A Logic to Reason About f-Indices of Inclusion over $Ł_n$

Tommaso Flaminio[1(✉)], Lluis Godo[1], Nicolás Madrid[2], and Manuel Ojeda-Aciego[2]

[1] IIIA-CSIC, 08193 Bellaterra, Spain
{tommaso,godo}@iiia.csic.es

[2] Departamento Matemática Aplicada, Universidad de Málaga, 29071 Málaga, Spain
{nicmadlab,aciego}@uma.es

Abstract. In this paper we provide a sound and complete logic to formalise and reason about f-indices of inclusion. The logic is based on finite-valued Łukasiewicz logic $Ł_n$ and its S5-like modal extension $S5(Ł_n)$ with additional unary operators.

1 Introduction

Inclusion is one of the most fundamental relations between sets. In previous work [10], it was shown how the degree of inclusion between two L-fuzzy sets can be represented in terms of a function that specifically determines the minimal modifications required in one L-fuzzy set to be included (in Zadeh's sense) in another.

The key idea was the notion of f-inclusion, which defines a family of crisp binary relations between L-fuzzy sets that are used as indexes of inclusion and, subsequently, we define the f-index of inclusion as the most suitable f-inclusion under certain criteria. In addition, it was shown that the f-index of inclusion satisfies versions of many common axioms usually required for measures of inclusion in the literature, namely the axiomatic approaches of Kitainik [8] and Sinha-Dougherty [14].

In [11], the f-index was shown to be definable by means of a fuzzy conjunction which is part of an adjoint pair. Moreover, it is also proven in [11] that when the undelying structure in the modus ponens inference rule is given by adjoint pairs, the f-index provides the maximum possible truth-value in the conclusion obtained by fuzzy modus ponens using any other possible adjoint pair.

In this paper, we continue the study of the logical properties of the f-index of inclusion. Specifically, we provide a first step towards a logical account of the notion of f-index of inclusion for fuzzy sets in the frame of an S5-like modal logic over the n-valued Łukasiewicz logic with truth-constants $Ł_n^c$. We take advantage of the good logical and expressive properties of this logical setting to define the logic $IŁ_n$ to reason about f-indexes of inclusion between n-valued propositions.

The paper is structured as follows. After this introduction, we first provide the necessary background on finite-valued Łukasiewicz logic and its S5-like modal

S. Massanet et al. (Eds.): EUSFLAT 2023/AGOP 2023, LNCS 14069, pp. 1–10, 2023.
https://doi.org/10.1007/978-3-031-39965-7_44

extension $S5(Ł_n^c)$, and on the f-index of inclusion of fuzzy sets. Then in Sect. 3 we define a logic $IŁ_n$ based on $S5(Ł_n^c)$ with additional unary operators to formalise and reason about f-indices of inclusion. We finish with some prospects for future work.

2 Preliminaries

2.1 The Finite-Valued Łukasiewicz Logic

Consider the propositional language $\mathcal{L}$ whose set of formulas $Fm_{\mathcal{L}}$ is built from a *finite* set of propositional variables Var, the connective $\rightarrow$ (implication) and truth constants $\overline{r}$ for each $r \in VŁ_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ for some fixed natural $n \geq 2$. Further connectives are defined as follows:

$$\begin{aligned} \neg\phi &:= \phi \rightarrow \overline{0} & \phi \wedge \psi &:= \phi \otimes (\phi \rightarrow \psi) \\ \phi \otimes \psi &:= \neg(\phi \rightarrow \neg\psi) & \phi \oplus \psi &:= \neg(\neg\phi \otimes \neg\psi) \\ \phi \vee \psi &:= ((\phi \rightarrow \psi) \rightarrow \psi) & \phi \equiv \psi &:= (\phi \rightarrow \psi) \otimes (\psi \rightarrow \phi) \end{aligned}$$

with ϕ and ψ being arbitrary formulas.[1]

