On extending fuzzy preorders to sets and their corresponding strict orders

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Abstract. In this paper we first consider the problem of extending a fuzzy (weak) preorder on a set W to a fuzzy relation (preorder) on subsets of W, and consider different possibilities using different forms of quantification. For each of them we propose possible definitions of corresponding indistinguishability and strict preorder relations associated to the initial preorder, both on W and on its power set $\mathcal{P}(W)$. We compare them and we study conditions under which the strict relation is transitive.

Keywords: fuzzy preorder, indistinguishability relation, strict fuzzy order, reflexive, irreflexive, anti-symmetric, $\odot$-transitive relation

1 Introduction

In the classical setting, from a preorder $\leqslant$ on a universe W we can define:

- an equivalence relation $\equiv$, where $x \equiv y$ if $x \leqslant y$ and $y \leqslant x$,
- a strict order $<$, where $x < y$ if $x \leqslant y$ and $y \nleqslant x$.

Observe that, so defined, these relations satisfy the condition $x \leqslant y$ iff $x \equiv y$ or $x < y$, so rougly speaking we can say that $\leqslant$ is the union of $\equiv$ and $<$.

An interesting topic is how we can obtain relations on the power set $\mathcal{P}(\mathcal{W})$ from a preorder on the universe W. With a logic-based approach (using quantifiers), there are six ways of doing such an extension, see for example [1, 11].

Definition 1. *Given a set W together with a preorder $\leqslant$, one can define the following six relations on $\mathcal{P}(\mathcal{W})$:*

- $A \leqslant_{\exists\exists} B$ *iff there exist* $u \in A$ *and* $v \in B$ *such that* $u \leqslant v$
- $A \leqslant_{\exists\forall} B$ *iff there exists* $u \in A$*, such that for all* $v \in B$, $u \leqslant v$
- $A \leqslant_{\forall\exists} B$ *iff for all* $u \in A$*, there exists* $v \in B$ *such that* $u \leqslant v$
- $A \leqslant_{\forall\forall} B$ *iff for all* $u \in A$ *and* $v \in B$*, then* $u \leqslant v$
- $A \leqslant_{\exists\forall 2} B$ *iff there exists* $v \in B$ *such that, for all* $u \in A$, $u \leqslant v$
- $A \leqslant_{\forall\exists 2} B$ *iff for all* $v \in B$*, there exists* $u \in A$ *such that* $u \leqslant v$

Notice that additional different preorders over subsets can be obtained as combination of the previously defined relations. As an example take a totally pre-ordered set $(W, \leqslant)$ and suppose we want to extend the preorder in W to an ordering on the set of intervals of W. Two very usual preorders on intervals are the following ones:

(i) $[a, b] \leqslant_1 [c, d]$ when $a \leqslant c$ and $b \leqslant d$,
(ii) $[a, b] \leqslant_2 [c, d]$ when $b \leqslant c$.

The relation $\leqslant_1$ coincides with the intersection of the relations $\leqslant_{\forall\exists}$ and $\leqslant_{\forall\exists 2}$, while the second, $\leqslant_2$, directly coincides with the relation $\leqslant_{\forall\forall}$. Observe that, strictly speaking, $\leqslant_{\forall\forall}$ is not a preorder because it is only reflexive for singletons.

The above six relations can be compared with respect to set inclusion.

Proposition 1. *[1, 5] The following inclusions hold:*

$$\leqslant_{\forall\forall} \subseteq \leqslant_{\forall\exists} \subseteq \leqslant_{\exists\exists}, \ \leqslant_{\forall\forall} \subseteq \leqslant_{\exists\forall} \subseteq \leqslant_{\exists\exists}, \ \leqslant_{\forall\forall} \subseteq \leqslant_{\forall\exists 2} \subseteq \leqslant_{\exists\exists}, \ \leqslant_{\forall\forall} \subseteq \leqslant_{\exists\forall 2} \subseteq \leqslant_{\exists\exists}$$

Moreover, the four intermediate relations are not comparable, except for the following inclusions:

$$\leqslant_{\exists\forall 2} \subseteq \leqslant_{\forall\exists}, \quad \leqslant_{\exists\forall} \subseteq \leqslant_{\forall\exists 2} .$$

