\pi)} = \varphi_T \quad \textit{and} \quad \models \varphi_{\mathsf{init}(\pi)} \to \varphi_{\mathsf{goals}(\pi)}$$

$\texttt{Input}$: $\mathbb{M} = (\varphi_T, A_{\mathsf{E}} \cup A_{\cup}, \varphi_G)$.
$\texttt{LET}$ $\mathsf{Plans} = \langle \pi_{\varnothing} \rangle$ and $\pi = \pi_{\varnothing}$
$\texttt{WHILE}$ π does not satisfy Terminating Condition
    $\texttt{DELETE}$ π from Plans. $\texttt{SET}$ $\mathsf{Plans} \leftarrow \mathsf{Plans}^{\frown}\langle\, \pi' \mid \pi'$ refines π and π' not a leaf $\rangle$.
    $\texttt{SET}$ $\pi =$ the first element of Plans
$\texttt{Output}$: π (i.e. the $\mathbb{M}$-sequence defined by π)

Fig. 5. BFS algorithm for backward non-deterministic planning in $\mathrm{LCC}_{\cup\otimes n}$.

After a run-time indistinguishable action, e.g. coin tossing, conditional plans (depending on the outcome) can still be made after an observation. Let us finally address the properties of non-deterministic planning based on BFS in the space of plans, for an arbitrary planning domain $\mathbb{M} = (\varphi_T, A_{\mathsf{E}} \cup A_{\cup}, \varphi_G)$.

Theorem 3. *Let the output of the BFS algorithm in Fig. 5 be* $[\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_k]$ *for a planning domain* $\mathbb{M}$*. Then,* $[\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_k]$ *is an* $\mathbb{M}$*-sequence and a solution for* $\mathbb{M}$*.*

Proof. Consider the logic $\mathrm{LCC}_{\cup\otimes n}$ where n is the maximum number $r+1$ (or $r'+1$) occurring among the $[\mathsf{U}, \mathsf{E}_k]$ expressions of the form $\mathsf{E}_j = (\mathsf{f} \otimes \mathsf{f}_1 \otimes \dots \otimes \mathsf{f}_r) \cup (\mathsf{f}' \otimes \mathsf{f}'_1 \otimes \dots \otimes \mathsf{f}_{r'})$ for $1 \leq j \leq k$.

On the one hand, it is obvious by construction that each $[\mathsf{U}, \mathsf{E}_j]$ is an $\mathbb{M}$-sequence. Those of the form $[\mathsf{U}, \mathsf{f} \otimes \dots \otimes \mathsf{f}']$ clearly are, since $\mathsf{f}, \dots, \mathsf{f}' \in A_{\mathsf{E}}$. Those of the form $[\mathsf{U}, \mathsf{f} \cup \mathsf{f}']$ for a run-time indist. action $\mathsf{f} \cup \mathsf{f}' \in A_{\cup}$ are also $\mathbb{M}$-sequences. And, finally, the same can be said about $[\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathsf{e}')]$, since the conditions: $\mathsf{f} \cup \mathsf{f}' \in A_{\cup}$ is run-time dist.; e and e' are elements of $A_{\mathsf{E}}^{<\omega}$. We conclude that $[\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_k]$ is an $\mathbb{M}$-sequence.

The proof that $[\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_k]$ is by induction on the length of the initial subsequences $[\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_j]$. Let π_j denote the $\mathbb{M}$-sequence $[\mathsf{U}, \mathsf{E}_j] \dots [\mathsf{U}, \mathsf{E}_k]$.

(Base Case) We show that $\models \varphi_T \to [\mathsf{U}, \mathsf{E}_1]\varphi_{\mathsf{goals}(\pi_2)}$ and $\models \varphi_T \to \langle \mathsf{U}, \mathsf{E}_1 \rangle \top$. Recall $[\mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_k]$ satisfies the Terminating Condition, so we have

$$\varphi_{\mathsf{init}(\pi_1)} = \varphi_T \quad \text{and} \quad \models \varphi_{\mathsf{init}(\pi_1)} \to \varphi_{\mathsf{goals}(\pi_1)}$$

so by definition of $\varphi_{\mathsf{goals}(\pi_1)}$ we obtain $\models \varphi_T \to t([\mathsf{U}, \mathsf{E}_1]\varphi_{\mathsf{goals}(\pi_2)} \wedge \langle \mathsf{U}, \mathsf{E}_1 \rangle \top)$. By the correctness of the translation t, the two claims above are obvious.