A *propositional evaluation* is a mapping $e : Var \rightarrow VŁ_n$ that is extended to formulas as follows: if ϕ and ψ are formulas and $r \in VŁ_n$, then

$$e(\phi \rightarrow \psi) = e(\phi) \Rightarrow e(\psi) \text{ and } e(\overline{r}) = r,$$

where $x \Rightarrow y = \min(1, 1 - x + y)$ for $x, y \in Ł_n$. Note that $x \Rightarrow y = 1$ iff $x \leq y$. The set of all such evaluations will be denoted by Ω_n. Notice that, in particular, for every formula ϕ and ψ and for every $e \in \Omega_n$, we have:

$$\begin{aligned} e(\neg\phi) &= 1 - e(\phi) & e(\phi \wedge \psi) &= \min(e(\phi), e(\psi)) \\ e(\phi \otimes \psi) &= \max(e(\phi) + e(\psi) - 1, 0) & e(\phi \oplus \psi) &= \min(1, e(\phi) + e(\psi)) \\ e(\phi \vee \psi) &= \max(e(\phi), e(\psi)) & e(\phi \equiv \psi) &= 1 - |e(\phi) - e(\psi)|. \end{aligned}$$

A formula ϕ is said to be *satisfiable* if there exists an $e \in \Omega_n$ such that $e(\phi) = 1$. In such a case we say that e is a *model* of ϕ and e is a model of a set of formulas T if e is a model of every formula in T. A *tautology* is a formula ϕ such that $e(\phi) = 1$ for each $e \in \Omega_n$. A formula ϕ is a *semantic consequence* of a set of formulas Γ, written as $\Gamma \models \phi$, if it holds that every model of Γ is also a model of ϕ.

This logic based on the language $\mathcal{L}$, which we will denote by $Ł_n^c$, has a sound and a strongly complete axiomatization, see e.g. [4] for details. In particular, the axioms of $Ł_n^c$ are

[1] For the sake of simplicity, along this paper we will use the same symbol to denote both a logical language $\mathcal{L}$ and its corresponding set of formulas $Fm_{\mathcal{L}}$ built in the usual way. This will be done with no danger of confusion.

(Ł1) $\varphi \to (\psi \to \varphi)$,
(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(Ł3) $((\varphi \to \overline{0}) \to (\psi \to \overline{0})) \to (\psi \to \varphi)$,
(Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$,
(Ł5) $(n-1)\varphi \equiv n\varphi$,
(Ł6) $(k\varphi^{k-1})^n \equiv n\varphi^k$, for each $k \in \{2, \dots, n-2\}$ not dividing $n-1$,
(Q1) $(\overline{r_1} \to \overline{r_2}) \equiv \overline{\min\{1, 1-r_1+r_2\}}$, for each $r_1, r_2 \in VŁ_n$,

and the only deduction rule is modus ponens (from ϕ and $\phi \to \psi$ infer ψ). Axioms (Ł1)-(Ł4) form an axiomatization for Łukasiewicz logic, and in axioms (Ł5) and (Ł6), $k\varphi$ is an abbreviation for $\varphi \oplus \overset{k}{\ldots} \oplus \varphi$ (k repetitions) and φ^k for $\varphi \otimes \overset{k}{\ldots} \otimes \varphi$ (k repetitions). Axiom (Q1) is a bookkeeping axiom for truth-constants. It is needed to derive how truth-constants are combined with the different connectives.

$Ł_n^c$ is strongly complete in the following sense: if $\vdash$ denotes the notion of proof defined from the set of axioms of $Ł_n^c$ and modus ponens, then we have that for any countable (possibly infinite) set of formulas $T \cup \{\psi\}$, it holds that $T \vdash \psi$ iff $T \models \psi$. A formula ψ that can be proven from the axioms of $Ł_n^c$ and modus ponens is called a *theorem*; in this case we will write $\vdash \psi$.

For each formula ϕ we will denote by $\Delta\phi$ the formula ϕ^n. Since we only have n truth values this formula is Boolean. Indeed, it is easy to check that

$$e(\Delta\phi) = \begin{cases} 1, \text{ if } e(\phi) = 1 \\ 0, \text{ if } e(\phi) < 1 \end{cases}$$

Note that Δ corresponds to the well-known Baaz-Monteiro projection operator [1,13].