In this paper we cope with the case where the initial preorder is fuzzy, as a follow-up of our previous papers [5, 6]. After this brief introduction, in Section 2 we recall different forms of extending a fuzzy preorder on a set W to fuzzy relations on the set $\mathcal{P}(\mathrm{W})$ of subsets of W, in a similar way to classical preorders. In Section 3 we consider the problem of defining an indistinguishability relation and a strict fuzzy order in a set from a given fuzzy preorder, while in Section 4 we deal with the problem of how to lift the strict fuzzy order to subsets. In this sense we have used and recovered some results in [2, 3] and in [7–10] and focus on the transitivity property of the strict fuzzy orders in both settings. The paper ends with some conclusions and comments on further research.

2 Extending a fuzzy preorder on a set W to a fuzzy relation on $\mathcal{P}(\mathcal{W})$

Let $\odot$ be a t-norm. In this section we study the extension of a fuzzy $\odot$-preorder on a set W to a relation on $\mathcal{P}(\mathcal{W})$. Remember that a fuzzy $\odot$-preorder is a relation $\leqslant: \mathrm{W} \times \mathrm{W} \longrightarrow [0, 1]$ satisfying reflexivity, i.e., $[u \leqslant u] = 1$ for all $u \in \mathrm{W}$, and $\odot$-transitivity, i.e., for all $u, v, w \in \mathrm{W}$, $[u \leqslant v] \odot [v \leqslant w] \leqslant [u \leqslant w]$, where $[u \leqslant v]$ denotes the value in $[0, 1]$ of the fuzzy relation $\leqslant$ applied to the ordered pair of elements $u, v \in \mathrm{W}$. Moreover we will assume that W is a finite set, and we will denote by δ_u the singleton $\{u\}$.

Generalizing the classical case, in [5] we have introduced the following fuzzy relations on $\mathcal{P}(\mathrm{W})$ (using inf and sup to interpret the universal and existential quantifiers).

Definition 2. *Given a fuzzy relation $\leqslant$ on* W*, we can define the following six fuzzy relations on* $\mathcal{P}(\mathrm{W})$ *by letting, for any* $A, B \in \mathcal{P}(\mathrm{W})$*:*

- $[A \leqslant_{\exists\exists} B] = \sup_{u \in A} \sup_{v \in B} [u \leqslant v]$
- $[A \leqslant_{\exists\forall} B] = \sup_{u \in A} \inf_{v \in B} [u \leqslant v]$
- $[A \leqslant_{\forall\exists} B] = \inf_{u \in A} \sup_{v \in B} [u \leqslant v]$
- $[A \leqslant_{\forall\forall} B] = \inf_{u \in A} \inf_{v \in B} [u \leqslant v]$
- $[A \leqslant_{\forall\exists 2} B] = \inf_{v \in B} \sup_{u \in A} [u \leqslant v]$
- $[A \leqslant_{\exists\forall 2} B] = \sup_{v \in B} \inf_{u \in A} [u \leqslant v]$.

In the same paper, we have proved similar comparisons to the classical case for these six relations.

Proposition 2. *For any sets* $A, B \in \mathcal{P}(\mathrm{W})$*, we have:*

- $[A \leqslant_{\forall\forall} B] \leqslant [A \leqslant_{\forall\exists} B] \leqslant [A \leqslant_{\exists\exists} B]$,
- $[A \leqslant_{\forall\forall} B] \leqslant [A \leqslant_{\forall\exists 2} B] \leqslant [A \leqslant_{\exists\exists} B]$,
- $[A \leqslant_{\forall\forall} B] \leqslant [A \leqslant_{\exists\forall} B] \leqslant [A \leqslant_{\exists\exists} B]$, *and*
- $[A \leqslant_{\forall\forall} B] \leqslant [A \leqslant_{\exists\forall 2} B] \leqslant [A \leqslant_{\exists\exists} B]$.

Moreover the four intermediate relations are not comparable, except for the same two cases of Prop. 1 changing inclusions by inequalities.

