

# Chapter 1

## Introduction

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### 1.1 Mathematical Fuzzy Logic

Since Petr Hájek, the scientist we are going to celebrate, is the main contributor to Mathematical Fuzzy Logic, we will first spend a few words about this subject.

Mathematical Fuzzy Logic is a mathematical study of a logical systems whose algebraic semantics involve some notion of truth degree. The origins of the discipline are both philosophical (modeling correct reasoning in some particular contexts like the treatment of vague predicates, for which classical logic may appear not adequate), Zadeh's Fuzzy Set Theory, which has been widely applied, and many-valued logics, which are logics with intermediate truth degrees, whose order is often assumed to be linear. Unlike Fuzzy Set Theory, which is mainly devoted to concrete applications, Mathematical Fuzzy Logic is a subdiscipline of Mathematical Logic, and hence it aims at a mathematical treatment of reasoning with intermediate truth degrees. Hence, as all known logics, Mathematical Fuzzy Logic deals with propositional and first-order formulas (and, in some cases, even with second-order formulas), it has several semantics, an algebraic semantics, a semantics given by chains, a semantics based on  $[0, 1]$ , and also a game-theoretical semantics. It deals with such

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problems as (un)decidability and computational complexity. Although it is questionable whether or not Mathematical Fuzzy Logic can satisfactorily capture vague concepts (and we tend to believe that it is not the case), for their treatment it seems at least more suitable than classical logic and than other non-classical logics. Finally, although Fuzzy Logic is different from probability, it is formally possible to treat probability (and hence, uncertainty) inside Fuzzy Logic enriched with a modality representing *Probably*. Hence, Mathematical Fuzzy Logic is a very beautiful mathematical theory with concrete applications. For more information, one can consult the *Handbook of Mathematical Fuzzy Logic* [14].

## 1.2 The beginning

When Petr Hájek began his work on Mathematical Fuzzy Logic, he and his collaborators immediately realized that several important fuzzy logics, like Łukasiewicz logic and Gödel logic, were already present in the literature. At the same time, the wide literature on t-norms suggested to him to associate to each continuous t-norm a logic, in which conjunction and implication are interpreted as the t-norm and its residuum, respectively. In particular, his attention was attracted by the logic of a very natural continuous t-norm, namely, the product t-norm. With F. Esteva and L. Godo, in the paper *A complete many-valued logic with product-conjunction* [53], the authors offered an axiom system for this product logic and proved that it is (sound and) complete with respect to the standard semantics on  $[0, 1]$ . To get this completeness result they introduced an algebraic semantics based upon product algebras in a way similar to the completeness proof which C. C. Chang gave for (the infinite valued) Łukasiewicz logic via MV-algebras [14].

The interest of product logic is also emphasized in the paper *Embedding logics into product logic* [3], by M. Baaz, J. Krajíček, P. Hájek and D. Švejda. In that paper, the authors construct a faithful interpretation of Łukasiewicz's logic in product logic (both propositional and predicate), as well as a faithful interpretation of Gödel logic into product logic with Baaz's projection connective  $\Delta$ . As a consequence, they prove that the set of standard first-order product tautologies is not recursively axiomatizable, and that the set of propositional formulas satisfiable in product logic (resp., in Gödel logic), is NP-complete.

A controversial problem in fuzzy logic is the notion of negation. Indeed, in the theory of fuzzy sets negation is always involutive. But if one defines  $\neg\varphi$  as  $\varphi \rightarrow \perp$ , as in intuitionistic logic, then the negation of several fuzzy logics like Gödel and product logic, is not involutive: over  $[0, 1]$  it is a function which exchanges 0 and 1 and sends to 0 any other value. Hence, in the paper *Residuated fuzzy logics with an involutive negation* [25] by Esteva, Godo, Hájek and Navara, the authors describe the logic arising from a residuated fuzzy logic with such a kind of negation by the addition of an involutive negation. In these logics, one has two negations: a classical (involutive) negation and the (strict) negation arising from residuation. Interestingly, for the case of Product logic, while one has standard completeness with respect to

the product usual connectives on  $[0, 1]$  and the class of all involutive negations, we do not have standard completeness with respect to the usual negation  $1 - x$  alone.

### 1.3 The monograph *Metamathematics of Fuzzy Logic*

All the above mentioned logics are treated in Hájek's monograph *Metamathematics of Fuzzy Logic* [38]. This book has played a fundamental role in the recent development of Mathematical Fuzzy Logic.

It is impossible to summarize the whole content of this book without overlooking something important. For example, the book contains an interesting preliminary discussion about the motivations of fuzzy logic and about their general semantic principles, which will not be reported here. However, in our opinion the main ideas contained in the book are the following:

