A fuzzy logic-based approach to reason with inconsistent probabilistic theories

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***Abstract*—In this paper we consider the probability logic over Rational Pavelka logic (RPL), denoted FP(RPL), and we explore two possible approaches to reason from inconsistent FP(RPL) theories in a non-trivial way. The first one amounts to replace the logic RPL, that is explosive, by its paraconsistent degree-preserving companion RPL$^{\leq}$. The second one consists of suitably weakening the formulas in an inconsistent theory T, depending on the degree of inconsistency of T.**

***Index Terms*—Probabilistic reasoning; Łukasiewicz fuzzy logic; Paraconsistent reasoning models; Inconsistency measures**

I. Introduction

Reasoning about probability can be properly handled in a fuzzy logical setting by expanding the language of Łukasiewicz fuzzy logic with a unary modality P and interpreting, for every classical formula φ, the modal formula $P\varphi$ as "φ is probable". Clearly, $P\varphi$ is a fuzzy proposition, whose truth-degree can be taken as the probability of φ. More precisely, the fuzzy modal logic FP(Ł), as firstly introduced in [4] and improved in [3], extends the language of Łukasiewicz logic Ł by the unary modal operator P that applies only to classical propositions and uses the ground logic Ł to express the basic properties of a probability function (in particular the finite additivity). Very recently, in [1] the authors have studied in depth the relationship of this fuzzy logic-based approach to more traditional probability logics after Halpern et al. see e.g. [5]. In this paper we will rather consider the probability logic FP(RPL) over Rational Pavelka logic (RPL), the expansion of Łukasiewicz fuzzy logic with rational truth-constants.

In this paper we explore two possible approaches to reason from inconsistent FP(RPL) theories in a non-trivial way. The first one amounts to replace the external logic RPL, that is explosive, by its paraconsistent companion RPL$^{\leq}$. The second one amounts to suitably weaken formulas of an inconsistent theory T depending on the degree of inconsistency of T.

II. The probability logics FP(Ł) and FP(RPL)

Based on a first formalisation in [4], the probability logic FP(Ł) (FP for Fuzzy Probability) was introduced in [3] and defined as a sort of modal extension with a unary operator P over the well-known Łukasiewicz fuzzy logic Ł (see e.g. [3] for details on both Ł and FP(Ł) logics). FP(Ł) allows for reasoning about the probability of classical propositions.

Let L denote the language of classical propositional logic (CPL) built from a countable set V of propositional variables using the classical binary connectives $\wedge$ and $\neg$ Then, the language of FP(Ł) is defined as follows. *Formulas* of FP(Ł) are of two types:

(1) Non-modal: they are the classical logic formulas of L and will be denoted by lower case Greek letters $\varphi, \psi, \ldots$.

(2) Modal: they are built from basic modal formulas of the form $P\varphi$, where $\varphi \in \mathsf{L}$ using the connectives of Ł $(\rightarrow_L, \neg)$, and denoted by upper case Greek letters $\Phi, \Psi, \ldots$.

This is a two-layer language, neither nested modalities nor formulas combining non-modal and modal subformulas are not allowed.

Axioms and rules of FP(Ł) are as follows:

(CPL) Axioms and rules of CPL for non-modal formulas;
(Ł) Axioms and rules of Ł for modal formulas;
(P) Axioms and rules for the modality P:
- (P1) $P(\varphi \rightarrow \psi) \rightarrow_L (P(\varphi) \rightarrow_L P(\psi))$
- (P2) $P(\neg\varphi) \leftrightarrow_L \neg P(\varphi)$
- (P3) $P(\varphi \vee \psi) \leftrightarrow_L [P(\varphi) \oplus (P(\psi) \ominus P(\varphi \wedge \psi))]$[1]
- (Nec) if $\vdash_{CPL} \varphi$, derive $P(\varphi)$.

