A modal logic for uncertainty: a completeness theorem

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Abstract

In the present paper, we axiomatize a logic that allows a general approach for reasoning about probability functions, belief functions, lower probabilities and their corresponding duals. The formal setting we consider arises from combining a modal $S5$ necessity operator $\Box$ that applies to the formulas of the infinite-valued Łukasiewicz logic with the unary modality P that describes the behaviour of probability functions. The modality P together with an S5 modality $\Box$ provides a language rich enough to characterise probability, belief and lower probability theories. For this logic, we provide an axiomatization and we prove that, once we restrict to suitable sublanguages, it turns out to be sound and complete with respect to belief functions and lower probability models.

Keywords: Fuzzy logic, Dempster-Shafer belief functions, probability functions, imprecise probabilities, modal logic.

1. Introduction

The relationship between modal logics, fuzzy logics and uncertainty measures is not new. In [21, 17, 19], see also [14] for a survey, probability functions are defined via a fuzzy modal operator P applied on classical propositional formulas. Thus, the probability of a boolean formula φ is taken to be the truth degree of the fuzzy proposition $P\varphi$ = "φ is probable". Remarkably, this modal fuzzy approach to probability has been proved in [2] to be equivalent to the possibly better known setting proposed and studied by Fagin, Halpern and Megiddo in [11]. The same approach has been then generalized to represent Dempster-Shafer belief functions and in [18] the belief degree of classical boolean φ is the truth degree of the modal formula $B\varphi = P\Box\varphi$, where $\Box$ is an S5 modality. In [24, 25], lower and upper probabilities have been formalized in a similar way. Furthermore, these setting have been also generalized to deal with nonclassical events in [13, 15, 12].

In the recent short paper [9], the authors propose an approach to deal with several uncertainty theories within a unique and general logical language that, in addition to the previously recalled modality P, also contains an additional S5 modal operator $\Box$. In the same paper [9], the problem of determining an axiomatization for that general logic was left as open. In the present paper we approach that issue showing an axiomatization for our logic. More precisely, the language proposed in the aforementioned paper, in addition to the probability formulas of the form $P(\varphi)$, was claimed to allow expressing "belief function formulas" by combining P and $\Box$ as $P(\Box\varphi)$ and "lower probability formulas" as $\Box P(\varphi)$. Although belief function formulas as the above were already considered in the literature (see [18] for instance), the models presented in [9] are slightly more general as they also allow to interpret lower probability formulas.

In the present paper we show that if we restrict to belief function formulas, our logic is sound and complete with respect to belief function models, while if we restrict to lower probability formulas, the same logic is complete with respect to lower probability evaluations. As the former will be a direct consequence of the completeness theorem shown in [18], the latter is entirely new. Having a unique logic to deal with uncertainty theories, and with belief functions and lower probability in particular, is interesting also in light of the result presented in [8] showing that in some non trivial situations, these two uncertainty measures cannot be distinguished. Therefore, having a common ground on where we can have a comparative analysis of these two theories, can pave the way to future interesting results. A first step towards such comparative analysis will be presented in the last part of the present document.

The rest of this paper is organized as follows: In the next Section 2 we will briefly recall our logical and measuretheoretical settings. Section 3 is dedicated recall the setting of [9]. There, we will hence define the measure-based and relational-based models that allow to represent uncertainty measures as a whole. In Section 4 we will axiomatize the logic S5(FP(Ł)) and show the promised completeness theorems. In Section 5 we will show a first comparative analysis and we will end with Section 6 in which we briefly present the next step we intend to go through.

2. Modal logics and uncertainty measures

2.1. Logical preliminaries

The propositional logic on which our approach is grounded is the [0, 1]-valued Łukasiewicz calculus. Let us briefly recall that Łukasiewicz infinite-valued logic Ł is a fuzzy logic, in the sense of [19], whose algebraic semantics is the variety of *MV-algebras*. Those are structures of the form $\mathbf{A} = (A, \oplus, \neg, 1)$ where A is a nonempty set, $\oplus$ is a binary and $\neg$ a unary operation on A, while 1 is a constant. Thus, an algebra in that signature is an MV-algebra iff $(A, \oplus, 1)$ is a commutative monoid, and the following equations are satisfied: $\neg\neg x = x$; $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$. These algebras form a variety, denoted by $\mathbb{MV}$, that is the equivalent algebraic semantics for Ł. This implies, among other things, that formulas of Łukasiewicz logic can be regarded as terms in the language of MV-algebras and we will henceforth use this convention without danger of confusion. $\mathbb{MV}$ is generated, both as a variety and a quasivariety by the so called *standard MV-algebra*, $[0, 1]_{MV} = ([0, 1], \oplus, \neg, 1)$ where $[0, 1]$ denotes the real unit interval, and for all $a, b \in [0, 1]$, $a \oplus b = \min\{1, a + b\}$ and $\neg a = 1 - a$.

As shown by Chang in [7], Łukasiewicz logic, whose axiomatization can be found in [19] (see also [26]) turns out to be sound and complete with respect to evaluations to $[0, 1]_{MV}$, i.e., with respect to maps e from the variables to $[0, 1]$ that interpret connectives by the operations of $[0, 1]_{MV}$ recalled above. The notion of *theorem* ($\vdash \varphi$) and *deduction from a theory* ($T \vdash \varphi$) are defined as usual in algebraic logic.

Theorem 1 *For every finite set of formulas $T \cup \{\varphi\}$ in the Łukasiewicz logic language, $T \vdash \varphi$ iff for every evaluation e into $[0, 1]_{MV}$ that maps all $\psi \in T$ to 1, $e(\varphi) = 1$ as well.*

The logic Ł features not only the strong disjunction $\oplus$ and negation $\neg$, but also a weak conjunction $\wedge$, a non-idempotent strong conjunction denoted by &, a weak disjunction $\vee$ and an implication $\to$. The standard semantics of the above-mentioned connectives is given by extending the evaluations e on [0, 1] as follows:

$$
\begin{aligned}
e(\varphi \oplus \psi) &= \min\{1, e(\varphi) + e(\psi)\};\\
e(\neg\varphi) &= 1 - e(\varphi);\\
e(\varphi \wedge \psi) &= \min\{e(\varphi), e(\psi)\};\\
e(\varphi \& \psi) &= \max\{0, e(\varphi) + e(\psi) - 1\};\\
e(\varphi \vee \psi) &= \max\{e(\varphi), e(\psi)\};\\
e(\varphi \to \psi) &= \min\{1 - e(\varphi) + e(\psi), 1\}.
\end{aligned}
$$

Łukasiewicz infinite-valued logic, together with modus ponens as inference rule, is axiomatized by the following set of axioms:

$(L1)$ $\varphi \to (\psi \to \varphi)$;
$(L2)$ $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;
$(L3)$ $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$;
$(L4)$ $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

Let us now recall some basic concepts of standard (classical) modal logic. The language of classical propositional logic, built up from a countable set of propositional variables Var, is enriched by a modal operator $\Box$ and its dual $\Diamond = \neg\Box\neg$. A *Kripke model* for the logic $S5$ is a structure $\mathcal{K} = (W, R, \{e_w\}_{w\in W})$, where W is a non-empty set of *worlds* (or states), $R \subseteq W \times W$ is the *accessibility relation* that is assumed to be an equivalence relation, and for every $w \in W$ e_w is a classical *evaluation* $e_w : Var \longrightarrow \{0, 1\}$ that assigns a truth value to each propositional variable in each state. The evaluations extend to classical formulas as usual. For a formula ϕ and world w we denote by $\|\phi\|_{\mathcal{K},w}$ the truth-value of ϕ in $\mathcal{K}$ at w. In particular for propositional formulas φ we have:

$$\|\varphi\|_{\mathcal{K},w} = e_w(\varphi).$$

For atomic modal formulas $\Box\varphi$ we have:

$$\|\Box\varphi\|_{\mathcal{K},w} = 1 \text{ iff for each } w' \in W,\ wRw' \text{ implies } e_{w'}(\varphi) = 1.$$

Truth-values of compound modal formulas are computed by truth-functionality.