1. Fuzzy logics are presented as logics of continuous t-norms and their residuals.
2. Since every continuous t-norm is the ordinal sum of Łukasiewicz, Gödel and product t-norms, the corresponding logics (Łukasiewicz, Gödel and product logics) are of fundamental importance.
3. One can look for a common fragment of the three fundamental fuzzy logics, as well as for the logic of all continuous t-norms. Then Hájek proposed a logic, called Basic (Fuzzy) Logic (in symbols, BL), which later on turned out to be the logic of all continuous t-norms and of their residuals.
4. Fuzzy logics are considered as logics of a comparatively graded notion of truth, indeed a formula  $\varphi \rightarrow \psi$  is 1-true whenever the degree of truth of  $\psi$  is greater or equal to that of  $\varphi$ . The ability of explicitly reasoning about truth-degrees motivates the study of the so called Rational Pavelka Logic, which has constants for all rational truth-values.
5. The general semantics of fuzzy logics is constituted by totally ordered commutative, integral and divisible residuated lattices, BL-chains for short. As noted by Baaz in his article in the volume *Witnessed years* [19], Hájek raised the problem of the independence of the axiom  $(\varphi \& (\varphi \rightarrow \psi)) \leftrightarrow (\psi \& (\psi \rightarrow \varphi))$ , corresponding to divisibility. This axiom turns out to be independent, but interestingly, if we remove it, we get another interesting logic, namely, the Monoidal T-norm-based Logic MTL of Esteva and Godo.
6. Every schematic extension L of BL has a first-order expansion  $L\forall$ , which is strongly complete with respect to the class of all safe interpretations on L-chains. The idea is that the existential quantifier and the universal quantifier are interpreted by suprema and infima, and an interpretation on an L-chain is said to be *safe* when all suprema and infima needed to interpret quantifiers exist in the L-chain. Interestingly, Hájek didn't require the L-chains to be complete. Indeed, with the remarkable exception of Gödel logic, for every continuous t-norm logic L, the set of first-order formulas which are valid in all complete L-chains is not recursively axiomatizable, while the set of formulas which are

valid in all safe interpretation over arbitrary L-chains is axiomatizable over L by a finite set of axiom schemata. Yet another interesting feature of this book is the discovery of the axiom  $\forall x(\varphi(x) \vee \psi) \rightarrow ((\forall x\varphi(x)) \vee \psi)$ , which in the case of intuitionistic first-order logic characterizes Kripke models with constant domain. It turns out that in the case of fuzzy logic, this axiom characterizes the semantics by chains.

7. The last part of the book deals with application aspects: e.g., fuzzy modal logics, a logical understanding of fuzzy if-then rules and fuzzy quantifiers like *many* and *probably* are discussed. Interestingly, although Hájek emphasizes the differences between fuzzy logic and probability theory (the former is truth functional, the latter is not, the former deals with vague concepts that may have an intermediate truth degree, while the latter deals with events which are unknown now but will be either completely true or completely false later), the author introduces an interpretation of the logic of probability into fuzzy logic enriched with the modality *Probably*. In this way, the probability of an event  $\varphi$  becomes the truth value of the sentence *Probably*  $\varphi$ .

Although the book is full of interesting results, it doesn't exhaust Petr's research in Mathematical Fuzzy Logic. Here below, we list some problems which are somehow addressed in the book and which have been further investigated by Petr and by his coauthors:

1. First-order fuzzy logics, and in particular: Supersound logics, complexity of standard tautologies or of standardly satisfiable formulas and witnessed models.
2. Computational complexity of propositional fuzzy logics.
3. Logics weaker than BL (MTL, hoop logics, ps-BL, flea-logics).
4. Logics with truth constants for the rationals.
5. Logics of probability, of possibility and of belief.
6. Logics with truth-hedges.
7. Fuzzy modal logics
8. Fuzzy description logic.
9. Mathematical theories (arithmetic, set theory) over fuzzy logic.

## 1.4 First-order fuzzy logics

As said before, an important contribution by Petr Hájek to first-order fuzzy logic is the discovery of the right semantics for it. Indeed, the first-order version of any schematic extension L of BL is strongly complete with respect to the class of all safe interpretations on L-chains (totally ordered models of L), and the same can be easily proved, essentially by the same proof, for extensions of first-order MTL. In general, we do not have completeness with respect to interpretations over completely ordered L-chains. That is, the class of all structures on completely ordered L-chains is a too narrow class to get completeness. One may try to do the opposite way, that is, to

enlarge the class of interpretations, and to define a formula *valid* if it is true in all (possibly unsafe) interpretations in L-chains in which its truth value is defined. But in this way we may lose correctness. A predicate fuzzy logic  $L\forall$  is said to be *supersound* if every theorem  $\varphi$  of  $L\forall$  is valid in all (possibly unsafe) interpretations on any L-chain in which its truth-value is defined.

In the paper *A note on the notion of truth in fuzzy logic* [67], P. Hájek and J. Shepherdson show that among the logics given by continuous t-norms, Gödel logic is the only one that is supersound. All other continuous t-norm logics are (sound but) not supersound. This supports the view that the usual restriction of semantics to safe interpretations (in which the truth assignment is total) is very natural.

Another semantics for first-order fuzzy logics for which completeness in general fails is the standard semantics on  $[0, 1]$ . In some cases, the failure is obtained in a very strong sense: for instance, for product logic, both the set of 1-tautologies and the set of 1-satisfiable formulas are not arithmetical. The arithmetical complexity of the standardly satisfiable formulas or of standard tautologies of the most prominent fuzzy logics is summarized in P. Hájek's paper *Arithmetical complexity of fuzzy predicate logics—a survey, II*, [53]

Among all logics of continuous t-norms, Gödel first-order logic is the only logic which is complete with respect to the standard semantics on  $[0, 1]$ . However, Gödel first-order logic is no longer complete if instead of  $[0, 1]$  we take an arbitrary closed subset of  $[0, 1]$  containing 0 and 1. Now in P. Hájek's paper *A non-arithmetical Gödel logic* [47], the following surprising result is proved: Let  $G\downarrow$  denote the first-order Gödel logic with truth degree set  $V\downarrow = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \dots\}$ . Then the sets of satisfiable formulas as well as of tautologies of  $G\downarrow$  are non-arithmetical. This is contrasted with the fact that for the similar system  $G\uparrow$  with truth degree set  $V\uparrow = \{1\} \cup \{\frac{n}{n+1} : n = 0, 1, \dots\}$ , whose set of tautologies is shown to be  $\Pi_2$ -complete.