Remark 1. For every formula φ of $Ł_n^c$ and for every nonempty subset S of Ω_n we can associate a fuzzy subset f_φ of S. Precisely, $f_\varphi : S \to Ł_n$ is defined by the stipulation

$$f_\varphi : w \in S \mapsto w(\varphi) \in VŁ_n.$$

Conversely, given S and a fuzzy set $f : S \to VŁ_n$, we can define a formula φ_f of $Ł_n^c$ such that $f = f_{\varphi_f}$. Precisely, let

$$\varphi_f := \bigwedge_{w \in S} (\mathbf{1}_w \to \overline{f(w)})$$

where $\mathbf{1}_w := \bigwedge_{p \in Var} \Delta(p \equiv \overline{w(p)})$ is such that $f_{\mathbf{1}_w}$ is the characteristic function of w in S.

This description of formulas as fuzzy sets will allow us to describe mathematical properties of fuzzy sets in the logical framework. In the present paper, we will deal with a logical treatment of the f-index of inclusion between fuzzy sets that we will recall in Subsect. 2.2.

Now, we recall the logic $S5(Ł_n^c)$ from [2], an S5-like modal extension of the logic $Ł_n^c$. To this end, let $\mathcal{L}_n^{\boxdot}$ be the expansion of the language $\mathcal{L}_n$ of the logic $Ł_n^c$

by a unary modal operator $\boxdot$. An $S5(\text{Ł}_n^c)$-*interpretation* for formulas in $\mathcal{L}_n^{\boxdot}$ is a mapping σ determined by a pair (w, S),[2] where $w \in \Omega_n$ is a Ł_n-evaluation and $S \subseteq \Omega_n$ is a set of Ł_n-evaluations such that $w \in S$. Formally, each pair (w, S) determines the map $\sigma : \mathcal{L}_n^{\boxdot} \to \text{VŁ}_n$ by the following stipulations:

– if $\varphi \in \mathcal{L}_n$, $\sigma(\varphi) = w(\varphi)$
– $\sigma(\boxdot\psi) = \inf\{(w', S)(\psi) \mid w' \in S\}$; (in particular, if $\psi \in \mathcal{L}_n$, $\sigma(\boxdot\psi) = \inf\{w'(\psi) \mid w' \in S\}$)
– $\sigma(\varphi \star \psi) = \sigma(\varphi) \,\overline{\star}\, \sigma(\psi)$, for $\star$ being a connective of Łukasiewicz logic

We will denote by Σ the set of $S5(\text{Ł}_n^c)$-interpretations, i.e. $\Sigma = \{\sigma = (w, S) \in \Omega_n \times 2^{\Omega_n} \mid w \in S\}$. We say that $\sigma \in \Sigma$ is a model of a formula φ, written $\sigma \models \varphi$, when $\sigma(\varphi) = 1$.

Now, let us recall from [2] the definition of the logic $S5(\text{Ł}_n^c)$ as the modal logic over Ł_n^c whose axioms and rules are:

(Łn) Axioms of Ł_n^c
(M1) $\boxdot(\varphi \wedge \psi) \to (\boxdot\varphi \wedge \boxdot\psi)$
(M2) $\boxdot(\overline{r} \to \varphi) \equiv (\overline{r} \to \boxdot\varphi)$
(M3) $\boxdot(\varphi \oplus \varphi) \equiv (\boxdot\varphi \oplus \boxdot\varphi)$
(K) $\boxdot(\varphi \to \psi) \to (\boxdot\varphi \to \boxdot\psi)$
(T) $\boxdot\varphi \to \varphi$
(4) $\boxdot\varphi \to \boxdot\boxdot\varphi$
(5) $\neg\boxdot\varphi \to \boxdot\neg\boxdot\varphi$
Rules: modus ponens and necessitation for $\boxdot$

The logic $S5(\text{Ł}_n^c)$ is proved in [2, Theorem 2, Proposition 2] to be strongly complete with respect to the class of structures Σ defined above.

Theorem 1. *Let $T \cup \{\varphi\}$ be a countable set of formulas in $\mathcal{L}_n^{\boxdot}$. Then $\Gamma \vdash \varphi$ iff for all $\sigma \in \Sigma$ such that $\sigma \models \gamma$ for all $\gamma \in \Gamma$, then $\sigma \models \varphi$.*

2.2 The *f*-Index of Inclusion

The f-index of inclusion represents the inclusion between fuzzy sets by means of mappings from $[0, 1]$ to $[0, 1]$. This feature is an important difference from the standard approaches [6,8,14,16], where the inclusion of one fuzzy set into another is given, in general, by a value in the unit interval $[0, 1]$. Not any mapping from $[0, 1]$ to $[0, 1]$ can be used to represent the f-index of inclusion: the set of possible assignable mappings is introduced below, together with the basic notion of f-inclusion.

