Connecting systems of mathematical fuzzy logic with fuzzy concept lattices

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Abstract. In this paper our aim is to explore a new look at formal systems of fuzzy logics using the framework of (fuzzy) formal concept analysis (FCA). Let L be an extension of MTL complete with respect to a given L-chain. We investigate two possible approaches. The first one is to consider fuzzy formal contexts arising from L where attributes are identified with L-formulas and objects with L-evaluations: every L-evaluation (object) satisfies a formula (attribute) to a given degree, and vice-versa. The corresponding fuzzy concept lattices are shown to be isomorphic to quotients of the Lindenbaum algebra of L. The second one, following an idea in a previous paper by two of the authors for the particular case of Gödel fuzzy logic, is to use a result by Ganter and Wille in order to interpret the (lattice reduct of the) Lindenbaum algebra of L-formulas as a (classical) concept lattice of a given context.

Keywords: Mathematical Fuzzy Logics, MTL, Concept Lattices, FCA, Łukasiewicz logic

1 Introduction

In this paper our aim is to explore a new look at formal systems of fuzzy logics using the framework of (fuzzy) formal concept analysis (FCA).

The possibility of connecting descriptions of real-world contexts with powerful formal instruments is what makes (fuzzy) FCA a promising framework, merging the intuitions of intended semantics with the advantages of formal semantics. In the case of classical logic, a first attempt has been done in [8].

To build a bridge between systems of fuzzy logic and FCA, we explore two possible approaches. In the first one, given a fuzzy logic L we consider fuzzy FCA tables where attributes are described by formulas of the logic L, while L-evaluations play the role of objects: every object (L-evaluation) satisfies attributes (formulas) to a given degree, and vice-versa, every attribute (formula) is satisfied to a given extent by objects (evaluations).

The second one is, following the idea in [5] for the particular case of Gödel fuzzy logic [12], is to use Ganter and Wille's result [10, Theorem 3] in order to interpret the lattice reduct of the Lindenbaum algebra of L-formulas as a lattice of the set of formal concepts of a given context. Then, in order to endow the lattice of concepts with a structure of L-algebra, suitable operations on formal concepts have to be defined.

The paper is structured as follows. After this brief introduction, we recall some background notions in Section 2, in Section 3 we introduce concept lattices of formulas and evaluations, and in Section 4 we recall the construction of [5]. Both approaches will be used to obtain formal concepts for formulas of the 3-valued Łukasiewicz logic.

2 Preliminaries

2.1 Basic notions on Formal Concept Analysis

We recollect the basic definitions and facts about formal concept analysis needed in this work. For further details on this topics we refer the reader to [10].

Recall that an element j of a distributive lattice H is called a *join-irreducible* if j is not the bottom of H and if whenever $j = a \sqcup b$, then $j = a$ or $j = b$, for $a, b \in L$. Meet-irreducible elements are defined dually. Given a lattice $\mathbf{H} = (H, \sqcap, \sqcup, 1)$, we denote by $\mathfrak{J}(H)$ the set of its join-irreducible elements, and by $\mathfrak{M}(H)$ the set of its meet-irreducible elements.

Let G and M be arbitrary sets of *objects* and *attributes*, respectively, and let $I \subseteq G \times M$ be an arbitrary binary relation. Then, the triple $\mathbb{K} = (G, M, I)$ is called a *formal context*. For $g \in G$ and $m \in M$, we interpret $(g, m) \in I$ as "the object g has attribute m". For $A \subseteq G$ and $B \subseteq M$, a Galois connection between the powersets of G and M is defined through the following operators:

$$A^* = \{m \in M \mid \forall g \in A : gIm\} \qquad B^\circ = \{g \in G \mid \forall m \in B : gIm\}$$

Every pair (A, B) such that $A^* = B$ and $B^\circ = A$ is called a *formal concept*. A and B are the *extent* and the *intent* of the concept, respectively. Given a context $\mathbb{K}$, the set $\mathfrak{B}(\mathbb{K})$ of all formal concepts of $\mathbb{K}$ is partially ordered by $(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ (or, equivalently, $B_2 \subseteq B_1$). The *basic theorem on concept lattices* [10, Theorem 3] states that the set of formal concepts of the context $\mathbb{K}$ is a complete lattice $(\mathfrak{B}(\mathbb{K}), \sqcap, \sqcup)$, called *concept lattice*, where meet and join are defined by:

$$\begin{aligned} \bigsqcap_{j \in J} (A_j, B_j) &= \left(\bigcap_{j \in J} A_j, \left(\bigcup_{j \in J} B_j \right)^{\circ *} \right), \\ \bigsqcup_{j \in J} (A_j, B_j) &= \left(\left(\bigcup_{j \in J} A_j \right)^{* \circ}, \bigcap_{j \in J} B_j \right), \end{aligned} \tag{1}$$

for a set J of indexes. The following proposition is fundamental for our purposes.

Proposition 1 ([10, Proposition 12]). *For every finite lattice H there is (up to isomorphisms) a unique context $\mathbb{K}_H$, with $L \cong \mathfrak{B}(\mathbb{K}_H)$:*

$$\mathbb{K}_H := (\mathfrak{J}(H), \mathfrak{M}(H), \leq).$$

The context $\mathbb{K}_H$ is called the *standard context* of the lattice H.