Moreover, in [5] we have also given characteristic properties for each one of these relations. All of them are reflexive (at least for singletons) and transitive, i.e, they are very close to be fuzzy preorders. As a matter of example, we give next the characterization results for the relation $\forall\exists$. The other relations can be characterized in a similar way.

Proposition 3. *The relation* $\leqslant_{\forall\exists}$ *satisfies the following properties, for all* $A, B, C \in \mathcal{P}(\mathrm{W})$*:*

1. *Inclusion:* $[A \leqslant_{\forall\exists} B] = 1$, *if* $A \subseteq B$
2. $\odot$*-Transitivity:* $[A \leqslant_{\forall\exists} B] \odot [B \leqslant_{\forall\exists} C] \leqslant [A \leqslant_{\forall\exists} C]$
3. *Left-OR:* $[(A \cup B) \leqslant_{\forall\exists} C] = \min([A \leqslant_{\forall\exists} C], [B \leqslant_{\forall\exists} C])$
4. *Restricted Right-OR:* $[A \leqslant_{\forall\exists} (B \cup C)] \geqslant \max([A \leqslant_{\forall\exists} B], [A \leqslant_{\forall\exists} C])$. *The inequality becomes an equality if A is a singleton.*

Theorem 1. *Let* $\preceq_{AE}$ *be a relation between sets of* $\mathcal{P}(\mathrm{W})$ *satisfying Properties 1, 2, 3 and 4 of Prop. 3. Then there exists a fuzzy* $\odot$*-preorder* $\leqslant$ *on the set* W *such that* $\preceq_{AE}$ *coincides with* $\leqslant_{\forall\exists}$ *as defined in Def. 2.*

3 About the decomposition of a fuzzy preorder and its associated strict fuzzy order

In this section we recall from [5] a possible generalization of the decomposition of a crisp preorder given in the introduction to the case that the preorder be fuzzy, and we prove new results about the $\odot$-transitivity of the strict associated order.

In the fuzzy setting (see for example [2, 7]), from a fuzzy $\odot$-preorder $\leqslant: W \times W \to [0, 1]$ we can define:

– the maximal indistinguishability relation $\equiv$ contained in the fuzzy preorder, defined by $[x \equiv y] = [x \leqslant y] \wedge [y \leqslant x]$;
– the minimal strict fuzzy $\odot$-order $<$ that satisfies the following equation

$$[x \leqslant y] = [x < y] \oplus [x \equiv y] \tag{1}$$

where $\oplus$ is a t-conorm (for example the maximum or the bounded sum).

So defined, the relation $\equiv$ is reflexive, symmetric and $\odot$-transitive, and thus it is a $\odot$-indistinguishability relation.

On the other hand, regarding (1), the minimal solution for b of the equation $a \leqslant b \oplus c$ in $[0, 1]$, is the so-called *dual residuated implication*, or implication associated to the t-conorm $\oplus$, which is defined as,

$$a \to^{\oplus} c = \inf\{b \mid a \oplus b \geqslant c\}.$$

Therefore, one can define the strict fuzzy order relation $<_{\oplus}$ associated to $\leqslant$ and to the t-conorm $\oplus$ as the fuzzy relation defined as

$$[x <_{\oplus} y] = [x \equiv y] \to^{\oplus} [x \leqslant y] = [y \leqslant x] \to^{\oplus} [x \leqslant y].$$

The following are the particular expressions of $[x <_{\oplus} y]$ for the three most prominent examples of $\oplus$.

(i) An easy computation shows that the strict fuzzy order relation for $\oplus = \max$ is defined as

$$[x <_{\max} y] = \begin{cases} [x \leqslant y], & \text{if } [x \leqslant y] > [y \leqslant x], \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

(ii) For $\oplus$ being the bounded sum (i.e. Łukasiewicz t-conorm), the corresponding strict fuzzy order is:[1]

$$[x <_{\oplus} y] = \left\{ \begin{array}{ll} [x \leqslant y] - [y \leqslant x], & \text{if } [x \leqslant y] > [y \leqslant x] \\ 0, & \text{otherwise} \end{array} \right\} = \max([x \leqslant y] - [y \leqslant x], 0)$$