The axioms of $S5$ are the following:

(CPL) Axioms of classical propositional logic;
(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
(T) $\Box\varphi \to \varphi$;
(4) $\Box\varphi \to \Box\Box\varphi$;
(B) $\varphi \to \Box\Diamond\varphi$.

and the inference rules are modus ponens and necessitation for $\Box$ (from φ derive $\Box\varphi$).

In recent years the community of fuzzy logicians has put forward several attempts to extend modal logic from the classical to the many-valued setting, and the approach followed by that community is mainly semantic-based. For instance, in [20] Hájek proposed a generalization of the modal logic S5 to the Łukasiewicz many-valued setting by defining its relational semantics as Kripke-like models of the form $\mathcal{K} = (W, R, \{e_w\}_{w\in W})$ where W is nonempty, $R = W \times W$ is the total relation on W, and for every $w \in W$, e_w evaluates Łukasiewicz (non-modal) formulas into the standard MV-algebra $[0, 1]_{MV}$ with their corresponding truth-functions, while the truth value of a modal formula $\Box\varphi$ at $w \in W$ is computed as follows:

$$\|\Box\varphi\|_{\mathcal{K},w} = \inf\{e_{w'}(\varphi) \mid wRw'\} = \inf\{e_{w'}(\varphi) \mid w' \in W\}.$$

We will henceforth denote the class of these models by $\mathcal{M}^t$, t for total. He shows (see also [5]) that the following Hilbert-style calculus, denoted by S5Ł and based on a language that expands that of Ł by a unary modality $\Box$ and its dual $\Diamond = \neg\Box\neg$, is sound and complete w.r.t. the above-defined class of relational models $\mathcal{M}^t$:

($\Box$1) $\Box\varphi \to \varphi$;
($\Box$2) $\Box(\psi \to \varphi) \to (\psi \to \Box\varphi)$;
($\Box$3) $\Box(\psi \vee \varphi) \to (\psi \vee \Box\varphi)$;
($\Diamond$1) $\Diamond(\varphi\&\varphi) \equiv \Diamond\varphi\&\Diamond\varphi$;
(MP) the modus ponens rule;
($N\Box$) the necessitation rule: from φ infer $\Box\varphi$.

Since extending Ł by the law of excluded middle,

$$(LEM)\ \varphi \vee \neg\varphi,$$

yields classical logic, if we add the axiom $\varphi \vee \neg\varphi$ to S5Ł then we obtain the classical modal logic S5 [4]. Moreover, analogously to the classical case, S5Ł turns out to be not only sound and complete w.r.t. the class $\mathcal{M}^t$ of Kripke models $(W, R, \{e_w\}_{w\in W})$ where $R = W \times W$, but also w.r.t. the class $\mathcal{M}^e$ of Kripke models $(W, R, \{e_w\}_{w\in W})$ in which $R \subseteq W \times W$ is an equivalence relation, i.e., it is reflexive, symmetric and transitive.

Definition 2 *$\mathcal{M}^e$ is the class of models $(W, R, \{e_w\}_{w\in W})$ where W is nonempty, R is an equivalence relation on W and for all $w \in W$, e_w is a $[0, 1]_{MV}$-evaluation of Łukasiewicz formulas.*

If ϕ is any formula in the language of S5Ł, $\mathcal{K} = (W, R, \{e_w\}_{w\in W}) \in \mathcal{M}^e$ and $w \in W$, the truth value of ϕ in $\mathcal{K}$ at w, denoted $\|\phi\|_{\mathcal{K},w}$, is defined as usual.

Lemma 3 *The classes $\mathcal{M}^t$ and $\mathcal{M}^e$ have the same tautologies.*

Proof Since every total relation on any W is in particular en equivalence relation, if ϕ is a tautology of $\mathcal{M}^e$, ϕ also is a tautology of $\mathcal{M}^t$, i.e. $\mathcal{M}^e \subseteq \mathcal{M}^t$. Conversely, assume that ϕ is not a tautology of $\mathcal{M}^e$, i.e., assume there exists $\mathcal{K} = (W, R, \{e_w\}_{w\in W}) \in \mathcal{M}^e$ and $w \in W$ such that $\|\phi\|_{\mathcal{K},w} < 1$. Since R is an equivalence relation, the restriction of R to $R[w] = \{w' \in W \mid wRw'\}$ is total. Thus, take $W' = R[w]$ (where w is the same as above), $R' = W' \times W'$ and for all $w^* \in W'$, e_{w*} is as in $\mathcal{K}$. Call $\mathcal{K}' = (W', R', \{e_{w^*}\}_{w^*\in W'})$. Since $w \in W'$, $\|\phi\|_{\mathcal{K}',w} = \|\phi\|_{\mathcal{K},w} < 1$ (easy to see by induction). It then follows that ϕ is not a tautology of $\mathcal{M}^t$ and $\mathcal{M}^t \subseteq \mathcal{M}^e$. ■

Thus, the following immediately follows.

Theorem 4 ([6]) *The logic S5(Ł) is sound and finitely strong complete with respect to the class $\mathcal{M}^e$. In other words, for every finite set of formulas $T \cup \{\varphi\}$, $T \vdash_{S5(Ł)} \phi$ iff $\|\phi\|_{\mathcal{K},w} = 1$ for all $\mathcal{K} \in \mathcal{M}^e$ such that $\|\tau\|_{\mathcal{K},w} = 1$ for all $\tau \in T$.*

2.2. Uncertainty Measures

As we assume the reader to be familiar with finitely additive probability measures (for otherwise, see for instance [22]), this section is meant to recall more general measures for uncertainty, namely belief functions [27] and lower probabilities [28]. Along the whole paper we will always work on finite boolean algebras. Here below we just recall the definitions of belief functions, possibility and necessity measures and lower probabilities.

Definition 5 (Belief functions [27]) *A* belief function *on a boolean algebra* **A** *is a* $[0, 1]$*-valued map* $B : \mathbf{A} \to [0, 1]$ *satisfying:*

(B1) $B(\top) = 1$, $B(\bot) = 0$;

(B2) $$B\left(\bigvee_{i=1}^{n} \psi_i\right) \geq \sum_{i=1}^{n} \sum_{\{J\subseteq\{1,\ldots,n\}:|J|=i\}} (-1)^{i+1} B\left(\bigwedge_{j\in J} \psi_j\right),$$ *for* $n \in \mathbb{N}$.

An element φ of a boolean algebra **A** is said to be *covered m times by a multiset $\{\{\psi_1, \ldots, \psi_n\}\}$ of elements of A* if every homomorphism of **A** to $\{0, 1\}$ that maps φ to 1, also maps to 1 at least m propositions from $\psi_1, \ldots, \psi_n$ as well. An (m, k)-cover of $(\varphi, \top)$ is a multiset $\{\{\psi_1, \ldots, \psi_n\}\}$ that covers $\top$ k times and covers φ $n + k$ times.

Definition 6 (Possibility and necessity measures [10]) *A* possibility measure *on a Boolean algebra* **A** *is a map* $\Pi : \mathbf{A} \to [0, 1]$ *such that*

(Π1) $\Pi(\top) = 1$, $\Pi(\bot) = 0$;

(Π2) $\Pi(\psi_1 \vee \psi_2) = \max\{\Pi(\psi_1), \Pi(\psi_2)\}$ *for all* $\psi_1, \psi_2 \in \mathbf{A}$.