Since H is finite, $\mathfrak{J}(H)$ is finite as well. Hence, the concept $(\mathfrak{J}(H), \emptyset)$ is the top element of $\mathfrak{B}(\mathbb{K}_H)$. We denote it by $\top_G$, emphasizing the fact that the join-irreducible elements of L are the objects of our context. Analogously, the concept $(\emptyset, \mathfrak{M}(H))$ is the bottom element of $\mathfrak{B}(\mathbb{K}_H)$, and we denote it by $\perp_M$.

2.2 On t-norm based fuzzy logics

In this paper we investigate logical systems based on left continuous *t-norms*, that are binary, commutative, associative and monotonically non-decreasing operations over $[0,1]$ that have 1 as unit element. A t-norm operator $\odot$ is used to interpret a conjunction connective, while its corresponding implication connective $\rightarrow$ is modelled by the *residuum* of $\odot$, that is, defined by $x \rightarrow y = max\{z \mid x \odot z \leq y\}$ for all $x, y, z \in [0,1]$. It has been shown that the necessary and sufficient condition for a t-norm $\odot$ to have a residuum (i.e. satisfying the condition $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for all $x, y, z \in [0,1]$) is the left-continuity $\odot$.

In [7] the authors introduce MTL, the logic of all left-continuos t-norms and their residua [13]. MTL encompasses the Basic fuzzy Logic BL of Hájek [12], which is the logic of continuous t-norms and their residua. For axiomatisations of MTL and BL, we refer the reader to [7] and [12] respectively.

Other relevant t-norm based fuzzy logics can be obtained as schematic extensions of MTL or BL. Gödel logic G is the schematic extension of BL obtained by adding the *idempotency* axiom, $\varphi \rightarrow (\varphi \odot \varphi)$. Łukasiewicz logic Ł is the schematic extension of BL obtained by adding the *double negation* axiom $\neg\neg\varphi \rightarrow \varphi$. Adding $\varphi \odot \varphi \leftrightarrow \varphi \odot \varphi \odot \varphi$ to Ł we obtain the 3-valued Łukasiewicz logic $Ł_3$.

Our interest in $Ł_3$ is given by the recent paper [6], where authors characterize this logic as the logic of prototypes and counterexamples. Gödel logic will be used as a stepping stone for developing the methodology to be applied to the case of $Ł_3$.

Each schematic extension L of MTL determines a subvariety $\mathbb{V}(\mathrm{L})$ of the variety of MTL algebras $\mathbb{MTL}$, that is the class of algebras $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, \perp, \top)$ of type $(2,2,2,2,0,0)$ such that $(A, \wedge, \vee, \perp, \top)$ is a bounded lattice, with top $\top$ and bottom $\perp$, $(A, \odot, \top)$ is a commutative monoid, satisfying the *residuation* equivalence, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$, and the *prelinearity* equation $(x \rightarrow y) \vee (y \rightarrow x) = \top$ [3]. Negation is usually defined as $\neg x = x \rightarrow \perp$.

The notion of logical consequence for a logic L relative to a class $\mathcal{A} \subseteq \mathbb{V}(\mathrm{L})$ is defined as follows: for any set of formulas $T \cup \{\varphi\}$, φ is a logical consequence

[3] MTL algebras are commutative integral bounded residuated lattices satisfying prelinearity [9].

of T, written $T \models_{\mathcal{A}} \varphi$, whenever for all algebra $\mathbf{A} \in \mathcal{A}$ and each evaluation e of formulas on $\mathbf{A}$, if $e(\psi) = 1$ for all $\psi \in T$, then $e(\varphi) = 1$ as well.

Given a logic L, two formulas φ and ψ are logically equivalent, in symbols $\varphi \equiv_{\mathrm{L}} \psi$, if and only if $\varphi \leftrightarrow \psi$ is a L-tautology, that is, if $\models_{\mathbb{V}(\mathrm{L})} \varphi \leftrightarrow \psi$. The *Lindenbaum Algebra* $\mathbf{Lind}(\mathrm{L})$ of L is the algebra whose elements are the equivalence classes of formulas of L, with respect to $\equiv_{\mathrm{L}}$. The free k-generated algebra $\mathbf{F}_k(\mathbb{V}(\mathrm{L}))$ in $\mathbb{V}(\mathrm{L})$ is the subalgebra of the Lindenbaum algebra $\mathbf{Lind}(\mathrm{L})$ of the formulas over the first k variables. Combinatorial representations of $\mathbf{F}_k(\mathbb{G})$ and $\mathbf{F}_k(\mathbb{MV}_3)$, where $\mathbb{G} = \mathbb{V}(G)$ and $\mathbb{MV}_3 = \mathbb{V}(Ł_3)$, can be found in [1].

3 The Concept Lattice of Formulas and Evaluations

Suppose L is an axiomatic extension of MTL that is complete with respect to a given L-chain M, that is, $\models_{\mathbb{V}(\mathrm{L})} = \models_M$. In what follow, we will denote by $\mathcal{L}$ the set of propositional L-formulas built from a *finite* set of propositional variables V, and by Ω the set of truth-evaluations of propositinal variables into the L-chain M, that is, $\Omega = \{e : V \to M\}$. Of course, every evaluation of variables uniquely extends to an evaluation of any propositional formula using the truth-functions interpreting the connectives.