(iii) And for $\oplus$ being the probabilistic sum (i.e. the dual of product t-norm by the negation $N(x) = 1 - x$), we have:

$$[x <_{\oplus} y] = \begin{cases} \frac{[x \leqslant y] - [y \leqslant x]}{1 - [y \leqslant x]}, & \text{if } [x \leqslant y] > [y \leqslant x], \\ 0, & \text{otherwise.} \end{cases}$$

It is well known (see, for example [8]) that the strict relation $<_{\oplus}$ obtained by the dual residuated implication satisfies the following form $\oplus$-transitivity:

$$[x <_{\oplus} y] \oplus [y <_{\oplus} z] \geqslant [x <_{\oplus} z].$$

But in general it is not $\odot$-transitive, even in cases where $\odot$ is a continuous t-norm and $\oplus = \max$, as the following examples show.

[1] This is the strict order companion defined and studied in [7].

Example 1. Take a set $A = \{u, v, w\}$ and let $\odot$ be either the Łukassiewicz or product t-norm. Let $a, b, c, d \in (0, 1]$, with $a > c, b > d$ and such that $a \odot b = c \odot d > 0$. Suppose now $\leqslant$ is a fuzzy preorder on A defined by reflexivity plus $[u \leqslant v] = a > b = [v \leqslant u], [v \leqslant w] = c > d = [w \leqslant v]$ and $[u \leqslant w] = a \odot b = [w \leqslant u]$. This relation is transitive if $a \odot a \odot c \leqslant d$ and $a \odot c \odot c \leqslant b$ (for example if $a = c = 0.9$ and $b = d = 0.8$). Then it is obvious that the strict relation w.r.t. $\oplus = \max$ is defined as $[u <_{\max} v] = a, [v <_{\max} w] = b$ and $[u <_{\max} w] = 0 < a \odot b = [u <_{\max} v] \odot [v <_{\max} w]$.

Example 2. Take a set $A = \{u, v, w\}$ and let $\odot$ be the t-norm which is the ordinal sum of a copy of Łukasiewicz t-norm plus a copy of an arbitrary continuous t-norm, with e being the idempotent element separating the two components. Let $a, b, c, d \in (e, 1]$, with $a > c, b > d$ and such that $a \odot b = c \odot d = e$. Suppose now $\leqslant$ is a fuzzy preorder on A defined by reflexivity plus the conditions $[u \leqslant v] = a > c = [v \leqslant u], [v \leqslant w] = b > d = [w \leqslant v]$ and $[u \leqslant w] = a \odot b = e = c \odot d = [w \leqslant u]$. Then it is obvious that the strict relation w.r.t. $\oplus = \max$ is defined as $[u <_{\max} v] = a, [v <_{\max} w] = b$ and $[u <_{\max} w] = 0 < e = a \odot b = [u <_{\max} v] \odot [v <_{\max} w]$.

Nevertheless, as we show in the next proposition, we have the following positive results for the cases: (i) $\odot = \min$ and $\oplus = \max$ and (ii) $\odot$ and $\oplus$ being Łukasiewicz t-norm and t-conorm respectively. The case $\odot = \min$ and $\oplus =$ Łukasiewicz t-conorm is already proven in [7, Theorem 15].

Proposition 4. *Let $\leqslant$ be a $\odot$-preorder on a universe* W *and let $<_\oplus$ be the associated strict relation w.r.t. $\oplus$. Then*

(i) $<_{\max}$ *is* min*-transitive.*
(ii) If $\odot$ and $\oplus$ are Łukasiewicz t-norm and t-conorm, then $<_\oplus$ is $\odot$-transitive.

Proof. To show (i), i.e. to show that $\min([u <_{\max} v], [v <_{\max} w]) \leqslant [u <_{\max} w])$, it is enough to check:

(1) $\min([u <_{\max} v], [v <_{\max} w]) \leqslant \min([u \leqslant v], [v \leqslant w]) \leqslant [u \leqslant w]$,

(2) if $[u <_{\max} v] = [u \leqslant v]$ and $[v <_{\max} w] = [v \leqslant w]$ then $[u <_{\max} w] = [u \leqslant w]$.