A necessity measure *is the dual notion of a possibility measure, and it is a map* $N : \mathbf{A} \to [0, 1]$ *such that*

(N1) $N(\top) = 1$, $N(\bot) = 0$*;*

(N2) $N(\psi_1 \wedge \psi_2) = \min\{N(\psi_1), N(\psi_2)\}$ *for all* $\psi_1, \psi_2 \in \mathbf{A}$.

Definition 7 (Lower Probability functions [28]) *A lower probability on a boolean algebra* $\mathbf{A}$ *is a monotone* $[0,1]$*-valued map* $\underline{P} : \mathbf{A} \to [0,1]$ *satisfying:*

(L1) $\underline{P}(\top) = 1$, $\underline{P}(\bot) = 0$*;*

(L2) For all natural numbers n, m, k *and all* $\psi_1, \ldots, \psi_n$*, if* $\{\{\psi_1, \ldots, \psi_n\}\}$ *is an* (m,k)*-cover of* $(\varphi, \top)$*, then* $k + m\underline{P}(\varphi) \geq \sum_{i=1}^{n} \underline{P}(\psi_i)$.

Although this definition does not make the name *lower probabilities* particularly obvious, [1, Theorem 1] puts forward the following enlightening characterisation, anticipated by [28]. Let $\underline{P} : \mathbf{A} \to [0,1]$ be a lower probability, let $\mathbb{P}$ be the set of all probability measures on $\mathbf{A}$, and denote with $\mathcal{M}(\underline{P})$ the set of probability functions which bound $\underline{P}$ from above, i.e.

$$\mathcal{M}(\underline{P}) = \{P \in \mathbb{P} \mid \underline{P}(\psi) \leq P(\psi),\ \forall \psi \in A\}. \quad (1)$$

Then, for all $\psi \in A$,

$$\underline{P}(\psi) = \inf_{P \in \mathcal{M}(\underline{P})} P(\psi).$$

In addition, whenever the algebra $\mathbf{A}$ is finite, by [23, Lemma A3], for every $\psi \in A$, there exists a probability function $P_\psi : A \to [0,1]$ such that $\underline{P}(\psi) = P_\psi(\psi)$. Therefore, and recalling that A is finite, $\underline{P}$ can be actually defined as a local minimum. In other words, one can find a set $\mathcal{P}'$ of probability functions $P' : A \to [0,1]$ such that, for all

$$\underline{P}(\psi) = \min_{P' \in \mathcal{P}'} P'(\psi).$$

This fact just recalled, will be used in Lemma 16 below.

Interestingly, belief functions and necessity measures can be seen as lower probabilities. Indeed, if B is a belief function then, considering the set

$$\mathcal{M}(B) = \{P \in \mathbb{P} \mid B(\psi) \leq P(\psi),\ \forall \psi \in A\},$$

it holds that the map $\eta : \mathbf{A} \to [0,1]$ such that, for all $\psi \in A$,

$$\eta(\psi) = \min_{P \in \mathcal{M}(B)} P(\psi)$$

is a belief function that coincides with B, see e.g. [29]. Analogously, as it has been proved in [16], if N is a necessity measure on $\mathbf{A}$ and we consider (1) above applied to N as follows,

$$\mathcal{M}(N) = \{P \in \mathbb{P} \mid N(\psi) \leq P(\psi),\ \forall \psi \in A\},$$

then the map $\eta : \mathbf{A} \to [0,1]$ such that, for all $\psi \in A$,

$$\eta(\psi) = \min_{N \in \mathcal{M}(N)} P(\psi)$$

is a necessity measure coinciding with N.

Remark 8 *From an axiomatic point of view, lower probabilities are more general measures than belief functions and necessity measures. In fact, a lower probability* $\underline{P}$ *on an algebra* $\mathbf{A}$ *is a belief function iff* $\underline{P}$ *satisfies (B2), namely*

$$\underline{P}\left(\bigvee_{i=1}^{n} \psi_i\right) \geq \sum_{i=1}^{n} \sum_{\{J \subseteq \{1,\ldots,n\} : |J| = i\}} (-1)^{i+1} \underline{P}\left(\bigwedge_{j \in J} \psi_j\right) \quad (2)$$

for all $n = 1, 2, \ldots$*. And* $\underline{P}$ *is necessity measure iff* $\underline{P}$ *satisfies*

$$\underline{P}(\varphi \wedge \psi) = \min(\underline{P}(\varphi), \underline{P}(\psi)) \quad (3)$$

and in that case, (2) is also satisfied, in accordance with the fact that necessity measures are a particular subclass of belief functions.

3. A Unified Logic for Uncertainty

Let us start from $\mathscr{L}$, the language of Łukasiewicz logic over finitely many (say n) propositional variables, and let the language $\mathscr{L}_{\Box,P}$ be the expansion of $\mathscr{L}$ by two additional unary modalities: $\Box$ and P. We will mainly focus in the following subclasses of formulas of $\mathscr{L}_{\Box,P}$:

CF: The set of *classical formulas*. Those are definable in $\mathscr{L}$ from variables, constants $\top$ and $\bot$, and the connectives $\wedge$, $\vee$, $\neg$.

CMF: The set of *classical modal formulas* is defined by closing **CF** by the unary modality $\Box$ as usual in a modal language. Notice that in **CMF**, for every $\varphi \in$ **CMF**, $\Diamond\varphi$ stands for $\neg\Box\neg\varphi$.

PMF: The set of *probabilistic modal formulas* is obtained by the following two steps:

1. *Atomic probabilistic formulas* are all those in the form $P(\varphi)$ for $\varphi \in$ **CMF**;
2. *Compound probabilistic formulas* are defined by composing atomic ones with connectives of the Łukasiewicz language.

UMF: Finally, the set of *uncertainty modal formulas* is the smallest set of formulas that contains **PMF** and is closed under $\Box$ and connectives of Łukasiewicz logic.

Remark 9 *In order to clarify with what formulas we are dealing with, in the following table, for each class of formulas, we show some example.*

CF	$\varphi, \psi, \varphi \wedge \psi$
CMF	$\Box\varphi, \varphi \to \Box\psi, \Box\varphi \wedge \Diamond\psi, \Box(\Box\varphi \wedge \Diamond\psi)$
PMF	$P(\varphi \to \Box\psi)$, $P(\varphi) \to P(\Box\psi)$
UMF	$\Box P(\varphi)$, $\Box P(\varphi) \to P(\Diamond\psi)$, $\Box(P(\varphi) \to P(\Diamond\psi))$, $\Box(\Box P(\varphi) \to P(\Diamond\psi))$

Table 1: Example of formulas of the corresponding class.

Neither $\Box\varphi \to P(\psi)$, nor $P(P(\varphi) \to P(\Box\psi))$ are examples of probabilistic modal formulas. In fact, we ask no interaction between classical modal and probabilistic modal formulas and P cannot occur nested.

Note that, by definition, **CF** $\subseteq$ **CMF** and **PMF** $\subseteq$ **UMF**. In what follows we will be mainly interested in the following subclasses of formulas from **UMF**:

BF: the set of *belief function formulas* is the smallest subset of **UMF** that contains all basic formulas of the form $P(\Box\varphi)$ for every classical formula φ and that is closed under Łukasiewicz connectives. Examples of formulas in **BF** are $P\Box\varphi \to P\Box\psi$ or $\neg P\Box\varphi$.