In our FCA-based analysis of the notion of consequence in the logic L, we will consider attributes described as propositional formulas from $\mathcal{L}$, and objects as evaluations from Ω. In this setting, a formal context will be specified by a triple

$$K = (\Omega_0, \mathcal{L}_0, R),$$

where $\Omega_0 \subseteq \Omega$ and $\mathcal{L}_0 \subseteq \mathcal{L}$ are finite sets, and $R : \Omega_0 \times \mathcal{L}_0 \to M$ is a M-valued fuzzy relation defined as $R(e, \varphi) = e(\varphi)$.

In this way, each attribute or formula $\varphi \in \mathcal{L}_0$ determines a fuzzy set of objects $\varphi^* : \Omega_0 \to M$, with $\varphi^*(e) = R(e, \varphi)$, for all $e \in \Omega_0$, and vice-versa, each object or evaluation $e \in \Omega_0$ determines a fuzzy set of attributes $e^\circ : \mathcal{L}_0 \to M$, with $e^\circ(\varphi) = R(e, \varphi)$, for all $\varphi \in \mathcal{L}_0$. More than that, following Pollandt [14] and Bělohlávek's [2] models of FCA, this correspondence is extended to a Galois connection between fuzzy sets of formulas and fuzzy sets of evaluations as follows.

Definition 1. *Let $F \in \mathcal{F}(\mathcal{L}_0)$ be a fuzzy subset of formulas (fuzzy theory) and let $E \in \mathcal{F}(\Omega_0)$ be a fuzzy set of evaluations. Define:*

- *F^* is the fuzzy subset of Ω_0 defined as $F^*(e) = \inf_{\varphi \in \mathcal{L}_0} F(\varphi) \to R(e, \varphi)$, for all $e \in \Omega_0$,*
- *E° is the fuzzy subset of $\mathcal{L}_0$ defined as $E^\circ(\varphi) = \inf_{e \in \Omega_0} E(e) \to R(e, \varphi)$, for all $\varphi \in \mathcal{L}_0$.*

A pair (E, F) is a logic fuzzy concept *if $F^* = E$ and $E^\circ = F$.*

In other words, F^* is the fuzzy set of models of the fuzzy theory F, and E° is the fuzzy set of formulas satisfied by the fuzzy set of evaluations E. Moreover, as it is known, $^{*\circ}$ is a closure operation on the set $\mathcal{F}(\mathcal{L}_0)$ of M-valued fuzzy sets of formulas, hence $F \leq F^{*\circ}$. Actually the mapping ${}^{*\circ} : \mathcal{F}(\mathcal{L}_0) \to \mathcal{F}(\mathcal{L}_0)$, defined by

$$F^{*\circ}(\varphi) = \inf_{e \in \Omega_0} [\inf_{\psi \in \mathcal{L}_0} F(\psi) \to e(\psi)] \to e(\varphi)$$

can be considered as a *graded* logical consequence relation, that it is even a bit more general than the one central to the so-called *graded approach to fuzzy logic*, developed by authors like J. A. Goguen, J. Pavelka, V. Nóvak and G. Gerla, as discussed e.g. in [11].

In what follows, we will denote by $\mathbf{C}(K) = (C(K), \preceq)$ the lattice of fuzzy concepts induced by a context K, where the ordering $\preceq$ is defined as

$$(E, F) \preceq (E', F') \text{ iff } E \leq E' \text{ and } F \geq F' ,$$

and the meet and join operations are defined as:

$$(E, F) \sqcap (E', F') = (E \cap E', (F \cup F')^{*\circ}), \quad (E, F) \sqcup (E', F') = ((E \cup E')^{\circ *}, F \cap F') ,$$

where $\cap$ and $\cup$ denote intersection and union of fuzzy sets, defined with the min and max operations respectively.

This lattice is bounded and the bottom element is the concept $\bot_K = (\emptyset, \mathcal{L}_0)$, while the top element is $\top_K = (\Omega_0, T_{\Omega_0})$, where T_{Ω_0} if the fuzzy set of formulas defined by $T_{\Omega_0}(\psi) = \inf_{e \in \Omega_0} e(\psi)$.

Let us see how it looks like the fuzzy concept in $\mathbf{C}(K)$ induced by (the crisp set of) a single formula $\varphi \in \mathcal{L}_0$, i.e. the pair $(\varphi^*, \varphi^{*\circ})$, where for the sake of a simpler notation we have used φ^* for $\{\varphi\}^*$ and $\varphi^{*\circ}$ for $(\{\varphi\}^*)^\circ$. An easy computation shows that:

- $\varphi^*(e) = R(e, \varphi) = e(\varphi)$, for all $e \in \Omega_0$;
- $\varphi^{*\circ}(\psi) = \inf_{e \in \Omega_0} R(e, \varphi) \to R(e, \psi) = \inf_{e \in \Omega_0} e(\varphi \to \psi)$, for all $\psi \in \mathcal{L}_0$.

Further, if we consider a finite set of formulas or theory T, using the same notation convention as above, the corresponding concept $(T^*, T^{*\circ})$ is as follows, where $\bigwedge T$ denotes the $\wedge$-conjunction of all the formulas in T, i.e. $\bigwedge T = \bigwedge_{\varphi \in T} \varphi$:

- $T^*(e) = \inf_{\varphi \in T} R(e, \varphi) = \inf_{\varphi \in T} e(\varphi) = e(\bigwedge T)$, for all $e \in \Omega_0$;
- $T^{*\circ}(\psi) = \inf_{e \in \Omega_0} T^*(e) \to R(e, \psi) = \inf_{e \in \Omega_0} e(\bigwedge T \to \psi)$, for all $\psi \in \mathcal{L}_0$.