(Ind. Case) Assume (Ind. Hyp.) that $\models \varphi_T \to [\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_j]\varphi_{\mathsf{goals}(\pi_{j+1})}$ and $\models \varphi_T \to \langle \mathsf{U}, \mathsf{E}_1 \rangle \langle \mathsf{U}, \mathsf{E}_j \rangle \top$. We proceed to show the two claims:

$$\models \varphi_T \to [\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_j][\mathsf{U}, \mathsf{E}_{j+1}]\varphi_{\mathsf{goals}(\pi_{j+2})} \quad \text{and} \quad \models \varphi_T \to \langle \mathsf{U}, \mathsf{E}_1 \rangle \langle \mathsf{U}, \mathsf{E}_j \rangle \langle \mathsf{U}, \mathsf{E}_{j+1} \rangle \top$$

The first claim can be shown as follows

$$\begin{array}{lll}
\models & \varphi_{\mathsf{goals}(\pi_{j+1}])} \to t([\mathsf{U}, \mathsf{E}_{j+1}]\varphi_{\mathsf{goals}(\pi_{j+2})} \wedge \langle \mathsf{U}, \mathsf{E}_{j+1} \rangle \top) & \text{(Def. plan)} \\
\models & \varphi_{\mathsf{goals}(\pi_{j+1}])} \to [\mathsf{U}, \mathsf{E}_{j+1}]\varphi_{\mathsf{goals}(\pi_{j+2})} & \text{(correctness } t\text{)} \\
\multicolumn{2}{l}{\models [\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_j](\varphi_{\mathsf{goals}(\pi_{j+1}])} \to [\mathsf{U}, \mathsf{E}_{j+1}]\varphi_{\mathsf{goals}(\pi_{j+2})})} & \text{(Nec.)} \\
\multicolumn{2}{l}{\models \varphi_T \to [\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_j]\varphi_{\mathsf{goals}(\pi_{j+1})}} & \text{(Ind. Hyp.)} \\
\multicolumn{2}{l}{\models [\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_j][\mathsf{U}, \mathsf{E}_{j+1}]\varphi_{\mathsf{goals}(\pi_{j+2})})} &
\end{array}$$

For the second claim above,

$$\models \varphi_T \to \langle \mathsf{U}, \mathsf{E}_1 \rangle \dots \langle \mathsf{U}, \mathsf{E}_j \rangle \top \text{ (Ind. Hyp.)}$$
$$\models \varphi_T \to [\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_j] \varphi_{\mathsf{goals}(\pi_{j+1})} \text{ (Ind.Hyp.)}$$
$$\models \varphi_T \to [\mathsf{U}, \mathsf{E}_1] \dots [\mathsf{U}, \mathsf{E}_j] t(\dots \wedge \langle \mathsf{U}, \mathsf{E}_{j+1} \rangle \top) \text{ (Def. } \varphi_{\mathsf{goals}(\pi_{j+1})})$$
$$\models \varphi_T \to \langle \mathsf{U}, \mathsf{E}_1 \rangle \dots \langle \mathsf{U}, \mathsf{E}_j \rangle \langle \mathsf{U}, \mathsf{E}_{j+1} \rangle \top \text{ (by the last 2 claims)}$$

The proof concludes by noticing that $\varphi_{\mathsf{goals}(\pi_\varnothing)}$ are the effects of $[\mathsf{U}, \mathsf{E}_k]$.

Theorem 4. *For a given planning domain $\mathbb{M}$, if some $\mathbb{M}$-sequence exists that is a solution to $\mathbb{M}$, then the BFS algorithm in Fig. 5 terminates (with a solution).*

Proof. Notice first that only actions in the finite sets A_E or $A_\cup$ are selected for refinement of plans. Hence, the search space is a finitely-branching tree (with root $\pi_\varnothing$), and thus the algorithm terminates provided some plan π exists that satisfies the Terminating Condition. Assuming a solution exists, let us check that some plan π exists that is an $\mathbb{M}$-sequence satisfying the Terminating Condition.