LF: the set of *lower probability formulas* is the smallest subset of **UMF** that contains all basic formulas of the form $\Box P(\varphi)$ for every classical formula φ and that is closed under Łukasiewicz connectives. Examples of formulas in **LF** are $\Box P\varphi \to \Box P\psi$ or $\neg\Box P\varphi$.

Now, let us define a semantics for the language $\mathscr{L}_{\mathbf{UM}}$ = **CMF** $\cup$ **UMF** containing all the sets of formulas defined above.

Definition 10 *An* S5 probability model *is a tuple*

$$\mathcal{U} = (W, R, \{e_w\}_{w\in W}, \{p_w\}_{w\in W})$$

where:

1. *W is a non-empty countable set;*
2. $(W, R, \{e_w\}_{w\in W})$ *is a classical S5-Kripke model;*
3. *For all $w \in W$, p_w is a probability distribution on W.*

Given an S5 probability model $(W, R, \{e_w\}_{w\in W}, \{p_w\}_{w\in W})$, for every $w \in W$, we will sometimes denote by μ_w the probability function on the measurable subsets of W induced by the distribution p_w.

If $\mathcal{U} = (W, R, \{e_w\}_{w\in W}, \{p_w\}_{w\in W})$ is an S5 probability model, $w \in W$ is any world and φ is a **UMF** formula, we define the truth-value of φ in $\mathcal{U}$ at w (denoted $\|\varphi\|_{\mathcal{U},w}$) in the following way:

(1) If $\varphi \in$ **CF**, then $\|\varphi\|_{\mathcal{U},w} = e_w(\varphi)$;

(2) If $\varphi \in$ **CMF**, then whenever $\varphi = \Box\psi$ we have

$$\|\varphi\|_{\mathcal{U},w} = \|\Box\psi\|_{\mathcal{U},w} = \inf\{\|\psi\|_{\mathcal{U},w'} \mid wRw'\}.$$

If φ is compound, then $\|\varphi\|_{\mathcal{U},w}$ is computed by truth-functionality using classical connectives. Note that for any $\varphi \in$ **CMF**, $\|\varphi\|_{\mathcal{U},w} \in \{0, 1\}$.

(3) If $\varphi \in$ **PMF** and φ is atomic, i.e., $\varphi = P(\psi)$ with $\psi \in$ **CMF**, then

$$\|\varphi\|_{\mathcal{U},w} = \|P(\psi)\|_{\mathcal{U},w} = \sum\{p_w(w') \mid \|\psi\|_{\mathcal{U},w'} = 1\}.$$

If $\varphi \in$ **PMF** and is compound, then its truth-value is computed by truth-functionality using Łukasiewicz connectives.

(4) If $\varphi \in$ **UMF** and $\varphi = \Box\psi$, and thus with $\psi \in$ **PMF**, then

$$\|\varphi\|_{\mathcal{U},w} = \|\Box\psi\|_{\mathcal{U},w} = \inf\{\|\psi\|_{\mathcal{U},w'} \mid wRw'\}.$$

Again, if φ is compound then we will compute $\|\varphi\|_{\mathcal{U},w}$ by truth-functionality using Łukasiewicz connectives.

Let $T \cup \{\phi\}$ be a set of **UMF** formulas. We will write $T \models_{S5P} \phi$ if for every S5 probability model $\mathcal{U}$ and world w, $\|\tau\|_{\mathcal{U},w} = 1$ for every $\tau \in T$ implies that $\|\phi\|_{\mathcal{U},w} = 1$.

The cases (1) and (2) above are indeed as usual. Let us hence provide some examples for the possibly less clear cases (3) and (4) and precisely to clarify the interpretation of belief function and lower probability formulas.

Let us start with the case (3) and let $\varphi = P(\psi)$ be an atomic **PMF** formula, with $\psi \in$ **CF**. Then,

$$\|P(\psi)\|_{\mathcal{U},w} = \sum\{p_w(w') \mid \|\psi\|_{\mathcal{U},w'} = 1\}$$
$$= \sum\{p_w(w') \mid e_{w'}(\psi) = 1\}.$$

In other words $\|P(\psi)\|_{\mathcal{U},w}$ is the probability of ψ computed in w.

If $\varphi = P(\Box\psi)$ is a belief function formula from **BF**, so that $\psi \in$ **CF**, then $\|P(\Box\psi)\|_{\mathcal{U},w}$ is, by definition, $\sum\{p_w(w') \mid \|\Box\psi\|_{\mathcal{U},w'} = 1\}$. In this case, notice that $\|\Box\psi\|_{\mathcal{U},w'} = 1$ iff $\|\psi\|_{\mathcal{U},w''} = 1$ for all w'' such that $w'Rw''$. Therefore, if we denote by μ_w the probability function on 2^W induced by the distribution p_w,

$$\|P(\Box\psi)\|_{\mathcal{U},w} =$$
$$= \mu_w(\{w^* \in W \mid \forall w' \in W\ (w^*Rw' \Rightarrow \|\psi\|_{\mathcal{U},w'} = 1)\}).$$

If $\varphi = \Box P(\psi)$ is a lower probability formula from **LF** with $\psi \in$ **CF**, then $\|\Box P(\psi)\|_{\mathcal{U},w}$ is defined as follows.

$$\|\Box P(\psi)\|_{\mathcal{U},w} = \inf\{\|P(\psi)\|_{\mathcal{U},w'} \mid wRw'\}$$
$$= \inf\{\mu_{w'}(\{w^* \in W \mid \|\psi\|_{\mathcal{U},w^*} = 1\}) \mid wRw'\}.$$

Clearly, more complex formulas in **UMF** could be considered but we will not go into details.

Remark 11 (On compound modal formulas) *While we can regard basic modal formulas of the form $P(\varphi)$, $P(\Box\varphi)$, or $\Box P(\varphi)$ as logical representations of the uncertainty on the event φ specified under different theories, compound modal formulas can be thought as dealing with* meta-logical *properties about the uncertain quantification of (a class of) events. For instance, in the setting of our extended language, the compound formula $\Box P(\varphi) \to P(\Box\varphi)$ might be used to express the fact that the lower probability value of the event φ is less or equal than the belief function value of the same event.*

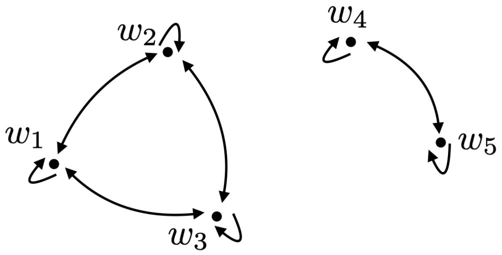

Figure 1: An S5 Kripke frame (W, R) on 5 possible worlds.

The following example is taken from [9].

Example 1 *Let us consider a language with three propositional variables p, q, r and the S5 Kripke model as in Figure 1. Let the evaluations e_w be as follows:*

$w_1 \models p, q, r$;
$w_2 \models p, \neg q, r$;
$w_3 \models \neg p, q, \neg r$;
$w_4 \models p, \neg q, \neg r$ *and*
$w_5 \models \neg p, \neg q, \neg r$.

Let us consider the following probability distributions.

	w_1	w_2	w_3	w_4	w_5
p_1	1/5	1/5	1/5	1/5	1/5
p_2	1/3	1/3	1/3	0	0
p_3	0	1/4	1/4	1/2	0
p_4	0	1/3	0	1/3	1/3
p_5	1/4	1/4	0	1/4	1/4

Thus, let $\mathcal{U} = (W, R, \{e_1, \ldots, e_5\}, \{p_1, \ldots, p_5\})$ and $\varphi = r \to (p \wedge q)$. Its models are w_1, w_3, w_4, w_5. Then, for every $i = 1, \ldots, 5$, it is easy to compute $\|P(\varphi)\|_{\mathcal{U},w_i}$ as $p_i(w_1) + p_i(w_3) + p_i(w_4) + p_i(w_5)$. For instance, $\|P(\varphi)\|_{\mathcal{U},w_1} = 4/5$ and $\|P(\varphi)\|_{\mathcal{U},w_2} = 2/3$.