Definition 1 (cf. [12]).

– *The set of indexes of inclusion, denoted by $\mathcal{F}$, consists of every monotonically increasing mapping $f : [0, 1] \to [0, 1]$ such that $f(x) \leq x$ for all $x \in [0, 1]$.*

[2] Actually, we will henceforth identify both notations σ and (w, S) to indicate this map, and we can even write $\sigma = (w, S)$.

– *Let A and B be two fuzzy sets over the same universe $\mathcal{U}$, and consider $f \in \mathcal{F}$. We say that A is* f-included *in B (denoted by $A \subseteq_f B$) if and only if the inequality $f(A(u)) \leq B(u)$ holds for all $u \in \mathcal{U}$.*

The suitability of the set $\mathcal{F}$ as a proper set of indexes to represent the inclusion is explained in [9,10,12]. In order to choose a convenient index among all in $\mathcal{F}$ to represent a specific inclusion between two fuzzy sets, in [12], we introduced the following definition.

Definition 2 (f-index of inclusion [12]). *Let A and B be two fuzzy sets over a same domain. The* f-index of inclusion *of A in B, denoted by $Inc(A, B)$, is defined as*

$$Inc(A, B) = \max\{f \in \mathcal{F} \mid A \subseteq_f B\}$$

The previous definition is correct, in the sense that it can be proved that the set $\{f \in \mathcal{F} \mid A \subseteq_f B\}$ has always a maximum for every pair of fuzzy sets A and B. An interesting interpretation of the f-index of inclusion is given by considering mappings $f \in \mathcal{F}$ as modifiers of membership degrees. Accordingly, the lesser pointwisely the mapping $f \in \mathcal{F}$ is, the greater the modification is. Therefore, taking the maximum $f \in \mathcal{F}$ such that $A \subseteq_f B$ is equivalent to consider the minimal modifications of membership degrees in A to include it into B in the Zadeh's sense. This interpretation brings the f-index of inclusion closer to the notion of truth stressers in fuzzy logic [3,5,7,15], since they modify truth degrees. This relation is used in the next section to define a unary operator in VŁ$_n$.

Lastly, we recall two interesting results of the f-index of inclusion that will be used in the next section. The first one determines an analytical structure of the f-index of inclusion.

Theorem 2 (cf. [10]). *Let A and B be two fuzzy sets over $\mathcal{U}$, then*

$$Inc(A, B)(x) = \bigwedge_{u \in \mathcal{U}} \{B(u) \wedge x \mid x \leq A(u)\},$$

for all $x \in [0, 1]$.

The second result provides some properties that support the use of the f-index of inclusion as a representation of the inclusion between fuzzy sets.

Theorem 3 (cf. [10]). *Let A, B and C be fuzzy sets over $\mathcal{U}$. The following properties hold:*

1. *(Full inclusion) $Inc(A, B) = id$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.*
2. *(Null inclusion) $Inc(A, B) = \bot$ if and only if there is a set of elements in the universe $\{u_i\}_{i \in I} \subseteq \mathcal{U}$ such that $A(u_i) = 1$ for all $i \in I$ and $\bigwedge_{i \in I} B(u_i) = 0$.*
3. *(Pseudo transitivity) $Inc(B, C) \circ Inc(A, B) \leq Inc(A, C)$.*
4. *(Monotonicity) If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $Inc(C, A) \leq Inc(B, A)$.*
5. *(Monotonicity) If $B(u) \leq C(u)$ for all $u \in \mathcal{U}$ then, $Inc(A, B) \leq Inc(A, C)$.*
6. *(Transformation Invariance) Let $T : \mathcal{U} \to \mathcal{U}$ be a bijection on $\mathcal{U}$, then $Inc(A, B) = Inc(A \circ T, B \circ T)$.*
7. *(Relationship with intersection) $Inc(A, B \cap C) = Inc(A, B) \wedge Inc(A, C)$.*
8. *(Relationship with union) $Inc(A \cup B, C) = Inc(A, C) \wedge Inc(B, C)$.*