On the one hand, (1) holds since we assume $\leqslant$ is min-transitive. We will prove (2) by contradiction. Suppose there exist elements $u, v, w \in W$ such that $[u <_{\max} v] > 0, [v <_{\max} w] > 0$ and $[u <_{\max} w] = 0$ which is equivalent that $[u \leqslant v] = a > b = [v \leqslant u]$, $[v \leqslant w] = c > d = [w \leqslant v]$ and $[u \leqslant w] = [w \leqslant u] = f$. Thus we have five values a, b, c, d, f and we know that

$$a > b \text{ and } c > d. \qquad (*)$$

We can now reason by cases:

(1) Suppose $a \geqslant c$ and $b \geqslant d$. Combining this assumption with (*) we have that $a \geqslant c > d$. By transitivity, $f \geqslant \min(a, c) = c$ and $f \geqslant \min(d, b) = d$ by hypothesis. Moreover $\min([w \leqslant u], [u \leqslant v]) = \min(f, a) \leqslant d = [w \leqslant v]$. This implies that $a \leqslant d$, in contradiction with the fact that $d < a$.

(2) Suppose $a \geqslant c$ and $b < d$. Combining this assumption with (*) we have that $d < c \leqslant a$. By transitivity, $f \geqslant \min(a, c) = c$ and $f \geqslant \min(d, b) = b$ by hypothesis. Moreover $\min([w \leqslant u], [u \leqslant v]) = \min(f, a) \leqslant d = [w \leqslant v]$. This implies that $f \leqslant d$, and by hypothesis $f \leqslant d < c$, in contradiction with $f \geqslant c$ previously proved.
(3) Suppose $a \leqslant c$ and $b \geqslant d$. Combining this assumption with (*) we have that $b < a \leqslant c$. By transitivity, $f \geqslant \min(a, c) = a$ and $f \geqslant \min(d, b) = d$ by hypothesis. Moreover $\min([v \leqslant w], [w \leqslant u]) = \min(c, f) \leqslant b = [v \leqslant u]$. This imply that $f \leqslant b$ and by hypothesis $f \leqslant b < a$, in contradiction with $f \geqslant a$ previously proved.
(4) Suppose $a \leqslant c$ and $b \leqslant d$. Combining this assumption with (*) we have that $b < a \leqslant c$. By transitivity, $f \geqslant \min(a, c) = a$ and $f \geqslant \min(d, b) = b$ by hypothesis. Moreover $\min([v \leqslant w], [w \leqslant u]) = \min(c, f) \leqslant b = [v \leqslant u]$. This implies that $f \leqslant b$, and by hypothesis $f \leqslant b < a$, in contradiction with $f \geqslant a$ previously proved.

Now we will prove (ii). For all $u, v, w \in W$, suppose that $[u \leqslant v] = a, [v \leqslant w] = b, [v \leqslant u] = c, [w \leqslant v] = d, [u \leqslant w] = e, [w \leqslant u] = f$. We have to prove transitivity of $<_{\oplus}$ in case that $a > c, b > d$ and $e > f$. The other cases are obviously transitive. Then the associated strict relation $<_{\oplus}$ contains only the pairs, $[u <_{\oplus} v] = a - c, [v <_{\oplus} w] = b - d, [u <_{\oplus} w] = e - f$. Then $<_{\oplus}$ is Łukasiewicz transitive if $(a - c) \odot (b - d) \leqslant (e - f)$. We have two cases:

- If $a - c \leqslant 1 - (b - d)$ then $(a - c) \odot (b - d) = 0$, and thus the inequality holds.
- Otherwise $(a - c) \odot (b - d) = ((a - c) + (b - d)) - 1 = (a + b) - 1 - (c + d)$. Since $\leqslant$ is Łukasiewicz transitive, then $a \odot b \leqslant e$ and $d \odot c \leqslant f$. Therefore $(a - c) \odot (b - d) \leqslant e - (c + d) \leqslant e - (c + d - 1) \leqslant e - f$.

□

Related results about the transitivity of the strict relation associated to a fuzzy preorder can be found in [2, 3, 7–10].