Note that, as discussed above, $T^{*\circ}$ accounts for a certain notion of graded consequence from T, in the sense that $T^{*\circ}(\psi)$ provides the degree in which ψ is implied by T, relative to the set of interpretations Ω_0. It is a graded consequence that resembles Pavelka's notion of *truth degree of a formula in a theory* (see e.g. [12, 11]), although they do not coincide. It is also related to the so-called degree preserving logic $\models_{\text{L}}^{\leq}$ companion of L, see e.g. [4]. Indeed, it is easy to check the following lemma.

Lemma 1. *For any $\psi \in \mathcal{L}_0$, $T^{*\circ}(\psi) = 1$ iff $e(\bigwedge T \to \psi) = 1$ for all $e \in \Omega_0$.*

Therefore, when $\Omega_0 = \Omega$, $T^{*\circ}(\psi) = 1$ holds if, and only if, $\inf_{\varphi \in T} e(\varphi) \leq e(\psi)$, i.e. iff $T \models_{\text{L}}^{\leq} \psi$. That is, the core of $T^{*\circ}$ is nothing but the set of consequences of T (restricted to $\mathcal{L}_0$) under the degree preserving logic companion of L.

Lemma 2. *$T_1^{*\circ} = T_2^{*\circ}$ iff $\bigwedge T_1$ and $\bigwedge T_2$ are logically equivalent relative to Ω_0, i.e. $e(\bigwedge T_1) = e(\bigwedge T_2)$ for every evaluation $e \in \Omega_0$.*

Proof. The direction right-to-left is trivial. As for the converse, if $T_1^{*\circ} = T_2^{*\circ}$, then in particular, for all χ, $T_1^{*\circ}(\chi) = 1$ iff $T_2^{*\circ}(\chi) = 1$. Take $\chi = \bigwedge T_1$. Since $T_1^{*\circ}(\bigwedge T_1) = 1$, then $T_2^{*\circ}(\bigwedge T_1) = 1$ as well, and by Lemma 1 this happens iff for all $e \in \Omega_0$, $e(\bigwedge T_2) \leq e(\bigwedge T_1)$. Analogously, if we take $\chi = \bigwedge T_2$, we would get that, for all $e \in \Omega_0$, $e(\bigwedge T_1) \leq e(\bigwedge T_2)$. □

Notice again that in case $\Omega_0 = \Omega$, then $T_1^{*\circ} = T_2^{*\circ}$ iff $\bigwedge T_1$ and $\bigwedge T_2$ are logically equivalent in the usual sense.

The set $C^{cg}(K)$ of concepts of the form $(T^*, T^{*\circ})$, with $T \subseteq \mathcal{L}_0$ a finite (crisp) set of formulas, is in fact what is known as the set of *crisply generated concepts* in the fuzzy concept lattice $\mathbf{C}(K)$ [3]. As already mentioned, for the purpose of building concepts, we can always replace a finite theory T by the $\wedge$-conjunction of its formulas $\bigwedge T$. Indeed, for every concept of the form $(T^*, T^{*\circ})$ with T a finite set of formulas, there is always a formula φ (e.g. $\bigwedge T$) such that $(T^*, T^{*\circ}) = (\varphi^*, \varphi^{*\circ})$. Thus $C^{cg}(K) = \{(\varphi^*, \varphi^{*\circ}) \mid \varphi \in \mathcal{L}_0\}$ and we can safely restrict ourselves to deal with concepts induced by a *single* formula.

The lattice operations in $\mathbf{C}(K)$ over concepts from $C^{cg}(K)$ take the following form.

Lemma 3. *For any $\varphi, \psi \in \mathcal{L}$,*

$$(\varphi^*, \varphi^{*\circ}) \sqcap (\psi^*, \psi^{*\circ}) = ((\varphi \wedge \psi)^*, (\varphi \wedge \psi)^{*\circ}), \tag{2}$$

$$(\varphi^*, \varphi^{*\circ}) \sqcup (\psi^*, \psi^{*\circ}) = ((\varphi \vee \psi)^*, (\varphi \vee \psi)^{*\circ}). \tag{3}$$

Proof. By definition, $(\varphi^*, \varphi^{*\circ}) \sqcap (\psi^*, \psi^{*\circ}) = (\varphi^* \cap \psi^*, (\varphi^* \cap \psi^*)^\circ)$, but since $(\varphi^* \cap \psi^*)(e) = \min(\varphi^*(e), \psi^*(e)) = \min(e(\varphi), e(\psi)) = e(\varphi \wedge \psi) = (\varphi \wedge \psi)^*(e)$, we have $(\varphi^* \cap \psi^*, (\varphi^* \cap \psi^*)^\circ) = ((\varphi \wedge \psi)^*, (\varphi \wedge \psi)^{*\circ})$.