3 A Logic to Reason About the F-Index of Inclusions over Ł_n^c

In this section we introduce an axiomatic extension of the finite-valued fuzzy modal logic $\mathrm{S5}(\text{Ł}_n^c)$ with new unary operators $\Box_{\varphi,\psi}$, one for every pair of formulas φ, ψ from $\mathcal{L}_n^c$, that will provide us with a logical formalisation of the f-index of inclusion between fuzzy concepts represented as propositions in Ł_n^c. Recall the representation of formulas as fuzzy sets from Remark 1. In this section, by *truth-stresser* we will mean a non-decreasing function $\tau : \mathrm{V}\text{Ł}_n \to \mathrm{V}\text{Ł}_n$ such that $\tau(x) \leq x$ for all $x \in \mathrm{V}\text{Ł}_n$.

We start by defining the syntax and semantics of our logic, and later we axiomatise it.

3.1 Syntax and Semantics

Let $\mathcal{IL}$, where $\mathcal{I}$ stands for *inclusion*, be the expansion of the modal language $\mathcal{L}_n^{\boxdot}$ by adding to its signature a unary operator $\Box_{\varphi,\psi}$ for every pair of formulas φ, ψ from $\mathcal{L}_n$.

The semantics for $\mathcal{IL}$ is still given by pairs $\sigma = (w, S) \in \Sigma$, which now further interpret the new operators $\Box_{\varphi,\psi}$.

- If $\varphi \in \mathcal{L}_n^{\boxdot}$, then $\sigma(\varphi)$ is defined as in $\mathrm{S5}(\text{Ł}_n^c)$ (see Sect. 2.1). Moreover, for each $\varphi \in \mathcal{L}_n^{\boxdot}$ we denote by φ_σ its corresponding fuzzy set on S defined as: $\varphi_\sigma(w') = \sigma'(\varphi)$, where $\sigma' = (w', S)$. Note that if $\varphi \in \mathcal{L}_n$ then $\sigma'(\varphi) = w'(\varphi)$ and hence the fuzzy set associated to φ is defined as in Remark 1.
- σ interprets operators $\Box_{\varphi,\psi}$ as one-place functions $\sigma(\Box_{\varphi,\psi}) : \mathrm{V}\text{Ł}_n \to \mathrm{V}\text{Ł}_n$ defined as

$$\sigma(\Box_{\varphi,\psi}) = \max\{\tau : \mathrm{V}\text{Ł}_n \to \mathrm{V}\text{Ł}_n \text{ truth-stresser} \mid \tau(\varphi_\sigma) \leq \psi_\sigma\}.$$

- Finally, as customary, the interpretation by σ of a formula $(\Box_{\varphi,\psi}\chi)$ is defined as follows: $\sigma(\Box_{\varphi,\psi}\chi) = \sigma(\Box_{\varphi,\psi})(\sigma(\chi))$. In particular, if $\chi \in \mathcal{L}_n$, then $\sigma(\Box_{\varphi,\psi}\chi) = \sigma(\Box_{\varphi,\psi})(w(\chi))$.

Obviously, we can give a similar meaning to $\Box_{\varphi,\psi}$ than the one given to the f-index of inclusion in the previous section. Firstly, note that the inequality $\tau(\varphi_\sigma) \leq \psi_\sigma$ holds if, and only if, σ validates the implication $\varphi_\tau \to \psi$, where φ_τ is the formula defined as

$$\varphi_\tau := \bigvee_{s \in \mathrm{V}\text{Ł}_n} \Delta(\varphi \equiv \overline{s}) \wedge \overline{\tau(s)},$$

and, as it can be easily checked, it is such that $(\varphi_\tau)_\sigma = \tau(\varphi_\sigma)$.[3] Secondly, the larger the truth stresser (as a mapping), the smaller the degree of truth stress

[3] Indeed, if $w(\varphi) = r_0$, then $w(\psi) = \max_r \min(w(\Delta(\varphi \equiv \overline{r}), \tau(r)) =$ $= \max(\max_{r \neq r_0} w(\Delta(\varphi \equiv \overline{r}) \wedge \overline{\tau(r)}), w(\Delta(\varphi \equiv \overline{r_0}) \wedge \overline{\tau(r_0)})) = \max(0, \min(1, \tau(r_0)) =$ $= 0 \vee \tau(r_0) = \tau(w(\varphi))$.