Models of FP(Ł) are probability Kripke structures $K = \langle W, e, \mu\rangle$, where: W is a non-empty set of possible worlds; $e : V \times W \rightarrow \{0, 1\}$ provides for each world a *Boolean* (two-valued) evaluation of the proposition variables, that is, $e(p, w) \in \{0, 1\}$ for each propositional variable $p \in Var$ and each world $w \in W$; and $\mu : 2^W \rightarrow [0, 1]$ is a finitely additive probability measure on a Boolean algebra of subsets of W such that for each p, the set $\{w \mid e(p, w) = 1\}$ is measurable (cf. [3] 8.4.1). A truth-evaluation e is extended to non-modal formulas in the classical way, to elementary modal formulas as follows:

$$e(P\varphi, w) = \mu(\{w' \in W \mid e(\varphi, w') = 1\}),$$

and to compound modal formulas by using the truth-functions of Ł logic. Actually $e(P\varphi, w)$ does not depend on w and we will write $e(\Phi)$.[2] We will also denote by e_μ the truth-evaluation on modal formulas determined by the model $\langle \Omega, e, \mu\rangle$, where Ω is the set of classical models for L.

[1] Recall that $\Phi \oplus \Psi := \neg\Phi \rightarrow_L \Psi$ and $\Phi \ominus \Psi := \neg(\Phi \rightarrow_L \Psi)$.

[2] Recall $e(\Phi \rightarrow_L \Psi) = \min(1 - e(\Phi) + e(\Psi), 0)$, $e(\neg\Phi) = 1 - e(\Phi)$, $e(\Phi \oplus \Psi) = \min(e(\Phi) + e(\Psi), 1)$ and $e(\Phi \ominus \Psi) = \max(e(\Phi) - e(\Psi), 0)$.

Soundness and completeness of the logic FP(Ł) w.r.t. to the class of probability Kripke models reads as follows: if $T \cup \{\Phi\}$ is a finite set of modal FP(Ł)-formulas, then T proves Φ in FP(Ł), written $T \vdash_{FP} \Phi$, iff for any probability μ on Ω, $e_\mu(\Phi) = 1$ whenever $e_\mu(\Psi) = 1$ for all $\Psi \in T$.

If one wants to formalise reasoning with numeric probabilistic expressions, then one has to replace in FP(Ł) the (external) logic Ł by its expansion with rational truth-constants, the so-called Rational Pavelka logic (RPL for short). So we add to the language of Ł a rational truth constant $\overline{r}$ for every rational $r \in [0, 1]$. As Hájek shows [3], Rational Pavelka logic can be then axiomatized by adding to the axioms Łukasiewicz the following bookkepping axioms:

$$\text{(BK)} \quad \overline{r} \to \overline{s} \equiv \overline{\min(1, 1 - r + s)}$$

for any rational numbers $r, s \in [0, 1]$. The resulting probability logic, FP(RPL), inherits the soundness and completeness results from FP(Ł), where now FP(RPL)-evaluations e_μ are further equired to correctly interpret the truth-constants, that is, $e(\overline{r}) = r$ for every rational $r \in [0, 1]$.

III. Dealing with inconsistent FP(RPL) theories

As the probability logic FP(RPL) is grounded on the RPL logic, the latter being explosive, the logic FP(RPL) is explosive as well. This means that, for any formula Φ in the language of FP(RPL), $\{\Phi, \neg\Phi\} \vdash_{FP} \bot$, and thus $\{\Phi, \neg\Phi\} \vdash_{FP} \Psi$ for any Ψ. Our contribution consists in presenting two approaches to escape the explosion principle in FP(RPL) and to handle inconsistent probabilistic theories in a non-trivial way, briefly introduced below.

A. A paraconsistent probability logic

The first approach consists in replacing RPL by its "degree preserving companion", denoted by RPL$^{\leq}$. Conforming to the usual way of defining deductions in degree-preserving logics, given two modal formulas we define Φ and Ψ, $\Phi \vdash_{FP^{\leq}} \Psi$ iff for every probabilistic Kripke model $\mathcal{M} = (W, e, \mu)$ of FP$^{\leq}(RPL)$, $\|\Phi\|_{\mathcal{M}} \leq \|\Psi\|_{\mathcal{M}}$. This generalises to the more general case in which $T = \{\Phi_1, \ldots, \Phi_n\}$ is any finite set of modal formulas by defining $T \vdash_{FP^{\leq}} \Psi$ iff for all probabilistic Kripke model $\mathcal{M}$,

$$\|\Phi_1 \wedge \ldots \wedge \Phi_n\|_{\mathcal{M}} \leq \|\Psi\|_{\mathcal{M}}.$$