Now, consider the belief function formula $P(\Box\varphi)$. In order to compute $\|P(\Box\varphi)\|_{\mathcal{U},w_i}$, we first need to notice that the models of $\Box\varphi$, are w_4 and w_5. Indeed, for all $i = 1, 2, 3$, $w_i R w_2$ and $w_2 \not\models \varphi$. Thus, for all $i = 1, \ldots, 5$,

$$\|P(\Box\varphi)\|_{\mathcal{U},w_i} = p_i(w_4) + p_i(w_5).$$

Thus, $\|P(\Box\varphi)\|_{\mathcal{U},w_1} = 2/5$, $\|P(\Box\varphi)\|_{\mathcal{U},w_2} = 0$ and $\|P(\Box\varphi)\|_{\mathcal{U},w_5} = 1/2$. Note that, for all $i = 1, \ldots, 5$, $\|P(\Box\varphi)\|_{\mathcal{U},w_i} \leq \|P(\varphi)\|_{\mathcal{U},w_i}$, i.e. the **PMF** *formula $P(\Box\varphi) \to P(\varphi)$ is valid in $\mathcal{U}$.*

Finally, let us consider the lower probability formula $\Box P(\varphi)$. In this case the $\Box$ is external to P, thus the above is an uncertain modal formula. For all $i = 1, \ldots, 5$,

$$\|\Box P(\varphi)\|_{\mathcal{U},w_i} = \min\{\|P(\varphi)\|_{\mathcal{U},w_j} \mid w_i R w_j\}.$$

For instance,

$$\|\Box P(\varphi)\|_{\mathcal{U},w_1} = \min\{\|P(\varphi)\|_{\mathcal{U},w_1}, \|P(\varphi)\|_{\mathcal{U},w_2}, \|P(\varphi)\|_{\mathcal{U},w_3}\} = \min\{4/5, 2/3, 3/4\} = 2/3.$$

Again, since R is reflexive in all S5 Kripke frames, one has that $\Box P(\varphi) \to P(\varphi)$ is valid in $\mathcal{U}$. From this observation, together with the fact that $x \to y$ and $z \to y$ imply, in Łukasiewicz logic, that $(x \wedge z) \to y$,

$$(P(\Box\varphi) \wedge \Box P(\varphi)) \to P(\varphi)$$

holds in $\mathcal{U}$. In the following table, we summarize the evaluations of $\|P(\varphi)\|_{\mathcal{U},w_i}$, $\|\Box\varphi\|_{\mathcal{U},w_i}$, $\|P(\Box\varphi)\|_{\mathcal{U},w_i}$, $\|\Box P(\varphi)\|_{\mathcal{U},w_i}$ computed on all worlds in the given model $\mathcal{U}$.

	w_1	w_2	w_3	w_4	w_5
$\|P(\varphi)\|_{\mathcal{U},w_i}$	4/5	2/3	3/4	2/3	3/4
$\|\Box\varphi\|_{\mathcal{U},w_i}$	0	0	0	1	1
$\|P(\Box\varphi)\|_{\mathcal{U},w_i}$	2/5	0	1/2	2/3	1/2
$\|\Box P(\varphi)\|_{\mathcal{U},w_i}$	2/3	2/3	2/3	2/3	2/3

In this example, $\|\Box P(\varphi)\|_{\mathcal{U},w_i}$, the modal formula representing the "lower probability degree of φ" is always greater or equal to $\|P(\Box\varphi)\|_{\mathcal{U},w_i}$, the modal formula standing for the "belief of φ". However, this is not always the case and by changing the probability distributions on W there are cases in which $\|\Box P(\varphi)\|_{\mathcal{U},w_i} \leq \|P(\Box\varphi)\|_{\mathcal{U},w_i}$.

4. The general logic S5(FP(Ł)): completeness results

In this section we introduce the logic S5(FP(Ł)) over the language $\mathscr{L}_{\mathbf{UM}}$ and prove some completeness results.

Definition 12 *The axioms of the logic S5(FP(Ł)) are the following:*

(CPL) *The axioms and rules of classical propositional logic for formulas in* **CF**.

(S5) *The axioms of S5 applied to* **CMF**.

(FP(Ł)) *Axioms and rules of FP(Ł) for* **PMF** *formulas, i.e. the axioms of Ł recalled in Section 2.1 and the axioms for the modality P and Łukasiewicz implication:*

$(P1)$ $P(\varphi \to \psi) \to (P(\varphi) \to P(\psi))$;
$(P2)$ $\neg P(\varphi) \equiv P(\neg\varphi)$;
$(P3)$ $P(\varphi \vee \psi) \equiv [(P(\varphi) \to P(\varphi \wedge \psi)) \to P(\psi)]$;
(NP) *necessitation: from φ infer $P(\varphi)$.*

(S5(Ł)) *Axioms and rules of S5(Ł) recalled in Section 2.1 for* **UMF** *formulas (the modal language of S5(Ł) built over FP(Ł)-formulas closed by Ł-connectives.)*

The axiom (P1), which can be read as "whenever both $\varphi \to \psi$ and φ are probable, also ψ is probable", is the axiom (K) for the modality P and Łukasiewicz implication. Axiom (P2) says that if φ is not probable, then $\neg\varphi$ is probable, and vice versa. Axiom (P3) expresses finite additivity.

The above axiom schema for FP(Ł), has already been shown to define a complete calculus for (finitely additive) probability functions [19, 14]. The interaction between P and an S5 $\Box$ in formulas like $P(\Box\varphi)$ gives a way to represent belief functions [18].

Definition 13 *A belief function model is a triple $(W, \{e_w\}_{w\in W}, B)$ where W is a nonempty set of worlds, e_w are classical evaluations for each $w \in W$ and B is a belief function on 2^W. A belief function model evaluates a basic belief function formula $P(\Box\varphi)$ as $\|P(\Box\varphi)\|_B = B(\{w \mid e_w(\varphi) = 1\})$, and extends to compound formulas in* **BF** *by truth-functionality using Łukasiewicz connectives.*

The models considered in [18] differ from ours as they include only one probability function. These are indeed systems $(W, \{e_w\}_{w\in W}, R, \mu)$ in which the truth value of a belief function formula like $P(\Box\varphi)$ is evaluated as $\|P(\Box\varphi)\| = \mu(\{w^* \mid \forall w' \in W\ (w^*Rw' \Rightarrow \|\varphi\|_{w'} = 1\})$ and hence it is independent of the specific world we are in. However, it is clear that each such model determines an S5 probability model by letting, for all $w, w' \in W$, $e_w = e_{w'}$ and $\mu_w = \mu_{w'}$. Therefore the following adaptation of the completeness theorem from [18] applies to our case. In the next statement, for a set $T \cup \{\phi\}$ of formulas, we will write $T \models_{BF} \phi$ if for all pair (W, B) where W is a nonempty set and B is a belief function on 2^W,

Theorem 14 *For every finite subset of formulas $T \cup \phi \subseteq$ (**BF**), $T \vdash_{S5(FP(Ł))} \phi$ iff for all belief model $(W, \{e_w\}_{w\in W}, B)$, $\|\tau\|_B = 1$ implies $\|\phi\|_B = 1$.*

In the following theorem we show that the axioms of S5(FP(Ł)) also capture lower probabilities once we restrict to modal formulas from (**LF**).