Analogously, by definition $(\varphi^*, \varphi^{*\circ}) \sqcup (\psi^*, \psi^{*\circ}) = ((\varphi^* \cup \psi^*)^{\circ *}, \varphi^{*\circ} \cap \psi^{*\circ})$, but $(\varphi^{*\circ} \cap \psi^{*\circ})(\chi) = \min(\inf_e e(\varphi \to \chi), \inf_e e(\psi \to \chi)) = \inf_e \min(e(\varphi \to \chi), e(\psi \to \chi)) = \inf_e e(\varphi \vee \psi \to \chi) = (\varphi \vee \psi)^{*\circ}(\chi)$. Therefore $((\varphi^* \cup \psi^*)^{\circ *}, \varphi^{*\circ} \cap \psi^{*\circ}) = ((\varphi^* \cup \psi^*)^{\circ *}, (\varphi \vee \psi)^{*\circ}) = ((\varphi \vee \psi)^*, (\varphi \vee \psi)^{*\circ})$. □

As it proven in [3], $C^{cg}(K)$ is indeed a $\sqcap$-subsemilattice of $\mathbf{C}(K)$ in the general case. Indeed, notice that the $\sqcap$ operation is closed in $C^{cg}(K)$, since the concept induced by the conjunction $\bigwedge T$ of a set of formulas $T \subseteq \mathcal{L}_0$, even if $\bigwedge T$ does not belong to $\mathcal{L}_0$, is the same concept induced by the crisp set of formulas T, and hence it belongs to $C^{cg}(K)$. However, this is not the case for a disjunction of a set of formulas. However, if we can guarantee that the concept induced by a disjunction also belongs to $C^{cg}(K)$, then $C^{cg}(K)$ is actually a sublattice of $C(K)$.

Lemma 4. *If $\mathcal{L}_0$ is closed by $\vee$ (modulo logical equivalence) then $\sqcup$ is closed in $C^{cg}(K)$, and $\mathbf{C}^{cg}(K) = (C^{cg}(K), \sqcap, \sqcup, \top_K, \bot_k)$ is a sublattice of $\mathbf{C}(K)$.*

In the following we will assume $\mathcal{L}_0 = \mathcal{L}$ to avoid any problem. In such a case, we can also enrich the lattice $\mathbf{C}^{cg}(K)$ with some further operations in a natural way so to come up with a residuated lattice structure, inherited from the L-algebras.

Definition 2. *We define the following two operations on fuzzy concepts from $C^{cg}(K)$. For any $\varphi, \psi \in \mathcal{L}$, let us define:*

$$(\varphi^*, \varphi^{*\circ}) \boxtimes (\psi^*, \psi^{*\circ}) = ((\varphi \odot \psi)^*, (\varphi \odot \psi)^{*\circ}), \quad (4)$$

$$(\varphi^*, \varphi^{*\circ}) \Rightarrow (\psi^*, \psi^{*\circ}) = ((\varphi \to \psi)^*, (\varphi \to \psi)^{*\circ}). \quad (5)$$

It is easy to check that $\boxtimes$ and $\Rightarrow$ endow the lattice $C^{cg}(K)$ with a structure of residuated lattice, in particular with the structure of a L-algebra.

Proposition 2. $\mathbf{C^{cg}(K)} = (C^{cg}(K), \sqcap, \sqcup, \boxtimes, \Rightarrow, \top_K, \bot_K)$ *is an L-algebra that is isomorphic to the quotient algebra $\mathcal{L}/\equiv_{\Omega_0}$, where $\varphi \equiv_{\Omega_0} \psi$ iff $e(\varphi) = e(\psi)$ for all $e \in \Omega_0$.*

Proof. Elements of $\mathcal{L}/\equiv_{\Omega_0}$ are equivalence classes of formulas from $\mathcal{L}$, according to the congruence relation $\equiv_{\Omega_0}$. Given a formula $\varphi \in \mathcal{L}$, let us denote by $[\varphi]$ its equivalence class. Since the class of L-algebras is a variety, it is closed under quotients, hence $\mathcal{L}/\equiv_{\Omega_0}$ is an L-algebra as well. Now consider the mapping $\lambda : \mathcal{L}/\equiv_{\Omega_0} \to C^{cg}(K)$ defined by $\lambda([\varphi]) = (\varphi^*, \varphi^{*\circ})$. It is easy to check that this mapping is one-to-one thanks to Lemma 2, and moreover it is an algebraic homomorphism with respect to the operations involved: $\lambda([\varphi] \wedge [\psi])) = \lambda([\varphi]) \sqcap \lambda([\psi])$, etc. Therefore, $\mathbf{C^{cg}(K)}$ is an L-algebra as well, isomorphic to $\mathcal{L}/\equiv_{\Omega_0}$. □

Corollary 1. *If $\Omega_0 = \Omega$, then* $\mathbf{C^{cg}(K)}$ *is isomorphic to the Lindenbaum algebra* $\mathbf{Lind}(\mathrm{L}) = \mathcal{L}/\equiv_{\mathrm{L}}$.

3.1 An example: the case of Ł$_3$

In this section, we provide an example of the construction of the concept lattice of formulas and evaluations for the Łukasiewicz 3-valued logic Ł$_3$.

Let $\mathcal{L}_0 = \{\varphi_1, \varphi_2, \ldots, \varphi_{12}\}$ be the set of all Ł$_3$-formulas (up to logical equivalence) on one variable x, where[4]

$$\begin{array}{lll}
\varphi_1 = x^2 \wedge (\neg x)^2 = \bot, & \varphi_2 = (\neg x)^2, & \varphi_3 = x \wedge \neg x, \\
\varphi_4 = x^2, & \varphi_5 = \neg x, & \varphi_6 = (x \vee \neg x)^2, \\
\varphi_7 = \neg x^2 \wedge \neg(\neg x)^2, & \varphi_8 = x, & \varphi_9 = \neg x^2, \\
\varphi_{10} = x \vee \neg x, & \varphi_{11} = \neg(\neg x)^2, & \varphi_{12} = \neg x^2 \vee \neg(\neg x)^2 = \top\,.
\end{array}$$

Further, let us consider all possible 3-valued evaluations on the variable x as the set of objects: $\Omega_0 = \{e_0, e_1, e_2\}$, where $e_0(x) = 0, e_1(x) = \frac{1}{2}, e_2(x) = 1$. The following table shows the values of each formula of $\mathcal{L}_0$ under each evaluation.