(semantically). For example, the identity mapping imposes no truth stress, while the mapping $\perp$ (which always takes the value 0) imposes a drastic truth stress that makes false even true statements. Therefore, $\Box_{\varphi,\psi}$ determines the minimal amount of truth stress in φ we need to make the formula $\varphi \to \psi$ valid. In other words, we can rewrite the definition of the semantics of $\Box_{\varphi,\psi}$ as

$$\sigma(\Box_{\varphi,\psi}) = \max\{\tau : \mathrm{V}Ł_n \to \mathrm{V}Ł_n \text{ truth-stresser} \mid \sigma \models \varphi_\tau \to \psi\}.$$

We will use the notation Σ^I to refer to the set of interpretations $(w, S) \in \Sigma$ when applied to the expanded language $\mathcal{IL}$ as prescribed above.

Two remarks are in order here:

(i) As in the case of modal formulas $\boxdot\varphi$, the interpretation of formulas of the type $\Box_{\varphi,\psi}$ by a pair $\sigma = (w, S)$ does not actually depend on the particular world w but only on the set S.
(ii) By Theorem 2, we have that $\sigma(\Box_{\varphi,\psi}\varphi) = \min\{w'(\psi) \mid w' \in S, w(\varphi) \leq w'(\varphi)\} \wedge w(\varphi) = \min\{\psi_\sigma(w') \mid w' \in S, \varphi_\sigma(w) \leq \varphi_\sigma(w')\} \wedge \varphi_\sigma(w)$.

3.2 The Logic IŁ$_n$: Axiomatic System, Soundness and Completeness

Based on the properties of the f-index of inclusion recalled in Sect. 2.2, we axiomatically define the logic IŁ$_n$ as an axiomatic expansion of $S5(Ł_n^c)$ as follows, where we make use of the intended semantics of the modal S5 operator $\boxdot$ as a sort of universal quantifier over the set of interpretations.

Definition 3. *Axioms and rules of IŁ$_n$ are those of $S5(Ł_n^c)$ plus:*

(A1) $\boxdot(\Box_{\varphi,\psi}\chi \to \chi)$
(A2) $\boxdot(\Box_{\varphi,\psi}\varphi \to \psi)$
(A3) $\Delta \boxdot (\Box_{\gamma,\delta}\varphi \to \psi) \to \boxdot(\Box_{\gamma,\delta}\chi \to \Box_{\varphi,\psi}\chi)$
(A4) $\boxdot(\Delta(\gamma \to \delta) \to (\Box_{\varphi,\psi}\gamma \to \Box_{\varphi,\psi}\delta))$
(A5) $\bigvee_{s \in VŁ_n} \boxdot(\Box_{\varphi,\psi}\overline{r} \equiv \overline{s})$*, for any* $r \in VŁ_n$
(A6) $\boxdot(\Delta(\varphi \equiv \overline{r}) \to (\overline{\tau(r)} \to \Box_{\varphi,\varphi_\tau}\varphi))$*, for any truth-stresser* τ

The above mentioned fact that the modal S5 operator $\boxdot$ behaves as a universal quantifier over the set of evaluations, shows that the above axioms force a behavior of $\Box_{\varphi,\psi}$ that reflects that of the f-indexes of inclusion functions. In particular:

- Axiom (A1) states that for all evaluations the value of $\Box_{\varphi,\psi}$ in χ takes a lower value than χ itself. This hence reflects the property that every index of inclusion function f satisfies $f(x) \leq x$.
- Axiom (A2), encodes the fact the result of applying the index of inclusion of φ in ψ to the fuzzy set given by φ is indeed included into the fuzzy set given by ψ.

– Axiom (A3) captures the maximality property of the index of inclusion; that is the function associated to $\Box_{\varphi,\psi}$ is the maximal one among those that, applied to φ, give a fuzzy set included into ψ.
– Axiom (A4) is monotonicity, while axiom (A5) states that the index of inclusion of a constant function is constant as well. These two axioms are needed to prove that $\Box_{\varphi,\psi}$ is indeed interpreted as a function.
– Axiom (A6) expresses a technical property of the functions like $\Box_{\varphi,\psi}$ that will be used below to prove that the truth-stressers defined in this way are sufficiently many to ensure that the maximal stresser is attained within the set of functions $\Box_{\varphi,\psi}$.