Definition 15 *A lower probability model is a triple $(W, \{e_w\}_{w\in W}, \underline{P})$ where W is a nonempty set of worlds, e_w are classical evaluations for each $w \in W$ and $\underline{P}$ is a lower probability on 2^W. A lower probability model evaluates a lower probability formula $\Box P(\varphi)$ as $\|\Box P(\varphi)\|_{\underline{P}} = \underline{P}(\{w \mid e_w(\varphi) = 1\})$, and extends to compound formulas in* **LF** *by truth-functionality using Łukasiewicz connectives.*

In the proof of the next result we will employ a usual translation of modal formulas for uncertainty to propositional Łukasiewicz language (see [14] for instance) and a recent result contained in [6] and precisely, the variant of finite strong standard completeness of S5(Ł) we showed in Theorem 4 above. Also, we will employ the next lemma showing that finite lower probability models and finite S5 probability model satisfy the same formulas in **LF**. In the lemma we assume we will work with lower probability models $\mathcal{M} = (W, \{e_w\}_{w\in W}, \underline{P})$ such that $e_w \neq e_{w'}$ whenever $w \neq w'$.

Lemma 16 *For every $\Phi \in$ **LF** and for every finite lower probability model $\mathcal{M} = (W, \{e_w\}_{w\in W}, \underline{P})$ there exists a finite S5 probability model $\mathcal{M}^0 = (W^0, R^0, \{e_{w^0}\}_{w^0\in W^0}, \{p_{w^0}\}_{w^0\in W^0})$ such that $\|\Phi\|_{\mathcal{M}} = \|\Phi\|_{\mathcal{M}^0}$.*

Proof Assume $W = \{w_1, \ldots, w_n\}$. We prove the claim for the case of $\Phi = \Box P(\varphi)$. Let us start recalling that $\|\Box P(\varphi)\|_{\mathcal{M}} = \underline{P}([\varphi]_W) = \underline{P}(\{w \in W \mid e_w(\varphi) = 1\})$. Denoting by **A** the finite algebra of subsets of W, by [23, Lemma A3] there exists a family of probability functions $\{\mu_a\}_{a\in A}$ on **A** such that, for each $a \in A$, $\underline{P}(a) = \mu_a(a)$. Therefore, for each $\varphi \in$ **CF**,

$$\underline{P}([\varphi]_W) = \mu_{[\varphi]_W}([\varphi]_W). \tag{4}$$

Now, let us define $W^0 = A = 2^W$, the set of subsets of W so that, for every classical formula ψ, $[\psi]_W \in W^0$. Moreover, the assumptions made right above this lemma ensures that for every $w^0 \in W^0$ there exists a classical formula ψ s.t. $[\psi]_W = w^0$. In what follows we list the elements of W^0 as $w^1_1, w^1_2, \ldots, w^1_n, \ldots, w^{2^n-1}_1, \ldots, w^{2^n-1}_n$, where the first n elements of this list correspond to the elements of W, i.e. for $k = 1, \ldots, n$, we take $w^1_k = \{w_k\}$. Moreover, for each $k = 1, \ldots, n$, we also take $e_{w^1_k} = e_{w_k}$. Finally, for every $[\psi]_W, w^i_k \in W^0$, we define

$$\mu_{[\psi]_W}(w^i_k) = \begin{cases} \mu_{[\psi]_W}(w_k) & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now, we have the following set of equalities:

$$\begin{aligned}
\|\Box P(\varphi)\|_{\mathcal{M}^0} &= \inf_{w^j_l \in W^0} \mu_{w^j_l}(\{w^i_k \in W^0 \mid e_{w^i_k}(\varphi) = 1\}) \\
&= \inf_{[\psi]_W \in W^0} \mu_{[\psi]_W}(\{w^i_k \in W^0 \mid e_{w^i_k}(\varphi) = 1\}) \\
&= \inf_{[\psi]_W \in W^0} \sum \{\mu_{[\psi]_W}(w^i_k) \mid e_{w^i_k}(\varphi) = 1\} \\
&= \inf_{[\psi]_W \in W^0} \sum \{\mu_{[\psi]_W}(w^1_k) \mid e_{w^1_k}(\varphi) = 1\} \\
&= \inf_{[\psi]_W \in W^0} \sum \{\mu_{[\psi]_W}(w_k) \mid e_{w_k}(\varphi) = 1\} \\
&= \inf_{[\psi]_W} \mu_{[\psi]_W}([\varphi]_W) \\
&= \mu_{[\varphi]_W}([\varphi]_W) \\
&= \underline{P}([\varphi]_W) \\
&= \|\Box P(\varphi)\|_{\mathcal{M}}
\end{aligned}$$

where the three-last equality follows from (4). ■

Now we can hence prove the desired completeness theorem.

Theorem 17 *For every finite subset of formulas $T \cup \phi \subseteq$ (**LF**), the following conditions are equivalent:*

(i) $T \vdash_{S5(FP(Ł))} \phi$

(ii) for all finite (universal) S5 probability model $\mathcal{U} = (W, R, \{e_w\}_{w\in W}, \{\mu_w\}_{w\in W})$ *with* $R = W \times W$, $\|\tau\|_{\mathcal{U}} = 1$ *for each* $\tau \in T$ *implies* $\|\phi\|_{\mathcal{U}} = 1$.

(iii) for all finite lower probability model $(W, \{e_w\}_{w\in W}, \underline{P})$, $\|\tau\|_{\underline{P}} = 1$ *for each* $\tau \in T$ *implies* $\|\phi\|_{\underline{P}} = 1$.

Proof $(i) \Rightarrow (ii)$ is soundness and it holds because all axioms and rules of $S5(FP(Ł))$ holds in every (universal) S5 probability model; $(ii) \Leftrightarrow (iii)$ is Lemma 16. It is hence left to prove that $(ii) \Rightarrow (i)$.

To this end, let $\circ$ be a translation from S5(FP(Ł))-formulas to S5(Ł)-formulas defined in the usual way, i.e. introducing a new propositional variable s_A for each FP(Ł)-formula A. Let Var° denote the new set of propositional variables in s_A's and $AxFP^\circ$ instances of FP(Ł). Thus, we have to prove that

$$T \vdash_{S5(FP(Ł))} \phi \text{ iff } T^\circ \cup AxFP^\circ \vdash_{S5(Ł)} \phi^\circ.$$

Since there are only finitely-many non-equivalent formulas in $AxFP^\circ$, by Theorem 4, we have that

$$T \vdash_{S5(FP(Ł))} \phi \text{ iff } T^\circ \cup AxFP^\circ \models_{S5(Ł)} \phi^\circ.$$

Suppose that T is finite and that $T \nvdash \phi$ in S5(FP(Ł)). Thus, $T^\circ \cup AxFP^\circ \not\models_{S5(Ł)} \phi^\circ$. Therefore, there exists an S5(Ł) universal model $M^c = (W^c, R^c, e^c)$, where $R^c = W \times W$ and $e^c : W^c \times Var^\circ \to [0, 1]$, that extends to compound S5(Ł)-formulas as usual, such that $\|T^\circ \cup AxFP^\circ\|_{M,w} = 1$ and $\|\phi^\circ\|_{M,w} < 1$.

Let V_0 be the (finite) set of propositional variables appearing in $T \cup \phi$, and let Ω be the set of $\{0, 1\}$-valued Boolean interpretations of the set of propositional formulas $\mathcal{L}_0$ built from V_0, i.e. $\Omega = \{\omega \mid \omega\colon V_0 \to \{0, 1\}\}$, which is finite, that are extended to $\mathcal{L}_0$ using Boolean truth-functions.