Several new and original ideas about the semantics of first-order fuzzy logics are presented in P. Hájek and P. Cintula's paper *On theories and models in fuzzy predicate logics* [52]. There, a general model theory is presented for predicate logics, and a more general version of the completeness theorem is proved, using doubly Henkin theories. Moreover, the (very interesting) concept of witnessed model is introduced. These are models in which suprema and infima used to interpret existential and universal quantifiers are actually maxima and minima. The logic of witnessed models is obtained by the adding of the axioms  $\exists x\forall y(P(x) \rightarrow P(y))$  and  $\exists x(\exists yP(y) \rightarrow P(x))$ . Interestingly, although these axioms are valid in classical logic, they are not intuitively valid. For instance, the first axiom says that there is individual  $x$  such that if  $x$  gets drunk, then everybody gets drunk.

Although the paper by P. Hájek and F. Montagna, *A note on the first-order logic of complete BL-chains* [65], is probably not one of the most important papers by Petr, I will mention it because it has a nice story. The paper discusses an error in another paper by Sacchetti and Montagna. The error was based on the wrong assumption that in a complete BL-chain, the fusion operator distributes over arbitrary infima. This property clearly holds in any standard BL-algebra, but is not true in general (Felix Bou found a counterexample). As a consequence of that error, Mon-

tagna and Sacchetti claimed that the predicate logics of all complete BL-chains and of all standard BL-chains coincide. During a meeting, Petr told Montagna that he was going to do the same error. Then Petr and Montagna discussed this problem by e-mail, and arrived to the following result: a complete BL-chain  $\mathbf{B}$  satisfies all standard BL-tautologies iff for any transfinite sequence  $(a_i : i \in I)$  of elements of  $\mathbf{B}$ , the condition  $\bigvee_{i \in I} a_i^2 = (\bigvee_{i \in I} a_i)^2$  holds in  $\mathbf{B}$ . It is nice to observe that Montagna was going to repeat the error in another paper, but fortunately he noticed it before submitting the paper for publication.

## 1.5 Computational complexity of fuzzy logics

Propositional logics may have quite different complexities. For instance, classical logic is coNP-complete, intuitionistic logic is PSPACE-complete, as well as many modal logics, and linear logic is even undecidable. The most important many-valued logics extending BL are coNP-complete, and Hájek greatly contributed to the proof of this general claim. The book *Metamathematics of Fuzzy Logic* already contains a proof of coNP-completeness of Łukasiewicz, Gödel and product logics. The first result has been proved independently by Mundici [75] and Hähnle [37]. The coNP-completeness of Gödel logic is easy and the coNP-completeness of product logic follows from the above mentioned paper [3].

Another important result about computational complexity of fuzzy logics is the coNP-completeness of BL, which was proved by M. Baaz, P. Hájek, F. Montagna and H. Veith in the paper *Complexity of t-tautologies* [4].

In P. Hájek's paper *Computational complexity of t-norm based propositional fuzzy logics with rational truth constants* [50], the author discusses the complexity of Gödel logic, Łukasiewicz logic, and product logic added with constants for the rational numbers in  $[0, 1]$  along with bookkeeping axioms. For these logics the complexity remains the same as for their fragments without the constants. However, there are t-norms such that the complexity when one adds the rational constants may fall outside the arithmetical hierarchy.

Finally, in the paper *Complexity issues in axiomatic extensions of Łukasiewicz logic* [16] P. Cintula and P. Hájek show that all axiomatic extensions of propositional Łukasiewicz logic are coNP-complete.

It is worth noticing that Zuzana Haniková in the paper *A note on the complexity of propositional tautologies of individual t-algebras* [69] proved that all logics of continuous t-norms on  $[0, 1]$  are coNP-complete.

## 1.6 Logics weaker than BL

There are three types of fragments of BL, namely, the logics in a weaker language which are extended by BL conservatively, the logics in the language of BL whose

axiom set is properly included in the axiom set of BL, and the logics which have a weaker language than BL and are extended by BL, but not conservatively. Remarkable examples of fragments in the first sense are the logic BH of basic hoops, which has been investigated by F. Esteva, L. Godo, P. Hájek, and F. Montagna in the paper *Hoops and fuzzy logic* [21] and the logic BHCK of basic hoop BCK-algebras [1]. The first logic is the fragment of BL in the language  $\{\&, \rightarrow, \top\}$ , while the latter logic is the fragment of BL in the language  $\{\rightarrow, \top\}$ .