[4] We use φ^2 as a shorcut for $\varphi \odot \varphi$.

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}
$e_0(\cdot)$	0	1	0	0	1	1	0	0	1	1	0	1
$e_1(\cdot)$	0	0	1/2	0	1/2	0	1	1/2	1	1/2	1	1
$e_2(\cdot)$	0	0	0	1	0	1	0	0	0	1	1	1

As described before in this section, the triple $K = \{\Omega_0, \mathcal{L}_0, R\}$, where $R : \Omega_0 \times \mathcal{L}_0 \to \{0, \frac{1}{2}, 1\}$ is a 3-valued fuzzy relation defined as $R(e, \varphi) = e(\varphi)$, identifies a formal context.

First of all, we aim at obtaining all the concepts induced by a single formula. For instance, consider the formula $\varphi_8 = x$. Then, $\varphi_8^*(e_0) = e_0(\varphi_8) = 0$, $\varphi_8^*(e_1) = \frac{1}{2}$, and $\varphi_8^*(e_2) = 1$. We denote the fuzzy set of objects (evaluations) φ_8^* by the tuple $(0, \frac{1}{2}, 1)$. Let us compute the fuzzy set of attributes (formulas) $\varphi_8^{*\circ}$:

$$\varphi_8^{*\circ}(\varphi_1) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_1) = 0\,, \qquad \varphi_8^{*\circ}(\varphi_2) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_2) = 0\,,$$
$$\varphi_8^{*\circ}(\varphi_3) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_3) = 0\,, \qquad \varphi_8^{*\circ}(\varphi_4) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_4) = 1/2\,,$$
$$\varphi_8^{*\circ}(\varphi_5) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_5) = 0\,, \qquad \varphi_8^{*\circ}(\varphi_6) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_6) = 1/2\,,$$
$$\varphi_8^{*\circ}(\varphi_7) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_7) = 0\,, \qquad \varphi_8^{*\circ}(\varphi_8) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_8) = 1\,,$$
$$\varphi_8^{*\circ}(\varphi_9) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_9) = 0\,, \qquad \varphi_8^{*\circ}(\varphi_{10}) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_{10}) = 1\,,$$
$$\varphi_8^{*\circ}(\varphi_{11}) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_{11}) = 1\,, \qquad \varphi_8^{*\circ}(\varphi_{12}) = \inf_{e\in\Omega_0} e(\varphi_8 \to \varphi_{12}) = 1\,.$$

We indicate the value of $\varphi_8^{*\circ}$ by the tuple $(0, 0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 1, 0, 1, 1, 1)$. The pair $(\varphi_8^*, \varphi_8^{*\circ})$ is the formal concept induced by the furmula φ_8. In the same way, we can compute all the formal concepts induced by single formulas of $\mathcal{L}_0$, obtaining:

$$\begin{aligned}
(\varphi_1^*, \varphi_1^{*\circ}) &= ((0,0,0),(1,1,1,1,1,1,1,1,1,1,1,1))\,,\\
(\varphi_2^*, \varphi_2^{*\circ}) &= ((1,0,0),(0,1,0,0,1,1,0,0,1,1,0,1))\,,\\
(\varphi_3^*, \varphi_3^{*\circ}) &= ((0,1/2,0),(1/2,1/2,1,1/2,1,1/2,1,1,1,1,1,1))\,,\\
(\varphi_4^*, \varphi_4^{*\circ}) &= ((0,0,1),(0,0,0,1,0,1,0,1,0,1,1,1))\,,\\
(\varphi_5^*, \varphi_5^{*\circ}) &= ((1,1/2,0),(0,1/2,0,0,1,1/2,0,0,1,1,0,1))\,,\\
(\varphi_6^*, \varphi_6^{*\circ}) &= ((1,0,1),(0,0,0,0,0,1,0,0,0,1,0,1))\,,\\
(\varphi_7^*, \varphi_7^{*\circ}) &= ((0,1,0),(0,0,1/2,0,1/2,0,1,1/2,1,1/2,1,1))\,,\\
(\varphi_8^*, \varphi_8^{*\circ}) &= ((0,1/2,1),(0,0,0,1/2,0,1/2,0,1,0,1,1,1))\,,\\
(\varphi_9^*, \varphi_9^{*\circ}) &= ((1,1,0),(0,0,0,0,1/2,0,0,0,1,1/2,0,1))\,,\\
(\varphi_{10}^*, \varphi_{10}^{*\circ}) &= ((1,1/2,1),(0,0,0,0,0,1/2,0,0,0,1,0,1))\,,\\
(\varphi_{11}^*, \varphi_{11}^{*\circ}) &= ((0,1,1),(0,0,0,0,0,0,0,1/2,0,1/2,1,1))\,,\\
(\varphi_{12}^*, \varphi_{12}^{*\circ}) &= ((1,1,1),(0,0,0,0,0,0,0,0,0,1/2,0,1))\,.
\end{aligned}$$

Note that all the formal concepts above are precisely all the crisply generated concepts. Indeed, the concept generated by $\{\psi_1, \ldots, \psi_k\} \subseteq \mathcal{L}_0$ coincides with the concept generated by the single formula $\bigwedge_{i=1,\ldots,k} \psi_i$, which is logically equivalent to a formula of $\mathcal{L}_0$. We also observe that $\varphi_{12}^{*\circ} = (\inf_{e\in\Omega_0} e(\varphi_1), \ldots, \inf_{e\in\Omega_0} e(\varphi_{12})) = T_{\Omega_0} \neq \emptyset$.