4 Extending the decomposition to fuzzy relations on the power set of the universe

In this section we are interested in how to define a strict fuzzy order relation on sets of $\mathcal{P}(\mathrm{W})$ induced by a fuzzy preorder in W. Halpern notices in [11] that there are two different methods to define (in the crisp case) a strict relation on $\mathcal{P}(\mathrm{W})$ from a preorder on W. The extensions to the fuzzy case are straightforward and give us the following definitions (where $\leqslant_\circ$ denotes one of the six relations $\leqslant_{\exists\exists}$, $\leqslant_{\exists\forall}$, $\leqslant_{\exists\forall 2}$, $\leqslant_{\forall\exists}$, $\leqslant_{\forall\exists 2}$ or $\leqslant_{\forall\forall}$):

- The *standard method*, that amounts to define

$$[A <_\circ^{st} B] = \begin{cases} [A \leqslant_\circ^{st} B], & \text{if } [A \leqslant_\circ B] > [B \leqslant_\circ A] \\ 0, & \text{otherwise .} \end{cases}$$

This means in fact to define $[A <^{st}_{\circ} B]$ as the value of the strict order associated to the preorder $\leqslant_\circ$, .
- The *alternative method*, that first considers the strict order $<$ on W, and then defines $<^{alt}_{\circ}$ on $\mathcal{P}(\mathrm{W})$ according to Definition 2, but replacing $\leqslant$ by $<$.

In general, these two methods give rise to *two different* irreflexive and (restricted) antisymmetric strict relations. Nevertheless the following inequality always holds: $[A <^{alt}_{\circ} B] \leqslant [A \leqslant_\circ B]$. Therefore, by definition, if $[A \leqslant_\circ B] > [B \leqslant_\circ A]$ (when $[A <^{st}_{\circ} B] \neq 0$), then $[A <^{alt}_{\circ} B] \leqslant [A \leqslant^{st}_{\circ} B]$ holds as well. But different possibilities arise depending on the quantified extension of the original preorder on W as studied below.

Proposition 5. *Given a fuzzy preorder $\leqslant$ on a universe W, let $\leqslant_\circ$ be the induced relation on $\mathcal{P}(\mathcal{W})$ (where $\circ \in \{\exists\exists, \exists\forall, \forall\exists, \exists\forall 2, \forall\exists 2, \forall\forall\}$). Then the strict relations $<^{st}_{\circ}$ and $<^{alt}_{\circ}$ obtained by using standard and alternative method respectively are not comparable in general, but the following relationships hold:*

(i) Let $\circ = \exists\exists$. Then, for all $A, B \in \mathcal{P}(\mathcal{W})$, $[A <^{st}_{\exists\exists} B] \leqslant [A <^{alt}_{\exists\exists} B]$. In fact we have:
$$\begin{cases} [A <^{alt}_{\exists\exists} B] = [A <^{st}_{\exists\exists} B], & \text{if } [A <^{st}_{\exists\exists} B] \neq 0 \\ [A <^{alt}_{\exists\exists} B] \geqslant [A <^{st}_{\exists\exists} B], & \text{otherwise} \end{cases}$$
Moreover there have examples where $[A <^{st}_{\exists\exists} B] = 0$ and $[A <^{alt}_{\exists\exists} B] > 0$.

(ii) Let $\circ = \forall\forall$. Then, for all $A, B \in \mathcal{P}(\mathcal{W})$, $[A <^{st}_{\forall\forall} B] \geqslant [A <^{alt}_{\forall\forall} B]$. In fact we have:
$$\begin{cases} [A <^{alt}_{\forall\forall} B] \leqslant [A <^{st}_{\forall\forall} B], & \text{if } [A <^{st}_{\forall\forall} B] \neq 0 \\ [A <^{alt}_{\forall\forall} B] = [A <^{st}_{\forall\forall} B] = 0, & \text{otherwise} \end{cases}$$
Moreover there are examples where $[A <^{alt}_{\forall\forall} B] < [A <^{st}_{\forall\forall} B]$.

(iii) For all intermediate cases, i.e. when $\circ \in \{\exists\forall, \forall\exists, \exists\forall 2, \forall\exists 2l\}$, the values of $[A <^{alt}_{\circ} B]$ and $[A <^{st}_{\circ} B]$ are incomparable in general.