The most interesting fragment of the second type is probably the Monoidal t-norm Logic MTL by F. Esteva and L. Godo [23]. These authors, having in mind that in t-norm algebras the existence of the residual already yields the left continuity of the t-norm, conjectured that deleting the essential part  $a \wedge b \leq a * (a \rightarrow b)$  of the continuity condition, but maintaining the prelinearity condition, should yield the logic of all left continuous t-norms.<sup>1</sup> Although this interesting logic was not due to him, Hájek showed interest in this logic and in his paper *Observations on the monoidal t-norm logic* [41], he investigates some extensions of MTL. The leading idea was the following: BL has three well-known extensions: Łukasiewicz logic, Gödel logic, and product logic, which are axiomatized over BL by the axioms  $\neg\neg\varphi \rightarrow \varphi$ ,  $\varphi \rightarrow (\varphi \& \varphi)$  and  $\neg\psi \vee (((\psi \rightarrow (\varphi \& \psi)) \rightarrow \varphi)$ , respectively. Then it is natural to investigate the analogous extensions of MTL, namely MTL plus  $\neg\neg\varphi \rightarrow \varphi$ , denoted by IMTL, MTL plus  $\varphi \rightarrow (\varphi \& \varphi)$  and MTL plus  $\neg\psi \vee (((\psi \rightarrow (\varphi \& \psi)) \rightarrow \varphi)$ , which is denoted by  $\Pi$ MTL. While MTL plus  $\varphi \rightarrow (\varphi \& \varphi)$  is just Gödel logic, IMTL is weaker than Łukasiewicz logic, and MTL plus  $\neg\psi \vee (((\psi \rightarrow (\varphi \& \psi)) \rightarrow \varphi)$  is weaker than product logic.

While MTL is obtained from BL by removing divisibility, one may wonder what happens if one removes commutativity of the conjunction. BL deprived of commutativity has been investigated e.g. by G. Georgescu and A. Iorgulescu [34] and by P. Flondor, Georgescu and Iorgulescu [30], see also the book by S. Gottwald, *A treatise on many-valued logics* [9]. In his paper *Fuzzy logics with noncommutative conjunctions* [43], Hájek finds adequate axiomatizations for these logics and proves a completeness theorem for them. Moreover in his paper *Embedding standard BL-algebras into non-commutative pseudo-BL-algebras* [42], Hájek proves that each BL-algebra given by a continuous t-norm is a subalgebra of a non-commutative pseudo-BL-algebra on a ‘non-standard’ interval  $[0, 1]^*$ .

The logic BL was already an attempt to generalize the three main fuzzy logics, that is, Łukasiewicz, Gödel and product logics. Hence, probably Hájek didn’t imagine such an amount of generalizations obtained by removing either connectives or the divisibility axiom, or the commutativity axiom. In his paper *Fleas and fuzzy logic* [48], Hájek finds a common generalization of the logic of basic hoops and the logic psMTL of noncommutative pseudo-t-norms. He presents a general completeness theorem and he discusses the relations to the logic of pseudo-BCK algebras. The reference to fleas in the title is due to the following story:

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<sup>1</sup> Deleting even the prelinearity condition had given the monoidal logic of U. Höhle [71, 72]. This logic is characterized by the class of all residuated lattices, but seems to be too general as a logic for t-norms.

Some scientists make experiments on a flea: they remove one of its legs and tell it: *Jump!*. The flea can still jump. Then they repeat the experiment over and over again, and, although with some difficulty, the flea still jumps. But once all legs are removed, the flea is no longer able to jump. Then the doctors come to the conclusion that a flea without legs becomes deaf. Now the attitude of logicians who remove more and more axioms and symbols and still expect to be able to derive interesting properties, is compared to the attitude of the scientists of the story.

Another interesting paper about fragments is the one by P. Cintula, P. Hájek, R. Horčík, *Formal systems of fuzzy logic and their fragments* [17]. There, the authors investigate expansions of the logic BCK with the axiom of prelinearity which come about by the addition of further connectives, which are chosen in such a way that the resulting systems become fragments of well-known mathematical fuzzy logics. These logics are usually characterized by quasivarieties of lattice based algebraic structures, and in some cases by varieties. The authors give adequate axiomatizations for most of them.

## 1.7 Further logics related to BL

### 1.7.1 Rational Pavelka Logic

Besides the purely logical interest in mathematical fuzzy logics their consideration is motivated by the problem to search for suitable logics for fuzzy sets.

In this context it is natural to ask whether it is possible to generalize the standard entailment as well as provability considerations in logical systems to the case that one starts from *fuzzy sets of formulas*, and that one gets from them as consequence hulls again fuzzy sets of formulas. This problem was first treated by Jan Pavelka in 1979 in his three papers *On fuzzy logic I, II and III* [75]. Accordingly such approaches are sometimes called *Pavelka-style*, but they have also been coined approaches with *evaluated syntax*.

Such an approach has to deal with fuzzy sets  $\Sigma^\sim$  of formulas, i.e. besides formulas  $\varphi$  also their membership degrees  $\Sigma^\sim(\varphi)$  in  $\Sigma^\sim$ . And these membership degrees are just the truth degrees of the corresponding logic. This is an easy matter as long as the entailment relationship is considered. An evaluation  $e$  is a *model* of  $\Sigma^\sim$  iff  $\Sigma^\sim(\varphi) \leq e(\varphi)$  holds for each formula  $\varphi$ . Hence the semantic consequence hull of  $\Sigma^\sim$  should be characterized by the membership degrees  $\mathcal{C}^{\text{sem}}(\Sigma^\sim)(\psi) = \bigwedge \{e(\psi) \mid e \text{ model of } \Sigma^\sim\}$ .