As described in the previous part of the section, we can endow the set $\mathcal{C}^{cg}(K)$ with the operations defined in (2)–(5). We obtain in this way an algebra $\mathbf{C^{cg}(K)}$ of crisply generated concepts of $\mathcal{L}_0$ which is isomorphic to the free 1-generated Ł$_3$ algebra, depicted in Figure 1, via the isomorphism λ that associates each formula $\varphi \in \mathcal{L}_0$ with the concept $(\varphi^*, \varphi^{*\circ})$.

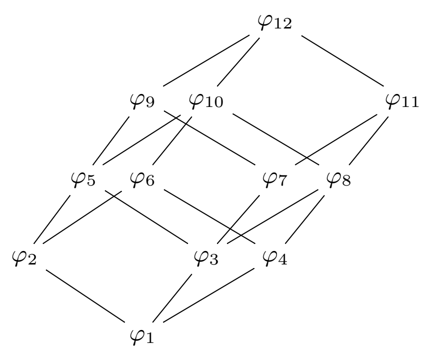

Fig. 1: The Lindenbaum-Tarski algebra of Ł$_3$ over one generator.

Consider now the set of objects (evaluations) $\Omega_B = \{e_0, e_2\} \subseteq \Omega_0$. Again, the triple $K_B = \{\Omega_B, \mathcal{L}_0, R_B\}$, where $R_B : \Omega_B \times \mathcal{L}_0 \to \{0, \frac{1}{2}, 1\}$ is a 3-valued fuzzy relation defined as $R(e, \varphi) = e(\varphi)$, identifies a formal context. Actually, the fuzzy relation R_B is in fact a crisp relation, since the evaluation e_0 and e_2 only evaluate x to either 0 or 1. In this new setting, we can compute all the formal concepts induced by single formulas of $\mathcal{L}_0$, obtaining:

$$(\varphi_1^*, \varphi_1^{*\circ}) = (\varphi_3^*, \varphi_3^{*\circ}) = (\varphi_7^*, \varphi_7^{*\circ}) = ((0,0), (1,1,1,1,1,1,1,1,1,1,1,1)),$$
$$(\varphi_4^*, \varphi_4^{*\circ}) = (\varphi_8^*, \varphi_8^{*\circ}) = (\varphi_{11}^*, \varphi_{11}^{*\circ}) = ((0,1), (0,0,0,1,0,1,0,1,0,1,1,1)),$$
$$(\varphi_2^*, \varphi_2^{*\circ}) = (\varphi_5^*, \varphi_5^{*\circ}) = (\varphi_9^*, \varphi_9^{*\circ}) = ((1,0), (0,1,0,0,1,1,0,0,1,1,0,1)),$$
$$(\varphi_6^*, \varphi_6^{*\circ}) = (\varphi_{10}^*, \varphi_{10}^{*\circ}) = (\varphi_{12}^*, \varphi_{12}^{*\circ}) = ((1,1), (0,0,0,0,0,1,0,0,0,1,0,1)),$$

which, in fact, they turn out to be classical concepts. Not surprisingly, endowing this set of concepts $\mathcal{C}^{cg}(K_B)$ with the operations defined in (2)–(5) we obtain an algebra of concepts which is isomorphic to the free 1-generated Boolean algebra. Such algebra is obtained as a quotient of $\mathbf{C^{cg}(K)}$. As it is easily seen using Proposition 2, this holds in general, that is, an algebra of concepts $\mathbf{C^{cg}(K')}$, with $K' = \{\Omega_0', \mathcal{L}_0, R\}$ and $\Omega_0' \subseteq \Omega_0$ is a quotient of the algebra $\mathbf{C^{cg}(K)}$.

4 The natural Concept Lattice of a Logic

In this section we recall the construction of concept lattices applied in [5] to characterize formal concept lattices associated to Gödel algebras.

Proposition 1 states that for every finite lattice H there is always a canonical way to build the *standard context* $\mathbb{K}_H$, whose concept lattice $\mathfrak{B}(\mathbb{K}_H)$ is isomorphic to H. Let $\mathbf{A} = (A, \wedge, \vee, \rightarrow, \top, \bot)$ be a finite algebra in a variety $\mathbb{V} \subseteq \mathbb{MTL}$, and let $C_{\mathbf{A}} = \mathfrak{B}((\mathfrak{J}(\mathbf{A}), \mathfrak{M}(\mathbf{A}), \leq))$ be the concept lattice of its standard context. Then, the lattice $\mathbf{C_A} = (C_{\mathbf{A}}, \sqcap, \sqcup, \top_G, \bot_M)$, is isomorphic to the lattice reduct of $\mathbf{A}$.