Proof. We prove (i) by cases, where for simplicity we will write $\leqslant$ instead of $\leqslant_{\exists\exists}$:

- Suppose that $[A <^{st} B] = [A \leqslant B] \neq 0$, equivalent to $[A \leqslant B] > [B \leqslant A]$. Therefore there exists $u_1 \in A, v_1 \in B$ such that $[u_1 \leqslant v_1] = \inf_{u \in A, v \in B}[u \leqslant v] > \inf_{v \in B, u \in A}[v \leqslant u]$, which implies that $[u_1 \leqslant v_1] > [v_1 \leqslant u_1]$. Therefore $[u_1 < v_1] = [u_1 \leqslant v_1]$, and so an easy computation proves that $[A <^{alt} B] = [u_1 < v_1] = [A \leqslant B] = [A <^{st} B]$.
- Suppose $[A <^{st} B] = 0$ due to the fact that $[A \leqslant B] \leqslant [B \leqslant A]$. In such a case it is possible that $[A <^{alt} B] > 0$, as the following example shows:
  Take $A = \{u_1, u_2\}, B = \{v_1, v_2\}$ and let $\leqslant$ be the relation that is reflexive and contains the following pairs, $[u_1 \leqslant v_1] = a \leqslant b = [v_2 \leqslant u_2]$ with $a \neq 0$. Then $[A \leqslant B] = a \leqslant b = [B \leqslant A]$ and so $[A <^{st} B] = 0$, but one can check that $[A <^{alt} B] = a > 0$.

We also prove item (ii) by cases, and as before, we will write $\leqslant$ instead of $\leqslant_{\forall\forall}$ for simplicity:

- Suppose that $[A <^{st} B] = 0$ due to the fact that $[A \leqslant B] \leqslant [B \leqslant A]$. In such a case there exists $u_1 \in A, v_1 \in B$ such that $[u_1 \leqslant v_1] = \inf_{u \in A, v \in B}[u \leqslant v] \leqslant \inf_{v \in B, u \in A}[v \leqslant u]$, which implies that $[u_1 \leqslant v_1] \leqslant [v_1 \leqslant u_1]$. Therefore $[u_1 < v_1] = 0$ and thus $[A <^{alt} B] = 0 = [A <^{st} B]$.
- Suppose that $[A <^{st} B] = [A \leqslant B]$. In such a case the following example shows that there exist cases where $[A <^{alt} B] < [A <^{st} B]$:
  Take $\mathrm{W} = \{w_1, w_2\}$ with the preorder defined by reflexivity plus $[w_1 \leqslant w_2] = 1$. Further, take $A = \{w_1\}$ and $B = \mathrm{W}$. Then it is obvious that $[A \leqslant B] = 1 > 0 = [B \leqslant A]$. Therefore we have $[A <^{st} B] = 1$, while $[A <^{alt} B] = \inf_{u \in A} \sup_{v \in B}[u < v] = 0$, since $[u < u] = 0$.

As for item (iii), and unlike the previous cases where we have shown that $<^{st}_{\circ}$ and $<^{alt}_{\circ}$ are comparable for $\circ \in \{\exists\exists, \forall\forall\}$, we will show the incomparability of the relations for the intermediate cases. Actually, we will prove it for the case $\circ = \forall\exists$, but similar results can be obtained for the other intermediate cases. As above, in the rest of the proof we will write $\leqslant$ instead of $\leqslant_{\forall\exists}$ for simplicity.

a) First we give an example where $[A <^{alt} B] < [A <^{st} B]$. Take $A = \{u_1, u_2\}$, $B = \{v_1, v_2\}$ with the following fuzzy preorder: reflexivity ($[x \leqslant x] = 1$) plus $[u_1 \leqslant v_1] = [v_1 \leqslant u_1] = a$ and $[u_2 \leqslant v_2] = b$, with $a, b \neq 0$. The associated strict relation on W is the one having only $[u_2 < v_2] = b$. Let $A = \{u_1, u_2\}$ and let $B = \{u_3, u_4\}$. Then it is clear that $[A \leqslant B] = a \wedge b > 0 = [B \leqslant A]$ and thus , by definition, $[A <^{st} B] = a \wedge b \neq 0$. Finally $[A <^{alt} B] = \inf_{u \in A} \sup_{v \in B}[u < v] = 0$. Thus $[A <^{alt} B] < [A <^{st} B]$.