For a syntactic characterization of this entailment relation it is necessary to treat *evaluated formulas*, i.e. ordered pairs consisting of a truth degree symbol and a formula in a logical calculus  $\mathbb{K}$ . Also the rules of inference have to deal with evaluated formulas. Each derivation of an evaluated formula  $(\bar{a}, \varphi)$  counts as a derivation of  $\varphi$  to the degree  $a$ . The *provability degree* of  $\varphi$  from  $\Sigma^\sim$  in  $\mathbb{K}$  is the supremum over all these degrees. The syntactic consequence hull of  $\Sigma^\sim$  is the fuzzy set



$\mathcal{C}_{\mathbb{K}}^{\text{syn}}$  of formulas characterized by the membership function  $\mathcal{C}_{\mathbb{K}}^{\text{syn}}(\Sigma^{\sim})(\psi) = \bigvee \{a \mid \mathbb{K} \text{ derives } (\bar{a}, \psi) \text{ out of } \Sigma^{\sim}\}$ .

Already Pavelka proved soundness and completeness saying  $\mathcal{C}^{\text{sem}}(\Sigma^{\sim}) = \mathcal{C}_{\mathbb{L}}^{\text{syn}}(\Sigma^{\sim})$ , but only for the case that the many-valued logic under consideration here is the (infinite valued) Łukasiewicz logic  $\mathbb{L}$ . (This restriction comes from the fact that the completeness proof needs the continuity of the residuation operation.) Because the truth degree symbols have to be part of the derivations, here one needs to refer to an uncountable language with constants for all the reals of the unit interval.

Petr Hájek realized the following important facts: (i) it is sufficient to have constants for the rationals from the unit interval; (ii) instead of working with evaluated formulas one can consider implications of the forms  $\bar{r} \rightarrow \varphi$  and  $\varphi \rightarrow \bar{r}$ ; (iii) the semantic degree  $\mathcal{C}^{\text{sem}}(\Sigma^{\sim})(\psi)$  is the infimum of all rationals  $r$  such that  $\bar{r} \rightarrow \psi$  is satisfiable in models of  $\Sigma^{\sim}$ , and the provability degree  $\mathcal{C}_{\mathbb{L}}^{\text{syn}}(\Sigma^{\sim})(\psi)$  is the supremum of all rationals  $r$  such that  $\bar{r} \rightarrow \psi$  is provable from  $\Sigma^{\sim}$ . All together this led him to an expanded version of  $L$ , expanded by truth degree constants for the rationals from the unit interval and by corresponding “book keeping” rules to treat these constants well, which he coined *Rational Pavelka Logic*. Hence, in a certain sense, Rational Pavelka Logic is equally powerful as the original Pavelka style extension of Łukasiewicz logic.

One may wonder what is the relationship between the Rational Pavelka Logic and other mathematical fuzzy logics, and in particular, whether Rational Pavelka Logic is conservative over Łukasiewicz logic. In the paper *Rational Pavelka Logic is a conservative extension of Łukasiewicz logic* [66] by P. Hájek, J. Paris and J. Shepherdson, the this last question is solved affirmatively. Besides this result, it is shown that the provability degree of a formula can also be defined within the framework of Łukasiewicz logic, i.e. without truth-constants in the language.

### 1.7.2 Logics of probability, of possibility and of belief

Already in a 1994, Petr Hájek and Dagmar Harmancová [60] noticed that one can safely interpret a probability degree on a Boolean proposition  $\varphi$  as a truth degree, not of  $\varphi$  itself but of another (modal) formula  $P\varphi$ , read as “ $\varphi$  is *probable*”. The point is that “being probable” is actually a fuzzy predicate, which can be more or less true, depending on how much probable is  $\varphi$ . Hence, it is meaningful to take the truth-degree of  $P\varphi$  as the probability-degree of  $\varphi$ . The second important observation is the fact that the standard Łukasiewicz logic connectives provide a proper modelling of the Kolmogorov axioms of finitely additive probabilities. For instance, the following axiom

$$P(\varphi \vee \psi) \leftrightarrow ((P\varphi \rightarrow P(\varphi \wedge \psi)) \rightarrow P\psi)$$

faithfully captures the finite-additive property when  $\rightarrow$  is interpreted by the standard Łukasiewicz logic implication. Indeed, these were the key issues that are behind the first probability logic defined as a theory over Rational Pavelka logic in

the paper by Hájek, Esteva and Godo, *Fuzzy Logic and Probability* [57]. This was later described with an improved presentation in Hájek's monograph [38] where  $P$  is introduced as a (fuzzy) modality. Exactly the same approach works to capture uncertainty reasoning with necessity measures, replacing the above axiom by  $N\varphi \wedge N\psi \rightarrow N(\varphi \wedge \psi)$ . More interesting was the generalization of the approach to deal with Dempster-Shafer belief functions proposed in the paper by Godo, Hájek and Esteva, *A fuzzy modal logic for belief functions* [35]. There, to get a complete axiomatization, the authors use one of possible definitions of Dempster-Shafer belief functions in terms of probability of knowing (in the epistemic sense), and hence they combine the above approach to probabilistic reasoning with the modal logic S5 to introduce a modality  $B$  for belief such that  $B\varphi$  is defined as  $P\Box\varphi$ , where  $\Box$  is a S5 modality and  $\varphi$  is a propositional modality-free formula. The complexity of the fuzzy probability logics over Łukasiewicz and ŁΠ logics was studied by Hájek and Tulipani in [68].