Let $[\mathsf{U}, \mathsf{E}_k] \dots [\mathsf{U}, \mathsf{E}_1]$ be a solution to $\mathbb{M}$. We show that it is generated by the BFS algorithm and satisfies the Terminating Condition -if no other plan is generated first, in which case we would also be done. Without loss of generality, we can also assume no A_E or $A_\cup$ action in this solution can be deleted, while preserving success and executability. (Otherwise, we would just define a constructible solution $[\mathsf{U}, \mathsf{E}'_{k'}] \dots [\mathsf{U}, \mathsf{E}'_1]$ with $k' \leq k$ and iteratively deleting such an action from A_E or $A_\cup$ for each $\mathbb{M}$-sequence $[\mathsf{U}, \mathsf{E}_j]$ at a time, and re-enumerating $[\mathsf{U}, \mathsf{E}_k] \dots [\mathsf{U}, \mathsf{E}_1]$ accordingly whenever some E_j becomes completely deleted.)

The proof for this is by induction on the construction of the solution. As usual we will refer by $\pi_{k'}$ to the $\mathbb{M}$-sequence (or plan) $[\mathsf{U}, \mathsf{E}_{k'}] \dots [\mathsf{U}, \mathsf{E}_1]$.

Since the Base Case for $[\mathsf{U}, \mathsf{E}_1]$ and the Ind. Case $[\mathsf{U}, \mathsf{E}_{k'}] \dots [\mathsf{U}, \mathsf{E}_1] \mapsto [\mathsf{U}, \mathsf{E}_{k'+1}][\mathsf{U}, \mathsf{E}_{k'}] \dots [\mathsf{U}, \mathsf{E}_1]$ are similar (just replace φ_G by $\varphi_{\mathsf{goals}(\pi_{k'})}$) we only prove the former.

(Case $\mathsf{E}_1 \in A_\mathsf{E}^{<\omega}$.) This is exactly the case for deterministic planning in the domain $(\varphi_{\mathsf{goals}(\pi_1)}, A_\mathsf{E} \cup A_\cup, \varphi_G)$ giving as solution $[\mathsf{U}, \mathsf{f}_1] \dots [\mathsf{U}, \mathsf{f}_r]$, which is logically equivalent to the present form $[\mathsf{U}, \mathsf{f}_1 \otimes \mathsf{f}_r]$.

(Case $\mathsf{E}_1 \in A_\cup$ is run-time indist.) This case $\mathsf{E}_1 = \mathsf{f} \cup \mathsf{f}'$ is obviously constructible as the plan $\pi_\varnothing(\mathsf{f} \cup \mathsf{f}')$.

(Case $\mathsf{E}_1 = [\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathsf{e}')]$.) By definition of $\mathbb{M}$-sequence, this action must be of the form $\mathsf{e} \in A_\mathsf{E}^{<\omega}$ and $\mathsf{e}' \in A_\mathsf{E}$ and $\mathsf{f} \cup \mathsf{f}' \in A_\cup$ a run-time dist. action. Let $\mathsf{e} = \mathsf{f}_1 \otimes \dots \otimes \mathsf{f}_r$ and $\mathsf{e}' = \mathsf{f}'_1 \otimes \dots \otimes \mathsf{f}'_{r'}$. We show the construction of $[\mathsf{U}, \mathsf{E}_1]$ as a sequence of refinements from $\pi_\varnothing$ to $\pi_\varnothing(\mathsf{f}_r) \cdots (\mathsf{f}_1)(\mathsf{f} \cup \mathsf{f}')(\mathsf{f}'_{r'}) \cdots (\mathsf{f}'_1)$.

The refinements with deterministic actions $\pi_\varnothing(\mathsf{f}_r) \cdots (\mathsf{f}_1)$ are successively shown to be constructible as in the deterministic case (by the above assumption, no such f_j can be a leaf or a solution). Now, these refinements result in the plan $[\mathsf{U}, \mathsf{f}_1 \otimes \dots \otimes \mathsf{f}_r]$.

At this point, the conditions for a refinement with $(\mathsf{f} \cup \mathsf{f}')$ are obviously met, so the plan $\pi = [\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathbf{x})]$ is constructible. Its goals are by definition $\varphi_{\mathsf{goals}(\pi)} = \varphi_{\pi_\varnothing} = \varphi_G$.