b) Finally we give an example where $[A <^{alt} B] > [A <^{st} B]$.
Take the same sets as in the previous example and the relation defined by reflexivity plus $[u_1 \leqslant v_1] = [u_2 \leqslant v_1] = [u_1 \leqslant u_2] = a$, $[v_1 \leqslant u_2] = [u_2 \leqslant v_2] = b$ and $[v_2 \leqslant v_1] = c$ where $0 < a < b < c$. Then it is easy to compute that $[A \leqslant_{\forall\exists} B] = a < b = [B \leqslant_{\forall\exists} A]$ and thus $[A <^{st}_{\forall\exists} B] = 0$ while $[A \leqslant^{alt}_{\forall\exists} B] = \min([u_1 < v_1], [u_2 < v_1]) = a$. □

Notice that if the strict order on W is $\odot$-transitive, so are the strict relations obtained by the alternative method (they are strict orders), but this is not clear for strict relations obtained by the standard method. In fact we have the following open problems:

- Let $\leqslant$ be a fuzzy preorder on W and let $\leqslant_{\circ}$ be one of the fuzzy preorders defined on $\mathcal{P}(\mathrm{W})$ considered in the previous sections. Is the strict relation obtained by the standard method $\odot$-transitive?
- It is obvious that the strict order $<$ on W and the strict order on $\mathcal{P}(\mathrm{W})$ obtained from the preorder by the standard method satisfies the following anti-symmetry property: for all $A, B \in \mathcal{P}(\mathrm{W})$, $\min([A <_{\circ} B], [B <_{\circ} A]) = 0)$. It is clear that for singletons the strict order obtained by the alternative method satisfies the same anti-symmetry property but, is this true for the strict order obtained by the alternative method in general? Otherwise, what type of anti-symmetry property does it satisfy?

Therefore, as far as we are interested in obtaining strict fuzzy orders (irreflexive and $\odot$-transitive relations), it seems reasonable to consider the strict relations obtained by the *alternative method* from a strict order over W and its characteristics properties. Next theorem provides a characterization result for these strict orders. Like for the relations associated to a fuzzy preorder, we give the characterization for the case of $\forall\exists$. The other cases can be characterized in a similar way.

Theorem 2. *Let $\prec_{AE}$ be a relation on $\mathcal{P}(\mathrm{W})$ satisfying Properties 2, 3 and 4 of Prop. 3 plus irreflexivity ($[A \prec_{AE} A] = 0$) and restricted antisymmetry ($\min([A \prec_{AE} B], [B \prec_{AE} A]) = 0$ for all singletons $A, B \in \mathcal{P}(\mathrm{W})$). Then there exists a strict fuzzy order $<$ on the set W such that $\prec_{AE}$ coincides with the strict fuzzy order associated to $\leqslant_{\forall\exists}$ obtained by the alternative method.*

5 Conclusions and further research

In this paper we have explored (crisply quantified) extensions of fuzzy preorders on a universe W to relations on $\mathcal{P}(\mathcal{W})$. Moreover, extending [5, 6], we have further studied the decomposition of a fuzzy preorder in an indistinguishability relation and a strict (order) relation as a generalization of the well known decomposition in the crisp case.

As for future work, we are interested in applications to preference modelling and reasoning, see [5] for some initial ideas in this direction. Moreover we plan the use of fuzzy quantifications like "nearly all" or "someone", etc. to obtain new extensions of the initial preorder to relations on $\mathcal{P}(\mathcal{W})$. This seems to be a challenging topic, specially related to applications.

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Dedication This paper is our humble homage to the memory of Pedro Gil. Excellent researcher and better person, he has been one of the pioneers of fuzzy sets in Spain and founder and driving force of the research group on fuzzy sets at the University of Oviedo. Our contribution is devoted to fuzzy preorders and their decomposition, a subject that was very close to the research interests of Pedro Gil. Along many years, we jointly participated in many events and we have enjoyed his friendship and shared many unforgettable moments.

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