Pushing further this approach, when $\mathbb{V}$ is a locally finite variety the k-generated free algebras $\mathbf{F}_k(\mathbb{V})$ are finite, and hence we can apply to them the above construction. As the elements of $\mathbf{F}_k(\mathbb{V})$ are equivalence classes of logical formulas in k variables, this amount to associate every logical formula to its natural formal concept.

For some cases it is possible to extend the lattice isomorphism to a full isomorphism of algebras by defining suitable operations between the formal concepts. In [5] the authors use this methodology to obtain formal concepts for every Gödel logic formula. This is possible because in Gödel algebras lattice and monoidal conjunctions coincide, and hence it is natural to define an implication operator between concepts by using the residum of the concepts meet.

Comparing to Section 3.1, in the next subsection we apply the above sketched construction to $\mathbf{F}_1(\mathbb{V}(\text{Ł}_3))$.

4.1 Constructing the concept lattice of the logic Ł$_3$

Consider the set $\mathcal{L}_0 = \{\varphi_1, \varphi_2, \ldots, \varphi_{12}\}$ of all Ł$_3$-fomulas (up to logical equivalence) on one variable x. The formulas of $\mathcal{L}_0$ are exhibited in Figure 1. Let $H = (\mathcal{L}_0, \leq)$ be the lattice reduct of the free 1-generated Ł$_3$ algebra $\mathbf{F}_1$ depicted in Figure 1. The sets of join irreducible elements and meet irreducible elements of L are $\mathfrak{J}(H) = \{\varphi_2, \varphi_3, \varphi_4, \varphi_7\}$, and $\mathfrak{M}(H) = \{\varphi_6, \varphi_9, \varphi_{10}, \varphi_{11}\}$, respectively. By Proposition 1, we can identify $\mathfrak{J}(H)$ and $\mathfrak{M}(H)$ with the set of objects and attributes, respectively, of a standard context $\mathbb{K}_H = (\mathfrak{J}(H), \mathfrak{M}(H), \leq)$. The following table shows the relation $\leq$

$\leq$	φ_6	φ_9	φ_{10}	φ_{11}
φ_2	×	×	×	
φ_3		×	×	×
φ_4	×		×	×
φ_7		×		×

The corresponding standard context lattice is depicted in Figure 2. Clearly, by Proposition 1, it is isomophic to the lattice reduct of the free 1-generated Ł$_3$ algebra of Figure 1, via a lattice isomorphim f defined as follows. For each $\varphi \in \mathcal{L}_0$, let J_φ be the maximal subset of $\mathfrak{J}(H)$ such that $\varphi = \bigvee J_\varphi$, and M_φ be the maximal subset of $\mathfrak{M}(H)$ such that $\varphi = \bigwedge M_\varphi$. Then the map f associates each $\varphi \in \mathcal{L}_0$ with the formal concept $(J_\varphi, M_\varphi) \in \mathbb{K}_H$.

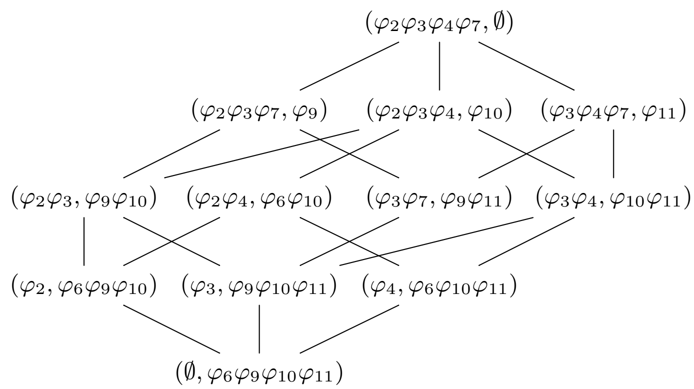

Fig. 2: The concept lattice associated with the lattice reduct of $\mathbf{F}_1$

To extend the above defined lattice isomorphism to an algebraic isomorphism between $\mathbf{F}_1(Ł_3)$ and the concept lattice of the standard context $\mathbb{K}_H$, it is necessary to define a *proper* monoidal conjunction between concepts of $\mathbb{K}_H$. Of course, an obvious way to define such an operation is through the isomorphism f, that is, for each pair of concepts $(E, F), (E', F') \in \mathbb{K}_H$, to define $(E, F) \otimes (E', F') = (J_{\varphi\odot\psi}, M_{\varphi\odot\psi})$, where $f^{-1}((E, F)) = \varphi$ and $f^{-1}((E', F')) = \psi$. However, this does not shed any light on how the operation works on the elements of the concepts. To have a much better insight in the operation seems not to be an easy task, even in the case of locally finite subvarieties of $\mathbb{MTL}$ (such as $Ł_3$), and it will be faced in some future paper.

5 Conclusions and further developments

To obtain a direct relation between a formal concept and a fuzzy logic formula, in this work we have explored two ways to obtain concept lattices isomorphic to Lindenbaum algebras of many-valued logics. The first approach naturally gives the desired isomorphism between the concept lattice and the algebra of formulas, while to complete the second approach additional research has to be done.

To depict the two constructions we have chosen the logic $Ł_3$. In [6], $Ł_3$ has been characterized as a logic of prototypes and counterexamples. The construction of *possible worlds* in [6] gives a lattice of functions $\Omega_0^{\Omega_0^n}$ very similar to the concept lattice of our first approach in Section 3.1. Hence, putting together the characterization of [6] with the constructions presented here, it will be ideally possible to build a formal concept semantics of prototypes and counterexamples for the logic $Ł_3$.

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