This line of research has been followed in a number of papers where analogs of these uncertainty logics have been extended over different fuzzy logics, mainly Łukasiewicz and Gödel logics, see e.g. [26, 28, 29, 27]. Hájek himself wrote another very interesting paper [51], generalising [68], about the complexity of general fuzzy probability logics defined over what he calls *suitable* fuzzy logics, i.e. logics whose standard set of truth values is the real unit interval  $[0, 1]$  and the truth functions of its (finitely many) connectives are definable by open formulas in the ordered field of reals.

### 1.7.3 Fuzzy modal logics

Another related field where Petr Hájek has made significant contributions is on the study of modal extensions of fuzzy logics and has also paved the way for further studies in this field. Inspired by the pioneer work of Fitting [24, 25] on many-valued modal logic valued on finite Heyting algebras, in a 1996 conference paper with Dagmar Harmanová [61] there is already a first study of a generalization of the modal logic S5 over Łukasiewicz logic. This topic is later developed in Hájek's monograph [38], where he considers modal logics  $S5(\mathcal{L})$ , where  $\mathcal{L}$  stands for any recursively axiomatized fuzzy propositional logic extending BL. The language of  $S5(\mathcal{L})$  is that of fuzzy propositional calculus (the language of  $\mathcal{L}$ ) extended by modalities  $\Box$  and  $\Diamond$ . The semantics is given by Kripke models of the form  $K = (W, e, A)$  where  $W$  is a set of possible worlds,  $A$  is a BL-chain and  $e(\cdot, w)$  is an evaluation of propositional variables in  $A$ , for each possible world  $w \in W$ . As usual,  $e(\cdot, w)$  extends to arbitrary formulas interpreting propositional connectives by the corresponding operations in  $A$ , and to modal formulas as  $\Box\varphi$  and  $\Diamond\varphi$  as universal and existential quantifiers over possible worlds, that is,  $e(\Box\varphi, w) = \inf_{v \in W} e(\varphi, v)$ , and  $e(\Diamond\varphi, w) = \sup_{v \in W} e(\varphi, v)$ . This is clearly a fuzzy variant of classical S5 modal semantics with total accessibility relations. In his book [38], Hájek proposes a set of axioms but leaves open the problem of proving its completeness. This problem is positively solved in his 2010

paper [54] by relating  $S5(\mathcal{C})$  to the monadic fragment  $m\mathcal{C}^\forall$  with just one variable (but with possibly countably-many constants) of the first order logic  $\mathcal{C}^\forall$ , and shows that the monadic axioms of  $\mathcal{C}^\forall$  provides an axiomatization of  $m\mathcal{C}^\forall$  that is strongly complete with respect to the general semantics. In [38] Hájek had already shown that, for  $\mathcal{C}$  being Łukasiewicz ( $\mathbb{L}$ ) or Gödel ( $G$ ) logics,  $S5(\mathcal{C})$  standard tautologies coincide with the general tautologies. Therefore one gets as a direct consequence the standard completeness of the  $S5(\mathbb{L})$  and  $S5(G)$  logics (the problem is left open for other choices of  $\mathcal{C}$ ). In this paper Petr Hájek also considers other kinds of Kripke models, namely witnessed and interval-valued models, besides some complexity results.

Petr Hájek has also studied other systems of fuzzy (or many-valued) modal logic [14, 64, 31]. In particular, in [14] a logic called MVKD45 is defined to provide a modal account of a certain notion of necessity and possibility of fuzzy events. MVKD45 is developed over a finitely-valued Łukasiewicz logic  $\mathbb{L}_k$  expanded with some unary operators to deal with truth-constants and its semantics is given by Kripke models of the form  $K = (W, e, \pi)$ , where  $W$  and  $e$  are as above (but evaluations are now over the  $(k + 1)$ -valued Łukasiewicz chain  $S_k$ , and  $\pi : W \rightarrow S_k$  is a possibility distribution on possible worlds. This semantics can be thus considered as a many-valued variant of the classical KD45 modal semantics.

As it has happened in other areas, Hájek ideas have been the seed for further investigations on fuzzy modal logics. Particular relevant are the papers by Caicedo and Rodriguez [10, 9] and by Metcalfe and Olivetti [74] on general modal logics over Gödel logics, the paper by Hansoul and Teheux [70] on modal logics over Łukasiewicz logic, and the paper by Bou et al. [2] on minimal modal logics over a finite residuated lattice.