Now, for every $w \in W$, define $\mu^c_w : \mathcal{L}_0 \to [0, 1]$ by putting $\mu^c_w(\varphi) = \|s_{P\varphi}\|_{M,w}$. Since $\|AxFP^\circ\|_{M,w} = 1$, μ^c_w is a probability on $\mathcal{L}_0$-formulas. Therefore, for each $w \in W$, there is a probability distribution p_w on Ω such that $\mu_w(\varphi) := \sum\{p_w(\omega) \mid \omega \in \Omega,\ \omega(\varphi) = 1\} = \mu^c_w(\varphi)$, for all $\varphi \in \mathcal{L}_0$.

Now, let us define the finite S5 probability model U as follows:

$$U = (\Omega,\ R,\ \{e_\omega\}_{\omega\in\Omega},\ \{p_\omega\}_{\omega\in\Omega})$$

where:

$R = \Omega \times \Omega$,

$e_\omega : CF \longrightarrow \{0, 1\}$ is such that

$$e_\omega(v) = \begin{cases} \omega(v) & \text{if } v \in V_0, \\ 1 & \text{otherwise.} \end{cases}$$

Then, one can check that, for every $\varphi \in \mathcal{L}_0$, we have, for every $\omega \in \Omega, w \in W^c$,

$$\begin{aligned} \|\Box P\varphi\|_{U,\omega} &= \min\{p_\omega(\varphi) \mid \omega \in \Omega\} = \\ &= \min\{\mu^c_w(\varphi) \mid w \in W^c\} = \\ &= \|\Box P\varphi\|_{M^c,w}. \end{aligned}$$

Therefore, for every $\Psi \in T \cup \{\phi\}$, $\|\Psi\|_{U,\omega} = \|\Psi\|_{M^c,w}$. Thus, in particular, U is a model for T, while U does not satisfy ϕ. ■

By direct inspection on the above proof it is immediate to see that, at least for the formulas in **LF**, the logic S5(FP(Ł)) has the finite model property and hence it is decidable.

5. Recovering belief function and probability logics

As is well-known, belief functions (and probability and necessity) measures are particular examples of lower probabilities. For exemplifying purposes, we can show how one defines an axiomatic theory over the general logic S5(FP(Ł)) to capture reasoning about belief functions, probability and possibilistic necessity.

Let us consider the following recursive definition of the formulas $Ad(\Box P(\varphi_1 \vee \ldots \vee \varphi_k))$, for every $k \in \mathbb{N}$:

$k = 2$ $\quad Ad(\Box P(\varphi_1 \vee \varphi_2)) := \Box P\varphi_1 \oplus (\Box P\varphi_2 \ominus \Box P(\varphi_1 \wedge \varphi_2))$

$k = 3$ $\quad Ad(\Box P(\varphi_1 \vee \varphi_2 \vee \varphi_3)) := Ad(\Box P(\varphi_1 \vee \varphi_2)) \oplus (\Box P\varphi_3 \ominus Ad(\Box P((\varphi_1 \wedge \varphi_3) \vee (\varphi_2 \wedge \varphi_3)))$

...

$k = n$ $\quad Ad(\Box P(\varphi_1 \vee \ldots \vee \varphi_n)) := Ad(\Box P(\varphi_1 \vee \ldots \vee \varphi_{n-1})) \oplus [\Box P\varphi_n \ominus Ad(\Box P((\varphi_1 \wedge \varphi_n) \vee \ldots \vee (\varphi_{n-1} \wedge \varphi_n))]$

Now, we define the logic S5(FB) as the axiomatic extension of S5(FP(Ł)) with the following countable set of axioms:

$$AxBel = \{Ad(\Box P(\varphi_1 \vee \ldots \vee \varphi_k)) \to \Box P(\varphi_1 \vee \ldots \vee \varphi_k)\}_{k\in\mathbb{N}}$$

Lemma 18 *A S5 probability model* $\mathcal{U} = (W, R, \{e_w\}_{w\in W}, \{p_w\}_{w\in W})$ *satisfies the set of axioms* $AxBel$ *iff there is a belief function* bel *on* 2^W *such that, for every proposition* φ, $bel(\{w \in W \mid \|\varphi\|_{\mathcal{U},w} = 1\}) = \|\Box P\varphi\|_{\mathcal{U}}$.

Proof As we recalled in Remark 8, a lower probability is a belief function iff it satisfies the axiom (B2) and the schema $AxBel$ indeed encode (B2) in our logical language. Thus the claim follows. ■

By the above lemma it is hence immediate to show that the logic S5(FB) is complete with respect to belief function models as in Definition 13.

Theorem 19 *S5(FB) is strong finite complete wrt to belief function models.*

Furthermore, we can consider the logic S5(P) defined as the axiomatic extension of the logic S5(FP(Ł)) with the following axiom

$$Ad(\Box P(\varphi_1 \vee \varphi_2)) \equiv \Box P(\varphi_1 \vee \varphi_2)$$

Then, one can show that S5(P) is in fact a probabilistic logic whose S5 probability models are of the form $\mathcal{U} = (W, R, \{e_w\}_{w\in W}, \{p_w\}_{w\in W})$ where all the probabilities p_w are the same.

Similarly, one could consider the logic S5(N) which is the axiomatic extension of S5(FP(Ł)) with the following axiom

$$\Box P(\varphi_1 \wedge \varphi_2) \equiv (\Box P\varphi_1) \wedge (\Box P\varphi_2)$$

It is clear that the S5 probability models for this logic are those whose corresponding lower probabilities are in fact necessity measures.

Finally, let us comment on some specific S5 probability models that is worth to consider.

- Let $\mathcal{U} = (W, R, \{e_w\}_{w\in W}, \{p_w\}_{w\in W})$ be an S5 probability model in which, for all $w, w' \in W$, $\mu_w = \mu_{w'} = \mu$. In this case we have that $\|P(\varphi)\|_{\mathcal{U},w} = \|P(\varphi)\|_{\mathcal{U},w'} = \mu(\{w \in W \mid \|\varphi\|_w = 1\})$; $\|P(\Box\varphi)\|_{\mathcal{U},w} = \mu(\{w \in W \mid \|\Box\varphi\|_w = 1\}) = B(\varphi)$; $\|\Box P(\varphi)\|_{\mathcal{U},w} = \min\{\mu_w(\varphi) \mid w \in W\} = \mu(\varphi)$. Thus, for all φ this type of models validate the formulas

  $$P(\Box\varphi) \to P(\varphi) \text{ and } \Box P(\varphi) \equiv P(\varphi).$$

  Moreover, the values of the formulas $P(\Box\varphi)$, $P(\varphi)$ and $\Box P(\varphi)$ in $\mathcal{U}$ do not depend on the world w.

- Let $\mathcal{U} = (W, R, \{e_w\}_{w\in W}, \{p_w\}_{w\in W})$ be an S5 probability model in which the accessibility relation is trivial in the sense that wRw' iff $w = w'$. In this case, $\|\Box\varphi\|_{\mathcal{U},w} = \|\varphi\|_{\mathcal{U},w}$; $\|P(\Box\varphi)\|_{\mathcal{U},w} = \|P\varphi\|_{\mathcal{U},w} = \|\Box P\varphi\|_{\mathcal{U},w}$. Thus this type of models validate that in every world,

  $$P(\varphi) \equiv P(\Box\varphi) \equiv \Box P(\varphi).$$

- Let $\mathcal{U} = (W, R, \{e_w\}_{w\in W}, \{p_w\}_{w\in W})$ be an S5 probability model in which the accessibility relation is trivial and $\mu_w = \mu_{w'} = \mu$ for all $w, w' \in W$. Then in these models $P(\varphi) \equiv P(\Box\varphi) \equiv \Box P(\varphi)$ and their value in $\mathcal{U}$ does not depend on the world w we are in.