Let us notice that the logic FP$^{\leq}(RPL)$ is not explosive, and hence *paraconsistent*. Indeed, for each classical formula φ that is neither a classical theorem nor a contradiction, $P(\varphi), \neg P(\varphi) \not\vdash_{FP^{\leq}} \bot$ because, semantically, one can find a probability μ that assigns $\mu(\varphi) = 1/2$ and this gives

$$\min\{\mu(\varphi), \mu(\neg\varphi)\} = 1/2 > 0.$$

B. An inconsistency-tolerant probabilistic logic

Recall that, from a semantical point of view, the logic $FP(Ł)$ is defined as follows: for any set of FP(Ł)-formulas $T \cup \{\Phi\}$, $T \models_{FP} \Phi$ if, for every probability μ on Boolean formulas, if μ is a model of T then $e_\mu(\Phi) = 1$, where by μ being a model of T we mean that $e_\mu(\Psi) = 1$ for every $\Psi \in T$. We will denote by $\|T\|$ the set probability measures on formulas that are models of T.

Of course, the above definition trivializes in the case T is inconsistent, i.e. when $\|T\| = \emptyset$. But in FP(Ł) one can take advantage of its fuzzy setting and consider the notion of (in)consistency as being fuzzy as well. Indeed, even if T has no models, a situation where, for every probability μ there is always a formula Φ in T such that $e_\mu(\Phi) = 0$, is qualitatively different from a situation where there is a probability μ such that $e_\mu(\Phi) \geq \alpha$ for all $\Phi \in T$, for some value α close to 1. In the former case T is clearly inconsistent, while in the latter case one could say that T is close to being consistent.

This observation justifies to define, for each threshold α, the set of *α-generalised models* of T as follows:

$$\|T\|_\alpha = \{\text{probability } \mu \mid \text{for all } \Psi \in T, e_\mu(\Psi) \geq \alpha\}.$$

Note that the set $\|T\|_1$ coincides with the set of usual models of T. Moreover $\|T\|_\alpha$ is a convex set of probabilities.

Definition 3.1: Let T be a theory of $FP(Ł)$. The consistency degree of T is defined as $Con(T) = \sup\{\beta \in [0, 1] \mid \|T\|_\beta \neq \emptyset\}$. Dually, the inconsistency degree of T is defined as $Incon(T) = 1 - Con(T) = \inf\{1 - \beta \in [0, 1] \mid \|T\|_\beta \neq \emptyset\}$.[3]

The idea we explore in this paper is to use α-generalised models instead of usual models to define a context-dependent inconsistent-tolerant notion of probabilistic entailment.

Definition 3.2: Let T be a theory such that $Con(T) = \alpha > 0$. We define: $T \models\!\approx^* \Phi$ if $e_\mu(\Phi) = 1$ for all probabilities $\mu \in \|T\|_\alpha$.

Note that if $Con(T) > 0$, then $T \not\models\!\approx^* \bot$, hence $\models\!\approx^*$ does not trivialize even if T is inconsistent ($Con(T) < 1$). As an example, if $T = \{P\varphi \leftrightarrow_L \overline{0.4}, P\varphi \leftrightarrow_L \overline{0.3}\}$, that is inconsistent, then $Con(T) = 0.95$ and $T \models\!\approx^* \overline{0.35} \leftrightarrow_L P\varphi$.

The following are some interesting properties of the consequence relation $\models\!\approx^*$: clearly, $\models\!\approx^*$ is not monotonic, while $\models\!\approx^*$ is idempotent, that is, if $S \models\!\approx^* \varphi$ and $T \models\!\approx^* \psi$ for all $\psi \in S$, then $T \models\!\approx^* \varphi$.

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[3] $Incon(\cdot)$ can be seen as a particular case of distance-based inconsistency mesure, see e.g. [2].