Since f'_r is not a leaf, we also have that $\pi(\mathsf{f}'_{r'}) = [\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathbf{x} \otimes \mathsf{f}'_{r'})]$ is constructible, with $\varphi_{\mathsf{goals}(\pi(\mathsf{f}'_{r'}))} = t([\mathsf{U}, \mathsf{f}'_{r'}]\varphi_G \wedge \langle \mathsf{U}, \mathsf{f}' \rangle \top)$, so we are again in the case of deterministic planning. Thus, $\pi(\mathsf{f}'_{r'}) \cdots (\mathsf{f}'_1)$ is also constructible, since no action $\mathsf{f}'_{r'-1} \ldots \mathsf{f}'_1$ is a leaf. Finally, consider the last refinement with f'_1. The condition $\models [\mathsf{U}, \mathsf{f}'][\mathsf{U}, \mathsf{f}'_1 \otimes \cdots \otimes \mathsf{f}'_{r'}]\varphi_G$ is met, so the resulting plan is finally

$$[\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathsf{e}')] \qquad (= [\mathsf{U}, \mathsf{E}_1])$$

The proof concludes by observing that after constructing $\pi_k = [\mathsf{U}, \mathsf{E}_k] \ldots [\mathsf{U}, \mathsf{E}_1]$, the Terminating Condition is satisfied. On the one hand, by definition we have $\varphi_{\mathsf{init}(\pi_k)} = \varphi_T$ for plans of the form of π_k. The claim $\models \varphi_T \to \varphi_{\mathsf{goals}(\pi_k)}$ also follows from inspection of the definition in each case $\mathsf{E}_k \in A_{\mathsf{E}}$, $\mathsf{E}_k \in A_{\cup}$ and E_k of the form $[\mathsf{U}, (\mathsf{f} \otimes \mathsf{e}) \cup (\mathsf{f}' \otimes \mathsf{e}')]$.

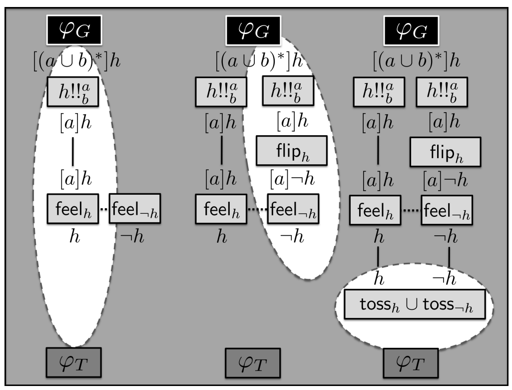

Fig. 6. Plan search in Example 1. Incremental construction of a solution for the coin example.

Example 6. Recall Example 1, where the planner agent a must show heads, denoted h, to win the prize. The action flip_h is secret in the sense of $(\mathsf{flip}_h, \mathsf{skip}) \in \mathsf{R}_b$, i.e. agent b believes nothing is happening; this secrecy is known by a provided flip_h is only R_a-related to itself. The construction of a solution is shown in Figure 6, where: (Left) a deterministic plan is being built, consisting of a's demonstration $h!!^a_b$ that h to b (with a knowing a priori that h); a plan-time indistinguishable action $\mathsf{feel}_h \cup \mathsf{feel}_{\neg h}$ is added. (Center) The planner proceeds to solve the rightmost case where $\mathsf{feel}_{\neg h}$ is executed (due to a $\neg h$ state). This planning sub-problem is solved by a flip_h action, followed by the same demonstration $h!!^a_b$. (Right) Finally, the algorithm stops after adding the run-time indistinguishable action of tossing $\mathsf{toss}_h \cup \mathsf{toss}_{\neg h}$. Note the remaining of the plan is executable no matter the result of the coin toss. The slightly different plan construction from [10] can also be built with two deterministic sensing actions (for h and $\neg h$).

8 Conclusions and Future Work

We presented backward planning algorithms for a planner-reasoner agent enabling her to find deterministic or (non-deterministic) strong plans in multi-agent scenarios.

We considered dynamic epistemic logics with ontic actions, further extended with composition and choice. Planners in these logics are sensitive to others' beliefs and may contain communications and observations as well as the usual fact-changing actions. As for future work, we would like to study more complex plan structures, or new kinds of actions like belief revision announcements. Another direction would be the study of (logical) heuristics to improve the performance of LCC planners.

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