#### 1.7.4 Fuzzy description logic

Computer scientists in Artificial Intelligence are interested in weakened but tractable versions of first-order logics. Description Logics (DLs) [2] are knowledge representation languages particularly suited to specify formal ontologies. DLs are indeed a family of formalisms describing a domain through a knowledge base (KB) where relevant concepts of the domain are defined (terminology, TBox) and where these defined concepts can be used to specify properties of certain elements of the domain (description of the world, ABox). The vocabulary of DLs consists of *concepts*, which denote sets of individuals, and *roles*, which denote binary relations among individuals and could be interpreted both in a multi-modal system and in first order logic: concepts as formulas and roles as accessibility relations in the modal setting and concepts as unary predicates and roles as binary predicates in the first order setting. A first approach toward fuzzified versions of description logics (FDLs from now on), i.e. versions referring to fuzzy logics instead of classical logic, was introduced in several papers, for instance in [84, 83, 80, 79, 77, 73]. However, the logic framework behind these initial works is very limited. The fuzzy logic con-

text consisted essentially only of the min-conjunction, the max-disjunction, and the Łukasiewicz negation.

In his 2005 paper *Making fuzzy description logic more general* [46], Petr Hájek proposes to deal with FDLs taking as basis  $t$ -norm based fuzzy logics with the aim of enriching their expressive possibilities (see also [49, 50]). This change of view gives rise to a wide number of choices on which a FDL can be based: for every particular problem we can consider the fuzzy logic that seems to be more adequate. As an example, Hájek studies an  $\mathcal{ALC}$ -style description logic with a suitable fragment of  $\text{BL}\forall$  as background logic. He proves, e.g. that the satisfiability of a concept when taking Łukasiewicz infinite-valued logic as background logic is decidable, as well as the validity of a concept in all finite models. The proof makes use of the fact that Łukasiewicz infinite-valued logic is complete w.r.t. witnessed models and it is based on a reduction of the satisfiability problem of a concept in description logic (or modal formula) to a satisfiability problem of a family of formulas of propositional logic, which is a decidable problem. In fact the result is valid for any description logic over any axiomatic extension of BL that satisfies the witnessed axioms, which is proved to be equivalent to a finite model property. But the main interest of Hájek's work was to bring a new view into Fuzzy description logics that took advantage of the recent advances of Mathematical Fuzzy logic, giving birth to a large family of FDLs.

From then, several papers on FDLs have followed Hájek ideas, for instance, [32, 5, 6, 12, 33, 13, 6, 7]. The results about translation of FDLs into first order fuzzy logics and modal many-valued logics given in Cerami's PhD thesis [11] are an explicit proof that Hájek's proposal for FDLs is the one that is actually in correspondence with the systems studied in mathematical fuzzy logic.

### 1.7.5 Logics with truth hedges

Truth hedges are clauses which directly refer to the truth of some sentence like *it is very true that*, *it is quite true that*, *it is more or less true that*, *it is slightly true that*, etc. In this formulation, after Zadeh, they have been represented in fuzzy logic systems (in broad sense) as functions from the set of truth values (typically the real unit interval) into itself, that modify the meaning of a proposition by applying over the membership function of the fuzzy set underlying the proposition. In the setting of mathematical fuzzy logic, Petr Hájek proposes in a series of three papers [42, 62, 40] to understand them as truth functions of new unary connectives called *truth-stressing* or *truth-depressing hedges*, depending on whether they reinforce or weaken the meaning of the proposition they apply over. The intuitive interpretation of a truth-stressing hedge on a chain of truth-values is a subdiagonal non-decreasing function preserving 0 and 1.

In his paper *On very true* [40], Petr axiomatizes the truth-stresser *very true* as an expansion of BL logic (and of some of their prominent extensions like Łukasiewicz or Gödel logics,) by a new unary connective  $\text{vt}$  satisfying the above mentioned con-

ditions together with the K-axiom  $\nu t(\varphi \rightarrow \psi) \rightarrow (\nu t \varphi \rightarrow \nu t \psi)$  and the rule of necessitation for  $\nu t$ . The logics he defines are shown to be algebraizable and to be complete with respect to the classes of chains of their corresponding varieties, and in the case of the logic over Gödel logic he proves standard completeness. This approach was later followed by Vychodil in order to deal with *truth depressers* as well. Finally Esteva, Godo and Noguera have given in [23] a more general approach containing as particular cases those of Hájek and Vichodil.

## 1.8 Mathematical theories over fuzzy logic

Two particular elementary theories have found the interest of Petr Hájek: an axiomatic set theory FST for fuzzy sets, and formalized arithmetic.

A ZF-like axiomatic theory FST, based upon the first-order logic  $\text{BL}\forall\Delta$ , is discussed by Petr and Z. Haniková in the paper *A development of set theory in fuzzy logic* [20]. Its first-order language has the equality symbol  $=$  as a logical symbol, and  $\in$  as its only non-logical primitive predicate. The axioms are suitable versions of the usual ZF-axioms together with an axiom stating the existence of the support of each fuzzy set.

A kind of “standard” model  $V^{\mathbf{L}} = \bigcup_{\alpha \in \text{On}} V_{\alpha}^{\mathbf{L}}$  for this theory FST is formed, w.r.t. some complete BL-chain  $\mathbf{L}$ , completely similar to the construction of Boolean valued models for ZF, i.e. with the crucial iteration step  $V_{\alpha+1}^{\mathbf{L}} = \{f \in {}^{\text{dom}(u)}L \mid \text{dom}(u) \subseteq V_{\alpha}^{\mathbf{L}}\}$ .