All these intuitive semantic interpretations of the axioms are supported by the semantics of the operators $\Box_{\varphi,\psi}$ given above, which faithfully reflect in turn the properties of the f-indices of inclusion described in Sect. 2.2. Then, it is not difficult to show that the above axioms are indeed sound.

Proposition 1. *$I\text{Ł}_n$ is sound with respect to the class of structures Σ^I.*

Since IŁ_n is an axiomatic expansion of $\text{S5}(\text{Ł}_n^c)$, one can reduce proofs in IŁ_n to proofs in $\text{S5}(\text{Ł}_n^c)$ taking the axioms (A1)-(A6) as additional premises. In the following, $Ax(I\text{Ł}_n)$ will stand for all the instances of the additional axioms (A1)-(A6).

Lemma 1. *For any set of $I\text{Ł}_n$-formulas $T \cup \{\phi\}$, it holds that $T \vdash_{I\text{Ł}_n} \phi$ iff $T \cup \{Ax(I\text{Ł}_n)\} \vdash_{S5(\text{Ł}_n^c)} \phi$, where $\vdash_{S5(\text{Ł}_n^c)}$ stands for proof in pure $S5(\text{Ł}_n^c)$.*

Finally, we can prove that IŁ_n is (sound and) complete with respect to the semantics previously defined.

Theorem 4. *For any set of $I\text{Ł}_n$-formulas, $T \cup \{\phi\}$ we have that, $T \vdash_{I\text{Ł}_n} \phi$ iff $T \models_{I\text{Ł}_n} \phi$.*

Proof. (Sketch). Assume $T \not\vdash_{I\text{Ł}_n} \phi$. By the above Lemma 1, this means that $T \cup \{Ax(I\text{Ł}_n)\} \not\vdash_{S5(\text{Ł}_n^c)} \phi$, and by completeness of $S5(\text{Ł}_n^c)$, $T \cup \{Ax(I\text{Ł}_n)\} \not\models_{S5(\text{Ł}_n^c)} \phi$. Therefore, there exists an $S5(\text{Ł}_n^c)$-interpretation $\sigma = (w, S) \in \Sigma$ such that $\sigma(T) = \sigma(Ax(I\text{Ł}_n)) = 1$ and $\sigma(\phi) < 1$. It remains to prove that in fact σ belongs to Σ^I, that is, that σ correctly interprets formulas of the kind $\Box_{\varphi,\psi}\chi$ as specified in Sect. 3.1.

If $\varphi, \psi \in \mathcal{L}$ are propositional, the fact that σ evaluates to 1 all the axioms (A1)-(A6) implies a set of conditions on the evaluation by $\sigma = (w, S)$ of formulas of the kind $\Box_{\varphi,\psi}\chi$. In particular, axioms (A4) and (A5) allow us to interpret each operator $\Box_{\varphi,\psi}$ as a unique unary function on VŁ_n, and the rest of the axioms allows one to prove that such a function is indeed the f-inclusion index of φ into ψ, once they are interpreted as fuzzy sets on S. This shows that $\sigma \in \Sigma^I$, hence $T \not\models_{I\text{Ł}_n} \phi$, and the proof is completed.

4 Conclusions and Future Work

In this paper we have provided a first step towards a logical account of the notion of f-index of inclusion for fuzzy sets in the frame of an S5-like modal logic over the n-valued Łukasiewicz logic with truth-constants. We have taken advantage of the good logical and expressive properties of this logical setting to define the logic $IŁ_n$ to reason about f-indexes of inclusion between n-valued propositions. Actually, our goal has been to syntactically link a particular truth-stresser, corresponding to the modality $\Box_{\varphi,\psi}$, to each pair of formulas φ and ψ in order to represent "the minimal amount of truth stress in φ we need to make the formula $\varphi \to \psi$ valid", not just to consider different and arbitrary truth-stressers in a logic. The question whether we can do all this without this syntactical link between pairs of formulas φ, ψ and modalities $\Box_{\varphi,\psi}$ is left for further work, although we think it can be a difficult task.