6. Conclusion and future work

The present paper builds upon the recent [8] and [9], where we initiated the present line of research that aims at investigating uniform ways to deal with general uncertainty measures. In particular, here we prove a first completeness theorem that was conjectured in [9]. Although a general completeness theorem with respect to universal S5 probability models is still laking, the logical framework of S5(FP(Ł)) is adequate to handle probability, belief functions and lower probability within the same formal setting. This common ground where to represent different uncertainty theories is interesting for us because in [8] we observed the existence of partial assignments on finite sets of events that are extendable to lower probabilities and that cannot be distinguished from those which are extendable to belief functions. Therefore, as future work it would be interesting to characterize these assignments in logical terms within S5(FP(Ł)).

Since n-monotone capacities are uncertainty measures more general than belief functions and less general than lower probabilities, it is an open question investigate how they can be defined in S5 probability models. On the other hand, instead of the full-fledged S5Ł, one could also consider a non-nested fragment that could be endowed with a simpler semantics, without the need of accessibility relations, similar to what was done in the classical setting in [3] and in the fuzzy setting in [15]. This will be part of our future work.

Interestingly, the class of models presented here might result in the definition of possibly new ways for the uncertain quantification that can be identified first on a syntactic level, e.g. a formula of the form $\Box(P(\Box\varphi))$ might represent a *lower belief function*. This line of research deserves to be explored as well.

Acknowledgments

We are very grateful to the four ISIPTA reviewers whose careful reading and thorough comments helped us clarifying a number of important points. All authors acknowledge the support by the MOSAIC project (EU H2020-MSCA-RISE-2020 Project 101007627). Corsi and Hosni acknowledge the support of by the Department of Philosophy “Piero Martinetti” of the University of Milan under the Project “Departments of Excellence 2023–2027” awarded by the Ministry of Education, University and Research (MIUR). Flaminio and Godo acknowledge the support by the Spanish project PID2019-111544GB-C21.

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References

[1] B. Anger and J. Lembcke. Infinitely subadditive capacities as upper envelopes of measures. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 68(3):403–414, 1985.

[2] Paolo Baldi, Petr Cintula, and Carles Noguera. Classical and fuzzy two-layered modal logics for uncertainty: Translations and proof-theory. *International Journal of Computational Intelligence Systems*, 13(1):988–1001, 2020.

[3] Mohua Banerjee and Didier Dubois. A simple logic for reasoning about incomplete knowledge. *Int. J. Approx. Reason.*, 55(2):639–653, 2014.

[4] Patrick Blackburn, Maarten De Rijke, and Yde Venema. *Modal logic*. Cambridge University Press, 2002.

[5] Diego Castaño, Cecilia Cimadamore, José Patricio Díaz Varela, and Laura Rueda. Completeness for monadic fuzzy logics via functional algebras. *Fuzzy Sets and Systems*, 407:161–174, 2021.

[6] Diego Castaño, Gabriel Savoy, and Patricio Dıaz Varela. Strong standard completeness for S5-modal Łukasiewicz. *Submitted*, 2021.

[7] Chen Chung Chang. A new proof of the completeness of the łukasiewicz axioms. *Transactions of the American Mathematical Society*, 93(1):74–80, 1959.

[8] Esther Anna Corsi, Tommaso Flaminio, and Hykel Hosni. When belief functions and lower probabilities are indistinguishable. In *ISIPTA 2021 - Proceedings of Machine Learning Research (147)*, pages 83–89, 2021.

[9] Esther Anna Corsi, Tommaso Flaminio, and Hykel Hosni. Towards a unified view on logics for uncertainty. In Florence Dupin de Saint-Cyr, Meltem Öztürk-Escoffier, and Nico Potyka, editors, *Scalable Uncertainty Management*, pages 329–337, Cham, 2022. Springer International Publishing. ISBN 978-3-031-18843-5.

[10] Didier Dubois and Henri Prade. Possibility theory: Approach to computerized processing of uncertainty, plennm n. 4. *Plenum Press, New York*, 1988.

[11] Ronald Fagin, Joseph Y Halpern, and Nimrod Megiddo. A logic for reasoning about probabilities. *Information and computation*, 87(1-2):78–128, 1990.

[12] Martina Fedel, Hykel Hosni, and Franco Montagna. A logical characterization of coherence for imprecise probabilities. *International Journal of Approximate Reasoning*, 52(8):1147–1170, 2011.

[13] Tommaso Flaminio and Lluís Godo. A logic for reasoning about the probability of fuzzy events. *Fuzzy Sets and Systems*, 158(6):625–638, 2007.

[14] Tommaso Flaminio, Lluıs Godo, and Enrico Marchioni. Reasoning about uncertainty of fuzzy events: an overview. *Understanding Vagueness-Logical, Philosophical, and Linguistic Perspectives, P. Cintula et al.(Eds.), College Publications*, pages 367–400, 2011.

[15] Tommaso Flaminio, Lluís Godo, and Enrico Marchioni. Logics for belief functions on mv-algebras. *International Journal of Approximate Reasoning*, 54 (4):491–512, 2013.

[16] Robin Giles. Foudations for a theory of possibility. In M.M. Gupta and E. Sanchez, editors, *Fuzzy Information and Decision Processes*, pages 183–195. North-Holland, 1982.

[17] Lluis Godo, Francesc Esteva, and Petr Hájek. Reasoning about probability using fuzzy logic. *Neural Network World*, 10:811–824, 2000.

[18] Lluís Godo, Petr Hájek, and Francesc Esteva. A fuzzy modal logic for belief functions. *Fundamenta Informaticae*, 57(2-4):127–146, 2003.

[19] Petr Hájek. *Metamathematics of fuzzy logic*. Springer Science & Business Media, 1998.

[20] Petr Hájek. On fuzzy modal logics $S5(C)$. *Fuzzy Sets and Systems*, 161(18):2389–2396, 2010.

[21] Petr Hájek, Lluís Godo, and Francesc Esteva. Fuzzy logic and probability. In Philippe Besnard and Steve Hanks, editors, *UAI '95: Proceedings of the Eleventh Annual Conference on Uncertainty in Artificial Intelligence, Montreal, Quebec, Canada, August 18-20, 1995*, pages 237–244. Morgan Kaufmann, 1995.

[22] Joseph Y Halpern. *Reasoning about uncertainty*. MIT press, 2003.

[23] Joseph Y. Halpern and Riccardo Pucella. A logic for reasoning about upper probabilities. *J. Artif. Intell. Res.*, 17:57–81, 2002.

[24] Enrico Marchioni. Uncertainty as a modality over t-norm based logics. In *EUSFLAT Conf. (1)*, pages 169–176, 2007.

[25] Enrico Marchioni. Representing upper probability measures over rational Łukasiewicz logic. *Mathware & Soft Computing*, 15(2):159–173, 2008.

[26] Daniele Mundici. *Advanced Łukasiewicz calculus and MV-algebras*, volume 35. Springer Science & Business Media, 2011.

[27] Glenn Shafer. *A mathematical theory of evidence*, volume 42. Princeton university press, 1976.

[28] C.A.B. Smith. Consistency in statistical inference and decision. *Journal of the Royal Statistical Society. Series B (Methodological)*, 23(1):1–37, 1961.

[29] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Wiley, 1991.