For the primitive predicate  $\in$  the truth degree  $\llbracket x \in y \rrbracket$  is defined as  $\llbracket x \in y \rrbracket = y(x)$  for  $x \in \text{dom}(y)$  and as 0 otherwise. And  $=$  has the truth degree  $\llbracket x = y \rrbracket = 1$  for  $x = y$  and 0 otherwise.

The main results are that the structure  $V^{\mathbf{L}}$  is a model of all of the authors’ axioms, and that ZF is interpretable in FST.

Another generalized set theory Petr is interested in is *Cantorian set theory*  $\text{CL}_0$  over Łukasiewicz logic  $L_{\infty}$ . In the background there is an older approach toward a consistency proof for naive set theory, i.e. set theory with *comprehension* and *extensionality* only, via  $L_{\infty}$  initiated by Th. Skolem [78]. This approach resulted — after a series of intermediate steps mentioned e.g. in [9] — in a proof theoretic proof (in the realm of  $L_{\infty}$ ) of the consistency of naive set theory with *comprehension only* by R. B. White [39].

In this context, Petr’s goal is to study the arithmetics of natural numbers. In his paper *On arithmetic in the Cantor-Łukasiewicz fuzzy set theory* [18], he finds out that this is a rather delicate matter.

Two equality predicates come into consideration here—so called Leibniz equality  $x =_l y =_{\text{def}} \forall z(x \in z \leftrightarrow y \in z)$  and the usual extensional equality  $x =_e y =_{\text{def}} \forall z(z \in x \leftrightarrow z \in y)$ . Leibniz equality is shown to be a *crisp* predicate, but extensional equality is *not*.

$\text{CL}_0$  becomes *inconsistent* adding the coincidence assumption  $x =_l y \leftrightarrow x =_e y$ . A constant  $\omega$  can be introduced to denote a suitably defined crisp set of natural

numbers such that  $CL_0(\omega)$  is a conservative extension of  $CL_0$ . Even a weak form of induction might be added to  $CL_0(\omega)$  saving consistency, viz. the rule

$$\frac{\varphi(0) \quad \forall x(\varphi(x) \leftrightarrow \varphi(S(x)))}{(\forall x \in \omega)\varphi(x)}$$

for formulas  $\varphi$  which do *not* contain the constant  $\omega$ .

This restriction on the induction formulas is crucial, however: deleting this restriction makes the system *inconsistent*.

Yet another approach toward arithmetics within mathematical fuzzy logic is offered in Petr Hájek's papers *Mathematical fuzzy logic and natural numbers* [52], and *Towards metamathematics of weak arithmetics over fuzzy logic* [55]. The starting point is a slightly modified form  $Q^\sim$  of a weakened version  $Q^-$  of the Robinson arithmetic  $Q$ , designed by A. Grzegorzczuk, and introducing addition and multiplication as ternary relations. Seen as an elementary theory over  $BL\forall$  this theory is denoted  $FQ^\sim$ . The main results are that  $Q^\sim$  as a theory over Gödel logic (or also over intuitionistic logic) is essentially incomplete and essentially undecidable, and that  $FQ^\sim$  is essentially undecidable too.

## 1.9 Petr's failures

As noted by Matthias Baaz in the book *Witnessed years*, Petr Hájek had a special skill to obtain interesting results also from his failures. Here are some examples. After he invented his logic BL, Petr tried to prove that it is standard complete, that is, that BL is complete with respect to the class of continuous t-norms and their residuals. He didn't succeed (the result was proved by Cignoli, Esteva, Godo and Torrens in the paper *Basic fuzzy logic is the logic of continuous t-norms and their residuals* [11]), but he proved something which is very close to the desired result. Namely, he proved that BL added with two axioms which are sound in any continuous t-norm algebra is standard complete. Then Cignoli, Esteva, Godo and Torrens proved that these axioms are redundant, i.e., they are provable in BL, and got the result.

Another example was Petr's attempt to extend the Mostert and Shield's decomposition of a continuous t-norm as an ordinal sum of Łukasiewicz, Gödel and product t-norms. In his paper *Basic fuzzy logic and BL-algebras* [37], Petr did not get the full result, but he proposed a method which was crucial in the proof of Aglianò-Montagna's decomposition of a BL-chain as an ordinal sum of MV-algebras and negative cones of abelian  $\ell$ -groups. That is, he suggested to take a maximal decomposition, that is, a decomposition in which each component can no longer be decomposed as an ordinal sum. To conclude the proof of the Aglianò-Montagna decomposition it is sufficient to prove that any indecomposable component is either an MV-algebra or a negative cone of an abelian  $\ell$ -group.

Finally, Petr failed to invent MTL-algebras, which are due to Esteva and Godo [23], but he conjectured the independence of the axiom  $(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \& (\psi \rightarrow$

$\varphi$ )), which separates BL from MTL, as an open problem. The independence of this axiom from the other axioms of BL may have suggested the investigation of BL deprived of it (and with the obvious axioms for  $\wedge$ ), that is, of MTL.

Finally, Petr tried to prove the redundancy of the axiom  $\forall x(\varphi(x) \vee \psi) \rightarrow ((\forall x\varphi(x)) \vee \psi)$ . It turned out that this axiom is not redundant, for a proof see for instance [21]. However, a first-order fuzzy logic with this axiom is sound and complete with respect to its chains, while first-order fuzzy logic deprived of this axiom is sound and complete with respect to the class of its (possibly not linearly ordered) algebras.

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