Note that equivalence classes of formulas determine the same truth-stresser by the modalities (i.e., $\varphi \equiv \varphi', \psi \equiv \psi' \vdash_{IŁ_n} \Box_{\varphi,\psi}\chi \equiv \Box_{\varphi',\psi'}\chi$), thus in fact there are only finitely-many distinct modalities, but that fact does not invalidate the use of both modalities $\Box_{\varphi,\psi}$ and $\Box_{\varphi',\psi'}$ in the language of the logic $IŁ_n$.

As for future work, we plan the study in depth the connection of modalities with the deduction theorem of Łukasiewicz logic or the existence of truth stressers, if any, that cannot be represented as a modality of the type $\Box_{\varphi,\psi}$. We also plan to consider a more general many-valued logical setting, lifting the assumption of dealing with finitely-many truth-degrees.

Acknowledgments. Flaminio and Godo acknowledge partial support by the MOSAIC project (EU H2020-MSCA-RISE-2020 Project 101007627) and by the Spanish project PID2019-111544GB-C21/AEI/10.13039/501100011033. Madrid and Ojeda-Aciego acknowledge partial support by the project VALID (PID2022-140630NB-I00 funded by MCIN/AEI/10.13039/501100011033), and by Plan Propio de la Universidad de Málaga.

References

1. Baaz, M.: Infinite-valued Gödel logics with 0-1-projections and relativizations. Lect. Notes Logic **6**, 23–33 (1996)
2. Blondeel, M., Flaminio, T., Schockaert, S., Godo, L., Cock, M.D.: On the relationship between fuzzy autoepistemic logic and fuzzy modal logics of belief. Fuzzy Sets Syst. **276**, 74–99 (2015)
3. Ciabattoni, A., Metcalfe, G., Montagna, F.: Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions. Fuzzy Sets Syst. **161**(3), 369–389 (2010)
4. Cignoli, R., D'Ottaviano, I.M.L., Mundici, D.: Algebraic foundations of many-valued reasoning. Trends in Logic-Studia Logica Library, vol. 7. Kluwer Academic Publishers, Dordrecht (2000)
5. Esteva, F., Godo, L., Noguera, C.: A logical approach to fuzzy truth hedges. Inf. Sci. **232**, 366–385 (2013)

6. Fan, J., Xie, W., Pei, J.: Subsethood measure: new definitions. Fuzzy Sets Syst. **106**, 201–209 (1999)
7. Hájek, P.: On very true. Fuzzy Sets Syst. **124**(3), 329–333 (2001)
8. Kitainik, L.M.: Fuzzy inclusions and fuzzy dichotomous decision procedures. In: Kacprzyk, J., Orlovski, S.A. (eds.) Optimization Models Using Fuzzy Sets and Possibility Theory. Theory and Decision Library, vol. 4, pp. 154–170. Springer, Netherlands, Dordrecht (1987). https://doi.org/10.1007/978-94-009-3869-4_11
9. Madrid, N., Ojeda-Aciego, M.: A view of f-indexes of inclusion under different axiomatic definitions of fuzzy inclusion. In: Moral, S., Pivert, O., Sánchez, D., Marín, N. (eds.) SUM 2017. LNCS (LNAI), vol. 10564, pp. 307–318. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-67582-4_22
10. Madrid, N., Ojeda-Aciego, M.: Functional degrees of inclusion and similarity between L-fuzzy sets. Fuzzy Sets Syst. **390**, 1–22 (2020)
11. Madrid, N., Ojeda-Aciego, M.: The f-index of inclusion as optimal adjoint pair for fuzzy modus ponens. Fuzzy Sets Syst. **466**, 108474 (2023)
12. Madrid, N., Ojeda-Aciego, M., Perfilieva, I.: f-inclusion indexes between fuzzy sets. In: Alonso, J.M., et al. (eds.) Proceedings of IFSA-EUSFLAT 2015, AISR 89, Atlantis Press, pp. 1528–1533 (2015)
13. Monteiro, A.: Sur les algèbres de Heyting simétriques. Portugal. Math. **39**, 1–237 (1980)
14. Sinha, D., Dougherty, E.: Fuzzification of set inclusion: theory and applications. Fuzzy Sets Syst. **55**, 15–42 (1993)
15. Vychodil, V.: Truth-depressing hedges and BL-logic. Fuzzy Sets Syst. **157**(15), 2074–2090 (2006)
16. Young, V.: Fuzzy subsethood. Fuzzy Sets Syst. **77**, 371–384 (1996)